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DIFFERENTIAL ALGEBRA IN A TOPOS

by William F. KEIGHER

This paper is dedicated to the memory of Charles Ehresmann, whose contributions to categorical algebra and its applications were an inspiration to me.

It has been shown that in the topos *Sets* certain results in commutative algebra generalize easily to differential algebra, for example, results concerning rings of fractions, sheaves of rings, local ringed spaces and the prime spectrum of a ring [2]. However, there are other standard results in commutative algebra which do not generalize to differential algebra. For example, the radical of a differential ring may fail to be a differential ideal, and the differential ring of the differential affine scheme of a differential ring A may not be isomorphic to A. In this paper we seek to consider this situation in a topos $\mathcal{E}$ having a natural number object N. In the following we make strong use of N and its properties.

Let *Ann* denote the category of commutative rings in $\mathcal{E}$, and *DAnn* the category of differential rings in $\mathcal{E}$. A differential ring in $\mathcal{E}$ is a pair (A, d), where $A = (A, m, e, +, o)$ is a commutative ring in $\mathcal{E}$ (cf. [4]) and $d: A \to A$ is a derivation on A, i.e., the diagrams

$$\begin{tikzcd} A \times A \arrow[r, "+"] \arrow[d, "d \times d"'] & A \arrow[d, "d"] \\ A \times A \arrow[r, "+"] & A \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} A \times A \arrow[rr, "m"] \arrow[d, "\Delta"'] & & A \arrow[dd, "d"] \\ A \times A \times A \times A \arrow[d, "d \times 1 \times 1 \times d"'] & & \\ A \times A \times A \times A \arrow[r, "m \times m"] & A \times A \arrow[r, "+"] & A \end{tikzcd}$$

commute. A differential ring homomorphism in $\mathcal{E}$ is a ring homomorphism in $\mathcal{E}$ which commutes with the derivations. For example, a differential ring in *Sets* is an ordinary differential ring, and a differential ring in $Shv(X)$ for a topological space X is a sheaf of differential rings on X. Letting U denote the obvious forgetful functor $DAnn \to Ann$, we have the following

PROPOSITION 1. *U has a right adjoint C, and U is comonadic.*

PROOF. Define, for any commutative ring $\mathrm{A} = (A, m, e, +, o)$ in $\mathcal{E}$,

$$C\mathrm{A} = (G\mathrm{A}, \partial_{\mathrm{A}}), \quad \text{where} \quad G\mathrm{A} = (A^N, m', e', +', o').$$

Here $e': 1 \to A^N$ is the exponential transpose of the map

$$1 \times N \xrightarrow{\approx} N \xrightarrow{\approx} 1 \amalg N \xrightarrow{1 \amalg N} 1 \amalg 1 \xrightarrow{\binom{e}{o}} A,$$

$+': A^N \times A^N \to A^N$ is given directly by

$$A^N \times A^N \xrightarrow{\approx} (A \times A)^N \xrightarrow{+^N} A^N,$$

$o': 1 \to A^N$ is the exponential transpose of the map

$$1 \times N \xrightarrow{\pi_1} 1 \xrightarrow{o} A,$$

and $m': A^N \times A^N \to A^N$ is the exponential transpose of a map

$$\mu: A^N \times A^N \times N \to A$$

which, in the case $\mathcal{E} = Sets$, is given by the formula

$$\mu((a_n),(b_n),n) = \sum_{k=0}^{n} \left[{n \atop k} \right] a_k\, b_{n-k} \,.$$

In the general case, μ is defined using recursion and several natural maps together with both maps m and $+$ from A. Finally, $\partial_{\mathrm{A}}: A^N \to A^N$ is given by $\partial_{\mathrm{A}} = A^s$. The verification that $G\mathrm{A}$ is a commutative ring and that ∂_{A} is a derivation on $G\mathrm{A}$ proceeds in a straightforward fashion similar to the case $\mathcal{E} = Sets$ in [3]. If we define the adjunction transformations

$$\epsilon: UC \to Ann \quad \text{and} \quad \eta: DAnn \to CU$$

so that $\epsilon_{\mathrm{A}}: UC\mathrm{A} \to \mathrm{A}$ is given by the map

$$A^N \xrightarrow{A^o} A^1 \xrightarrow{\approx} A$$

and $\eta_{(\mathrm{A},d)}: (\mathrm{A}, d) \to CU(\mathrm{A}, d)$ is given by the exponential transpose of the map

$$A \times N \xrightarrow{\approx} N \times A \xrightarrow{t} A$$

where $\bar{t} : N \to A^A$ is the unique morphism making the diagram

$$\begin{tikzcd} 1 \arrow[r, "o"] \arrow[dr, "\bar{1}_A"'] & N \arrow[r, "s"] \arrow[d, "\bar{t}"] & N \arrow[d, "\bar{t}"] \\ & A^A \arrow[r, "d^A"] & A^A \end{tikzcd}$$

commute, then it is easy to verify that C is right adjoint to U. If G = (G, ϵ, δ) denotes the comonad on *Ann* induced by this adjunction, then the dual of Beck's Theorem [1, page 3] shows that the cocomparison functor $\Phi: DAnn \to Ann_{\mathrm{G}}$ is an equivalence of categories, i. e., U is comonadic. (Note that the counit $\delta_{\mathrm{A}} : G\mathrm{A} \to GG\mathrm{A}$ of the comonad G is given by the map

$$A^N \xrightarrow{A^+} A^{N \times N} \xrightarrow{\approx} (A^N)^N .)$$

Now let *LAnn* denote the subcategory of *Ann* consisting of local rings and local ring homomorphisms as in [4]. Defining a differential local ring to be a differential ring (A, d) such that A is local and taking differential local ring homomorphisms to be the obvious ones, we obtain the category *DLAnn*. The next result says that the comonad G of Proposition 1 restricts nicely to *LAnn*.

PROPOSITION 2. *If* A *is a local ring, then so is* $G\mathrm{A}$, *and* $\epsilon_{\mathrm{A}} : G\mathrm{A} \to \mathrm{A}$ *and* $\delta_{\mathrm{A}} : G\mathrm{A} \to GG\mathrm{A}$ *are local ring homomorphisms, as is any G-costructure* $a : \mathrm{A} \to G\mathrm{A}$ *on* A. *If* $f: \mathrm{A} \to \mathrm{B}$ *is a local ring homomorphism, so is* $Gf: G\mathrm{A} \to G\mathrm{B}$.

PROOF. First observe that if B is a local ring and $f: \mathrm{A} \to \mathrm{B}$ is a ring homomorphism such that the diagram

$$\begin{tikzcd} Z(A) \arrow[r] \arrow[d, tail] & Z(B) \arrow[d, tail] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

is a pullback, where $Z(A)$ denotes the subobject of units of A as defined in [4], then A is a local ring and f is local as well. Furthermore, if A, B and C are local and $f\colon A \to B$ and $g\colon B \to C$ are ring homomorphisms such that g is local, then gf is local if and only if f is local. So we are done if we can show that the diagram

$$\begin{tikzcd}
Z(A^N) \arrow[r] \arrow[d, tail] & Z(A) \arrow[d, tail] \\
A^N \arrow[r, "\epsilon_A"] & A
\end{tikzcd}$$

is a pullback. But consider the diagram

$$\begin{tikzcd}
I(A^N) \arrow[rr] \arrow[dr, "h"] \arrow[dd, tail] & & 1 \arrow[dr] \arrow[dd, "e'" near start] & \\
& I(A) \arrow[rr, crossing over] & & 1 \arrow[dd, "e"] \\
A^N \times A^N \arrow[rr, "m'" near start] \arrow[dr, "\epsilon_A \times \epsilon_A"'] & & A^N \arrow[dr, "\epsilon_A"] & \\
& A \times A \arrow[rr, "m"'] \arrow[from=uu, tail, crossing over] & & A
\end{tikzcd}$$

where h exists since the front square is a pullback. It follows that the left hand square is a pullback. Hence in the diagram

$$\begin{tikzcd}
I(A^N) \arrow[rr, "\approx"] \arrow[dr, "h"] \arrow[dd, tail] & & Z(A^N) \arrow[dr, "h'"] \arrow[dd, tail] & \\
& I(A) \arrow[rr, "\approx" near start, crossing over] & & Z(A) \arrow[dd, tail] \\
A^N \times A^N \arrow[rr, "\pi_2" near start] \arrow[dr, "\epsilon_A \times \epsilon_A"'] & & A^N \arrow[dr, "\epsilon_A"] & \\
& A \times A \arrow[rr, "\pi_2"'] \arrow[from=uu, tail, crossing over] & & A
\end{tikzcd}$$

where h' is defined by h and the isomorphisms, the right hand square is a pullback also.

COROLLARY. *The forgetful functor $U'\colon DLAnn \to LAnn$ has a right adjoint C', and U' is comonadic.*

Extending the definitions given in [4] to the differential case, we define the categories *DR-top* and *DLR-top* of differential ringed toposes and differential local ringed toposes, respectively. An object in *DR-top* is a pair $(\mathcal{E}, A)$ where $\mathcal{E}$ is a topos and A is a differential ring in $\mathcal{E}$, and a morphism $(\mathcal{E}', A') \to (\mathcal{E}, A)$ is a pair (f, ϕ) where $f: \mathcal{E}' \to \mathcal{E}$ is a geometric morphism and $\phi: f^*A \to A'$ is a differential ring homomorphism in $\mathcal{E}'$. If $(\mathcal{E}', A')$ and $(\mathcal{E}, A)$ are differential local ringed toposes, meaning A' and A are differential local rings, then (f, ϕ) is called local if ϕ is local. We then have the following

PROPOSITION 3. *The forgetful functors*

$$U_1: DR\text{-}top \to R\text{-}top \quad and \quad U_1': DLR\text{-}top \to LR\text{-}top$$

have left adjoints C_1 and C_1' respectively, and U_1 and U_1' are both monadic.

PROOF. Defining C_1 on objects by $C_1(\mathcal{E}, A) = (\mathcal{E}, CA)$ and on morphisms in a similar fashion, and similarly for C_1', we obtain the desired adjoints. Beck's Theorem applies to give the monadicity of U_1 and U_1'.

Note that the monadicity of U_1 and U_1', as compared with the comonadicity of U and U' of Proposition 1 and the corollary to Proposition 2, is due to the duality of the morphisms in these categories.

PROPOSITION 4. *The forgetful functor $DLR\text{-}top \to DR\text{-}top$ has a right adjoint DSpec.*

PROOF. Consider the diagram

$$\begin{tikzcd} DLR\text{-}top \arrow[r] \arrow[d, "U_1'"'] & DR\text{-}top \arrow[d, "U_1"] \\ LR\text{-}top \arrow[r, shift right, "U_0"'] & R\text{-}top \arrow[l, shift right, "Spec"'] \end{tikzcd}$$

where U_0 denotes the forgetful functor and *Spec* is the right adjoint of U_0 as constructed in [4]. Then U_0 commutes with the monads G_1' and G_1 on *LR-top* and *R-top* induced by the adjunctions of Proposition 3 respectively, so that by the dual of Corollary 2.3 [2] the adjunction $U_0 \dashv Spec$ extends to one between the categories of algebras for G_1' and G_1, i. e., between the categories *DLR-top* and *DR-top*.

We note that Proposition 4 also follows from Cole's Theorem [1, page 206] if we take S to be the theory of differential rings, T the theory of differential local rings and A the class of differential local ring homomorphisms. In this case the factorization lemma is proved by showing that if $f\colon A \to L$ is a differential ring homomorphism where L is a differential local ring and the diagram

$$\begin{tikzcd} S \arrow[r] \arrow[d, tail] & Z(L) \arrow[d, tail] \\ A \arrow[r, "f"] & L \end{tikzcd}$$

is a pullback, then the ring of fractions $A[S^{-1}]$ as defined in [4] is a differential ring in a canonical fashion (cf. Section 3 in [2] for details in the case $\mathcal{E} = Sets$).

We also note that Gavin Wraith has pointed out that Proposition 1 follows from a result in [5] if we take S to be the theory of commutative rings, T the theory of differential rings and observe that T is obtained from S by adjoining a 1-ary operation satisfying equations involving the other operations in which the 1-ary operation can be «pushed through» the equation in a natural manner. In this case the models of T are then co-algebras for a comonad on the models of S.

In closing, it is noted that those situations in commutative algebra which generalize nicely to differential algebra are very closely related to the fact that differential rings are coalgebras for a comonad in commutative rings, and the comonad behaves nicely with respect to the given situations. It appears quite likely that those situations which do not generalize as nicely are not related to the coalgebra description of differential algebra.

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