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LOCALIC GROUPS

by Gavin C. WRAITH

Perhaps the most significant recent development in topos theory since the publication of P.T. Johnstone's book [2] has been the understanding of the importance of locales. Charles Ehresmann [1] was the first to stress the localic aspect of topology, that open sets are more fundamental than points.

Recall that a locale is a complete lattice in which finite infima distribute over arbitrary suprema. In symbols, for any element x and subset S of the locale, we have the rule

$$x \wedge \bigvee S = \bigvee \{ x \wedge y \mid y \in S \}.$$

A map f of locales from L to M is defined to be a function f^* from M to L (note the reversal of direction) preserving finite infima and arbitrary suprema. We include among these the maximal element $\top$ and the minimal element $\perp$.

The lattice $O(X)$ of open subsets of a topological space X is a locale. O is evidently a functor. On the full subcategory of sober spaces and continuous maps it is full and faithful, so it is convenient to identify a sober space with its locale of open sets. The notion of locale is straightforward to formulate in a topos, and it is now well established that the localic approach is the right way to do topology in a topos [4].

To set out notation, let us denote by $Loc(\mathcal{E})$ the category of locales in the topos $\mathcal{E}$. The subobject classifier $\Omega_{\mathcal{E}}$ of $\mathcal{E}$ has a canonical locale structure, and is terminal in $Loc(\mathcal{E})$. For the map of locales

$$L \xrightarrow{h} \Omega_{\mathcal{E}}$$

we denote $h^*(u)$ by $[\![u]\!]$ so that, for example

$$[\![\mathit{true}]\!] = \top, \quad [\![\mathit{false}]\!] = \perp .$$

Direct image functors of geometric morphisms preserve locale structure [7]. It follows that for any geometric morphism

$$\mathcal{F} \xrightarrow{v} \mathcal{E}$$

we have a locale $v_*(\Omega_{\mathcal{F}})$ in $\mathcal{E}$. In fact, all locales in $\mathcal{E}$ are of this form [3], and we have an equivalence between $Loc(\mathcal{E})$ and the category of localic $\mathcal{E}$-toposes.

We shall write

$$Points(L) \quad \text{for} \quad Hom_{Loc(\mathcal{E})}(\Omega_{\mathcal{E}}, L),$$

so that *Points* is a functor from $Loc(\mathcal{E})$ to $\mathcal{E}$. It is right adjoint to the functor which takes an object A of $\mathcal{E}$ to the discrete locale $P(A)$. An open sublocale of a discrete locale is discrete, but in general a discrete locale may have sublocales that are neither open nor discrete.

We say that a locale L *has enough points* if the end adjunction

$$P(Points(L)) \xrightarrow{\epsilon} L$$

is surjective, i. e. if ϵ^* is faithful. In *Sets* a locale is spatial, i. e. is of the form $O(X)$ for some topological space X, if and only if it has enough points. Locales without enough points arise naturally. If $f: Y \to X$ is a continuous map between sober topological spaces, we obtain a locale $f_*(\Omega_Y)$ in $shv(X)$ which determines it completely. $Points(f_*(\Omega_Y))$ is the sheaf of sections of f, so $f_*(\Omega_Y)$ has enough points only if f has enough local sections. It should be clear that $f_*(\Omega_Y)$ is discrete only if f is a local homeomorphism.

The point that I want to make in this article is that certain mathematical structures often carry a natural topology - we shall see some examples below - which it is easy to overlook. For most purposes it may be harmless to do so. However, when it comes to formulating these structures in a topos, it is vital to take account of the topology, so that one gets localic structures. To forget the topology is tantamount to applying the *Points* functor, which sometimes throws the baby out with the bathwater in a manner which I hope the examples below will make clear.

If A and B are sets, we can think of them as discrete topological spaces and we can form the product space $\prod_A B$ whose points correspond to functions from A to B. If A is infinite, this space is not discrete. Analogously, if A and B are objects of a topos $\mathcal{E}$ we can form the locale $Map(A, B)$ in $\mathcal{E}$ by taking an $A \times B$-indexed family of generators

$$< a \mapsto b >, \quad a \in A, \ b \in B$$

satisfying the relations

$$< a \mapsto b > \wedge < a \mapsto b' > \leq [\![b = b']\!], \qquad \bigvee_b < a \mapsto b > = \mathrm{T}.$$

It should be clear that we have

$$Points(Map(A, B)) = B^A$$

and that in $Sets$ we have $Map(A, B) = \prod_A B$ with the product topology (note that we have suppressed the use of O).

By adding the further relations

$$< a \mapsto b > \wedge < a' \mapsto b > \leq [\![a = a']\!]$$

we get a locale $Inj(A, B)$, or by adding the further relations

$$\bigvee_a < a \mapsto b > = \mathrm{T}$$

we get a locale $Surj(A, B)$, or by adding both we get a local $Bij(A, B)$. It should be clear what their points are.

In the case $\mathcal{E} = Sets$, a completeness theorem of Makkai and Reyes [6] says that if B is finite or countable, then $Surj(\mathrm{N}, B)$ is spatial. However, if B is uncountable $Surj(\mathrm{N}, B)$ has no points, but is nonetheless a nontrivial locale.

We define $Perm(A)$ to be $Bij(A, A)$; it is clearly a localic group, i. e. a group object in $Loc(\mathcal{E})$. Let us describe its structure in slightly more detail. It has an identity

$$\Omega_{\mathcal{E}} \xrightarrow{e} Perm(A) \quad \text{given by} \quad e^* < a \mapsto a' > = [\![a = a']\!].$$

It has a multiplication

$$Perm(A) \times Perm(A) \xrightarrow{m} Perm(A)$$

given by

$$m^* < a \mapsto a'' > = \bigvee_{a'} p_1^* < a \mapsto a' > \wedge \ p_2^* < a' \mapsto a'' >$$

where p_1, p_2 are the projections from the product $Perm(A) \times Perm(A)$. The inverse map is clearly given by

$$< a \mapsto a' > \mapsto < a' \mapsto a > .$$

Let us consider localic groups acting continuously on objects. If A is an object of $\mathcal{E}$ and G is a localic group, this means that we have an action

$$G \times P(A) \xrightarrow{\alpha} P(A).$$

This is exponentially adjoint in $Loc(\mathcal{E})$ to a homomorphism of localic groups $\tilde{\alpha} : G \to Perm(A)$ and $\tilde{\alpha}^*$ is completely determined by its values on the generators $< a \mapsto a' >$ of $Perm(A)$. Thus α is completely determined by the map $A \times A \to G$ in $\mathcal{E}$ given by

$$(a, a') \mapsto \tilde{\alpha}^* < a \mapsto a' > .$$

The fixed point subobject of A under the G-action α is given by

$$Fix(G, A) = \{ a \in A \mid \tilde{\alpha}^* < a \mapsto a > = \top \}.$$

Dually,

$$\neg [\![\tilde{\alpha}^* < a \mapsto a' > = \bot]\!]$$

is an equivalence relation on A, whose classes we call the *orbits of the action.*

$Perm(A)$ has a canonical action on A, and since

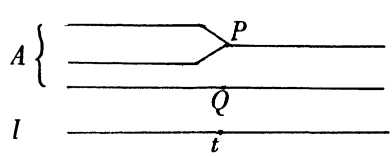
$$[\![< a \mapsto a' > = \bot]\!] = \mathit{false}$$

we deduce that every element of A belongs to the same orbit under $Perm(A)$. In other words, $Perm(A)$ acts transitively on A, as indeed it should.

By applying the functor *Points* we get from a continuous G-action on A an action of $Points(G)$ on A. It is *not* the case that

$$Aut(A) = Points(Perm(A))$$

acts transitively on A. To see this, consider the case $\mathcal{E} = shv(I)$, the category of sheaves on the unit interval, and take A to be the sheaf pictured below :



Suppose that P and Q lie in the stalk A_t at $t \in I$ and that P is a bifurcation point and that Q is not. Clearly, there is no open neighborhood U of t for which there is an automorphism A/U interchanging P and Q.

In general $Fix(G, A)$ is contained in $Fix(Points(G), A)$, but these two subobjects of A do not necessarily coincide. A convenient example, due originally to M. Fourman, and used by Kennison [5] and the author [8], concerns Galois groups.

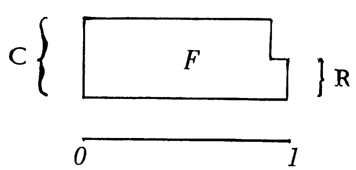
The fundamental theorem of Galois theory states the following: let $K \subset L$ be an algebraic separable normal extension of fields, and let $G(L/K)$ be the group of K-automorphisms of L with the Krull topology. Then there is a bijection between the intermediate extensions $K \subset F \subset L$ and the closed subgroups $H \subset G(L/K)$ given by

$$H \mapsto Fix(H), \quad F \mapsto G(L/F).$$

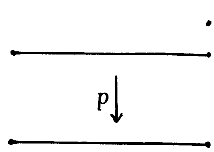
Provided we interpret $G(L/K)$ as the localic group of K-automorphisms of L, this results holds in a topos [8]. Consider the case when $\mathcal{E} = shv(I)$ again, and where $K \subset L$ is the inclusion of constant sheaves on I induced by the standard inclusion $\mathbb{R} \subset \mathbb{C}$. It is no surprise to find that $G(L/K)$ is then the discrete localic group $P(\mathbb{Z}_2)$. Let F be the intermediate sheaf of fields given by

$$F_t = \begin{cases} \mathbb{C} & 0 \leq t < 1 \\ \mathbb{R} & t = 1 \end{cases} .$$

In pictures:



Then $G(L/F)$ is $p_*(\Omega_Y)$ where $p: Y \to I$ is the group object in topological spaces over I with trivial fibre over $t < 1$ and fibre $\mathbb{Z}_2$ over 1. In pictures



It is clear that p has no non-zero local sections, so that

$$Aut_F(L) = Points(G(L/F)) = 1.$$

This example demonstrates two things:

a) that Galois theory does not work in a topos if one looks at the group of automorphisms objects, instead of the localic group of automorphisms,

b) that the fixed point subobject of L under $G(L/F)$ is not the same as that under $Points(G(L/F))$.

Of course this is a phenomenon which does not manifest itself with topological groups in *Sets*. If one has a topological group acting continuously on a set, the fixed point subset is clearly unaltered if we happen to forget the topology. As we have seen this circumstance is misleading when we look at the analogous situation in a topos.

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