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DIFFERENTIAL FORMS WITH VALUES IN GROUPS

(preliminary report)

by Anders KOCK

Summary. We compare the classical Lie-algebra valued differential *1*- and *2*-forms with a notion of group valued differential form, which occurs very naturally in the context of synthetic differential geometry. The main observation (Theorem 1) is that the expression $d\omega + \frac{1}{2}[\omega, \omega]$, which occurs so often for the Lie algebra valued forms is the classical version of a natural coboundary operator for group-valued *1*-forms.

In the formal manifolds M [4] which occur in synthetic differential geometry, we have around each point $x \in M$ a «*1*-monad» $\mathfrak{M}_1(x) \subset M$ which makes the notion «jet at x» representable: a jet at x (of a map into N, say) is a map $\mathfrak{M}_1(x) \to N$. The elements y of $\mathfrak{M}_1(x)$ are called the *1-neighbours of* x, or just (in the present note) *neighbours*, and we write $x \sim y$.

If G is a group (-object; we work consistently in some topos $\mathcal{E}$ where synthetic differential geometry makes sense), then we consider laws ω which to any pair $x \sim y$ of neighbours associates an element $\omega(x, y)$, with $\omega(x, x) = e$ (neutral element). If G is the (additive) group $(R, +)$ by which one measures the quantity «work», an example of such a law is «the amount of work required to go directly from x to a neighbour point y». If G is the (non-commutative) group of rotations, an example of an ω on a curve M in space is $\omega(x, y) =$ «the rotation which the Frenet frame gets by going from x to a neighbour point y». Similarly for the Darboux frame on a surface.

In both cases, we evidently have

$$(1) \qquad \omega(x, x) = e \qquad \forall x,$$

(2) $$\omega(x,y)\cdot\omega(y,x) = e \qquad \forall x \sim y,$$

and a law ω like this is what we shall call a *differential 1-form with value in* G. (Actually, for all value groups G we shall consider, (2) follows from (1).)

To compare this form notion with the classical notion of linear differential 1-form on M, we note that if M and N are formal manifolds, and $x \in M$, $y \in N$, then there is a natural bijective correspondence between

$$\text{maps } \mathfrak{M}_1(x) \to N \quad \text{taking } x \text{ to } y$$

and

$$R\text{-linear maps } T_x M \to T_y N$$

(see e.g. [4], Remark 6.1). In particular, if we take $N = (R, +)$ (which we now take to mean the number line), and $y = 0$, then we get a bijective correspondence between

(3) $$\text{maps } \mathfrak{M}_1(x) \to R \quad \text{taking } x \text{ to } 0$$

and

(4) $$R\text{-linear maps } T_x M \to T_0 R \approx R.$$

The data (3) is, when we have it for all $x \in M$, a differential form on M, as introduced here ((2) in this case being derivable from (1)), whereas the data (4) is a linear differential 1-form in the classical sense. Similarly, if we take N to be a group G and $y = e$ (the neutral element), we get a bijective correspondence between 1-forms in our sense with values in G, and classical 1-forms on M with values in the Lie algebra $T_e G$ (this is provided G is itself a formal manifold, or if $G = Diff(F)$, the group of all bijective maps $F \to F$ where F is a formal manifold).

There is a similar notion of 2-form θ on M with values in G; this is a law which to any «triangle» x, y, z in M (with $x \sim y$, $y \sim z$, $z \sim x$) associates an element $\theta(x,y,z) \in G$, and which is e if $x = y$ or $x = z$ or $y = z$. (For G commutative, one can similarly consider n-forms for any n, and this has, for the case $G = (R, +)$, been used by Bkouche and Joyal long ago.) There is also a bijective comparison between «our» 2-forms on M with values in G, and classical (bilinear alternating) 2-forms on M with values in $T_e G$, for G a formal manifold or $Diff(F)$ with F a formal manifold.

We shall consistently denote the linear classical version of a form ω (in our sense) by $\bar{\omega}$.

A 0-form on M with values in G is just a map $f: M \to G$.

The main point we want to make is the following: there are natural coboundary operators d from 0-forms to 1-forms and from 1-forms to 2-forms; and that, for 1-forms, the coboundary operator does *not* correspond exactly to the classical coboundary operator on the linearized forms, but rather, there is an «error» term in the latter involving the Lie bracket on $T_e G$:

THEOREM 1. *For ω a G-valued 1-form on M,*

$$(5) \qquad \overline{d\omega} = \bar{d}\bar{\omega} + \frac{1}{2}[\bar{\omega},\bar{\omega}] .$$

Here $\bar{\omega}$ is the linearized $T_e G$-valued version of ω, d and $\bar{d}$ denote respectively the coboundary operator for G-valued 1-forms introduced in (6) below, and the classical coboundary operator for linear forms; finally we remind the reader that, when $\bar{\omega}$ and $\bar{\theta}$ are (linear) 1-forms with values in a Lie algebra, $[\bar{\omega},\bar{\theta}]$ is a 2-form given by

$$[\bar{\omega},\bar{\theta}](u,v) = [\bar{\omega}(u),\bar{\theta}(v)] - [\bar{\omega}(v),\bar{\theta}(u)]$$

(u and v are tangent vectors at the same point of M).

The coboundary operator d has the simplest possible description : for $x \sim y$, $y \sim z$, $z \sim x$, we put

$$(6) \qquad (d\omega)(x,y,z) := \omega(x,y).\omega(y,z).\omega(z,x).$$

The right hand side here is evidently to be thought of as the curve integral of ω around the boundary of the triangle $\sigma = (x,y,z)$; so the formula yields Stokes's Theorem :

$$\int_\sigma d\omega = \int_{\partial\sigma} \omega$$

true by the very definition for such «infinitesimal» triangles σ.

For a 0-form $f: M \to G$, we get a 1-form df by putting, for $x \sim y$:

$$(df)(x,y) := f(x)^{-1}.f(y) ;$$

clearly, $d(df) = 0$ (where 0 denotes the 2-form with constant value e); with evident terminology, we express this: exact 1-forms are closed (for conditions for the converse, see Theorem 3 below).

If G itself is a formal manifold, we may talk about forms on G, with values in G. There is a very canonical 0-form, namely the identity map $i: G \to G$. Its coboundary di we denote Ω. It «is» the Maurer-Cartan form. Its linearized version $\overline{\Omega}$ (which is a $T_e\, G$-valued 1-form on G) is the classical Maurer-Cartan form. Now Ω being exact ($= di$) is closed, $d\Omega = 0$, whence we get:

COROLLARY 2. *The linear Maurer-Cartan form $\overline{\Omega}$ satisfies*

$$\overline{d\Omega} = -\tfrac{1}{2}[\overline{\Omega}, \overline{\Omega}] \,. \tag{7}$$

PROOF. Since $d\Omega = 0$, $\overline{d\Omega} = 0$, so that, substituting Ω for ω in (5) gives the result.

We say that G *admits integration* if, for any $a, b \in R$ with $a \leq b$, and any 1-form ω on $[a, b]$, we have

$$\exists !\ g: [a, b] \to G \quad \text{with} \quad g(a) = e \ \text{and} \ dg = \omega \,.$$

This generalizes (from $G = (R, +)$) the integration axiom of [5]; in the same well-adapted model we considered there, any Lie group satisfies the axiom, whereas $Diff(F)$ does not, in general.

We write

$$\int_a^b \omega \qquad \text{for } g(b) \quad (= g(a)^{-1} \cdot g(b)),$$

where g and ω are related as above, and prove certain standard rules of integration, just like in [5] except we have to be careful with the non-commutativity of G. In particular, for $a \leq b \leq c$,

$$\int_a^b \omega \cdot \int_b^c \omega = \int_a^c \omega \,. \tag{8}$$

Let I denote $[0, 1]$. We have two «piecewise smooth» paths in $I \times I$ from $(0, 0)$ to $(1, 1)$, and when ω is a 1-form on $I \times I$, we get correspondingly two curve integrals of ω along them (each of the two curve integrations being quickly describable as a G-product of two integrals $\int_0^1$).

We can then prove the following weak form of Stokes's Theorem:

LEMMA. *If* ω *is a closed 1-form on* $I \times I$, *then these two curve integrals agree.*

PROOF (sketch). There are two steps. The first is by means of (8) to reduce the question to «infinitesimal rectangles», and this is exactly as in [6]. But there, this was all we had to do, because coboundary of forms was essentially defined so as to make Stokes's Theorem true on such infinitesimal rectangles. In the present paper we used infinitesimal triangles instead, and the second step is therefore reduction from such rectangles to triangles. This is non-trivial, using coordinate calculations (one cannot pave the infinitesimal rectangles with infinitesimal triangles, it seems).

THEOREM 3. *If* M *is a (stably) pathwise connected, simply connected formal manifold, and* G *a group admitting integration, then any closed* G*-valued 1-form* ω *on* M *is exact.*

PROOF (sketch). Choose a point $x \in M$, and define $f(y)$ to be the curve integral of ω from x to y, along any curve $k: I \to M$ from x to y. Independence of choice of curves follows then from the lemma and the existence of a homotopy $I \times I \to M$ between the two given curves. To prove $df = \omega$, one has to use the lemma in conjunction with the fact that for a neighbour z of y, any path from x to y can be deformed by n infinitesimal homotopies to a curve from x to z (which is what we mean by *stably* pathwise connected); here $n = dim(M)$.

Any map $f: M \to N$ between formal manifolds takes monads to monads

$$f(\mathfrak{M}_1(x)) \subset \mathfrak{M}_1(f(x)) \qquad \forall x \in M.$$

Therefore a j-form ω $(j = 0, 1, 2)$ on N immediately gives a j-form $f^*\omega$ on M, and f^* commutes with d. Also $\overline{f^*\omega} = f^*\overline{\omega}$.

Note that, for any $f: M \to G$ (with G a group which is also a formal manifold),

$$df = d(f^*(i)) = f^*(di) = f^*\Omega$$

where $\Omega = di$ is the (non-linear) Maurer-Cartan form on G. Therefore also

$$(9) \qquad \overline{df} = f^*\overline{\Omega},$$

where $\overline{\Omega}$ is the $T_e G$-valued linear Maurer-Cartan form.

With M and G as in Theorem 3 (and G further a formal manifold) we have then

COROLLARY 4. *Let $\overline{\omega}$ be a (linear) $T_e G$-valued 1-form on M. Then if*

$$(10) \qquad d\overline{\omega} = -\frac{1}{2}[\overline{\omega}, \overline{\omega}],$$

we can find an $f: M \to G$ with $f^*\overline{\Omega} = \overline{\omega}$.

PROOF. Let ω be the G-valued 1-form on M corresponding to $\overline{\omega}$. Then by Theorem 1, (10) means that $\overline{d\omega} = 0$, hence $d\omega = 0$. By Theorem 3, therefore ω is exact, $\omega = df$ for some $f: M \to G$, so

$$\overline{\omega} = \overline{df} = f^*\overline{\Omega},$$

by (9).

This, of course, is a classical theorem, cf. e.g. [2]. An f with $df = \omega$ (i.e. $f^*\overline{\Omega} = \overline{\omega}$) can also in our context be proved unique up to left multiplication by a constant form g, due to

THEOREM 5. *If $f_1, f_2: M \to G$ have $df_1 = df_2$, then $f_1 = g.f_2$ for some unique $g \in M$.*

This follows easily from the uniqueness assertion in the integration axiom.

Let us finally consider a case where one naturally encounters G-valued 1-forms with value in a «big» group $Diff(F)$. Let $\mathfrak{D}$ be a distribution on $M \times F$ transversal to the fibres of $proj: M \times F \to M$. This means for each $(x,u) \in M \times F$, we have given a subset

$$\mathfrak{D}(x,u) \subset \mathfrak{M}_1(x,u) \text{ with } (x,u) \in \mathfrak{D}(x,u),$$

such that $\mathfrak{D}(x,y)$ by $proj$ maps bijectively onto $\mathfrak{M}_1(x)$. If $y \sim x$, there is a unique u' such that $(y,u') \in \mathfrak{D}(x,u)$, and to $x \sim y$ we associate the map $\omega(x,y): u \mapsto u'$ which is a bijective map $F \to F$. Then ω is a $Diff(F)$-valued 1-form on M. It is closely connected to the «déplacement infinitésimal» of [1] page 40. We can prove that if $\mathfrak{D}$ is a Frobenius distribution

(the commutator of two vectorfields subordinate to $\mathcal{D}$ is again in $\mathcal{D}$), then ω is closed. Because *Diff(F)* does not admit integration, we cannot assert ω exact (which would imply that we have *global* solutions of the problem $\mathcal{D}$).

In the classical case, there is a sense in which *Diff(F)* does admit curve integration if F is compact: this is precisely the statement that $\mathcal{D}$ is a «connexion infinitésimale» in $M \times F \to M$; cf. [1], Définition and Proposition on page 36.

The linearized version of ω is the differential *1*-form («the connection form of $\mathcal{D}$») with values in the (big) Lie algebra of all vector fields on F, as considered in [3].

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1. C. EHRESMANN, Les connexions infinitésimales dans un espace fibré différentiable, *Colloque Topologie Bruxelles 1950*, CBRM, 29-55.

2. P. GRIFFITHS, On Cartan's method of Lie groups and moving frames..., *Duke Math. J.* 41 (1974), 775-814.

3. A. HAEFLIGER, Cohomology of Lie algebras and foliations, in Proc. Rio de Janeiro Conf. (ed. P. A. Schweitzer), *Lecture Notes in Math.* 652, Springer (1976).

4. A. KOCK, Formal manifolds and synthetic theory of jet bundles, *Cahiers Topo. et Géom. Diff.* XXI-3 (1980), 227-241.

5. A. KOCK & G. E. REYES, Models for synthetic integration theory, *Math. Scand.* 48 (1981).

6. A. KOCK, G.E. REYES & B. VEIT, Forms and integration in synthetic differential geometry, *Aarhus Preprint Series* 1979/80, n° 31.

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