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FIBRATIONS IN BICATEGORIES

by Ross STREET

Dedicated to the memory of Professor Charles Ehresmann

INTRODUCTION.

Consider a monoidal category $\mathcal{V}$ which, for simplicity, we shall suppose is symmetric, closed, small complete and small cocomplete. For $\mathcal{V}$-categories A, B, a $\mathcal{V}$-module from B to A amounts to a $\mathcal{V}$-functor $A^{op} \otimes B \to \mathcal{V}$ (these have also been called bimodules, profunctors and distributors in the literature). In the first instance it seems that in order to speak of $\mathcal{V}$-modules one needs the extra operations $(\ )^{op}$ and $\otimes$ on the 2-category $\mathcal{V}$-$\mathcal{C}at$ of small $\mathcal{V}$-categories, and also the $\mathcal{V}$-category $\mathcal{V}$ itself which lives outside of $\mathcal{V}$-$\mathcal{C}at$.

It is shown here that this first impression is false and that $\mathcal{V}$-modules are accessible purely from the bicategory $\mathcal{V}$-$\mathcal{C}at$ and certain limits and colimits (in the sense appropriate to bicategories) therein. Furthermore, it is shown that composition (or tensor product) of $\mathcal{V}$-modules is also accessible and behaves well due to a certain commuting property between these limits and colimits.

More precisely, given a bicategory $\mathcal{H}$ satisfying certain finitary completeness conditions, it is possible (purely from these limits and colimits) to construct a bicategory $\mathcal{B}$ such that, when $\mathcal{H}$ is $\mathcal{V}$-$\mathcal{C}at$, $\mathcal{B}$ is essentially the bicategory $\mathcal{V}$-$\mathcal{M}od$ of $\mathcal{V}$-modules.

In Street [12,13], a construction on a 2-category $\mathcal{K}$ was given which generalized the construction of $\mathcal{S}et$-modules from the 2-category $\mathcal{C}at$. This involved defining the notion of two-sided fibrations in a 2-category. Perhaps surprisingly the same basic idea leads to a solution of the present problem: two variants are needed, one *minor* and one *radical*.

The minor variant is forced by the replacement of «2-category» by

«bicategory» (although a solution to the corresponding problem for 2-categories can be obtained from our work too). Functors which are equivalences of categories are not necessarily fibrations or opfibrations in the sense of Gray [4]. Isomorphisms of categories are both. This indicates that the notion of fibration (or fibred category) presently in use is a 2-categorical and not a bicategorical notion. In Section 3 we introduce the notion of two-sided fibration relevant to a bicategorical approach to category theory; that is, we define what it means for a span in a bicategory to be a fibration (this is quite different from the question of fibrations between bicategories which is not relevant here).

In recent years it has become even more obvious that, although the fundamental constructions of set theory are categorical, the fundamental constructions of category theory are bicategorical. Since the paper Bénabou [1] in which bicategories were introduced, little has been published on them explicitly. Much more has been written on 2-categories and there has been an attempt to introduce as few «pseudo-concepts» (terminology of Kelly-Street [6]) as necessary to convey the ideas of a given situation. Thus we have found it necessary to show systematically (Section 1) how the ideas of 2-category theory must be modified for bicategories. Little use is made here of general morphisms of bicategories (lax functors). It has been shown in Street [14] that lax limits can be calculated as indexed limits; we avoid lax limits for bicategories by considering what are called here *indexed bi-limits*. These are the limits pertinent to bicategories.

Doctrines on bicategories are dealt with in the second section. Problems of coherence are avoided by taking a global approach: a doctrine M on a bicategory $\mathcal{K}$ is a homomorphism of bicategories from the simplicial category Δ to the endo-bicategory $\mathcal{K}$ which preserves the monoid structures. After all, the simplicial diagrams that a monad (= standard construction) provided were the most important consideration in the work in which they were introduced (Godement [3]). Doctrines on bicategories generalize monads on categories. The doctrines which appropriately generalize idempotent monads are here called KZ-*doctrines* (see Kock

[7] and Zöberlein [17]). The theory of KZ-doctrines is simplified by the observation (which becomes our global definition) that a doctrine M is such precisely when it extends from the category Δ to the 2-category of finite ordinals.

The condition that a span should be a fibration is that it should bear a structure of algebra for an appropriate KZ-doctrine on the bicategory of spans with the same source and target. The KZ-doctrine is defined in terms of finite indexed limits (and so is first-order). With minor changes the work of Street [14] carries over to bicategories.

Recall that profunctors can be regarded as spans in $\mathcal{C}at$, and they are composed by first composing the spans using pullback and then forming a coequalizer. Fibrations in a bicategory are composed by first forming a bipullback and then a bicoequinverter. The details of this composition are made explicit in Section 4. The composition works best in a *fibrational bicategory.*

Section 5 gives the interpretation of fibrations in $\mathcal{C}at$ and the proof that $\mathcal{C}at$ is a fibrational bicategory.

The radical variant referred to in the fourth paragraph of this introduction is met in Section 6 : instead of looking at fibrations in $\mathcal{V}$-$\mathcal{C}at$, we should look at fibrations in $(\mathcal{V}\text{-}\mathcal{C}at)^{op}$; that is, at *cofibrations in* $\mathcal{V}$-$\mathcal{C}at$. The $\mathcal{V}$-modules turn out to amount to the bicodiscrete cofibrations in $\mathcal{V}$-$\mathcal{C}at$. The (not necessarily bicodiscrete) cofibrations are also identified (they amount to what we call $\mathcal{V}$-*gamuts* : certain diagrams in $\mathcal{V}$-$\mathcal{M}od$) and this is used to show that $(\mathcal{V}\text{-}\mathcal{C}at)^{op}$ is a fibrational bicategory. This means that cofibrations can be composed and that the composition amounts precisely to the usual composition of $\mathcal{V}$-modules.

The solution to our problem is then to take $\mathcal{H}^{op}$ to be a fibrational bicategory and to take $\mathcal{B}$ to be the bicategory with the same objects as $\mathcal{H}$ and the bidiscrete fibrations in $\mathcal{H}^{op}$ as arrows.

Two solutions to the problem have thus appeared for the case where $\mathcal{V}$ is $\mathcal{S}et$ so that $\mathcal{V}$-$\mathcal{C}at$ is $\mathcal{C}at$. The contradiction between these two solutions is resolved by the observation that $\mathcal{C}at$ possesses a very special exactness property: *every (bi)codiscrete cofibration in* $\mathcal{C}at$ *is*

a (bi)cocomma object. This is why $\mathcal{S}et$-modules can be captured by either starting with spans or starting with cospans in $\mathcal{C}at$.

This work also clenches the case for working with two-sided fibrations. For the example where $\mathcal{V}$ is $\mathcal{S}et$ this is not so obviously important. After all a fibration from B to A in $\mathcal{C}at$ corresponds (somewhat unnaturally) to a fibration over $A \times B^{op}$ and the former is (bi)discrete iff the latter is. On the other hand, the one-sided cofibrations $A \to S$ under A in $\mathcal{C}at$, which were considered in Gray [4], do not subsume the two-sided cofibrations in a similar way at all. A one-sided cofibration under A in Gray's sense is essentially a cofibration from the empty category 0 to A in our sense; this amounts to a $\mathcal{S}et$-module with target A. Such a cofibration is bicodiscrete iff the corresponding $\mathcal{S}et$-module is the unique one $0 \to A$. So there is essentially only one bicodiscrete cofibration under A. This means there is no hope of capturing two-sided bicodiscrete cofibrations from their one-sided counterparts.

Note that we agree with Gray [4] that the fibrations in $\mathcal{C}at^{op}$ should be called cofibrations in analogy with topology. Beware, however that the term «cofibration» is also used in the literature (after Grothendieck) for fibration in Cat^{co}. The two-sided approach avoids the need for a special terminology for the latter concept: if

$$A \xleftarrow{\;p\;} E \xrightarrow{\;q\;} B$$

is a fibration from B to A then $p: E \to A$ is a fibration from 1 to A and $q: E \to B$ is a fibration from B to 1; in Grothendieck's terminology, p is a «fibration» and q is a «cofibration».

The $\mathcal{V}$-module aspect of this work was presented during February-April 1977 in colloquia at the Universities of Chicago, Illinois (Urbana), and Montréal, and at Tulane University (New Orléans). The bicategorical aspect has been available in preprint form since February 1978.

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1. BICATEGORIES.

(1.1) Most of our terminology will be that of the introductory paper of Bénabou [1]. Suppose $T, S : \mathcal{A} \to \mathcal{K}$ are *morphisms of bicategories* as described in [1] Section 4. A *transformation* $a : T \to S$ consists of the data displayed in the diagrams

$$\begin{tikzcd} TA \arrow[r, "aA"] \arrow[d, "Tf"'] \arrow[dr, phantom, "af \Downarrow"] & SA \arrow[d, "Sf"] \\ TB \arrow[r, "aB"'] & SB \end{tikzcd}$$

in $\mathcal{K}$ as $f : A \to B$ runs over the arrows in $\mathcal{A}$, satisfying coherence conditions. A *modification* $s : a \Rightarrow \beta$ between transformations $a, \beta : T \to S$ is a family of 2-cells

$$sA : aA \Rightarrow \beta A \quad \text{such that} \quad \beta f.(Sf)\ (sA) = (sB)(Tf).af.$$

With the obvious compositions we obtain a bicategory $\mathcal{B}icat(\mathcal{A}, \mathcal{K})$ of morphisms, transformations and modifications. Morphisms of bicategories can be composed (Bénabou [1] Section 4.3), and we write $\mathcal{B}icat$ for the category of bicategories and morphisms.

(1.2) Categories will be regarded as special 2-categories with only identity 2-cells; 2-categories will be regarded as special bicategories in which composition of arrows is strictly associative and the identity arrows are identities. If $\mathcal{K}$ is a 2-category, then so is $\mathcal{B}icat(\mathcal{A}, \mathcal{K})$.

(1.3) For bicategories $\mathcal{A}$, $\mathcal{K}$, our main interest will be in *homomorphisms* $T : \mathcal{A} \to \mathcal{K}$; these preserve composition of arrows and identity arrows up to natural coherent invertible 2-cells. A transformation $a : T \to S$ of homomorphisms is called *strong* when the 2-cells af are invertible for all arrows f in $\mathcal{A}$. Write $\mathcal{H}om(\mathcal{A}, \mathcal{K})$ for the bicategory of homomorphisms, strong transformations and modifications. Write $\mathcal{H}om$ for the category of bicategories and homomorphisms.

(1.4) Recall the construction of the *classifying category* $C\mathcal{A}$ of a bicategory $\mathcal{A}$ (Bénabou [1] Section 7.2, page 56). The objects of $C\mathcal{A}$ are

the objects of $\mathcal{A}$ and the arrows of $C\mathcal{A}$ are isomorphism classes of arrows of $\mathcal{A}$. In fact, we have a functor $C: \mathcal{H}om \to \mathcal{C}at$.

(1.5) *Adjunction* in a 2-category has been discussed for example in Kelly-Street [6] Section 2. The adjustments necessary for a bicategory in order to allow for associativity of composition of arrows only up to isomorphism are minor (see Gray [5] page 137) and will not be made explicit here. An arrow $u: A \to B$ in a bicategory $\mathcal{A}$ is said to be an *equivalence* when there exists an arrow $f: B \to A$ and invertible 2-cells $1_B \Rightarrow uf$ and $fu \Rightarrow 1_A$. The invertible 2-cells may be chosen so as to obtain an adjunction $f \dashv u$ in $\mathcal{A}$; so f is unique up to an invertible 2-cell. When there exists an equivalence from A to B we say that *A and B are equivalent* and write $A \approx B$.

(1.6) Homomorphisms of bicategories preserve adjunctions and equivalences.

(1.7) An object A of a bicategory $\mathcal{K}$ is called *groupoidal* when each 2-cell

$$\begin{tikzcd} X \arrow[r, bend left=50] \arrow[r, bend right=50] \arrow[r, phantom, "\Downarrow"] & A \end{tikzcd}$$

is an isomorphism. Call A *posetal* when for each pair of arrows $u, v: X \to A$ there is at most one 2-cell $u \Rightarrow v$. Call A *bidiscrete* when it is both groupoidal and posetal. The groupoidal objects of $\mathcal{K}$ form a full sub-bicategory of $\mathcal{K}$ denoted by $G\mathcal{K}$. Note that $G\mathcal{K}$ is a local groupoid. The bidiscrete objects of $\mathcal{K}$ form a full sub-bicategory of $\mathcal{K}$ whose classifying category (1.4) is denoted by $D\mathcal{K}$. For example the groupoidal objects of $\mathcal{C}at$ are the groupoids and $G\mathcal{C}at$ is the 2-category of groupoids; the bidiscrete objects of $\mathcal{C}at$ are those categories which are equivalent to discrete categories, and $D\mathcal{C}at$ is equivalent to the category $\mathcal{S}et$ of sets.

(1.8) A category is called *trivial* when it is equivalent to the category 1 with one object 0 and one arrow 1_0. An object K of a bicategory $\mathcal{A}$ is called *biterminal* when, for each object A of $\mathcal{A}$, the category $\mathcal{A}(A, K)$ is trivial. Biterminal objects are unique up to equivalence. A *biinitial* ob-

ject of $\mathcal{A}$ is a biterminal object of $\mathcal{A}^{op}$ (we write $\mathcal{A}^{op}$, $\mathcal{A}^{co}$, $\mathcal{A}^{coop}$ for the duals $\mathcal{A}^{t}$, $\mathcal{A}^{c}$, $\mathcal{A}^{s}$ of Bénabou [1] page 26 in accord with the notion of Kelly-Street [6] page 82 for 2-categories).

(1.9) *Yoneda Lemma for bicategories. For a homomorphism* $T : \mathcal{A} \to \mathcal{C}at$ *of bicategories, evaluation at the identity for each object* X *of* $\mathcal{A}$ *provides the components*

$$\mathcal{H}om(\mathcal{A}, \mathcal{C}at)(\mathcal{A}(X, -), T) \xrightarrow[\approx]{} TX$$

of an equivalence in $\mathcal{H}om(\mathcal{A}, \mathcal{C}at)$.

(1.10) *Grothendieck construction.* For two bicategories $\mathcal{A}$, $\mathcal{B}$ and a homomorphism $R : \mathcal{A}^{op} \times \mathcal{B} \to \mathcal{C}at$, we shall describe a bicategory $\mathcal{E}\ell R$ (or, more correctly, $\mathcal{E}\ell(\mathcal{A}, R, \mathcal{B})$). The objects (A, r, B) consist of objects A, B, r of $\mathcal{A}$, $\mathcal{B}$, $R(A, B)$, respectively. The arrows

$$(a, \rho, b) : (A, r, B) \to (A', r', B')$$

consist of arrows

$$a : A \to A', \quad b : B \to B', \quad \rho : R(A, b)r \to R(a, B')r'$$

of $\mathcal{A}$, $\mathcal{B}$, $R(A, B')$, respectively. The 2-cells :

$$(\alpha, \beta) : (a, \rho, b) \Rightarrow (c, \sigma, d)$$

consist of 2-cells $\alpha : a \Rightarrow c$, $\beta : b \Rightarrow d$ of $\mathcal{A}$, $\mathcal{B}$ such that

$$\sigma . R(A, \beta)r = R(\alpha, B')r' . \rho .$$

Composition of arrows is given by

$$(a', \rho', b')(a, \rho, b) = (a'a, \tau, b'b)$$

where τ is the composite

$$R(A, b'b)r \approx R(A, b')R(A, b)r \xrightarrow{R(A, b')\rho} R(A, b')R(a, B'')r' \approx$$

$$R(a, B'')R(A', b')r' \xrightarrow{R(a, B'')\rho'} R(a, B'')R(a', B'')r'' \approx R(a'a, B'')r''$$

in $R(A, B'')$. The other compositions are the obvious ones. First and last projection give strict homomorphisms $P : \mathcal{E}\ell R \to \mathcal{A}$, $Q : \mathcal{E}\ell R \to \mathcal{B}$. If $\mathcal{A}$, $\mathcal{B}$ are 2-categories, so is $\mathcal{E}\ell R$, and P, Q are 2-functors. Write $\mathcal{E}\ell_0 R$ for $\mathcal{E}\ell R$ when $\mathcal{A} = 1$, and write $\mathcal{E}\ell_1 R$ for $\mathcal{E}\ell R$ when $\mathcal{B} = 1$. An arrow

(a, ρ, b) of $\mathcal{E}\ell_i R$ is called *i-cartesian* $(\, i = 0, 1\,)$ when ρ is invertible. Write $\mathcal{E}\ell_i^c R$ for the locally full sub-bicategory of $\mathcal{E}\ell_i R$ consisting of all the objects and the i-cartesian arrows.

(1.11) A *birepresentation* for a homomorphism $T : \mathcal{A} \to \mathcal{C}at$ of bicategories is an object K of $\mathcal{A}$ together with an equivalence

$$\mathcal{A}(K, -) \xrightarrow[\approx]{} T \quad \text{in} \quad \mathcal{H}om(\mathcal{A}, \mathcal{C}at)\,.$$

Using (1.9), we see that a birepresentation of T precisely amounts to an object K of $\mathcal{A}$ and an object k of TK such that:

- for each object A of $\mathcal{A}$ and object a of TA, there exist an arrow $u : K \to A$ in $\mathcal{A}$ and an isomorphism $(Tu)k \approx a$ in TA ; and

- for arrows $u, v : K \to A$ in $\mathcal{A}$ and $\alpha : (Tu)k \to (Tv)k$ in TA, there exists a unique 2-cell $\sigma : u \Rightarrow v$ in $\mathcal{A}$ such that $(T\sigma)k = \alpha$.

This implies that (K, k) is a biinitial object of $\mathcal{E}\ell_0^c T$. It follows that birepresentations for T are unique up to equivalence in $\mathcal{E}\ell_0^c T$. Call T *birepresentable* when it admits a birepresentation.

(1.12) Suppose $\mathcal{A}$ is a small bicategory and $J : \mathcal{A} \to \mathcal{C}at$, $S : \mathcal{A} \to \mathcal{K}$ are homomorphisms. A *J-indexed bilimit for* S is a birepresentation (K, k) for the homomorphism

$$\mathcal{H}om(\mathcal{A}, \mathcal{C}at)(J, \mathcal{K}(1_{\mathcal{K}}, S)) : \mathcal{K}^{op} \to \mathcal{C}at\,.$$

A particular choice of K is denoted by $\{J, S\}$ and is characterized up to equivalence by an equivalence

$$\mathcal{K}(X, \{J, S\}) \approx \mathcal{H}om(\mathcal{A}, \mathcal{C}at)(J, \mathcal{K}(X, S)) \tag{1.13}$$

which is a strong transformation in X.

(1.14) When $\mathcal{K}$ is a 2-category, both sides of (1.13) are 2-functorial in X and it sometimes happens that $\{J, S\}$ can be chosen so that the equivalence is in fact a 2-natural isomorphism ; in this case, we will write $psdlim(J, S)$ instead of $\{J, S\}$ and call it a *J-indexed pseudo-limit for* S . Of course, if $K \approx psdlim(J, S)$, then K is a J-indexed bilimit for S.

(1.15) PROPOSITION. *For any small bicategory* $\mathcal{A}$ *and homomorphisms*

$J, S: \mathcal{A} \to \mathcal{C}at$, a J-indexed pseudo-limit for S exists and is given by:

$$psdlim(J, S) \approx \mathcal{H}om(\mathcal{A}, \mathcal{C}at)(J, S). \quad \square$$

(1.16) There is another type of limit for bicategories which may appear to give some new constructions but in fact does not. For morphisms of bicategories $J: \mathcal{A} \to \mathcal{C}at$, $S: \mathcal{A} \to \mathcal{K}$ (1.1), it is natural to ask for an object L of $\mathcal{K}$ satisfying an equivalence

(1.17) $$\mathcal{K}(X, L) \approx \mathcal{B}icat(\mathcal{A}, \mathcal{C}at)(J, \mathcal{K}(X, S)).$$

Such an object can in fact be obtained as an indexed bilimit; this will not be needed here so we shall not give the details except to note that bicategorical versions of Street [14] Section 4 and Gray [5] pages 92-94 are relevant.

(1.18) There are important 2-categories (for example, $\mathcal{H}om(\mathcal{A}, \mathcal{C}at)$ and topoi over a base topos) which admit certain indexed bilimits which are not indexed limits or even indexed pseudo-limits. Therefore, in the absence of a coherence theorem allowing the replacement of a bicategory and its indexed bilimits by an equivalent 2-category and its indexed limits we are forced (even for 2-categories) into the more involved (yet more natural) context. Fortunately, the theory of Street [14] is altered only slightly by the need to insert isomorphisms in diagrams which previously commuted.

(1.19) A homomorphism $F: \mathcal{K} \to \mathcal{L}$ is said to *preserve the J-indexed bilimit (K, k)* for $S: \mathcal{A} \to \mathcal{K}$ when FK together with the composite

$$J \xrightarrow{k} \mathcal{K}(K, S) \xrightarrow{F} \mathcal{L}(FK, FS)$$

form a J-indexed bilimit for FS. By abuse of language we express this as an equivalence $F\{J, S\} \approx \{J, FS\}$. An indexed bilimit is called *absolute* when it is preserved by all homomorphisms.

(1.20) Combining (1.13), (1.11), (1.15), we obtain:

PROPOSITION. *Birepresentable homomorphisms from a bicategory $\mathcal{A}$ into $\mathcal{C}at$ preserve whatever indexed bilimits that exist in $\mathcal{A}$.*

(1.21) The analogue of the representability Theorem (Mac Lane [10] page 118; Schubert [11] page 88) will not be needed and is left to the reader.

(1.22) For a category C and an object A of a bicategory $\mathcal{K}$ regarded as homomorphisms $1 \to \mathcal{C}at$, $1 \to \mathcal{K}$, we call $\{C, A\}$ when it exists, a *cotensor biproduct of* C and A; then there is an equivalence

$$\mathcal{K}(X, \{C, A\}) \simeq [C, \mathcal{K}(X, A)].$$

(1.23) When $J: \mathcal{A} \to \mathcal{C}at$ is constant at the category 1, a J-indexed bilimit is just called a *bilimit* and we write $bilim\, S$ in place of $\{J, S\}$. In particular, if A is a set and S a family of objects of a bicategory $\mathcal{K}$ the bilimit of S is called the *biproduct of* S and denoted by $\underset{A}{\prod\!\!\prod} SA$ (the symbol $\prod\!\!\prod$ is to be read «bi-pi»). The notions of *biequalizer, bipullback, bikernel pair, ...* are now self-explanatory. (Some authors use «limit» for «bilimit», and some use «2-limit» in the case of 2-categories.)

(1.24) We can construct the indexed bilimit $\{J, S\}$ as the biequalizer of a pair of arrows

$$\underset{A}{\prod\!\!\prod} \{JA, SA\} \rightrightarrows \underset{A,B}{\prod\!\!\prod} \{\mathcal{A}(A, B) \times JA, SB\}$$

provided the biequalizer, cotensor biproducts, and biproducts involved exist.

(1.25) In a 2-category, indexed pseudo-limits (and hence indexed bilimits) can be constructed from cotensor products, products and equalizers. For, pseudo-equalizers can be constructed from cotensor products and pullbacks; so, if we replace the biproducts and cotensor biproducts in (1.24) by products and cotensor products and then take the pseudo-equalizer, we obtain $psdlim(J, S)$.

(1.26) For homomorphisms $J: \mathcal{A}^{op} \to \mathcal{C}at$, $S: \mathcal{A} \to \mathcal{K}$, we write $J * S$ for $\{J, S^{op}\}$ and call it the *J-indexed bicolimit of S in $\mathcal{K}$*.

(1.27) A bicategory $\mathcal{K}$ will be called *finitely bicategorically complete* («finitely bicomplete» would be misleading!) when it admits cotensor biproducts with the ordinal 2, biequalizers, biproducts of pairs of ob-

jects, and a biterminal object 1. There is a notion of *finitary homomorphism* $J : \mathcal{A} \to \mathcal{C}at$ which satisfies the appropriate version of Street [14] Theorem 9, page 163. For example, with such finite bicat. completeness, any monad (A, t) admits an *Eilenberg-Moore object* A^t satisfying the equivalence

$$\mathcal{K}(X, A^t) \approx \mathcal{K}(X, A)^{\mathcal{K}(X,t)} \tag{1.28}$$

(1.29) For a finitely bicat. complete bicategory $\mathcal{K}$, cotensor biproduct provides a homomorphism of bicategories $\{\ ,\ \} : \mathcal{C}at_{fp}^{op} \times \mathcal{K} \to \mathcal{K}$, where $\mathcal{C}at_{fp}$ is the full sub-2-category of $\mathcal{C}at$ consisting of the finitely presented categories (i.e. coequalizers of pairs of functors between free categories on finite graphs). For $C \in \mathcal{C}at_{fp}$, $A \in \mathcal{K}$, the homomorphisms

$$\{C, -\} : \mathcal{K} \to \mathcal{K}, \quad \{-, A\} : \mathcal{C}at_{fp}^{op} \to \mathcal{K}$$

preserve finitary indexed bilimits (1.19).

(1.30) A *left bilifting* of an object X of $\mathcal{L}$ through a homomorphism $T : \mathcal{K} \to \mathcal{L}$ of bicategories is a birepresentation of $\mathcal{L}(X, T-) : \mathcal{K} \to \mathcal{C}at$ (1.11); that is, an object SX of $\mathcal{K}$ and an equivalence

$$\mathcal{K}(SX, -) \approx \mathcal{L}(X, T-). \tag{1.31}$$

If each object of $\mathcal{L}$ has a left bilifting through T the axiom of choice allows us to extend the function $X \mapsto SX$ to a homomorphism $S : \mathcal{L} \to \mathcal{K}$ such that (1.31) becomes a strong transformation in X. This S is unique up to equivalence in $\mathcal{H}om(\mathcal{L}, \mathcal{K})$. When such a homomorphism S exists we say that T *has a left biadjoint* and write $S \dashv_b T$.

(1.32) A homomorphism with a left biadjoint preserves indexed bilimits.

(1.33) A homomorphism $T : \mathcal{K} \to \mathcal{L}$ of bicategories is said to be a *biequivalence* when:

- for each object X of $\mathcal{L}$, there exists an object A of $\mathcal{K}$ such that $TA \approx X$ (1.5); and
- for each pair A, B of objects of $\mathcal{K}$, the functor

$$T_{A,B} : \mathcal{K}(A, B) \to \mathcal{L}(TA, TB)$$

is an equivalence of categories.

Clearly a biequivalence T has a left biadjoint S which is also a biequivalence. So biequivalence is an equivalence relation written: $\mathcal{K} \sim \mathcal{L}$.

(1.34) For bicategories $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, there are natural biequivalences:

$$\mathcal{H}om(\mathcal{A} \times \mathcal{B}, \mathcal{C}) \sim \mathcal{H}om(\mathcal{A}, \mathcal{H}om(\mathcal{B}, \mathcal{C})), \quad \mathcal{H}om(1, \mathcal{A}) \sim \mathcal{A}.$$

(1.35) The biequivalences (1.34) show that $\mathcal{H}om$ is a weak kind of cartesian closed category. There are canonical homomorphisms

$$1 \to \mathcal{H}om(\mathcal{A}, \mathcal{A}), \quad \mathcal{H}om(\mathcal{B}, \mathcal{C}) \to \mathcal{H}om(\mathcal{H}om(\mathcal{A}, \mathcal{B}), \mathcal{H}om(\mathcal{A}, \mathcal{C})).$$

(1.36) There is a canonical natural isomorphism

$$\mathcal{H}om(\mathcal{A}, \mathcal{H}om(\mathcal{B}, \mathcal{C})) \approx \mathcal{H}om(\mathcal{B}, \mathcal{H}om(\mathcal{A}, \mathcal{C}))$$

in $\mathcal{H}om$ which is in fact the restriction of a strict isomorphism

$$\mathcal{B}icat(\mathcal{A}^{op}, \mathcal{B}icat(\mathcal{B}, \mathcal{C})^{op})^{op} \approx \mathcal{B}icat(\mathcal{B}, \mathcal{B}icat(\mathcal{A}^{op}, \mathcal{C}^{op})^{op}).$$

2. DOCTRINES.

(2.1) *The simplicial category.* The full sub-2-category $\mathcal{O}rd_f$ of $\mathcal{C}at$ consisting of the finite ordinals is generated under composition by the cosimplicial complex:

(2.2)

$$\begin{tikzcd}[column sep=large] 0 \arrow[r, "\partial_0"] & 1 \arrow[r, shift left=6, "\partial_0"] \arrow[r, shift right=6, "\partial_1"'] & 2 \arrow[l, "\iota_0"'] \arrow[r, shift left=10, "\partial_0"] \arrow[r, "\partial_1"] \arrow[r, shift right=10, "\partial_2"'] & 3 \arrow[l, shift right=5, "\iota_0"'] \arrow[l, shift left=5, "\iota_1"] \arrow[r, shift left=15, "\partial_0"] \arrow[r, shift left=5, "\partial_1"] \arrow[r, shift right=5, "\partial_2"] \arrow[r, shift right=15, "\partial_3"'] & 4 \arrow[l, shift right=10, "\iota_0"'] \arrow[l, "\iota_1"] \arrow[l, shift left=10, "\iota_2"] & \cdots \end{tikzcd}$$

and the natural transformations $\partial_m \leq \partial_{m+1}$, $\iota_{m+1} \leq \iota_m$; the functors ι_m are distinct epimorphisms and the functors ∂_m are distinct monomorphisms. Furthermore, the complex (2.2) is generated by adjunction and pushout in $\mathcal{C}at$ from the two unique functors

(2.3)
$$0 \xrightarrow{\partial_0} 1 \xleftarrow{\iota_0} 2.$$

For, at each stage in (2.2) we have the string of adjunctions

(2.4) $$\partial_0 \dashv \iota_0 \dashv \partial_1 \dashv \iota_1 \dashv \partial_2 \dashv \iota_2 \dashv \cdots$$

where the unit of $\partial_m \dashv \iota_m$ and the counit of $\iota_m \dashv \partial_{m+1}$ are identities; and for $n > 1$, the squares

(2.5) $$\begin{tikzcd} n-1 \arrow[r, "\partial_0"] \arrow[d, "\partial_{n-1}"'] & n \arrow[d, "\partial_n"] \\ n \arrow[r, "\partial_0"] & n+1 \end{tikzcd}$$

are pushouts. The cosimplicial identities:

$$\partial_r \partial_s = \partial_{s+1} \partial_r \quad \text{and} \quad \iota_s \iota_r = \partial_r \iota_{s+1} \quad \text{for } r \leq s,$$

(2.6) $$\iota_s \partial_r = \begin{cases} \partial_r \iota_{s-1} & \text{for } r < s \\ 1 & \text{for } r = s \text{ or } s+1 \\ \partial_{r-1} \iota_s & \text{for } r > s+1, \end{cases}$$

are consequences of these facts. The 2-category $\mathcal{O}rd_f$ is locally partially ordered, and there is a 2-functor

(2.7) $$+ : \mathcal{O}rd_f \times \mathcal{O}rd_f \to \mathcal{O}rd_f$$

given by the ordinal sum which enriches $\mathcal{O}rd_f$ with a structure of monoid in the category of 2-categories and 2-functors. The simplicial category Δ is the underlying category of $\mathcal{O}rd_f$.

(2.8) The pushouts (2.5) are also bipushouts in $\mathcal{C}at$.

(2.9) The arrow $\iota_0 : 2 \to 1$ is an absolute bicoequalizer (1.19) of the pair $\iota_0, \iota_1 : 3 \to 2$ in the subcategory of Δ consisting of the objects $1, 2, 3$ and the last-element-preserving arrows (and hence in Δ and $\mathcal{C}at$).

(2.10) *Doctrines on a bicategory* $\mathcal{K}$. The bicategory $\mathcal{H}om(\mathcal{K},\mathcal{K})$ with composition as multiplication and the category Δ with ordinal sum as multiplication are monoids in $\mathcal{H}om$. A *doctrine on* $\mathcal{K}$ is a homomorphism of bicategories $\mathrm{M} : \Delta \to \mathcal{H}om(\mathcal{K},\mathcal{K})$ which preserves the monoid structures. If we put M, η, μ equal to the respective images under M of 1, $\partial_0 : 0 \to 1$, $\iota_0 : 2 \to 1$, then it follows that the image of (2.2) under M is

the diagram

(2.11)
$$\begin{tikzcd}[column sep=large]
1_{\mathcal{K}} \arrow[r, "\eta"] & M \arrow[r, shift left=4, "M\eta"] \arrow[r, shift right=4, "\eta M"'] & M^2 \arrow[l, "\mu"'] \arrow[r, shift left=8, "M^2\eta"] \arrow[r, "M\eta M"] \arrow[r, shift right=8, "\eta M^2"'] & M^3 \arrow[l, shift right=4, "M\mu"'] \arrow[l, shift left=4, "\mu M"] \arrow[r, shift left=12, "M^3\eta"] \arrow[r, shift left=4, "M^2\eta M"] \arrow[r, shift right=4, "M\eta M^2"] \arrow[r, shift right=12, "\eta M^3"'] & M^4 \arrow[l, shift right=8, "M^2\mu"'] \arrow[l, "M\mu M"] \arrow[l, shift left=8, "\mu M^2"] & \cdots
\end{tikzcd}$$

in $\mathcal{H}om(\mathcal{K},\mathcal{K})$. The identities (2.6) are converted in $\mathcal{H}om(\mathcal{K},\mathcal{K})$ to invertible 2-cells which are coherent; for example, there are the invertible 2-cells r, l, a given by the composites:

$$r\colon 1_M \approx \mathrm{M}1_1 = \mathrm{M}(\partial_0\partial_0) \approx \mathrm{M}(\iota_0)\mathrm{M}(\partial_0) = \mu . M\eta,$$

$$l\colon \mu.\eta M = M(\iota_0)\mathrm{M}(\partial_1) \approx \mathrm{M}(\iota_0\partial_1) = \mathrm{M}1_1 \approx 1_M$$

$$a\colon \mu . M\mu = \mathrm{M}(\iota_0)\mathrm{M}(\iota_0) \approx \mathrm{M}(\iota_0\iota_0) = \mathrm{M}(\iota_0\iota_1) \approx \mathrm{M}(\iota_0)\mathrm{M}(\iota_1) = \mu.\mu M$$

and all the other invertible 2-cells can be obtained from these three. In fact, all the data for M are determined by M, η, μ, r, l, a.

(2.12) Let $\mathcal{O}rd_f^{suc}$ denote the sub-2-category of $\mathcal{O}rd_f$ consisting of the non-empty finite ordinals and the last-element-preserving arrows. Let Δ^{suc} denote the underlying category of $\mathcal{O}rd^{suc}$. As pointed out in Lawvere [8] page 150, Δ^{suc} is the Eilenberg-Moore category of algebras for the *successor-monad* suc on Δ described as follows. The category Δ is a strict monoidal category with ordinal sum as tensor product. The data (2.3) describe a monoid structure on the object 1 in this monoidal category. Thus a monad suc on Δ is induced with underlying functor: $-+1 : \Delta \to \Delta$. Clearly Δ^{suc} is generated under composition by the diagram

(2.13)
$$\begin{tikzcd}[column sep=large]
1 & 2 \arrow[l, shift right=3, "\iota_0"'] \arrow[r, shift left=9, leftarrow, "\iota_0"] \arrow[r, shift left=3, "\partial_1"] \arrow[r, shift right=3, leftarrow, "\iota_1"] \arrow[r, shift right=9, "\partial_2"'] & 3 \arrow[r, shift left=15, leftarrow, "\iota_0"] \arrow[r, shift left=9, "\partial_1"] \arrow[r, shift left=3, leftarrow, "\iota_1"] \arrow[r, shift right=3, "\partial_2"] \arrow[r, shift right=9, leftarrow, "\iota_2"] \arrow[r, shift right=15, "\partial_3"'] & 0 \; \cdots
\arrow[from=1-1, to=1-2, shift right=3, "\partial_1"']
\end{tikzcd}$$

(2.14) The monoids Δ, $\mathcal{H}om(\mathcal{K},\mathcal{K})$ in $\mathcal{H}om$ act on the left on the bicategories Δ^{suc}, $\mathcal{K}$, respectively, by ordinal sum and evaluation.

(2.15) An *algebra* for a doctrine M on a bicategory $\mathcal{K}$ is a homomorphism of bicategories $A : \Delta^{suc} \to \mathcal{K}$ such that the following square commutes.

$$\begin{tikzcd}
\Delta \times \Delta^{suc} \arrow[r, "M \times A"] \arrow[d, "+"'] & \mathcal{H}om(\mathcal{K}, \mathcal{K}) \times \mathcal{K} \arrow[d, "eval"] \\
\Delta^{suc} \arrow[r, "A"'] & \mathcal{K}
\end{tikzcd}$$

Let A, α be the image of 1, $\iota_0 : 2 \to 1$ under A. Then commutativity of the above square implies that the image of (2.13) under A is the diagram

$$\begin{tikzcd}[column sep=huge]
A \arrow[r, leftarrow, shift left=1.5, "\alpha"] \arrow[r, shift right=1.5, "\eta A"'] & MA \arrow[r, leftarrow, shift left=4.5, "Ma"] \arrow[r, shift left=1.5, "M\eta A"] \arrow[r, leftarrow, shift right=1.5, "\mu A"] \arrow[r, shift right=4.5, "\eta MA"'] & M^2 A \arrow[r, leftarrow, shift left=7.5, "M^2 \alpha"] \arrow[r, shift left=4.5, "M^2 \eta A"] \arrow[r, leftarrow, shift left=1.5, "M\mu A"] \arrow[r, shift right=1.5, "M\eta MA"] \arrow[r, leftarrow, shift right=4.5, "\mu MA"] \arrow[r, shift right=7.5, "\eta M^2 A"'] & M^3 A \quad \ldots
\end{tikzcd}$$

in $\mathcal{K}$. The identities (2.6) which do not involve ∂_0 's are converted in $\mathcal{K}$ to invertible 2-cells, which are coherent; for example, there are the 2-cells

$$l : \alpha \cdot \eta A \approx 1_M , \quad a : \alpha \cdot M\alpha \approx \alpha \cdot \mu A$$

obtained from the equations

$$\iota_0 \partial_1 = 1 , \quad \iota_0 \iota_1 = \iota_0 \iota_0 ;$$

all the other such 2-cells can be obtained from these two and those for M. Given M, all the data for A are determined by A, α, l, a.

(2.16) A *morphism* f : A → B *of algebras* for M is a transformation of morphisms of bicategories such that $\mathrm{f}+ = eval.(\mathrm{M} \times \mathrm{f})$. A 2-cell $\sigma : \mathrm{f} \to \mathrm{g}$ between such morphisms is a modification such that $\bar{\sigma} + = eval.(\mathrm{M} \times \sigma)$. A morphism f is determined by $f = \mathrm{f}1 : A \to B$ and $a = \mathrm{f}\iota_0$:

$$\begin{tikzcd}
MA \arrow[r, "Mf"] \arrow[d, "\alpha"'] \arrow[dr, phantom, "a \Downarrow"] & MB \arrow[d, "\beta"] \\
A \arrow[r, "f"'] & B
\end{tikzcd} ;$$

and a 2-cell $\bar{\sigma} : f \Rightarrow g$ is determined by $\sigma = \bar{\sigma} 1 : f \Rightarrow g$.

(2.17) Write $\mathcal{K}_\ell^{\mathrm{M}}$ for the bicategory of algebras for M, morphisms between these, and 2-cells between these.

(2.18) A *strong morphism of algebras for* M is a strong transformation $f : A \to B$ which is a morphism of algebras for M; this amounts to saying that $a : \beta \cdot Mf \Rightarrow f \cdot a$ is invertible.

(2.19) Write $\mathcal{K}^{\mathrm{M}}$ for the bicategory of algebras for M, strong morphisms and 2-cells. Alternatively, $\mathcal{K}^{\mathrm{M}}$ is the equalizer of the two homomorphisms

$$\begin{tikzcd}
\mathcal{H}om(\Delta^{suc}, \mathcal{K}) \arrow[rr, "+^*"] \arrow[dr, "M \times -"'] & & \mathcal{H}om(\Delta \times \Delta^{suc}, \mathcal{K}) \\
& \mathcal{H}om(\Delta \times \Delta^{suc}, \mathcal{H}om(\mathcal{K}, \mathcal{K}) \times \mathcal{K}) \arrow[ur, "eval_*"'] &
\end{tikzcd}$$

in the category $\mathcal{H}om$ (see Bénabou [1] 7.4.1, page 57).

(2.20) A doctrine M on $\mathcal{K}$ induces a doctrine $M*$ on $\mathcal{H}om(\mathcal{A}, \mathcal{K})$, namely the composite

$$\Delta \xrightarrow{M} \mathcal{H}om(\mathcal{K}, \mathcal{K}) \xrightarrow[(1.35)]{} \mathcal{H}om(\mathcal{H}om(\mathcal{A}, \mathcal{K}), \mathcal{H}om(\mathcal{A}, \mathcal{K})),$$

The isomorphism (1.36)

$$\mathcal{H}om(\Delta^{suc}, \mathcal{H}om(\mathcal{A}, \mathcal{K})) \approx \mathcal{H}om(\mathcal{A}, \mathcal{H}om(\Delta^{suc}, \mathcal{K}))$$

induces an isomorphism

(2.21) $$\mathcal{H}om(\mathcal{A}, \mathcal{K})^{M*} \approx \mathcal{H}om(\mathcal{A}, \mathcal{K}^{\mathrm{M}})$$

in $\mathcal{H}om$.

(2.22) Evaluation at 1 provides a homomorphism $U : \mathcal{K}^{\mathrm{M}} \to \mathcal{K}$. For each X of $\mathcal{K}$, the composite

$$\Delta^{suc} \longrightarrow \Delta \xrightarrow{\mathrm{M}} \mathcal{H}om(\mathcal{K}, \mathcal{K}) \xrightarrow{eval_X} \mathcal{K}$$

provides a left bilifting (1.30) of X through U; so we obtain a left bi-adjoint $F : \mathcal{K} \to \mathcal{K}^{\mathrm{M}}$ for U.

(2.23) For each algebra A for M, (2.9) yields the absolute bicoequalizer

$$MMA \underset{\mu_A}{\overset{Ma}{\rightrightarrows}} MA \xrightarrow{\ a\ } A$$

in $\mathcal{K}$.

(2.24) The functor $+ : \Delta^{suc} \to [\Delta^{suc}, \Delta^{suc}]$ corresponding under the cartesian-closed adjunction to ordinal sum induces a homomorphism

$$\begin{tikzcd} \mathcal{H}om(\Delta^{suc}, \mathcal{K}) \arrow[r, "(1.35)"] & \mathcal{H}om([\Delta^{suc}, \Delta^{suc}], \mathcal{H}om(\Delta^{suc}, \mathcal{K})) \arrow[d, "+^*"'] \\ & \mathcal{H}om(\Delta^{suc}, \mathcal{H}om(\Delta^{suc}, \mathcal{K})) \end{tikzcd}$$

which restricts to a homomorphism

(2.25) $$\mathcal{K}^{\mathrm{M}} \longrightarrow \mathcal{H}om(\Delta^{suc}, \mathcal{K})^{\mathrm{M}*}.$$

Thus we can regard each algebra A for M as an algebra for M$*$ (2.20) on $\mathcal{H}om(\Delta^{suc}, \mathcal{K})$. Applying (2.23) to A regarded as an algebra for M$*$ we obtain an absolute bicoequalizer

(2.26) $$FMA \rightrightarrows FA \longrightarrow A$$

in $\mathcal{H}om(\Delta^{suc}, \mathcal{K})$.

(2.27) A *KZ-doctrine* (Kock [7], Zöberlein [15]) on $\mathcal{K}$ is a doctrine M on $\mathcal{K}$ for which there is a homomorphism of bicategories

$$\overline{\mathrm{M}} : \mathcal{O}rd_f^{suc} \longrightarrow \mathcal{H}om(\mathcal{K}, \mathcal{K})$$

such that the following square commutes (2.12).

$$\begin{tikzcd} \Delta^{suc} \arrow[r] \arrow[d] & \mathcal{O}rd_f^{suc} \arrow[d, "\overline{\mathrm{M}}"] \\ \Delta \arrow[r, "\mathrm{M}"] & \mathcal{H}om(\mathcal{K}, \mathcal{K}) \end{tikzcd}$$

Since all the 2-cells in $\mathcal{O}rd_f^{suc}$ correspond under adjunction to equalities and $\overline{\mathrm{M}}$ must preserve adjunction (1.6), $\overline{\mathrm{M}}$ is unique if it exists. The image of (2.13) under $\overline{\mathrm{M}}$ is just (2.11) with all the top arrows $M^n\eta$ omitted. Moreover, in this image diagram each arrow is a left adjoint for the one below it (if it has one) (1.5), (2.4). In fact, it can be seen that M is a KZ-doctrine iff $\mu \dashv \eta M$ with counit $l : \mu \cdot \eta M \approx 1_M$.

(2.28) Any algebra A for a KZ-doctrine M has a unique extension to a homomorphism $\bar{\mathrm{A}} : \mathcal{O}rd_{\mathfrak{f}}^{suc} \to \mathcal{K}$. To see this, first note that $eval_X \bar{\mathrm{M}}$ provides the unique lifting of each «free» algebra FX (2.22). Then the bicoequalizer (2.26) can be used to define A on 2-cells giving an extension $\bar{\mathrm{A}}$ which is unique since all the 2-cells in $\mathcal{O}rd_{\mathfrak{f}}^{suc}$ correspond under adjunction to identities.

(2.29) It follows from (2.28), (1.5), (2.15) that, for any algebra A for a KZ-doctrine M, there is an adjunction $a \dashv \eta A$ with counit l, and the isomorphism a is uniquely determined.

(2.30) The homomorphism $U : \mathcal{K}_{\mathfrak{f}}^{\mathrm{M}} \to \mathcal{K}$ given by evaluation at 1 is fully faithful (= local isomorphism) when M is a KZ-doctrine. For algebras A, B and an arrow $f : A \to B$, the 2-cell $a : \beta . Mf \Rightarrow fa$ which corresponds under the adjunction $a \dashv \eta A$, $\beta \dashv \eta B$ to the isomorphism

$$\eta f : Mf . \eta A \Rightarrow \eta B . f$$

enriches f with a structure of morphism $\mathrm{f} : \mathrm{A} \to \mathrm{B}$ which is unique with the property that $U\mathrm{f} = f$. Any 2-cell $a : f \Rightarrow g$ will clearly respect such a's and so be a 2-cell in $\mathcal{K}_{\mathfrak{f}}^{\mathrm{M}}$.

3. FIBRATIONS.

Throughout this section we shall work in a fixed finitely bicategorically complete bicategory $\mathcal{K}$ (1.27).

(3.1) A *span* (u, S, v) from B to A is a diagram

$$A \xleftarrow{\ u\ } S \xrightarrow{\ v\ } B .$$

When u, v are understood, (u, S, v) is abbreviated to S. Identify A with the span $(1_A, A, 1_A)$ from A to A.

(3.2) A *homomorphism* $(\phi, f, \psi) : (u, S, v) \to (u', S', v')$ *of spans* is a diagram

$$\begin{tikzcd} & S \arrow[dl, "u"'] \arrow[dr, "v"] \arrow[dd, "f"] & \\ A \arrow[r, phantom, "\approx \Downarrow \phi"] & \arrow[r, phantom, "\psi \Downarrow \approx"] & B \\ & S' \arrow[ul, "u'"] \arrow[ur, "v'"'] & \end{tikzcd}$$

in which the 2-cells ϕ, ψ are invertible. When ϕ, ψ are understood (ϕ, f, ψ) is abbreviated to f. When ϕ, ψ are identities, f is called a *strict homomorphism* (or an *arrow of spans*).

(3.3) A *2-cell* $\sigma: (\phi, f, \psi) \Rightarrow (\gamma, g, \kappa)$ *of homomorphisms of spans* is a 2-cell $\sigma: f \Rightarrow g$ such that

$$u'\sigma \cdot \phi = \gamma, \quad v'\sigma \cdot \psi = \kappa .$$

(3.4) Let $(\mathcal{S}pn\mathcal{K})(B, A)$ denote the bicategory of spans from B to A, homomorphisms, and 2-cells, with the obvious compositions. When $\mathcal{K}$ is understood we write $\mathcal{S}pn(B, A)$.

(3.5) For each list of objects $A_0, \ldots, A_n$, we have a homomorphism of bicategories

$$\mathcal{C}mp_n : \prod_{i \in \mathrm{n}} \mathcal{S}pn(A_{i+1}, A_i) \to \mathcal{S}pn(A_n, A_0)$$

called *composition* whose value at $S_n, \ldots, S_1$ is the bilimit of the diagram

$$\begin{tikzcd}[column sep=small]
 & S_1 \arrow[dl] \arrow[dr] & & S_2 \arrow[dl] \arrow[dr] & & S_3 \arrow[dl] & & S_n \arrow[dr] & \\
A_0 & & A_1 & & A_2 & & \cdots & & A_n
\end{tikzcd}$$

denoted by $S_1 \circ \ldots \circ S_n$. Of course, it is understood that, when $n = 0$, the span A_0 from A_0 to A_0 is picked out by $\mathcal{C}mp_0$, and, when $n = 1$, $\mathcal{C}mp_1$ is the identity homomorphism.

(3.6) Suppose $n_1, \ldots, n_k$ is a list of integers and put

$$\mathrm{m}_j = \mathrm{n}_1 + \ldots + \mathrm{n}_j \quad \text{for } j = 0, \ldots, k.$$

For any list $A_0, \ldots, A_{m_k}$ of objects, there is an equivalence

$$\begin{tikzcd}
\prod_{i \in \mathrm{m}_k} \mathcal{S}pn(A_{i+1}, A_i) \arrow[rr, "\mathcal{C}mp_{m_k}"] \arrow[dr] & \arrow[d, phantom, "\alpha \Downarrow \approx"] & \mathcal{S}pn(A_{m_k}, A_0) \\
 & \prod_{j \in \mathrm{k}} \mathcal{S}pn(A_{m_{j+1}}, A_{m_j}) \arrow[ur] &
\end{tikzcd}$$

and any diagram involving expanded instances of α and α^{-1} commutes

up to a unique canonical invertible modification.

(3.7) For a span (u, S, v) from B to A and arrows $a: X \to A$, $b: Y \to B$, the composite span (3.5)

$$(1_X, X, a) \circ (u, S, v) \circ (b, Y, 1_Y)$$

from Y to X is denoted by $S(a, b)$ and called *the fibre of* S *over* a, b. The notion of «fibration» below arises from the question: for what S is $S(a, b)$ functorial in a, b?

(3.8) For each object A, we write HA for the span $(d_0, \{2, A\} d_1)$ from A to A, where

$$d_0 = \{\partial_0, 1\}, \quad d_1 = \{\partial_1, 1\}.$$

For arrows $a: X \to A$, $b: Y \to A$, the span $HA(a, b)$ (as defined in (3.7) with HA for S) is called *the bicomma object of* a, b. The canonical arrow $HA(a, b) \to \{2, A\}$ corresponds to a 2-cell

$$\begin{tikzcd} HA(a,b) \arrow[r, "d_1"] \arrow[d, "d_0"'] \arrow[dr, phantom, "\overset{\lambda}{\Rightarrow}"] & Y \arrow[d, "b"] \\ X \arrow[r, "a"'] & A \end{tikzcd}$$

and $HA(a, b)$ can also be characterized as appearing in the biuniversal such diagram with a, b fixed.

(3.9) In $\mathcal{S}pn(A, A)$, there is an equivalence

$$\underset{(n \text{ terms})}{(HA) \circ \dots \circ (HA)} \simeq (\{\text{first elt}, 1\}, \{n+1, A\}, \{\text{last elt}, 1\})$$

which is unique up to a unique isomorphism. Thus the functors $\iota_0 : 2 \to 1$, $\partial_1 : 2 \to 3$ induce homomorphisms of spans

(3.10) $$A \xrightarrow{\eta A} HA \xleftarrow{\mu A} (HA) \circ (HA)$$

for which there are unique canonical invertible 2-cells

$$rA: 1_{HA} \simeq (\mu A)(HA \circ \eta A), \quad lA: (\mu A)(\eta A \circ HA) \simeq 1_{HA},$$
$$aA: (\mu A)(HA \circ \mu A) \simeq (\mu A)(\mu A \circ HA).$$

These data uniquely determine a homomorphism of bicategories

$$\mathrm{H}A : \Delta \to \mathcal{S}pn(A, A)$$

which preserves the monoidal structures ; in particular, the image of (2.3) under $\mathrm{H}A$ is (3.10).

(3.11) A more global description of $\mathrm{H}A$ is as follows. First observe that Δ is isomorphic to the dual of the category of non-empty ordinals and first-and-last-element-preserving functors (leave off the top and bottom ∂'s from (2.2) and what remains is the dual of (2.2) after renaming). So we have a full monomorphism $\Delta \to \Delta^{op}$ given by

$$n \mapsto n+1 , \quad \partial_j \mapsto \iota_j , \quad \iota_j \mapsto \partial_{j+1} .$$

Refer now to (1.29) and consider the diagram

$$\begin{tikzcd}
 & \Delta \arrow[dl] \arrow[d] \arrow[dr] & \\
1 \arrow[dr] \arrow[r, phantom, "\overset{(\text{first elt})^*}{\Leftarrow}"] & \Delta^{op} \arrow[d, "\{-,A\}"] \arrow[r, phantom, "\overset{(\text{last elt})^*}{\Rightarrow}"] & 1 \arrow[dl] \\
 & \mathcal{K} &
\end{tikzcd}$$

This induces a homomorphism of bicategories $\Delta \to \mathcal{S}pn(A, A)$ which is equivalent to $\mathrm{H}A$.

(3.12) Composition of spans $\mathcal{C}mp_2$ (3.5) determines a homomorphism of bicategories

$$\mathcal{S}pn(A, A) \longrightarrow \mathcal{H}om(\mathcal{S}pn(B, A), \mathcal{S}pn(B, A)),$$

and composition of this homomorphism with $\mathrm{H}A$ yields a doctrine L on $\mathcal{S}pn(B, A)$. Dually (replacing $\mathcal{K}$ by $\mathcal{K}^{co}$), we obtain a doctrine R on $\mathcal{S}pn(B, A)$ using $\mathrm{H}B$. Composition of spans $\mathcal{C}mp_3$ (3.5) determines a homomorphism of bicategories

$$\mathcal{S}pn(B, B) \times \mathcal{S}pn(A, A) \longrightarrow \mathcal{H}om(\mathcal{S}pn(B, A), \mathcal{S}pn(B, A))$$

and composition of this homomorphism with

$$\begin{bmatrix} \mathrm{H}B \\ \mathrm{H}A \end{bmatrix} : \Delta \longrightarrow \mathcal{S}pn(B, B) \times \mathcal{S}pn(A, A)$$

yields a third doctrine M on $\mathcal{S}pn(B, A)$ (which is the «composite» of L, R in the sense of «distributive laws» whose theory we do not develop here).

(3.13) Write

$$(\mathcal{F}il\mathcal{K})(B, A), \quad (\mathcal{F}in\mathcal{K})(B, A), \quad (\mathcal{F}ib\mathcal{K})(B, A)$$

for the bicategories of algebras for L, R, M, respectively, where in each case the arrows are the strong morphisms (2.19). The objects of these three bicategories are respectively called *left fibrations, right fibrations, fibrations, from* B *to* A. For a given span E from B to A, a structure E of (left, right) fibration on E is called a *(left, right) cleavage for* E.

(3.14) The adjunctions $\iota_0 \dashv \partial_1 \dashv \iota_1$ between the two ordinals 2, 3 give rise to adjunctions

$$HB \circ \eta B \dashv \mu B \dashv \eta B \circ HB$$

with unit for the first being rB, and counit for the second being lB. From (2.27), (3.12), we obtain:

(3.15) R *is a KZ-doctrine in* $\mathcal{S}pn(B, A)$. It follows from (2.29) that *a right cleavage for a span* E *from* B *to* A *is unique up to isomorphism, if one exists at all.*

(3.16) For each span (p, E, q) from B to A, there is an arrow (3.8) $\hat{q}: HE \to HB(q, 1_B)$ which is unique up to isomorphism with the property that $\lambda \hat{q} \approx q\lambda$. In fact, $\hat{q}$ can be regarded as an arrow in $\mathcal{S}pn(1, A)$ where the necessary arrows into A are

$$HE \xrightarrow{d_0} E \xrightarrow{p} A, \quad HB(q, 1_B) \xrightarrow{d_0} E \xrightarrow{p} A.$$

(3.17) *Chevalley criterion. There exists a right cleavage for the span* (p, E, q) *from* B *to* A *iff the arrow* $\hat{q}: HE \to HB(q, 1_B)$ *in* $\mathcal{S}pn(1, A)$ *has a left adjoint with invertible unit.*

PROOF. Since $\{\partial_0, 1\}: \{2, B\} \to B$ has a left adjoint $\{\iota_0, 1\}$ with invertible unit, so does

$$d_0: HB(q, 1_B) \approx E \circ HB \to E.$$

The counit of the latter adjunction is a 2-cell beween endo-arrows of $HB(q, 1_B)$. This 2-cell yields an arrow

$$k: HB(q, 1_B) \to \{2, HB(q, 1_B)\}.$$

If (ζ, l, a) is a right cleavage for E then the composite

$$HB(q, 1_B) \xrightarrow[k]{} \{2, HB(q, 1_B)\} \xrightarrow[\{1, \zeta\}]{} \{2, E\} = HE$$

is a left adjoint for $\hat{q}$ with invertible unit. Conversely, if $h \dashv \hat{q}$ with invertible unit then the composite

$$\zeta: E \circ HB \approx HB(q, 1_B) \xrightarrow[h]{} HE \xrightarrow[d_1]{} E$$

is a left adjoint for $E \circ \eta B$ with invertible counit l; this leads to a right cleavage (ζ, l, a) for E. □

(3.18) It follows from (3.14) that L is a «dual» KZ-doctrine on $\mathcal{S}pn(B, A)$. In particular, left cleavages lead to right adjoints to $\eta A \circ E$ with invertible units. So left cleavages are unique up to isomorphism.

(3.19) The doctrine M (3.12) is a «composite» of the KZ-doctrine R and the dual KZ-doctrine L, yet it is not in any sense a «KZ-doctrine» itself. However, *cleavages on a span are unique up to isomorphism* as can be deduced from the following:

(3.20) PROPOSITION. *Suppose (ζ, l, a) is a cleavage for a span E from B to A. Then:*

(i) composition with $\eta A \circ 1: E \circ HB \to HA \circ E \circ HB$ yields a right cleavage (ζ_0, l_0, a_0) for E;

(ii) composition with $1 \circ \eta B: HA \circ E \to HA \circ E \circ HB$ yields a left cleavage (ζ_1, l_1, a_1) for E;

(iii) there are canonical invertible 2-cells

$$\begin{tikzcd}
HA \circ E \circ HB \arrow[rr, "1 \circ \zeta_0"] \arrow[d, "\zeta_1 \circ 1"'] \arrow[drr, "\zeta"] \arrow[dr, phantom, "\overset{\approx}{\Longrightarrow}"] & & HA \circ E \arrow[d, "\zeta_1"] \arrow[dl, phantom, "\overset{\approx}{\Longrightarrow}"] \\
E \circ HB \arrow[rr, "\zeta_0"'] & & E
\end{tikzcd}$$

which, together with l_0, a_0, l_1, a_1 *uniquely determine* l, a. □

(3.21) Suppose $f: E \to E'$ is a homomorphism of spans from A to B. From (2.30) we see that right cleavages for E, E' give rise to a unique structure of morphism of R-algebras (2.16) on f; when this morphism is strong (2.18), we say that f *is right cartesian*. The arrows of $\mathcal{F}ir(B, A)$ are precisely the right-cartesian homomorphisms. Dually, the arrows of $\mathcal{F}il(B, A)$ are called *left-cartesian* homomorphisms. Cleavages for E, E' give rise, by (3.20), to right and left cleavages for E, E'. The unique structure of morphisms of R-, and L-algebras on f yield a unique morphism of M-algebras on f. When this morphism is strong we say that f is *cartesian*; this amounts precisely to saying that f is both left and right cartesian. The arrows of $\mathcal{F}ib(B, A)$ are the cartesian homomorphisms.

(3.22) Suppose (p, E, q) is a right fibration from B to A. For each arrow $e: Y \to E$ and 2-cell $\beta: qe \Rightarrow b$, an arrow $Y \to HB(q, 1_B)$ is induced (3.8) (it is unique up to isomorphism), and composition with a left adjoint h of $\hat{q}$ (3.17) yields an arrow $Y \to HE$. This arrow corresponds to a 2-cell $\zeta_\beta: e \Rightarrow \beta_*(e)$, called *the direct image of* e *under* β with the following properties:

(i) $p\zeta_\beta: pe \Rightarrow p\beta_*(e)$ is an isomorphism;

(ii) there is an isomorphism $\gamma: b \Rightarrow q\beta_*(e)$ such that $\gamma . \beta = q\zeta_\beta$ and, for all 2-cells

$$\xi: e \Rightarrow e', \quad \beta': b \Rightarrow qe' \quad \text{with} \quad q\xi = \beta'\beta,$$

there exists a unique 2-cell

$$\xi': \beta_*(e) \Rightarrow e' \text{ such that } \xi' . \zeta_\beta = \xi \text{ and } q\xi' . \gamma = \beta';$$

(iii) for all $y: Y' \to Y$, the canonical 2-cell $(\beta y)_*(ey) \Rightarrow \beta_*(e)y$ is invertible.

Notice that the notion of direct image contains the heart of the Chevalley criterion (3.17); the left adjoint for $\hat{q}$ corresponds to the direct image of $d_0: HB(q, 1_B) \to E$ under $\lambda: qd_0 \Rightarrow d_1$ (3.8).

(3.23) Dually to (3.22), for a left fibration (p, E, q) we have the no-

tion of *inverse image* $\zeta^a : a^*(e) \Rightarrow e$ of an arrow $e : X \to E$ under a 2-cell $a : a \Rightarrow pe$.

(3.24) PROPOSITION. *For arrows* $a : X \to A$, $b : Y \to B$, *if* E *is a (left, right) fibration from* B *to* A *then the fibre* $E(a, b)$ *(3.7) is a (left, right) fibration from* Y *to* X.

PROOF. Consider the diagram which follows:

$$\begin{tikzcd}
 & & & HX \circ E(a,b) \arrow[ddll, dashed] \arrow[drr] & & & \\
 & & & & & E(a,b) \arrow[dll, dashed] \arrow[ddr, dashed] & \\
 & \{2, X\} \arrow[dl, "d_0"'] \arrow[dr, "d_1"] \arrow[d, phantom, "\overset{\lambda}{\Rightarrow}"] \arrow[rr, phantom, "\underset{\approx}{\Rightarrow}"] & & E(a,B) \arrow[dl] \arrow[dr] & & & \\
X \arrow[rr, "1_X"'] & {} & X \arrow[dr, "a"'] \arrow[rr, phantom, "\underset{\approx}{\Rightarrow}"] & & E \arrow[dl, dashed, "p"] \arrow[dr, dashed, "q"'] \arrow[rr, phantom, "\overset{\approx}{\Rightarrow}"] & & Y \arrow[dl, dashed, "b"] \\
 & & & A & & B &
\end{tikzcd}$$

If E is a left fibration we can apply inverse image (3.23) to the part of the diagram obtained by ignoring the dotted arrows. This yields a 2-cell

$$\begin{tikzcd}
HX \circ E(a,b) \arrow[r, bend left] \arrow[r, bend right, dashed] \arrow[r, phantom, "\Downarrow"] & E
\end{tikzcd}$$

which corresponds to an arrow $HX \circ E(a, b) \to E$. This, together with the canonical arrows

$$HX \circ E(a, b) \to X, \quad HX \circ E(a, b) \to Y,$$

gives a left cleavage $\zeta : HX \circ E(a, b) \to E(a, b)$ for $E(a, b)$. Dually, if E is a right fibration we obtain a right cleavage for $E(a, b)$. If E is a fibration, it can be seen that the left and right cleavages so obtained yield a cleavage (3.20) (iii). □

(3.25) For a left fibration E from B to A, we shall now describe a homomorphism of bicategories

$$E(-, B) : \mathcal{K}(X, A)^{op} \to \mathfrak{Fil}(B, X).$$

For $a: X \to A$, put $E(-, B)a = E(a, B)$, the fibre of E over a, 1_B. For any 2-cell $\alpha : a \Rightarrow a'$, we obtain a left cartesian homomorphism denoted by $E(\alpha, B): E(a', B) \to E(a, b)$ as follows. Apply inverse image to the diagram

$$\begin{tikzcd}
& E(a', B) \arrow[dl] \arrow[dr] & \\
X \arrow[dr, "a'"] \arrow[dr, bend right=50, "a"'] \arrow[rr, phantom, "\overset{\approx}{\Longrightarrow}"] & & E \arrow[dl, "p"] \\
& A &
\end{tikzcd}$$

(with $\overset{\Longrightarrow}{\alpha}$ between a and a')

to obtain a 2-cell whose source is an arrow $E(a', B) \to E$, which, together with $E(a', B) \to X$, determines the required arrow $E(\alpha, B)$.

(3.26) Dually, a right fibration E from B to A determines a homomorphism of bicategories

$$E(A, -): \mathcal{K}(Y, B) \to \mathcal{F}\mathit{in}(Y, A).$$

(3.27) A fibration E from B to A determines a homomorphism of bicategories

$$E(-, -): \mathcal{K}(X, A)^{op} \times \mathcal{K}(Y, B) \to \mathcal{F}\mathit{ib}(Y, X)$$

(as foreshadowed in (3.7)). By (3.24) we may take

$$E(-, -)(a, b) = E(a, b).$$

For $\alpha : a \Rightarrow a'$, $\beta : b \Rightarrow b'$, we take $E(\alpha, \beta): E(a', b) \to E(a, b')$ to be isomorphic to the composite

$$E(a', b) \approx E(a', B)(X, b) \xrightarrow{E(a', B)(X, \beta)} E(a', B)(X, b')$$
$$\approx E(A, b')(a', Y) \xrightarrow{E(A, b')(\alpha, Y)} E(A, b')(a, Y) \approx E(a, b')$$

(or the dual isomorphic composite).

(3.28) *The bicategories* $\mathcal{S}\mathit{pn}(B, A)$, $\mathcal{F}\mathit{il}(B, A)$, $\mathcal{F}\mathit{in}(B, A)$, $\mathcal{F}\mathit{ib}(B, A)$ *are finitely bicat. complete* (1.27). For a span (u, S, v) from B to A, the cotensor biproduct $\{2, (u, S, v)\}$ in $\mathcal{S}\mathit{pn}(B, A)$, denoted by $\{2, S\}_B^A$, is the bilimit in $\mathcal{K}$ of the diagram

$$A \xrightarrow{\{\iota_0, 1\}} \{2, A\} \xleftarrow{\{1, u\}} \{2, S\} \xrightarrow{\{1, v\}} \{2, B\} \xleftarrow{\{\iota_0, 1\}} B.$$

The biterminal object of $\mathcal{S}pn(B, A)$ is $(pr_1, A \Pi\!\!\Pi B, pr_2)$ (1.23). The biproduct $S \Pi\!\!\Pi_B^A T$ of spans S, T from B to A is the bilimit in $\mathcal{K}$ of the diagram

$$\begin{tikzcd} S \arrow[r] \arrow[d] & B \\ A & T \arrow[l] \arrow[u] \end{tikzcd}$$

The biequalizer of two homomorphisms $f, g: S \to T$ of spans from B to A is just the biequalizer of f, g in $\mathcal{K}$ made into a span in the obvious way. Any bicategory of algebras and strong morphisms for a doctrine on a finitely bicat. complete bicategory is finitely bicat. complete and the underlying homomorphism preserves the indexed bilimits.

(3.29) A (left, right) fibration E from B to A is called *groupoidal*, *posetal* or *bidiscrete* according as the underlying span has the property in the bicategory $\mathcal{S}pn(B, A)$ (1.7). Clearly (as is more generally true for bicategories of algebras of a doctrine) this amounts to saying E has the property in the appropriate bicategory $\mathcal{F}il(B, A)$, $\mathcal{F}ir(B, A)$, $\mathcal{F}ib(B, A)$. We have locally groupoidal bicategories

(3.30) $\qquad G\mathcal{F}il(B, A), \quad G\mathcal{F}ir(B, A), \quad G\mathcal{F}ib(B, A)$

and categories

(3.31) $\qquad D\mathcal{F}il(B, A), \quad D\mathcal{F}ir(B, A), \quad D\mathcal{F}ib(B, A)$

in the notation of (1.7).

(3.32) PROPOSITION. *Any homomorphism of spans (3.2) from a (left, right) fibration to a groupoidal (left, right) fibration is automatically (left, right) cartesian (3.20).*

PROOF. If $f: E \to E'$ is a homomorphism between right fibrations, then the unique structure of morphism of R-algebras on f (3.21) is a 2-cell between homomorphisms of spans from $E \circ HB$ to E'. So, if E' is groupoidal, this 2-cell is an isomorphism and hence f is right cartesian. Left

cartesian follows dually and cartesian conjunctively. □

(3.33) COROLLARY. *The forgetful homomorphisms from the bicategories (3.30) and the categories (3.31) to* $\mathcal{S}pn(B, A)$ *are fully faithful (that is, local isomorphisms).*

(3.34) Let Iso denote the category with two distinct objects and an isomorphism between them; and let Ppr be the category with two distinct objects, two arrows between them in one direction and no arrows in the other direction (the only other arrows being identities). The functors

$$2 \to 1, \quad \mathrm{Ppr} \to 2, \quad \mathrm{Ppr} \to \mathrm{Iso}$$

(the latter two being bijective on objects) induce arrows

$$A \to \{2, A\}, \quad \{2, A\} \to \{\mathrm{Ppr}, A\}, \quad \{\mathrm{Iso}, A\} \to \{\mathrm{Ppr}, A\}$$

for each object A of $\mathcal{K}$. *The object* A *is respectively groupoidal, posetal, bidiscrete (1.7) iff the first, second, third of these arrows is an equivalence* (1.5). This shows the bilimit-nature of these conditions and applied in the bicategories of spans yields the next two statements.

(3.35) *The bicategories (3.30) are finitely bicat. complete and the categories (3.31) are finitely complete; the bilimits and limits are constructed as for spans (3.28).*

(3.36) *If* S *is a groupoidal, posetal, bidiscrete span, then so is any fibre* $S(a, b)$ *(3.7) of* S.

(3.37) There is a homomorphism of bicategories

$$D\mathcal{F}i\ell : \mathcal{K}^{op} \times \mathcal{K}^{coop} \to \mathcal{C}at$$

described as follows. For objects B, A, the category $D\mathcal{F}i\ell(B, A)$ is the category of bidiscrete fibrations from B to A (3.31). For arrows $a : X \to A$, $b : Y \to B$, the functor

$$D\mathcal{F}i\ell(b, a) : D\mathcal{F}i\ell(B, A) \to D\mathcal{F}i\ell(Y, X)$$

is given on objects by $D\mathcal{F}i\ell(b, a)E = E(a, b)$ and on arrows by using the biuniversal property of the fibre. For 2-cells $\alpha : a \Rightarrow a'$, $\beta : b \Rightarrow b'$,

the natural transformation $D\mathcal{F}i\mathcal{b}(\alpha,\beta)$ has $E(\alpha,\beta)$ (3.27) as its component at E.

(3.38) Of course, for objects A, B, there are homomorphisms of bicategories

$$D\mathcal{F}i\ell(B,-): \mathcal{K}^{coop} \to \mathcal{C}at, \quad D\mathcal{F}in(-,A): \mathcal{K}^{op} \to \mathcal{C}at,$$

described similarly.

(3.39) REMARK. The 2-category $\mathcal{C}at$ appearing in (3.37), (3.38) is based on a category of sets large enough to contain the set of 2-cells of $\mathcal{K}$ as an object.

(3.40) When dealing with spans from B to 1 we write (S,v) instead of (u,S,v) since u is unique up to isomorphism. Note also that

$$\mathcal{F}in(B,1) = \mathcal{F}i\mathcal{b}(B,1), \quad \mathcal{F}i\ell(B,1) = \mathcal{S}pn(B,1).$$

(3.41) PROPOSITION. *A span (p,E,q) from B to A is a right fibration iff (E,q) is a fibration from B to A and $\binom{p}{q}: (E,q) \to (A\Pi B, pr_2)$ is a cartesian homomorphism.* □

(3.42) PROPOSITION. *Suppose $r: (E,q) \to (E',q')$ is a homomorphism from B to 1 and that (E',q') is a fibration from B to 1. Then (E,r) is a fibration from E' to 1 iff (E,q) is a fibration from B to 1 and $r: (E,q) \to (E',q')$ is a cartesian homomorphism. Under these conditions, $((E,q),r)$ is a fibration from (E',q') to $1 = (B, 1_B)$ in the bicategory $\mathcal{F}i\mathcal{b}(B,1)$.* □

(3.43) PROPOSITION. *For each object A, the span HA from A to A is a bidiscrete fibration.* The cleavage is $\{\partial_1\partial_1, A\}: \{0,A\} \to \{2,A\}$.

(3.44) *Each bicomma object (3.8) is a discrete fibration* (combine (3.24), (3.36), (3.43)). In general there are bidiscrete fibrations not equivalent to bicomma objects (see (4.10)).

4. FIBRATIONAL COMPOSITION.

(4.1) The set of path components of a category A can be constructed in $\mathcal{C}at$ as the coidentifier of the 2-cell

$$\begin{tikzcd}[2,A] \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\lambda\Downarrow"] & A\end{tikzcd}$$

(see Street [14] page 153 for the notion of identifier). This construction is not bicategorical. The most obvious analogue which comes to mind is the bicoinverter of

$$\begin{tikzcd}[2,A] \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\lambda\Downarrow"] & A\end{tikzcd} ;$$

but this only gives the reflection of A into the 2-category of groupoids. Further investigation leads one to see that the reflection of A into the 2-category of bidiscrete categories can be obtained as the biuniversal $A \to \pi A$ which renders the two birepresenting 2-cells

$$\begin{tikzcd}\{\mathrm{Ppr}, A\} \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow\ \Downarrow"] & A\end{tikzcd}$$

equal and invertible. This leads us to the following definition.

(4.2) Let 2_3 denote the 2-category whose distinct non-identity arrows and 2-cells are depicted below

$$\begin{tikzcd}0 \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow\ \Downarrow"] & 1\end{tikzcd}$$

Let $J : 2_3^{op} \to \mathcal{C}at$ denote the 2-functor which assigns the natural isomorphism

$$\begin{tikzcd}\mathrm{Iso} \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & 1\end{tikzcd}$$

(either non-identity such will do; see (3.34)) to both the non-identity 2-cells of 2_3. Any pair of 2-cells

$$\begin{tikzcd}A \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\sigma\Downarrow\ \Downarrow\tau"] & B\end{tikzcd}$$

in $\mathcal{K}$ determines a homomorphism of bicategories $S : 2_3 \to \mathcal{K}$. A J-indexed

bicolimit $J * S$ (1.26) of S is called a *bicoequinverter of* σ, τ ; it amounts to a biuniversal arrow $w: B \to C$ such that $w\sigma$, $w\tau$ are equal invertible 2-cells.

(4.3) The following lemma on bicoequinverters will be of use in our discussion of fibrational composition. Consider the data displayed in the diagram

$$\begin{tikzcd} X \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\sigma\Downarrow\ \tau\Downarrow"] \arrow[d, "u"'] & Y \arrow[r, "h"] \arrow[d, "v"] & Z \arrow[d, "w"] \\ X' \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\sigma'\Downarrow\ \tau'\Downarrow"] & Y' \arrow[r, "h'"] & Z' \end{tikzcd}$$

together with invertible arrows

$$(\theta, \theta'): \sigma' u \Rightarrow v\sigma, \quad (\phi, \phi'): \tau' u \Rightarrow v\tau \quad \text{in } [2, \mathcal{K}(X, Y')]$$

and an invertible 2-cell $\psi: h'v \Rightarrow wh$. Suppose that the top row is a bicoequinverter and, for each object K, the functor

$$\mathcal{K}(u, K): \mathcal{K}(X', K) \to \mathcal{K}(X, K)$$

is fully faithful (as is the case for example when u is the bicoequinverter of some pair of 2-cells). *Under these conditions, the bottom row is a bicoequinverter iff* h', w, ψ *form a bipushout for* h, v.

(4.4) Suppose E is a right fibration from B to A and F is a left fibration from C to B (3.13). Let $\zeta_0: E \circ HB \to E$, $\zeta_1: HB \circ F \to F$ be right, left cleavages for E, F, respectively. The Chevalley criterion (3.17), (3.22) gives 2-cells

$$\begin{tikzcd} E \circ HB \arrow[r, bend left, "pr_1"] \arrow[r, bend right, "\zeta_0"'] \arrow[r, phantom, "\Downarrow"] & E \end{tikzcd} \qquad \begin{tikzcd} HB \circ F \arrow[r, bend left, "\zeta_1"] \arrow[r, bend right, "pr_2"'] \arrow[r, phantom, "\Downarrow"] & F \end{tikzcd}$$

which compose with

$$pr_1: E \circ HB \circ F \to E \circ HB, \quad pr_2: E \circ HB \circ F \to HB \circ F$$

to yield two 2-cells which induce (using the bipullback property of $E \circ F$)

a 2-cell

$$(4.5)\qquad \begin{tikzcd} E\circ HB\circ F \arrow[r, bend left, "E\circ\zeta_1"] \arrow[r, bend right, "\zeta_0\circ F"'] \arrow[r, phantom, "\Downarrow"] & E\circ F \end{tikzcd}$$

in $\mathcal{S}pn(C, A)$. Composition of (4.5) with the arrow

$$E\circ\eta B\circ F : E\circ F \to E\circ HB\circ F$$

yields a 2-cell canonically isomorphic to the identity 2-cell of the identity arrow of $E\circ F$. Next, form the bilimit of the diagram

$$\begin{tikzcd} E\circ HB\circ F \arrow[r, "E\circ\zeta_1"] \arrow[dr, "\zeta_0\circ F"'] & E\circ F & E\circ HB\circ F \arrow[l, "E\circ\zeta_1"'] \arrow[dl, "\zeta_0\circ F"] \\ & E\circ F & \end{tikzcd}$$

which yields two arrows $pr_1, pr_2 : E\hat{\circ}F \to E\circ HB\circ F$ and invertible 2-cells

$$(E\circ\zeta_1)pr_1 \Rightarrow (E\circ\zeta_1)pr_2,\quad (\zeta_0\circ F)pr_1 \Rightarrow (\zeta_0\circ F)pr_2 .$$

Composing with each of pr_1, pr_2 yields two 2-cells

$$(4.6)\qquad \begin{tikzcd} E\hat{\circ}F \arrow[r, bend left, "\rho_1"] \arrow[r, bend right, "\rho_0"'] \arrow[r, phantom, "\Downarrow\ \Downarrow"] & E\circ F \end{tikzcd}$$

where

$$\rho_1 \approx (E\circ\zeta_1)pr_1 \approx (E\circ\zeta_1)pr_2,\quad \rho_0 \approx (\zeta_0\circ F)pr_1 \approx (\zeta_0\circ F)pr_2 .$$

(4.7) The *fibrational composite* $E\otimes F$ *of* E, F is the bicoequinverter of the two 2-cells (4.6) as a span from C to A; it is unique up to equivalence when it exists. There is a canonical homomorphism of spans

$$c = c_2 : E\circ F \to E\otimes F .$$

(4.8) PROPOSITION. *(a) For any left fibration F from B to A, the following diagram is an absolute bicoequinverter.*

$$\begin{tikzcd} HA\hat{\circ}F \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow(4.6)\Downarrow"] & HA\circ F \arrow[r, "\zeta_1"] & F . \end{tikzcd}$$

(b) For any right fibration E from B to A, the following diagram is an absolute bicoequinverter

$$\begin{tikzcd} E \hat{\circ} HB \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow (4.6) \Downarrow"] & E \circ HB \arrow[r, "\zeta_0"] & E. \end{tikzcd}$$

(c) A fibration (p, E, q) from B to A is bidiscrete iff a cleavage for E induces an equivalence

$$HA(A, p) \otimes HB(q, B) \approx E$$

in $\mathcal{S}pn(B, A)$. Furthermore, in this case, the following diagram is an absolute bicoequinverter.

$$\begin{tikzcd} HA(A,p) \hat{\circ} HB(q,B) \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow (4.6) \Downarrow"] & HA(A,p) \circ HB(q,B) \arrow[r, "\zeta"] & E. \end{tikzcd}$$

PROOF. Let

$$\begin{tikzcd} \cdot \arrow[r, bend left, "u"] \arrow[r, bend right, "v"'] \arrow[r, phantom, "\sigma \Downarrow \; \Downarrow \tau"] & \cdot \arrow[r, "w"] & \cdot \end{tikzcd}$$

denote any of the diagrams in (a), (b), (c). In case (a), one finds arrows $\xleftarrow{t} \cdot \xleftarrow{s}$ and coherent isomorphisms

$$ws \approx 1, \quad ut \approx 1, \quad vt \approx sw \quad \text{such that} \quad \sigma t = \tau t\ ;$$

these equations together with the fact that $w\sigma = w\tau$ and is invertible yield the bicoequinverter property which must thus be absolute. Case (b) is similar. In case (c) one sees that $w\sigma = w\tau$ and is invertible precisely when E is bidiscrete, and one always has arrows

$$\xleftarrow{t_i} \xleftarrow{s} \qquad i = 1, 2, 3,$$

with coherent isomorphisms

$$ws \approx 1, \quad ut_1 \approx 1, \quad vt_3 \approx sw$$

and 2-cells $t_1 \Rightarrow t_2 \Rightarrow t_3$ such that v inverts the first of these 2-cells, u inverts the second and each t_i equalizes σ, τ. □

(4.9) In the situation of (4.8) (a), for any arrow $a: A' \to A$, bipullback

along a preserves the bicoequinverter, so there is a canonical equivalence

$$HA(a, A) \otimes F \approx F(a, B).$$

Similarly, in the situation of (4.8) (b), for any arrow $b: B' \to B$, there is a canonical equivalence

$$E \otimes HB(B, b) \approx E(A, b).$$

(4.10) As pointed out in (3.44), not every bidiscrete fibration is a bicomma object; (4.8) (c) shows that *every bidiscrete fibration is a fibrational composite of two bicomma objects.*

(4.11) *Fibrational bicategories.* A bicategory $\mathcal{K}$ will be called *fibrational* when it is finitely bicategorically complete, each pair of 2-cells with the same source and target has a bicoequinverter, and bipullback along a leg of a fibration preserves bicoequinverters.

(4.12) *It will be assumed for the remainder of this section that we are working in a fibrational bicategory* $\mathcal{K}$.

(4.13) PROPOSITION. *If* E *is a (right) fibration from* B *to* A *and* F *is a fibration from* C *to* B *then* $E \otimes F$ *is a (right) fibration from* C *to* A.

PROOF. The bicoequinverter of (4.6) is taken to a bicoequinverter by $- \circ HB$ from (4.11). The right cleavage of F induces an arrow

$$(E \circ F) \circ HB \to E \circ F$$

whose composite with $E \circ F \to E \otimes F$ renders equal and invertible the two 2-cells obtained by applying $- \circ HB$ to (4.6). So an arrow

$$(E \otimes F) \circ HB \to E \otimes F$$

is induced which is a right cleavage for $E \otimes F$. □

(4.14) Suppose E is a right fibration from B to A, F is a fibration from C to B, and G is a left fibration from D to C. Define the span $E \otimes F \otimes G$ from D to A by the following bipushout, which exists by (4.3) and (4.11).

$$\begin{tikzcd}
E \circ F \circ G \arrow[rr, dashed, "(c \circ 1)a"] \arrow[d, "(1 \circ c)a"'] \arrow[drr, phantom, "\overset{\Rightarrow}{\approx}"] & & (E \otimes F) \circ G \arrow[d] \\
E \circ (F \otimes G) \arrow[rr] & & E \otimes F \otimes G
\end{tikzcd}$$

Moreover, the equivalences a of (3.6) induce equivalences

$$a^b \text{ of } E \otimes F \otimes G \text{ with } (E \otimes F) \otimes G \text{ and } E \otimes (F \otimes G)$$

which are left cartesian when E is a fibration and right cartesian when G is a fibration.

(4.15) More generally, for each list of objects $A_0, \dots, A_n$, we have a homomorphism of bicategories

$$\mathcal{F}\mathit{ibcmp}_n : \prod_{i \in n} \mathcal{F}\mathit{ib}(A_{i+1}, A_i) \to \mathcal{F}\mathit{ib}(A_n, A_0)$$

called *n-fold fibrational composition*. For $n = 0$, the homomorphism picks out the fibration HA_0 from A_0 to A_0. For $n = 1$, the homomorphism is the identity. For $n = 2$, the value at F, E is $E \otimes F$ (4.7). For $n = 3$, the value at G, F, E is $E \otimes F \otimes G$ (4.14). For $n \geq 3$, the value at $E_n, \dots, E_1$ is the bicolimit of the following $n - 1$ arrows with common source:

$$\begin{tikzcd}
E_1 \circ \dots \circ E_n \arrow[r, phantom, "\overset{a}{\cong}"] & E_1 \circ \dots \circ (E_i \circ E_{i+1}) \circ \dots \circ E_n \arrow[d, "1 \circ c \circ 1"] \\
& E_1 \circ \dots \circ (E_i \otimes E_{i+1}) \circ \dots \circ E_n .
\end{tikzcd}$$

Write $E_1 \otimes \dots \otimes E_n$ for $\mathcal{F}\mathit{ibcmp}_n(E_n, \dots, E_1)$. There is a canonical homomorphism of spans

$$c_n : E_1 \circ \dots \circ E_n \to E_1 \otimes \dots \otimes E_n ;$$

these provide the components of a strong transformation

$$c_n : \mathcal{C}\mathit{mp}_n \to \mathcal{F}\mathit{ibcmp}_n \qquad (3.5).$$

(4.16) The equivalences a of (3.6) lift through the underlying homomorphisms $\mathcal{F}\mathit{ib}(A_i, A_j) \to \mathcal{S}\mathit{pn}(A_i, A_j)$ to yield equivalences:

$$a^b : \mathcal{F}\mathit{ibcmp}_{m_k} \Rightarrow \mathcal{F}\mathit{ibcmp}_k . \prod_j \mathcal{F}\mathit{ibcmp}_{m_j} .$$

Any diagram involving expanded instances of these equivalences and their inverses commute up to a unique canonical invertible modification.

(4.17) A fibration E from B to A is said to be *Cauchy* (inspired by Lawvere [9] page 163) when it has a right biadjoint E^*; that is, a fibration E^* from A to B such that, for all objects C, the homomorphism of bicategories $E \otimes - : \mathcal{F}i\ell(C, B) \to \mathcal{F}i\ell(C, A)$ is a left biadjoint (1.30) for $E^* \otimes -$. Thus there are equivalences

(4.18) $$\mathcal{F}i\ell(C, A)(E \otimes F, G) \approx \mathcal{F}i\ell(C, B)(F, E^* \otimes G).$$

In more elementary terms, we have arrows

$$\eta : HB \to E^* \otimes E, \quad \epsilon : E \otimes E^* \to HA$$

in $\mathcal{F}i\ell(B, B)$, $\mathcal{F}i\ell(A, A)$, respectively and coherent invertible 2-cells

(4.19)
$$\begin{tikzcd} E \approx E \otimes HB \arrow[r, "E \otimes \eta"] \arrow[dr, "\approx"'] & E \otimes E^* \otimes E \arrow[d, "\epsilon \otimes E"] \arrow[dl, phantom, "\overset{\Rightarrow}{\approx}"] \\ & HA \otimes E \end{tikzcd} \qquad \begin{tikzcd} E^* \approx HB \otimes E^* \arrow[r, "\eta \otimes E^*"] \arrow[dr, "\approx"'] & E^* \otimes E \otimes E^* \arrow[d, "E^* \otimes \epsilon"] \arrow[dl, phantom, "\overset{\Rightarrow}{\approx}"] \\ & E^* \otimes HA \end{tikzcd}$$

It thus follows that we also have equivalences

(4.20) $$\mathcal{F}i\ell(A, D)(P \otimes E^*, Q) \approx \mathcal{F}i\ell(B, D)(P, Q \otimes E)$$

which also determine the biadjunction $E \dashv_b E^*$,

(4.21) PROPOSITION. *Cauchy fibrations are bidiscrete.*

PROOF. From (4.20) with $Q = HA$ we obtain the equivalence

$$\mathcal{F}i\ell(B, A)(P, E) \approx \mathcal{F}i\ell(A, A)(P \otimes E^*, HA).$$

From (3.29), (3.43), the right-hand side is equivalent to a discrete category whence the left is, for all P. By (3.29), E is bidiscrete. □

(4.22) A fibration E from B to A is said to be *convergent* when there exists an arrow $f : B \to A$ and an equivalence $E \approx HA(A, f)$ in $\mathcal{F}i\ell(B, A)$. By (3.44), convergent fibrations are bidiscrete, but we can go further and prove :

(4.23) PROPOSITION. *Convergent fibrations are Cauchy.*

PROOF. We claim that $HA(f,A)$ is a right biadjoint for $HA(A,f)$. The identity 2-cell of f induces an arrow $HB \to HA(f,f)$ whose composite with the equivalence

$$HA(f,f) \approx HA(f,A) \otimes HA(A,f)$$

(4.9) yields an arrow

$$\eta : HB \to HA(f,A) \otimes HA(A,f).$$

The free M-algebra on the span (f,B,f) from A to A is

$$M(f,B,f) \approx HA(A,f) \circ HA(f,A),$$

so the isomorphism of spans $(\eta A)f: (f,B,f) \to HA$ induces a cartesian homomorphism $HA(A,f) \circ HA(f,A) \to HA$ which can be shown to equalize and invert the appropriate two 2-cells (4.6). Hence a cartesian homomorphism

$$\epsilon : HA(A,f) \otimes HA(f,A) \to HA$$

is induced (4.7). Properties (4.19) for these η, ϵ can be verified. □

(4.24) An object A is said to be *Cauchy complete* when, for all objects B, all Cauchy fibrations from B to A are convergent.

(4.25) *The Cauchy extension* $\tilde{\mathcal{K}}$ of the bicategory $\mathcal{K}$ is the bicategory whose objects are the objects of $\mathcal{K}$, whose hom-categories $\tilde{\mathcal{K}}(B,A)$ are the full subcategories of the categories $D(\mathfrak{Fib}\mathcal{K})(B,A)$ consisting of the Cauchy fibrations from B to A, and whose composition is fibrational composition (4.7). We can identify $\mathcal{K}$ with the sub-bicategory of $\tilde{\mathcal{K}}$ consisting of the convergent arrows.

5. FIBRATIONS BETWEEN CATEGORIES.

(5.1) The 2-category $\mathcal{C}at$ of small categories is the example for which the theory of Sections 3 and 4 is *essentially* folklore [4, 12]. However, the familiar results do require minor alterations owing to our insistence that $\mathcal{C}at$ should be dealt with as a bicategory for our present purposes, although it does enjoy a strictly associative composition of course. The main point of deviation is that we are unable to capture bicategorically

the strict fibre of a functor $p: E \to A$ over an object a of A, and we must settle for the bipullback of $a: 1 \to A$ and p. To this extent our terms «fibre» and «fibration» differ from common usage. The advantage of our fibrations is that an equivalence of categories is both a left and right fibration. On the other hand, every left (right) fibration over A is isomorphic to a composite of an equivalence and a (co-)fibration over A in the sense of Grothendieck.

(5.2) Write $\mathcal{C}at/_b B$ for $(\mathcal{S}pn\,\mathcal{C}at)(B, 1)$. For a left fibration E from B to A in $\mathcal{C}at$, we have a homomorphism of bicategories (3.25)

$$E(-, B): \mathcal{C}at(X, A)^{op} \to (\mathcal{F}il\,\mathcal{C}at)(B, X)$$

which yields, on setting $X = 1$ and using (3.40), a homomorphism of bicategories

$$E(-, B): A^{op} \to \mathcal{C}at/_b B.$$

Similarly, when E is a right fibration, respectively, fibration, we obtain (3.26), (3.27) the homomorphisms

$$E(A, -): B \to \mathcal{C}at/_b A, \quad E(-,-): A^{op} \times B \to \mathcal{C}at.$$

These assignments of homomorphisms to fibrations are the object functions of biequivalences; this result is the appropriate modification of Grothendieck's result (see Gray [4] page 32) for our fibrations.

(5.3) *For categories A, B, there are biequivalences of bicategories*

$$\mathcal{F}il(B, A) \sim \mathcal{H}om(A^{op}, \mathcal{C}at/_b B), \quad \mathcal{F}ir(B, A) \sim \mathcal{H}om(B, \mathcal{C}at/_b A),$$
$$\mathcal{F}ib(B, A) \sim \mathcal{H}om(A^{op} \times B, \mathcal{C}at).$$

We shall indicate the first of these, the other two are similar. Suppose E, E' are left fibrations from B to A. Each homomorphism of spans $f: E \to E'$ induces a functor

$$f(a, B): E(a, B) \to E'(a, B) \quad \text{for} \quad a: 1 \to A$$

by the property of bipullback. When f is left cartesian (so that it commutes with inverse images) these are the components of a strong transformation $f(-, B): E(-, B) \Rightarrow E'(-, B)$ (3.25). The remainder of the

definition of the homomorphism

$$\mathcal{F}i\ell(B, A) \to \mathcal{H}om(A^{op}, \mathcal{C}at/_b B)$$

is easily supplied and it may be checked that is is a local equivalence. It remains to see that it is surjective on objects up to equivalence. A homomorphism $R: A^{op} \to \mathcal{C}at/_b B$ gives a strong transformation

$$\begin{tikzcd} A^{op} \arrow[r, "R"] \arrow[d] \arrow[dr, phantom, "\tau \Downarrow"] & \mathcal{C}at/_b B \arrow[d, "D_0"] \\ 1 \arrow[r, "B"'] & \mathcal{C}at \end{tikzcd}$$

Apply the Grothendieck construction (1.10) to $D_0 R$ to obtain a category $E = \mathcal{E}\ell_1(D_0 R)$ and a projection $p: E \to A$ which can be seen to be a left fibration. Then $\tau : D_0 R \Rightarrow B$ induces a functor $q: E \to B$ which makes (p, E, q) a left fibration from B to A for which there is an equivalence $E(-, B) \approx R$.

(5.4) Let $\mathcal{G}pd$ be the 2-category of small groupoids and let $\mathcal{S}et$ be the category of small sets. The biequivalences of (5.3) restrict to biequivalences of bicategories

$$G\mathcal{F}i\ell(B, A) \sim \mathcal{H}om(A^{op}, G(\mathcal{C}at/_b B)),$$
$$G\mathcal{F}ir(B, A) \sim \mathcal{H}om(B, G(\mathcal{C}at/_b A)),$$
$$G\mathcal{F}ib(B, A) \sim \mathcal{H}om(A^{op} \times B, \mathcal{G}pd),$$

and restrict further to equivalences of categories

$$D\mathcal{F}i\ell(B, A) \approx [A^{op}, D(\mathcal{C}at/_b B)],$$
$$D\mathcal{F}ir(B, A) \approx [B, D(\mathcal{C}at/_b A)],$$
$$D\mathcal{F}ib(B, A) \approx [A^{op} \times B, \mathcal{S}et],$$

where the square brackets on the right-hand sides denote the functor categories. In fact, if $\mathcal{CAT}$ is a 2-category of categories based on a category $\mathcal{SET}$ of sets which includes the set of 2-cells of $\mathcal{C}at$ (3.39), this last equivalence enriches to an equivalence in

$$\mathcal{H}om(\mathcal{C}at^{op} \times \mathcal{C}at^{coop}, \mathcal{CAT})$$

between the homomorphism $D(\mathcal{F}ib\,\mathcal{C}at)$ (see (3.37)) and the 2-functor

$$\begin{tikzcd} \mathcal{C}at^{op}\times\mathcal{C}at^{coop} \arrow[r, "1\times(-)^{op}"'] & \mathcal{C}at^{op}\times\mathcal{C}at^{op} \arrow[r, "\text{twist}"'] & \mathcal{C}at^{op}\times\mathcal{C}at^{op} \arrow[d, "-\times-"] \\ & & \mathcal{C}at^{op} \arrow[d, "{[-,\mathcal{S}et]}"] \\ & & \mathcal{CAT} \end{tikzcd}$$

(5.5) THEOREM. *Suppose $q: E \to B$ is a fibration from B to 1 in $\mathcal{C}at$. The homomorphism of bicategories*

$$E\underset{B}{\pi}-: \mathcal{C}at/_b B \to \mathcal{C}at/_b E$$

obtained by bipullback along q has a right biadjoint.

PROOF. In fact what we shall prove is that the 2-functor

$$E\underset{B}{\pi}-: \mathcal{C}at/B \to \mathcal{C}at/_b E$$

obtained by pseudopullback along q has a right adjoint whose value $(\hat{X},\hat{u})$ at an object (X,u) of $\mathcal{C}at/_b E$ has the property that the inclusion

(5.6) $\quad (\mathcal{C}at/B)((Z,w),(\hat{X},\hat{u})) \to (\mathcal{C}at/_b B)((Z,w),(\hat{X},\hat{u}))$

is an equivalence for all $w: Z \to B$.

An object (b,x,θ) of $\hat{X}$ consists of an object b of B, a functor $x: Eb \to X$ (where Eb is the pseudopullback of $b: 1 \to B$ along q), and a natural isomorphism θ:

$$\begin{tikzcd} Eb \arrow[rr, "x"] \arrow[dr] & \arrow[d, phantom, "\overset{\theta}{\Rightarrow}"] & X \arrow[dl, "u"] \\ & E & \end{tikzcd}$$

An arrow $(\beta,\xi): (b,x,\theta) \to (c,y,\theta)$ in $\hat{X}$ consists of an arrow $\beta: b \to c$ in B and a natural transformation

$$\begin{tikzcd} Eb \arrow[rr, "E\beta"] \arrow[dr, "x"'] & \arrow[d, phantom, "\overset{\xi}{\Rightarrow}"] & Ec \arrow[dl, "y"] \\ & X & \end{tikzcd}$$

such that the following equality holds:

$$\begin{tikzcd} Eb \arrow[rr, "E\beta"] \arrow[d] \arrow[drr, "x" description] \arrow[dr, phantom, "\overset{\theta}{\Rightarrow}" near start] \arrow[rr, phantom, shift right=4, "\overset{\xi}{\Rightarrow}" near end] & & Ec \arrow[d, "y"] \\ E \arrow[rr, "u"'] & & X \end{tikzcd} = \begin{tikzcd} Eb \arrow[rr, "E\beta"] \arrow[d] \arrow[dr, phantom, "\overset{\xi_\beta}{\Rightarrow}" near start] & & Ec \arrow[d, "y"] \arrow[dll] \arrow[dl, phantom, "\overset{\theta}{\Rightarrow}" near start] \\ E \arrow[rr, "u"'] & & X \end{tikzcd}$$

The functor $\hat{u}: \hat{X} \to B$ is given by $\hat{u}(b, x, \theta) = b$, $\hat{u}(\beta, \xi) = \beta$. Then we have an arrow

$$(\epsilon, \theta\, pr_1): (E \underset{B}{\pi} \hat{X}, pr_1) \to (X, u)$$

in $\mathcal{C}at/_b E$ given by

$$\epsilon(e, \mu, b, x, \theta) = x(e, \mu),$$

$$\epsilon(\lambda, \beta, \xi) = (\, x(e, \mu) \xrightarrow{\xi(e,\mu)} y(E\beta)(e, \mu) \xrightarrow{y\hat{\lambda}} y(d, \nu)\,),$$

for

$$(\lambda, \beta, \xi): (e, \mu, b, x, \theta) \to (d, \nu, c, y, \theta)$$

in $E \underset{B}{\pi} \hat{X}$, where $\hat{\lambda}: (E\beta)(e, \mu) \to (d, \nu)$ is induced by λ using the universal property of direct image (3.22) (ii).

The arrow $(\epsilon,\ \theta\, pr_1)$ induces a functor

$$\omega : (\mathcal{C}at/B)((Z, w), (\hat{X}, \hat{u})) \to (\mathcal{C}at/_b E)((E \underset{B}{\pi} Z, pr_1), (X, u))$$

which we shall describe explicitly and prove is an isomorphism. For $f: (Z, w) \to (\hat{X}, \hat{u})$, put

$$fz = (wz, x_z, \theta) \quad \text{and} \quad f\zeta = (w\zeta, \eta_\zeta)$$

for each object z and arrow $\zeta: z \to z'$ in Z. Define $h: E \underset{B}{\pi} Z \to X$ by $h(e, \mu, z) = x_z(e, \mu)$ and

$$h(\lambda, \zeta) = (x_z(e, \mu) \xrightarrow[\eta_\zeta(e,\mu)]{} x_{z'} E(w\zeta)(e, \mu) \xrightarrow[x_{z'}\hat{\lambda}]{} x_{z'}(e', \mu')).$$

Then

$$\omega(f) = (h, \psi) \quad \text{where} \quad \psi(e, \mu, z) = \theta(e, \mu): e \approx u x_z(e, \mu).$$

The value of ω at $\sigma: f \Rightarrow g$ is described as follows. For each z put

$$\sigma z = (1, \xi_z): (wz, x_z, \theta) \to (wz, y_z, \theta)\ ;$$

then the components of $\omega\sigma : \omega(f) \Rightarrow \omega(g)$ are the arrows

$$\xi_z(e, \mu): x_z(e, \mu) \to y_z(e, \mu).$$

With this description one sees clearly that ω is fully faithful and bijective on objects; and so ω is an isomorphism.

It remains now to prove that (5.6) is an equivalence. It is clearly fully faithful. Take an object (h, ψ):

$$\begin{tikzcd} Z \arrow[rr, "h"] \arrow[dr, "w"'] & \arrow[d, phantom, "\underset{\approx}{\psi}"] & \hat{X} \arrow[dl, "\hat{u}"] \\ & B & \end{tikzcd}$$

of the target of (5.6). For $z \in Z$, suppose $hz = (b_z, x_z, \theta)$. Then the diagram

$$\begin{tikzcd} E(wz) \arrow[r, "(E(\psi z)^{-1})^{-1}"] \arrow[dr, ""] \arrow[dr, phantom, bend left, "\approx"] & Eb_z \arrow[r, "x_z"] \arrow[d] \arrow[dr, phantom, "\underset{\approx}{\theta}"] & X \arrow[dl, "u"] \\ & E & \end{tikzcd}$$

gives an object $(wz, x'_z, \theta') = h'z$ of $\hat{X}$ and an isomorphism $hz \approx h'z$ in $\hat{X}$. Defining h' on arrows in Z in the obvious way, we get

$$h' : (Z, w) \to (\hat{X}, \hat{u}) \quad \text{in } \mathcal{C}at/B$$

and an isomorphism $(h, \psi) \approx (h', 1)$ in $(\mathcal{C}at/_b B)((Z, w), (\hat{X}, \hat{u}))$. $\square$

(5.7) COROLLARY. *$\mathcal{C}at$ is a fibrational bicategory (4.11).*

PROOF. If (p, E, q) is a fibration from B to A in $\mathcal{C}at$, then $p : E \to A$ is a fibration from 1 to A and $q : E \to B$ is a fibration from B to 1. So, by (5.6) and its dual (obtained by applying $(\ )^{op}$), bipullback along p and bipullback along q have right adjoints and so preserve all bicolimits including bicoequinverters. $\square$

(5.8) The results of this section for $\mathcal{C}at$ generalize to the 2-category $\mathcal{C}at(\mathcal{A})$ of categories in $\mathcal{A}$ where $\mathcal{A}$ is a finitely complete, finitely cocomplete, and internally complete [16] category.

6. HOM-ENRICHED CATEGORIES.

(6.1) The base monoidal category $\mathcal{V}$ will be assumed to satisfy the fol-

lowing conditions:

(a) $\mathcal{V}$ is finitely complete and locally small;

(b) $\mathcal{V}$ has all small colimits and, for each object X, the functors $X\otimes -$, $-\otimes X$ preserve them.

(6.2) Under these circumstances the 2-category $\mathcal{V}\text{-}\mathcal{C}at$ of small $\mathcal{V}$-categories is finitely complete and small cocomplete. It follows from (1.14) that $\mathcal{V}\text{-}\mathcal{C}at$ has finite bilimits and small bicolimits.

(6.3) *Our intention in this section is to interpret the work of Sections 3 and 4 in the case where* $\mathcal{K}$ *is the 2-category* $(\mathcal{V}\text{-}\mathcal{C}at)^{op}$. For this it is convenient to use dual terminology so that we can really work in $\mathcal{V}\text{-}\mathcal{C}at$ instead of $\mathcal{K}$.

(6.4) A span in $\mathcal{H}^{op}$ is called a *cospan in* $\mathcal{H}$. Write $(\mathcal{C}ospn\mathcal{H})(B,A)$ for $(\mathcal{S}pn\mathcal{H}^{op})(B,A)^{op}$, or simply $\mathcal{C}ospn(B,A)$ when $\mathcal{H}$ is understood. A fibration from B to A in $\mathcal{H}^{op}$ is called a *cofibration from* B *to* A *in* $\mathcal{H}$ (in agreement with the terminology of Gray [4] rather than with that of Grothendieck). Write $(\mathcal{C}ofib\mathcal{H})(B,A)$ for $(\mathcal{F}ib\mathcal{H}^{op})(B,A)^{op}$; it is the bicategory of coalgebras for a codoctrine M on $(\mathcal{C}ospn\mathcal{H})(B,A)$.

(6.5) Posetal, groupoidal and bidiscrete objects in $\mathcal{H}^{op}$ are called *coposetal, cogroupoidal and bicodiscrete objects in* $\mathcal{H}$. We write $Cog\mathcal{H}$, $Cod\mathcal{H}$ for $(G\mathcal{H}^{op})^{op}$, $(D\mathcal{H}^{op})^{op}$, respectively.

(6.6) For $\mathcal{V}$-categories A, B, we recall the definition of a $\mathcal{V}$-module θ *from* B *to* A (called «bimodule» by Lawvere [9] - we drop «bi» because two-sidedness is already apparent from «from B to A»; other authors have used the terms «distributor» and «profunctor»). The basic data involved in θ are objects $\theta(a,b)$ of $\mathcal{V}$ and arrows

$$A(a',a)\otimes\theta(a,b)\to\theta(a',b),\quad \theta(a,b)\otimes B(b,b')\to\theta(a,b')$$

in $\mathcal{V}$ satisfying the appropriate five axioms. Write $\mathcal{V}\text{-}\mathcal{M}od(B,A)$ for the category of $\mathcal{V}$-modules from B to A.

(6.7) Recall also that *tensor product of modules* provides a functor

$$\mathcal{V}\text{-}\mathcal{M}od(B, A) \times \mathcal{V}\text{-}\mathcal{M}od(C, B) \to \mathcal{V}\text{-}\mathcal{M}od(C, A)$$

whose value at (ϕ, ψ) is denoted by $\phi \otimes \psi$. This is the composition for a bicategory $\mathcal{V}$-$\mathcal{M}od$ whose objects are small $\mathcal{V}$-categories and whose arrows are $\mathcal{V}$-modules.

(6.8) There are two homomorphisms of bicategories

$$(\ )_* : \mathcal{V}\text{-}\mathcal{C}at \to \mathcal{V}\text{-}\mathcal{M}od, \quad (\ )^* : (\mathcal{V}\text{-}\mathcal{C}at)^{op} \to \mathcal{V}\text{-}\mathcal{M}od$$

which are the identity on objects, locally fully faithful, and, for $f : B \to A$ in $\mathcal{V}$-$\mathcal{C}at$, the module f_* from B to A consists of the objects $A(a, fb)$ and the module f^* from A to B consists of the objects $A(fb, a)$. Recall that $f_* \dashv f^*$ in $\mathcal{V}$-$\mathcal{M}od$.

(6.9) Suppose A, B are $\mathcal{V}$-categories. A $\mathcal{V}$*-gamut from* B *to* A is a diagram $(\theta, X, \xi, \tilde{\xi}, m)$:

$$\begin{tikzcd} & X \arrow[dr, "\xi"] \arrow[d, phantom, "m \Downarrow"] & \\ B \arrow[ur, "\tilde{\xi}"] \arrow[rr, "\theta"'] & {} & A \end{tikzcd}$$

in $\mathcal{V}$-$\mathcal{M}od$. An *arrow*

$$(t, f, z, \tilde{z}) : (\theta, X, \xi, \tilde{\xi}, m) \to (\theta', X', \xi', \tilde{\xi}', m')$$

of $\mathcal{V}$*-gamuts* consists of a $\mathcal{V}$-functor $f : X \to X'$ and arrows

$$t : \theta \Rightarrow \theta', \quad z : \xi \Rightarrow \xi' \otimes f_*, \quad \tilde{z} : f_* \otimes \tilde{\xi} \Rightarrow \tilde{\xi}'$$

of $\mathcal{V}$-modules such that:

$$\begin{tikzcd} & X \arrow[dr, "\xi"] \arrow[d, "f_*"] \arrow[d, phantom, shift right=4ex, "\tilde{z}\Downarrow"] \arrow[d, phantom, shift left=4ex, "\Downarrow z"] & \\ B \arrow[ur, "\tilde{\xi}"] \arrow[r, "\tilde{\xi}'"'] \arrow[rr, bend right=60, "\theta'"'] & X' \arrow[r, "\xi'"'] \arrow[d, phantom, "m'\Downarrow"] & A \\ & {} & \end{tikzcd} = \begin{tikzcd} & X \arrow[dr, "\xi"] \arrow[d, phantom, "\Downarrow m"] & \\ B \arrow[ur, "\tilde{\xi}"] \arrow[rr, "\theta"] \arrow[rr, bend right=60, ""] & {} \arrow[d, phantom, "t\Downarrow"] & A \\ & {} & \end{tikzcd}$$

In this way we obtain an obvious 2-category $\mathcal{V}$-$\mathcal{G}am(B, A)$ of $\mathcal{V}$-gamuts.

(6.10) *There is a locally fully faithful 2-functor*

$$\Sigma : \mathcal{V}\text{-}\mathcal{G}am(B, A) \to \mathcal{C}ospn(B, A)$$

described as follows. For each $\mathcal{V}$-gamut $(\theta, X, \xi, \tilde{\xi}, m)$, the 2-functor Σ gives a cospan S from B to A. The set of objects of S is the disjoint union of the sets of objects of A, X, B. For $a, a' \in A$, $x, x' \in X$, $b, b' \in B$,

$$S(a, a') = A(a, a'),\quad S(x, x') = X(x, x'),\quad S(b, b') = B(b, b'),$$
$$S(a, b) = \theta(a, b),\quad S(a, x) = \xi(a, x),\quad S(x, b) = \tilde{\xi}(x, b),$$
$$S(b, a) = S(x, a) = S(b, x) = 0 \text{ (initial object of } \mathcal{V}\text{)}.$$

Composition in S is given by composition in A, X, B, the actions of A, X, B on the modules θ, ξ, $\tilde{\xi}$, and the arrow of modules m. Routine diagrams show that S is a $\mathcal{V}$-category. There are fully faithful $\mathcal{V}$-functors $u : A \to S$, $v : B \to S$ which are given on objects by inclusion and which have identity components for their effects on homs. A strict homomorphism h of cospans is assigned by Σ to each arrow $(t, f, z, \tilde{z})$ of $\mathcal{V}$-gamuts; h leaves the objects of A, B fixed and applies f to the objects of X; the effect of h on homs is obtained from the effect of f on homs and from t, z, $\tilde{z}$.

(6.11) For any cospan S from B to A, the cospan MS from B to A (as described in (3.12) with $\mathcal{K} = (\mathcal{V}\text{-}\mathcal{C}at)^{op}$) can be obtained by applying Σ to the $\mathcal{V}$-gamut

$$\begin{tikzcd} & S \arrow[dr, "u^*"] \arrow[d, phantom, "1 \Downarrow"] & \\ B \arrow[ur, "v_*"] \arrow[rr, "u^* \otimes v_*"'] & {} & A \end{tikzcd}$$

In fact, if we take $2 * A$ to be the tensor product of 2 with A in $\mathcal{V}\text{-}\mathcal{C}at$ (and not just any tensor biproduct), one easily sees that Σ applied to this $\mathcal{V}$-gamut appears in the following diagram of pushouts in $\mathcal{V}\text{-}\mathcal{C}at$. Furthermore, these pushouts are also bipushouts (this is not a general phenomenon: pullbacks in $\mathcal{C}at$ are not bipullbacks in general!) so that MS is the MS of (3.12) for this $\mathcal{K}$. The objects of MS are elements of the disjoint union of the sets of objects of A, S, B; it is convenient

$$\begin{tikzcd}[column sep=small]
A \arrow[dr, "\partial_0"'] & & A \arrow[dl, "\partial_1"'] \arrow[dr, "u"] & & B \arrow[dl, "v"'] \arrow[dr, "\partial_0"] & & B \arrow[dl, "\partial_1"] \\
& 2*A \arrow[dr] & & S \arrow[dl] \arrow[dr] & & 2*B \arrow[dl] & \\
& & LS \arrow[dr] & & RS \arrow[dl] & & \\
& & & MS & & &
\end{tikzcd}$$

to denote the elements in the copy of A by $(0, a)$ and in B by $(1, b)$ where $a \in A$, $b \in B$. Similarly, the objects of LS will be taken to be of the form $(0, a)$ for $a \in A$ and s for $s \in S$.

The counit and comultiplication for the codoctrine M are actually strict homomorphisms of cospans $\epsilon S : MS \to S$, $\delta S : MS \to MMS$ and the 2-cells r, l, a (2.10) are identities. Indeed, M induces a comonad on the 2-category of cospans from B to A and strict homomorphisms.

(6.12) *The 2-functor* Σ *of (6.10) factors through the forgetful 2-functor* $\mathcal{C}\mathit{ofib}(B, A) \to \mathcal{C}\mathit{ospn}(B, A)$. For each cospan S from B to A constructed as in (6.10), we obtain a coalgebra structure $\chi : S \to MS$ which takes objects a, x, b to $(0, a)$, x, $(1, b)$, respectively, and which has identity components for its effect on homs; the 2-cells l, a of (2.15) are identities. Clearly h as constructed in (6.10) is a strict morphism of coalgebras (that is, a as in (2.18) is an identity).

(6.13) THEOREM. *For small* $\mathcal{V}$*-categories* A, B, *the 2-functor* Σ *of (6.10) induces a biequivalence:*

$$\mathcal{V}\text{-}\mathcal{G}\mathit{am}(B, A) \sim \mathcal{C}\mathit{ofib}(B, A).$$

PROOF. Let S denote a cofibration from B to A with M-coalgebra structure $\chi : S \to MS$, l, a. By (3.20), (3.18), (2.29), the composite

$$\chi_1 : S \xrightarrow{\chi} MS \longrightarrow LS$$

is a left adjoint for the counit $\epsilon_0 S : LS \to S$ of L at S. So there is a $\mathcal{V}$-natural isomorphism

$$LS(\chi_1 ua, (0, a')) \approx S(ua, \epsilon_0(0, a'))$$

which can be seen to be the effect of u on homs $A(a, a') \to S(ua, ua')$. So u, and similarly v, is fully faithful.

Suppose $S(s, ua')$ is not initial in $\mathcal{V}$. If $\chi_1 s = s'$ for some $s' \in S$, then

$$S(s, ua') \approx LS(\chi_1 s, (0, a')) \approx (LS)(s', (0, a')) \approx 0,$$

a contradiction. So $\chi_1 s = (0, a)$ for some $a \in A$; so

$$s \approx (\epsilon_0 S)\chi_1 s = \epsilon_0(0, a) \approx ua.$$

So s is in the replete image of u.

Similarly, if $S(vb', s)$ is not initial in $\mathcal{V}$, then s is in the replete image of v.

Let X denote the full sub-$\mathcal{V}$-category of S consisting of those objects which are not in the replete image of u and not in the replete image of v. Let θ, ξ, $\tilde{\xi}$ denote the $\mathcal{V}$-modules made up of the objects

$$\theta(a, b) = S(ua, vb), \quad \xi(a, x) = S(ua, x), \quad \tilde{\xi}(x, b) = S(x, vb),$$

with obvious actions. Composition in S gives an arrow of modules: $m: \xi \otimes \tilde{\xi} \Rightarrow \theta$. So we have a $\mathcal{V}$-gamut $(\theta, X, \xi, \tilde{\xi}, m)$ whose image under Σ is now easily seen to be equivalent to S.

The remainder of the proof is routine. □

(6.14) *The codiscrete objects of* $\mathcal{C}\mathit{ospn}(B, A)$ *are precisely those cospans*

$$A \xrightarrow{\ u\ } S \xleftarrow{\ v\ } B$$

for which each object of S *is isomorphic either to an object of the form* ua *or an object of the form* vb. This follows easily from (6.5), (1.7). Then (3.29), (6.13) give:

(6.15) *A* $\mathcal{V}$*-gamut* $(\theta, X, \xi, \tilde{\xi}, m)$ *from* B *to* A *is bicodiscrete in* $\mathcal{V}\text{-}\mathcal{G}\mathit{am}(B, A)$ *iff* X *is the empty* $\mathcal{V}$*-category. So a bicodiscrete* $\mathcal{V}$*-gamut amounts precisely to a* $\mathcal{V}$*-module.*

(6.16) COROLLARY. *For small* $\mathcal{V}$*-categories* A, B, *the 2-functor* Σ *of (6.10) induces an equivalence:* $\mathcal{V}\text{-}\mathcal{M}\mathit{od}(B, A) \to \mathit{Cod}\mathcal{C}\mathit{ofib}(B, A)$. □

(6.17) PROPOSITION. *Suppose* $j: A \to C$ *is a fully faithful* $\mathcal{V}$*-functor such that, for each* $a \in A$, $c \in C$, *the object* $C(c, ja)$ *is initial in* $\mathcal{V}$. *The pushout of* j *and any* $\mathcal{V}$*-functor* $u: A \to X$ *is obtained by applying the construction* Σ *to the codiscrete* $\mathcal{V}$*-gamut arising from the* $\mathcal{V}$*-module* $u_* \otimes j^*$ *from* C *to* X. *Furthermore, this pushout is also a bipushout.* □

(This generalizes the bipushout constructions used in (6.11).)

(6.18) PROPOSITION. *For a* $\mathcal{V}$*-functor* $j: A \to C$ *as in (6.17), bipushout along* j, *as a homomorphism of bicategories*

$$\mathcal{C}ospn(B, A) \to \mathcal{C}ospn(B, C),$$

preserves biequinverters.

PROOF. The biequinverter of a pair of 2-cells

$$\begin{tikzcd} S \arrow[r, bend left=50, "h"] \arrow[r, bend right=50, "k"'] \arrow[r, phantom, "\sigma\Downarrow\ \ \Downarrow\tau"] & T \end{tikzcd}$$

in $\mathcal{C}ospn(B, A)$ can be obtained as the full sub-$\mathcal{V}$-category R of S consisting of those objects s of S for which $\sigma s = \tau s$ and σs is an isomorphism. With the description of bipushout along j given in (6.17), one now easily verifies the result. □

(6.19) The last two propositions with their appropriate duals yield the fact that bipushout along a leg of a cofibration in $\mathcal{V}$-$\mathcal{C}at$ preserves biequinverters. This is what is needed to complete the proof of:

(6.20) THEOREM. *For* $\mathcal{V}$ *as in (6.1), the 2-category* $(\mathcal{V}\text{-}\mathcal{C}at)^{op}$ *is a fibrational bicategory.* □

(6.21) PROPOSITION. *Fibrational composition of bidiscrete fibrations in* $(\mathcal{V}\text{-}\mathcal{C}at)^{op}$ *corresponds, under the equivalence of (6.16), to tensor product of* $\mathcal{V}$*-modules.*

The proof of this is left to the reader.

(6.22) It is therefore possible to construct the bicategory $\mathcal{V}$-$\mathcal{M}od$, up to biequivalence, purely from the 2-category $\mathcal{V}$-$\mathcal{C}at$ by internal bicategorical constructions.

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