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LOCALLY INJECTIVE G-SHEAVES OF ABELIAN GROUPS

by Roswitha HARTING

The problem of the existence of enough injective abelian group objects in an elementary topos with a natural number object leads to the construction of the internal (parametrized) coproduct of abelian group objects [4]. From certain properties of this parametrized coproduct we earlier derived some further consequences [5], among them the surprising result that «all internal notions of injectivity for abelian group objects are equivalent».

In the following summary we shall apply this result to $Shv(X)^{G^{op}}$, the topos of set-valued sheaves on a topological space X with a left action of a group-valued sheaf G.

We require the following results and definitions [4, 5] (where $\mathfrak{S}$ denotes an elementary topos with natural number object and $Ab(\mathfrak{S})$ the category of abelian group objects in $\mathfrak{S}$).

(0.1) THEOREM AND DEFINITION. *For any object X in $\mathfrak{S}$ the functor $X^*: Ab(\mathfrak{S}) \to Ab(\mathfrak{S}/X)$ has a left adjoint $\oplus_X: Ab(\mathfrak{S}/X) \to Ab(\mathfrak{S})$ which respects monomorphisms and is faithful.*

For $A(x) \in Ob\, Ab(\mathfrak{S}/X)$ the abelian group object $\oplus_X A(x)$ in $\mathfrak{S}$ is called parametrized coproduct *of $A(x)$.* (We use «parametrized» to emphasize that the indexing object is in general not just a set but for example a set with an action of a group on it.)

A consequence of this theorem is the following proposition [5]:

(0.2) PROPOSITION. *If $Ab(\mathfrak{S})$ has enough injectives, then so does $Ab(\mathfrak{S}^{\underline{A}})$ for any internal category $\underline{A}$ in $\mathfrak{S}$.*

In the following the internal Hom-functor $Ab(\mathfrak{S})^{op} \times Ab(\mathfrak{S}) \to Ab(\mathfrak{S})$

is denoted by $Hom(-,-)$. For A, B abelian group objects in $\mathcal{E}$, $Hom(A, B)$ is the abelian group object in $\mathcal{E}$ that internalizes the abelian group of group-morphisms from A to B.

(0.3) DEFINITION. *An abelian group object B in $\mathcal{E}$ is called*

(i) internally injective *if for every monomorphism $A \rightarrowtail C$ in $Ab(\mathcal{E})$, $Hom(C, B) \twoheadrightarrow Hom(A, B)$ is an epimorphism in $\mathcal{E}$.*

(ii) locally injective *if for every diagram of the form*

$$\begin{tikzcd} A \arrow[r, tail, "m"] \arrow[d, "f"'] & C \\ B & \end{tikzcd}$$

in $Ab(\mathcal{E})$ there exists a cover $U \twoheadrightarrow 1$ of $\mathcal{E}$ and a morphism $g: U^*C \to U^*B$ in $Ab(\mathcal{E}/U)$ such that

$$\begin{tikzcd} U^*A \arrow[r, "U^*m"] \arrow[d, "U^*f"'] & U^*C \arrow[dl, "g"] \\ U^*B & \end{tikzcd}$$

commutes.

(0.4) PROPOSITION. *The following conditions on an abelian group object B in $\mathcal{E}$ are equivalent (for (iii) we suppose that $Ab(\mathcal{E})$ has enough injectives):*

(i) B is locally injective.

(ii) For every diagram of the form

$$\begin{tikzcd} A \arrow[r, tail, "m"] \arrow[d, "f"'] & C \\ B & \end{tikzcd}$$

in $Ab(\mathcal{E})$ there exist a cover $U \twoheadrightarrow 1$ of $\mathcal{E}$ and a morphism

$$g: U \to Hom(C, B) \text{ in } \mathcal{E}$$

such that the diagram

$$\begin{tikzcd} Hom(C,B) \arrow[rr, "Hom(m,B)"] & & Hom(A,B) \\ U \arrow[u, "g"] \arrow[rr, two heads] & & 1 \arrow[u, "\ulcorner f \urcorner"'] \end{tikzcd}$$

commutes in $\mathcal{E}$.

(iii) There exists a cover $U \twoheadrightarrow 1$ *of* $\mathcal{E}$ *such that* $\eta_B : B \rightarrowtail B^U$ *is an injective effacement* [2]. *Here* $\eta : id\, Ab(\mathcal{E}) \to \pi_U . U^*$ *denotes the unit of the adjunction* $U^* \dashv \pi_U$, *and the abelian group object structure of* B^U *is induced by that of* B.

(A monomorphism $m : A \rightarrowtail C$ in $Ab(\mathcal{E})$ is an injective effacement iff for every monomorphism $f : A \rightarrowtail B$ in $Ab(\mathcal{E})$ there exists a morphism $g : B \to C$ in $Ab(\mathcal{E})$ such that $g.f = m$.)

(0.5) LEMMA. *An abelian group object* B *in* $\mathcal{E}$ *is internally injective iff for every monomorphism* $m : A \rightarrowtail C$ *in* $Ab(\mathcal{E})$ *and every generalized element* $f : V \to Hom(A,B)$ *in* $\mathcal{E}$ *there exist an object* U, *an epimorphism* $h : U \twoheadrightarrow V$ *and a morphism* $g : U \to Hom(C,B)$ *in* $\mathcal{E}$ *such that*

$$\begin{tikzcd} Hom(C,B) \arrow[rr, "Hom(m,B)"] & & Hom(A,B) \\ U \arrow[u, "g"] \arrow[rr, two heads, "h"] & & V \arrow[u, "f"'] \end{tikzcd}$$

commutes.

(0.6) THEOREM. *An abelian group object* B *in* $\mathcal{E}$ *is locally injective iff* B *is internally injective.*

In the following we shall study the meaning of this result in the topos $\mathcal{E} = Shv(X)^{G^{op}}$, where X denotes a topological space (resp. a locale [7, 9]) and G a group-valued sheaf on X. Then $Ab(Shv(X)^{G^{op}})$ is the category of abelian group-valued sheaves on X equipped with a left action of G compatible with the abelian group structure. So $Ab(Shv(X)^{G^{op}})$ is the category of G-modules on X, and will be denoted from now on by $G\text{-}Mod(X)$.

(1.1) SOME REMARKS.

(i) The following diagram of forgetful functors commutes:

$$\begin{tikzcd} G\text{-}Mod(X) \arrow[r, "V''"] \arrow[d, "V"'] & Shv(X)^{G^{op}} \arrow[d, "\bar{V}"] \\ Ab(Shv(X)) \arrow[r, "V'"] & Shv(X) \end{tikzcd}$$

All these forgetful functors create epimorphisms and monomorphisms, they all have a left adjoint, and they respect injectives [10].

(ii) In $Ab(Shv(X))$ the notions of injectivity and internal injectivity coincide [5]. For A, B in $Ab(Shv(X))$ the internal Hom is obtained as follows: for U open in X, $Hom(A, B)(U)$ is defined to be

$$Hom_{Ab(Shv(U))}(A/U, B/U).$$

(iii) G-$Mod(X)$ has enough injectives. (This follows immediately from (0.2).)

(iv) In G-$Mod(X)$ the internal Hom is obtained as follows: For A, B G-modules on X, $Hom(A, B)$ is defined to be

$$Hom(VA, VB) \quad \text{in} \quad Ab(Shv(X))$$

equipped with the following action of G : for U open in X,

$$\begin{aligned} GU \times Hom_{AbShv(U)}(VA/U, VB/U) &\to Hom_{AbShv(U)}(VA/U, VB/U) \\ (s, h) &\longmapsto s \circ h \end{aligned}$$

is defined by:

$$(s \circ h)_W(x) := (s/W).h_W((s^{-1}/W).x),$$

where $W \subset U$, W open in X and $x \in AW$.

(1.2) PROPOSITION. *Let B be a G-module on X.*

(i) B is internally injective iff VB is injective in $Ab(Shv(X))$.

(ii) If B is internally injective, then B^G is injective.

PROOF. (i): Suppose B to be internally injective. To show that B is injective in $Ab(Shv(X))$, it is sufficient to show that B is internally injective in $Ab(Shv(X))$ (cf. (1.1)(ii)). So let $m: A \rightarrowtail C$ be a monomorphism in $Ab(Shv(X))$. Let G operate trivially on A and C ; then $m: A \rightarrowtail C$ becomes a monomorphism in $G\text{-}Mod(X)$. Since B is internally injective it follows that $Hom(C,B) \twoheadrightarrow Hom(A,B)$ is an epimorphism in $Shv(X)^{G^{op}}$, and hence an epimorphism in $Shv(X)$ (cf. (1.1)(i) and (iv)). So B is injective in $Ab(Shv(X))$.

The other implication is equally easy.

(ii): Let

$$\begin{tikzcd} A \arrow[r, tail, "m"] \arrow[d, "f"'] & C \\ B^G & \end{tikzcd}$$

be a diagram in $G\text{-}Mod(X)$ and suppose B to be internally injective. We have a sequence of natural bijections between the following sets:

$$\begin{array}{ll}
A \to B^G \text{ in } G\text{-}Mod(X) & \\
\hline
G \to V''(Hom(A,B)) \text{ in } Shv(X)^{G^{op}} & \\
\hline
1 \to \bar{V}V''(Hom(A,B)) \text{ in } Shv(X) & \bar{F} \dashv \bar{V},\ \bar{F}(1) = G \\
\hline
1 \to V'(Hom(VA,VB)) \text{ in } Shv(X) & \bar{V}V'' = V'V,\ (1.1)(iv) \\
\hline
VA \to VB \text{ in } Ab(Shv(X)). &
\end{array}$$

So $f: A \to B^G$ determines, and is determined by, a morphism $\bar{f}: VA \to VB$ in $Ab(Shv(X))$. B is supposed to be internally injective, so, by (i), VB is injective in $Ab(Shv(X))$. Hence there is a morphism h in $Ab(Shv(X))$ such that

$$\begin{tikzcd} VA \arrow[r, tail, "Vm"] \arrow[d, "\bar{f}"'] & VC \arrow[dl, "h"] \\ VB & \end{tikzcd}$$

commutes. As above h determines a morphism $\hat{h}: C \to B^G$ in $G\text{-}Mod(X)$, and it is easy to verify that $\hat{h}m = f$. So B^G is injective in $G\text{-}Mod(X)$.

(1.3) REMARK. Let ΔZ be the ring-valued sheaf on X associated to the constant presheaf with value Z. Then $U \mapsto \Delta Z(U)[GU]$ defines an abelian group-valued presheaf with the usual left action of G, where we denote by $\Delta Z(U)[GU]$ the group-ring over GU. The associated sheaf is a G-module on X and is denoted by $Z[G]$. Some obvious calculations show that there is a natural isomorphism

$$Hom(Z[G], A) \approx A^G .$$

In the following the composite

$$A \xrightarrow{\eta_A} A^G \xrightarrow{\approx} Hom(Z[G], A)$$

is again denoted by η_A (cf. (0.4) (iii)).

(1.4) PROPOSITION. *Let B be a G-module on X. The following conditions on B are equivalent:*

(i) $Hom(Z[G], B)$ is an injective G-module on X.

(ii) There exists an epimorphism $D \twoheadrightarrow 1$ in $Shv(X)^{G^{op}}$ such that B^D is an injective G-module on X.

(iii) $\eta_B : B \rightarrowtail Hom(Z[G], B)$ is an injective effacement in G-$Mod(X)$.

(iv) There exists an epimorphism $D \twoheadrightarrow 1$ in $Shv(X)^{G^{op}}$ such that $\eta_B : B \rightarrowtail B^D$ is an injective effacement in G-$Mod(X)$.

(v) B is injective in $Ab(Shv(X))$.

(vi) B is internally injective in G-$Mod(X)$.

(vii) There exists an open cover of X, $X = \bigcup_{i \in I} U_i$, such that B/U_i is injective in $Ab(Shv(U_i))$ for every $i \in I$.

PROOF. $(iii) \xRightarrow[(1.3)]{} (iv) \xRightarrow[(1.1)(iii),\ (0.4)(iii),\ (0.6)]{} (vi) \xRightarrow[(1.2)(ii)]{} (i) \Longrightarrow (iii)$.

$$(ii) \Longrightarrow (iv) \xRightarrow[\text{above}]{} (i) \xRightarrow[(1.3)]{} (ii).$$

$$(vi) \xLeftrightarrow[(1.2)(i)]{} (v) \xLeftrightarrow[[6]]{} (vii).$$

So they are all equivalent.

(1.5) REMARK. If X is reduced to a point, then $Shv(X)^{G^{op}}$ is the topos of G-sets, $Ab(Shv(X)^{G^{op}})$ is the category of left $Z[G]$-modules, and $Ab(Shv(X))$ is the category of abelian groups. For two $Z[G]$-modules A, B the internal Hom, $Hom(A,B)$, is the abelian group of all Z-linear maps from A to B equipped with the following G-action:

$$(s \circ f)(x) = sf(s^{-1}x) \quad \text{for} \quad s \in G,\ f \in Hom_Z(A,B),\ x \in A.$$

$\eta_B\colon B \rightarrowtail Hom(Z[G],B)$ is here defined by

$$\eta_B(b)(s) = b \quad \text{for every} \quad s \in G.$$

We remark that in the above case the Proposition (1.4) remains true if G is replaced by a monoid M and consequently $Hom(Z[G],B)$ by B^M (cf. [3]).

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