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**ESSENTIAL PROPERTIES OF PRO-OBJECTS IN  
GROTHENDIECK CATEGORIES**

*by Timothy PORTER*

There is a class of problems in Algebra, algebraic Topology and category Theory which can be subsumed under the question: When does a «structure» on a category  $C$  extend to a similar «structure» on the corresponding procategory  $pro(C)$ ?

Thus one has the work of Edwards and Hastings [7] and the author [27, 28, 29] on extending a «model category for homotopy theory» à la Quillen [35] from a category  $C$ , usually the category of simplicial sets or of chain complexes over some ring, to give a homotopy theory or homotopical algebra in  $pro(C)$ . (It is worth noting that the two approaches differ considerably, but they agree in the formation of the homotopy category/ category of fractions  $Hopro(C)$  which, in this case, is distinct from the «prohomotopy category»  $proHo(C)$ .)

In some separate work, the author tried to use an extension of the simple torsion theory on a category of modules to study the «stability» problem of promodules; that is, the problem of determining not only those promodules isomorphic to constant promodules, but of using this information to obtain other results on promodules, cf. [30-33]. One of the problems there was that of finding an extension of a localisation to promodules in such a way as to give «useful» information; this proved possible only if the category was semi-artinian (cf. Popescu [26]). Another result to note from those papers was an attempt to use a «stabilised» Krull-Gabriel dimension in procategories to give restrictions on the vanishing of the derived functors of  $\varprojlim$  [33]. Similar results have recently been obtained, by very different methods, by Jensen and Gruson [14, 15], who have shown that, if  $A$  is a noetherian ring and  $C = Mod-A$ , the category of finitely generated right modules over  $A$ , then any pro-object  $M$  in  $C$

which has the «lim sup» of the Krull dimensions of its constituent parts less than  $n$  satisfies

$$\varprojlim^{(i)} M = 0 \quad \text{for } i > n,$$

where  $\varprojlim^{(i)}$  is the  $i$ -th derived functor of the limit functor  $\varprojlim$ ; cf. Jensen [20] for earlier results of this nature.

To return to the problem mentioned at the start, the solution is known for various «structures»; for instance:

if  $C$  is additive, so is  $pro(C)$ ; if  $C$  is abelian, so is  $pro(C)$ , and so on. However the following interesting special cases do not seem to be known:

(i) If  $\Sigma$  is a calculus of fractions in  $C$ , one can form a class  $\bar{\Sigma}$  of morphisms in  $pro(C)$  which «locally» belong to  $\Sigma$ : is this class always a calculus of fractions?

(ii) If  $T = \langle T, \eta, \mu \rangle$  is a monad (or triple) on  $C$ , one can form a monad  $proT$  on  $pro(C)$  simply by extending  $T$  «pointwise» or «degree-wise»; suppose one forms the Eilenberg-Moore category  $C^T$  for  $T$  and then one forms  $pro(C^T)$ ; alternatively one could form  $pro(C)^{proT}$ . Is there any close relationship between  $pro(C^T)$  and  $pro(C)^{proT}$ ?

The reason why the form of  $\bar{\Sigma}$  is suggested as such in (i) is that, in the above mentioned homotopy theory, it was this «locally in  $\Sigma$ » idea which worked, but the proof it did so required structure in addition to the simple calculus of fractions. A similar result will be proved in this paper, namely when  $\Sigma$  is the class of morphisms inverted in a localisation, here again additional structure is used to simplify the problem.

As to (ii) the localisation result just mentioned gives such a connection if  $T$  is a left exact idempotent monad and  $C$  is abelian; in general this problem seems to be quite hard since for example it would give information on such things as:

Is a ring object in the category of pro-abelian groups naturally isomorphic to the underlying progroup of a pro-ring?

The answer would seem to depend on whether or not the ring satis-

fies some finiteness conditions, but again this is still vague. So in this paper we are not going to crack these basic problems of procategory theory, but we hopefully will scratch the surface. The main purpose of this note is to show that by an obvious adaptation of the «homotopy theory» methods previously used one can obtain results on certain types of localisation in pro-Grothendieck categories and that one can use these results to obtain a full picture of, categorically, why the result of Gruson and Jensen works.

To start with, we will provide an introduction to the necessary theory of pro-objects since the sources on procategories are reticent about some of the results we will need, and the proofs do not always illustrate the results in enough detail to be comparable with later developments ; however this Section contains little new material.

I should like to thank Chris Jensen, Laurent Gruson and Daniel Simson for encouragement ; Chris Jensen for providing me with a brief account of his joint work with Gruson ; and Anders Kock, Fred Linton and Gavin Wraith for numerous discussions either in Amiens in the summer of 1975 or by letter.

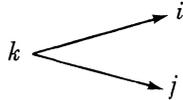
1. PRELIMINARIES ON PROCATEGORIES.

Let  $U$  be a (Grothendieck) universe.  $Ens$  will denote the category of  $U$ -sets and functions and  $Ab$  the category of  $U$ -small abelian groups and homomorphisms.  $C$  will denote a  $U$ -small category; initially there will be no other restriction on  $C$  but soon we will need  $C$  to be abelian and finally we will require it to be a Grothendieck category (i.e., AB5 plus a generator) satisfying AB4\* - products are exact.

A  $U$ -small category  $I$  will be called an *index category* provided:

(i) for each pair of objects  $i, j$  in  $I$ , either  $Hom(i, j) = \emptyset$  or it contains precisely one element; in this latter case, we write  $i \rightarrow j$  or  $i < j$ .

(ii) for each pair  $i, j$  of objects of  $I$ , there is an object  $k$  in  $I$  and maps



REMARK. In some of the sources on procategories, (i) is replaced by a weaker property:

If  $\alpha, \beta: i \rightarrow j$  are two maps in  $I$ , then there is some  $k$  and a map  $\gamma: k \rightarrow i$  with  $\alpha\gamma = \beta\gamma$ .

We do not need this added pseudo-generality in this work.

A functor  $F: I \rightarrow C$  will be called a *projective system in C* indexed by  $I$ . If  $C'$  is a subcategory of  $C$  and  $F$  factors through  $C'$ , then  $F$  will be said to be *of type C'*.

If  $\alpha: i \rightarrow j$  is a map in  $I$ , and  $F: I \rightarrow C$  is a projective system, then the map  $F(\alpha)$  will be called a *transition or transition map of F*.

If  $\phi: I \rightarrow J$  is a functor of index categories, then we say that  $\phi$  is an *initial functor* if for each  $j$  in  $J$  there is an  $i$  in  $I$  and a map (in  $J$ )  $\phi(i) \rightarrow j$ .

If  $F: I \rightarrow C$  is a projective system in  $C$ , then for each object  $T$  of  $C$  the composite

$$I^{op} \xrightarrow{F^{op}} C^{op} \xrightarrow{Hom(, T)} Ens$$

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defines a functor

$$\text{Hom}_C(F, T): I^{op} \rightarrow \text{Ens} : i \rightarrow \text{Hom}_C(F(i), T).$$

If  $f: T \rightarrow S$  is a morphism in  $C$ , then there is a natural transformation  $\text{Hom}_C(F, T) \rightarrow \text{Hom}_C(F, S)$  of functors from  $I^{op}$  to  $\text{Ens}$ .

Since  $\text{Ens}$  is complete and cocomplete, we can form the colimit of these functors; we define

$$h_F(T) = \varinjlim \text{Hom}_C(F, T)$$

and similarly for  $h_F(S)$ . The natural transformation induced from  $f$  in its turn induces a function  $h_F(T) \rightarrow h_F(S)$ , so that  $h_F$  is a functor - the properties are easily verified.

If  $C$  is an additive category, then  $\text{Hom}_C(F(i), T)$ , and hence  $h_F(T)$ , have natural abelian group structures, so the functor  $h_F: C \rightarrow \text{Ens}$  factors through a functor from  $C$  to  $\text{Ab}$ .

We next seek to identify these  $h_F$  functors amongst the functors from  $C$  to  $\text{Ens}$  (or  $\text{Ab}$ ).

A functor  $M: C \rightarrow \text{Ens}$  is said to be *prorepresentable* provided it is naturally isomorphic to a functor of the form  $h_F$  for some projective system  $F$  in  $C$ .

It is clear that a given functor  $M: C \rightarrow \text{Ens}$  may be isomorphic to many  $h_F$  and yet the projective systems may not be isomorphic in the usual classical sense. For instance, if  $\phi: I \rightarrow J$  is initial and  $F: I \rightarrow C$  is such that  $M \approx h_F$ , then also  $M \approx h_{F\phi}$  since, for each  $T$ ,  $h_F(T)$  is naturally isomorphic to  $h_{F\phi}(T)$  and yet  $F$  and  $F\phi$  are indexed by possibly different categories. (A proof that  $h_F = h_{F\phi}$  can be obtained by dualising Theorem 1, page 213, of MacLane [24]; it is also available in many other sources on category theory.)

Since, if  $C$  is additive, each  $h_F$  factors through the forgetful functor  $\text{Ab} \rightarrow \text{Ens}$ , we have that any prorepresentable functor  $M$  must also factor through  $\text{Ab}$ . (To simplify the characterisation of prorepresentable functors we will assume that  $C$  is abelian from now on - some of the results go through without this, for these see Duskin [6].)

Each Hom-functor  $Hom_C(X, \_): C \rightarrow Ab$  is left exact and  $U$ -small colimits in  $Ab$  are exact; we have that  $h_F$  is always left exact and hence that any prorepresentable functor  $M: C \rightarrow Ab$  is left exact. In fact, the converse is true; every left exact functor from an abelian category  $C$  to  $Ab$  is prorepresentable. The priority for this result is difficult to give; it is «well-known» in categorical circles, being of the status of «folklore». Duskin [6] gives a version which he assigns to Deligne [3] and Lazard, which characterises prorepresentable functors from a general category  $C$  to  $Ens$  in terms of left exactness and an additional smallness condition, but the only explicit proof that I have been able to find is in Stauffer [37] which seems unusually late as I am certain the result was well-known before 1970 when that paper was written. However the proof that Stauffer gives is easy to follow so I won't try to better it and will merely comment that he uses  $C^{op}$  instead of  $C$ . Adapting his result to allow for that, we quote his Theorem 3.5 page 379 of [37]:

*Let  $C$  be a  $U$ -small abelian category; then a functor  $M: C \rightarrow Ab$  is left exact iff  $M$  is a direct limit of representable functors over a directed index category, i. e.,  $M = \varinjlim Hom_C(F(i), -)$ .*

REMARKS. a) If one relaxes the condition that  $C$  be  $U$ -small to  $C$  being merely a  $U$ -category, then  $I$  may not be  $U$ -small.

b) It should be noted that in a later paper, Stauffer [38], it is shown that the dual of the construction of  $pro(C)$  produces another category,  $ind(C)$ ; and moreover  $ind(C)$  is a right completion of  $C$  whilst  $pro(C)$  is a left completion of  $C$ . In this later paper Stauffer proves the duals of many of the results we will be discussing here. In fact many of these results are well-known as he says.

Of particular interest to us here is a result of Stauffer [37], page 375, Proposition 2.1, where he shows that one can always replace an arbitrary indexing category by a p. f. p. indexing category - «p. f. p.» stands for «pointwise finitely preceded» and  $I$  is p. f. p. iff for each  $i_0$  in  $I$ ,

$$\{ i \mid Hom_I(i_0, i) \neq \emptyset \}$$

is finite. This result seems to have been discovered independently by Mardesič about the same time as [37] was written, but, in its use of shape theory, some writers have not seemed fully aware of the statement of this result since there are often references to the fact that the procategories being used by them are not exactly the same as those used by Artin and Mazur in [1]. The two definitions are, in fact, equivalent even though they are not equally easy to check in different situations.

It has become usual to denote the category of left exact functors from an abelian category  $C$  to the category  $Ab$  by  $Sex(C, Ab)$  - for the reason, see Gabriel [8], Page 348. The additive Yoneda embedding:  $Y: C^{op} \rightarrow Ab^C$ , where  $Ab^C$  denotes the category of additive functors from  $C$  to  $Ab$  factors through

$$Y': C^{op} \rightarrow Sex(C, Ab).$$

Since it is inconvenient to embed  $C^{op}$ , it has become customary to take the opposite of both categories and hence to obtain a natural embedding

$$h: C \rightarrow (Sex(C, Ab))^{op}.$$

The category  $(Sex(C, Ab))^{op}$  is then taken as the category  $pro(C)$  of pro-objects in  $C$ .

We next develop an intrinsic definition of  $pro(C)$ . For our objects we will take all projective systems  $F: I \rightarrow C$  and for our morphisms between  $F$  and  $G: J \rightarrow C$  we take

$$Hom_{pro(C)}(F, G) = \lim_{\leftarrow J} \lim_{\rightarrow I} Hom_C(F(i), G(j)).$$

To justify this we note that

$$\begin{aligned} Hom_{pro(C)}(h_F, h_G) &= Hom_{Sex(C, Ab)}(h_G, h_F) \\ &= Hom_{Sex(C, Ab)}(\lim_{\leftarrow J} h_{G(j)}, \lim_{\rightarrow I} h_{F(i)}) \\ &= \lim_{\leftarrow J} Hom_{Sex(C, Ab)}(h_{G(j)}, \lim_{\rightarrow I} h_{F(i)}) \\ &= \lim_{\leftarrow J} (\lim_{\rightarrow I} Hom_{Sex(C, Ab)}(h_{G(j)}, h_{F(i)})) \\ &= \lim_{\leftarrow J} (\lim_{\rightarrow I} Hom_C(F(i), G(j))). \end{aligned}$$

The last stage of this string of equalities follows from the Yoneda Lemma and the fact that  $Sex(C, Ab)$  is a full subcategory of  $Ab^C$ .

Clearly the category constructed above is equivalent to  $pro(C)$  and in fact some authors have defined  $pro(C)$  in this way - notably Artin and Mazur [1], Appendix. It has the advantage that it works for non-abelian categories but the definition of composition of morphisms is, it must be admitted, rather strained in this context. The definition can be made in terms of pro-representable functors but, as mentioned before, the characterisation of prorepresentables in non-abelian categories is less elegant and is more messy than even the composition of morphisms for this intrinsic definition.

Our view is that it is convenient to have both definitions especially when one has some of the reindexing results which will be proved soon. The intrinsic definition is more useful in our context, but the definition as  $(Sex(C, Ab))^{op}$  or as the completion of  $C$  provides a far better algebraic motivation for studying  $pro(C)$  than the intrinsic definition.

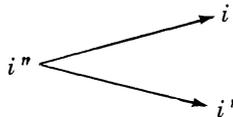
To use the intrinsic definition to the full, we need to examine the morphisms in it more closely. We have the set of morphisms between  $F$  and  $G$  defined to be

$$\lim_{\leftarrow J} (\lim_{\rightarrow I} Hom_C(F(i), G(j))).$$

An element of  $\lim_{\rightarrow I} Hom_C(F(i), G(j))$  consists of an index  $i$  plus a morphism  $f_i: F(i) \rightarrow G(j)$  modulo the equivalence relation that

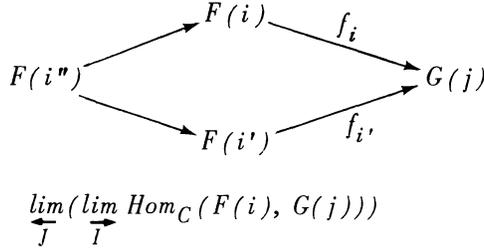
$$f_i: F(i) \rightarrow G(j) \quad \text{and} \quad f_{i'}: F(i') \rightarrow G(j)$$

are equivalent if there is an  $i''$  with

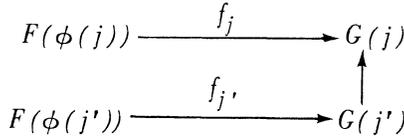


(whose existence is provided for by the axioms of an indexing category) such that the following diagram commutes. Thus it is possible to represent a map in

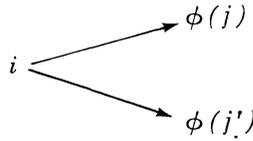
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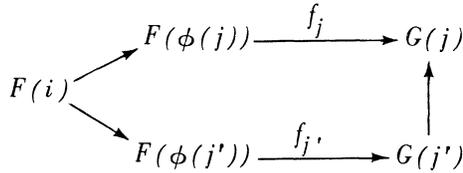
as a pair  $(\phi, \{f_j\}_{j \in J})$  where  $\phi: \text{Ob } J \rightarrow \text{Ob } I$  is a function and where each  $f_j: F(\phi(j)) \rightarrow G(j)$  is a morphism in  $C$ . If  $j' \rightarrow j$  in  $J$ , then there is a diagram



and since  $(\phi, \{f_j\})$  is to represent a map, these two morphisms with common codomain  $G(j)$  must be equivalent, i. e., there is some  $i$  in  $I$  and



such that

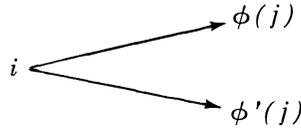


(all unlabelled maps are the transitions relevant to that position). In order to obtain exactly

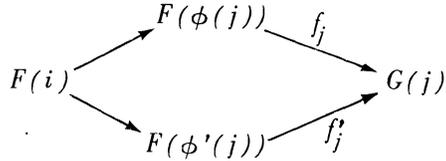
$$\lim_{\leftarrow J} \lim_{\rightarrow I} \text{Hom}(F(i), G(j))$$

by such a representation we put an equivalence relation on the set of all  $(\phi, \{f_j\})$  by stating that

$(\phi, \{f_j\}) R (\phi', \{f'_j\})$  if and only if for each  $j$  in  $J$  there is an  $i$  in  $I$  with



such that the following diagram commutes :



The set of equivalence classes is fairly easily seen to be exactly

$$\lim_{\leftarrow J} (\lim_{\rightarrow I} \text{Hom}_C(F(i), G(j))).$$

This description of the morphisms in (intrinsic)  $pro(C)$  makes the composition of such morphisms easier to define, i. e., if

$$F: I \rightarrow C, \quad G: J \rightarrow C, \quad H: K \rightarrow C$$

and we have  $f: F \rightarrow G$  represented by a pair  $(\phi, \{f_j\})$  and  $g: G \rightarrow H$  represented by  $(\psi, g_k)$ , then  $g \circ f: F \rightarrow H$  is the morphism represented by  $(\phi\psi, \{g_k \circ f_{\psi(k)}\})$ .

When we have obtained the reindexing results later on, the description of composition of morphisms will simplify even further. It is worth noting that Hilton and Deleanu [2] have obtained a result which suggests that, by adapting this representation of maps to non-filtered index categories, one obtains a weaker form of procategory which may be very useful in the study of comma categories. I do not know to what extent the results of this paper might extend to this weaker version of  $pro(C)$ .

At several places in the preceding pages, we have stated that  $pro(C)$  has (projective) limits; we can now prove this using the above description of maps in  $pro(C)$ .

Suppose  $F: J \rightarrow pro(C)$  is a projective system in  $pro(C)$  and for each  $j$  in  $J$  suppose  $F(j)$  is the projective system  $F(j): I_j \rightarrow C$ . We form a fresh indexing category  $L$  with  $Ob L = \prod_{j \in Ob J} Ob I_j$  (where  $\prod$  de-

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notes disjoint union of sets) and, by the usual method of forming disjoint unions, we will indicate an object of  $L$  by a pair  $(i, j)$  where  $i \in \text{Ob}L_j$ . With this notation we say:

$(i_1, j_1) \leq (i_2, j_2)$  iff  $j_1 \leq j_2$  and, if  $(\phi, \{f_j\})$  represents the map  $F(j_1) \rightarrow F(j_2)$ , then  $i_1 \leq \phi(i_2)$ .

With this new category  $L$  as indexing category, we form a new pro-object  $\bar{F}$  by letting

$$\bar{F}(i, j) = F(j)(i)$$

and if  $(i_1, j_1) \rightarrow (i_2, j_2)$  is a map in  $L$ ,

$$\bar{F}(i_1, j_1) \rightarrow \bar{F}(i_2, j_2) = F(j_1)(i_1) \rightarrow F(j_1)(\phi(i_2)) \xrightarrow{f_{i_2}} F(j_2)(i_2)$$

(here the first map in the composite is a transition in  $F(j_1)$  and the second is  $f_{i_2}$ , where  $(\phi, \{f_i\})$  is the transition promap  $F(j_1) \rightarrow F(j_2)$ ).

We claim  $\bar{F}$  is the (projective) limit of  $F$ . Clearly  $\bar{F}$  is a pro-object (in the intrinsic definition) and the limiting cone is given by the promaps

$$\mu_j: \bar{F} \rightarrow F(j) \quad \text{with representation } (in, \{id\}),$$

where  $in: \text{Ob}L_j \rightarrow \text{Ob}L$  is the inclusion, and the maps  $\bar{F}(in(j)) \rightarrow F(j)$  are the relevant identities; note that  $\bar{F}(in(j))$  is the same as  $F(j)$  under the obvious identification. If

$$G: K \rightarrow C \quad \text{and } \{ \lambda_j: G \rightarrow F(j) \}$$

gives another cone on  $F: J \rightarrow \text{pro}(C)$ , then each  $\lambda_j$  is represented by, say,  $(\psi_j, \{l_j\})$  where  $\psi_j: \text{Ob}L_j \rightarrow \text{Ob}K$  and

$$l_j(i): G(\psi_j(i)) \rightarrow F(j)(i) = \bar{F}(i, j)$$

is a morphism in  $C$ . Since  $L$  is a disjoint union, the  $\psi_j$ 's together form

$$\psi: L \rightarrow K \quad \text{with } \lambda(i, j) = \psi_j(i)$$

and then the  $l_j$ 's give morphisms

$$l_j(i): G(\psi(i, j)) \rightarrow \bar{F}(i, j)$$

which represent the unique map  $\lambda: G \rightarrow F$  in  $\text{pro}(C)$ . Thus  $\bar{F}$  is a limit of  $F$ .

The above demonstration sheds some more light on the relationship between the two definitions of  $pro(C)$ . Suppose  $F: I \rightarrow C$  is a projective system in  $C$ , then we can use the Yoneda embedding  $h: C \rightarrow pro(C)$  to consider  $F$  as the projective system

$$hF: I \rightarrow pro(C) \text{ in } pro(C);$$

since  $pro(C)$  has projective limits, we can find a limit  $\bar{F}$  for  $hF$ . In this case we have the same indexing category for each  $hF(i)$  and so  $L$  is essentially the same as  $I$ .

Using the description in terms of  $(Sex(C, Ab))^{op}$  we find that the functor corresponding to  $h_{\bar{F}}$  is exactly the same as  $h_F$ . Thus even when  $F$  has no limit within  $C$ , within  $pro(C)$  one can find a limit, namely another interpretation of  $F$  itself. Although this process of «double think» may seem like cheating, it, in fact, provides a very useful insight into, and method for, the study of pro-objects. We illustrate this in the ideas which lead up to the reindexing results.

Suppose we are given a functor  $F: I \rightarrow C$  as usual; then  $F$  is a pro-object. If we denote the constant functor from  $pro(C)$  to  $pro(C)^I$  by  $c: pro(C) \rightarrow pro(C)^I$  given by

$$c(F)(i) = F, \quad c(F)(i \rightarrow j) = id_F,$$

then the fact that  $F$  is itself the limit of the functor  $hF: I \rightarrow pro(C)$  is expressed by the adjunction equation

$$Hom_{pro(C)}(G, F) \approx Hom_{pro(C)^I}(c(G), hF)$$

for all pro-objects  $G$  and so the adjunction equations (or limiting cone according to your interpretation) gives a map  $\mu: c(F) \rightarrow hF$  in  $pro(C)^I$ , or equivalently a set

$$\{\mu_i: c(F)(i) \rightarrow hF(i)\}_{Ob I}$$

of promaps such that for  $i \rightarrow j$  the diagram

$$\begin{array}{ccc} c(F)(j) & \xrightarrow{\mu_j} & hF(j) \\ \uparrow & & \uparrow \\ c(F)(i) & \xrightarrow{\mu_i} & hF(i) \end{array}$$

commutes, the vertical maps being transitions. Omitting the «double think» this is merely a set of promaps  $\{\mu_i: F \rightarrow F(i)\}$  where  $F(i)$  is considered as a pro-object indexed by the category with one morphism, and the diagram is then merely

$$\begin{array}{ccc}
 & & F(j) \\
 & \nearrow^{\mu_j} & \uparrow F(a) \\
 F & & F(i) \\
 & \searrow_{\mu_i} & \\
 & & i \\
 & & \uparrow a \\
 & & j
 \end{array}$$

We now will drop once and for all the «double think» used above and will consider  $c, h$  etc... as inclusions rather than embeddings whenever this is feasible. Thus, for example, the object  $M$  of  $C$  and the pro-object  $F_M: I \rightarrow C$  (where  $F_M(*) = M$  and  $I$  is the one-morphism category) will be identified. By doing this we will effectively ignore the distinction between the two interpretations (or definitions) of  $pro(C)$ . This will not give rise to any confusion.

We now will prove the following reindexing result:

Given pro-objects  $F: I \rightarrow C, G: J \rightarrow C$  and a map  $f: F \rightarrow G$  in  $pro(C)$ , there is an indexing category  $M_f$  with «initiality» functors  $\phi_I: M_f \rightarrow I, \phi_J: M_f \rightarrow J$  and a map  $\bar{f}: F\phi_I \rightarrow G\phi_J$  in  $C^{M_f}$  such that the diagram

$$\begin{array}{ccc}
 F & \xrightarrow{f} & G \\
 \cong \updownarrow & & \updownarrow \cong \\
 F\phi_I & \xrightarrow{\bar{f}} & G\phi_J
 \end{array}$$

commutes, the vertical maps being initiality isomorphisms.

PROOF. First notice that any functor category  $C^I$  where  $I$  is an indexing category can be considered as part of  $pro(C)$  possibly with the identification of some morphisms, thus the fact that  $\bar{f}$  is in  $C^{M_f}$  means that it can be considered also as a promap. The category  $M_f$  is constructed as follows:

- The objects of  $M_f$  are the morphisms in  $C$  which «represent»  $f$  in

the sense that  $f_j^i: F(i) \rightarrow G(j)$  represents  $f$  if the diagram

$$\begin{array}{ccc} F & \xrightarrow{f} & G \\ \mu_i \downarrow & & \downarrow \mu_j \\ F(i) & \xrightarrow{f_j^i} & G(j) \end{array}$$

in  $pro(C)$  commutes.

- We say  $f_{j_1}^{i_1} \leq g_{j_2}^{i_2}$  (or  $f_{j_1}^{i_1} \rightarrow g_{j_2}^{i_2}$ ) if the diagram

$$\begin{array}{ccc} F(i_2) & \xrightarrow{g_{j_2}^{i_2}} & G(j_2) \\ \uparrow p_{F, i_2}^{i_1} & & \uparrow p_{G, j_2}^{j_1} \\ F(i_1) & \xrightarrow{f_{j_1}^{i_1}} & G(j_1) \end{array}$$

commutes. (The vertical maps are the obvious transitions).

It is easily checked that  $M_f$  is an indexing category and that the projections

$$\phi_I(f_j^i) = i \quad \text{and} \quad \phi_J(f_j^i) = j$$

are initial. It remains to define  $\bar{f}: F\phi_I \rightarrow G\phi_J$  to be the map given by  $\bar{f}(f_j^i) = f_j^i$ . Checking that this works is simple.

There are more sophisticated versions of this result available, proved in a similar fashion. For example :

(i) Given  $F \xrightarrow{f} G \xrightarrow{g} H$  in  $pro(C)$ , where

$$F: I \rightarrow C, \quad G: J \rightarrow C \quad \text{and} \quad H: K \rightarrow C,$$

there is an indexing category  $M_{f,g}$ , initial functors  $\phi_I, \phi_J, \phi_K$  and a pair of maps

$$F\phi_I \xrightarrow{\bar{f}} G\phi_J \xrightarrow{\bar{g}} H\phi_K$$

in  $C^{M_{f,g}}$  such that the «obvious» diagram commutes.

(Take  $M_{f,g}$  to be the category of maps «representing» the pair  $(f, g)$ .)

(ii) Given a finite diagram scheme  $D$  with no loops and a diagram

$X: D \rightarrow \text{pro}(C)$  in  $\text{pro}(C)$  of type  $D$ , there is an indexing category  $M_X$ , initial functors

$$\phi_d: M_X \rightarrow I_d, \text{ where } X(d): I_d \rightarrow C \text{ and } d \in \text{Ob } D,$$

and a pro-object  $\bar{X}: M_X \rightarrow C^D$  in  $C^D$  such that, using the obvious forgetful functor  $\text{pro}(C^D) \rightarrow \text{pro}(C)^D$ , the natural map  $\bar{X} \rightarrow X$  is an isomorphism. (Take  $M_X$  to be the category of maps in  $C^D$  «representing» the diagram of promaps  $X$ .)

[ For a complete proof of (ii) see Artin and Mazur [1], Appendix.]

These reindexing results allow one to think of  $\text{pro}(C)$  as being made up of various copies of functor categories  $C^I$  for varying indexing categories  $I$ , linked or «glued» together by initiality relations. This view enables one to suggest methods of lifting «structures» from  $C$  to  $\text{pro}(C)$  - first lift to  $C^I$  and then see how to «glue» these structures together, much as in the manner of the construction of «structures» on differentiable manifolds.

To illustrate this idea we show that  $\text{pro}(C)$  is abelian (if  $C$  is abelian). That  $\text{pro}(C)$  is additive follows from the equation

$$\text{Hom}_{\text{pro}(C)}(F, G) \approx \lim_{\leftarrow I} (\lim_{\rightarrow I} \text{Hom}_C(F(i), G(j))),$$

since each  $\text{Hom}_C(F(i), G(j))$  is an abelian group and any zero object in  $C$  furnishes a zero for  $\text{pro}(C)$ .

To show the existence of a direct sum of pro-objects  $F: I \rightarrow C$ ,  $G: J \rightarrow C$ , we merely have to take

$$F \oplus G: I \times J \rightarrow C, \quad F \oplus G(i, j) = F(i) \oplus G(j),$$

where the projection maps of  $I \times J$  towards  $I$  and  $J$  furnish the index maps underlying the inclusions

$$F(i) \rightarrow F(i) \oplus G(j) \quad \text{and} \quad G(j) \rightarrow F(i) \oplus G(j).$$

To find the kernel of a morphism  $f: F \rightarrow G$ , we first represent it, by re-indexing, by a promap

$$\bar{f}: \bar{F} \rightarrow \bar{G} \quad \text{in } C^M$$

so that

$$\begin{array}{ccc}
 F & \xrightarrow{f} & G \\
 \uparrow \cong & & \uparrow \cong \\
 \bar{F} & \xrightarrow{\bar{f}} & \bar{G}
 \end{array}$$

commutes and as usual the vertical maps are initiality isomorphisms ; next we take the kernel of  $\bar{f}$  in the category  $C^M$ , i. e., for each  $m$  in  $M$ ,

$$(Ker \bar{f})(m) = Ker(\bar{f}(m)).$$

This pro-object together with the monomorphism  $Ker \bar{f} \rightarrow \bar{F} \cong F$  gives the kernel of  $f$ . To see this, suppose the composite

$$E \xrightarrow{g} F \xrightarrow{f} G$$

is the zero morphism; by (i) above we can represent this as a composition of promaps in some  $C^N$ :

$$E_1 \xrightarrow{g_1} F_1 \xrightarrow{f_1} G_1 ;$$

$N$  consists of all map pairs

$$E(k) \xrightarrow{x} F(i) \xrightarrow{y} G(j)$$

such that the diagram

$$\begin{array}{ccccc}
 E & \xrightarrow{f} & F & \xrightarrow{g} & G \\
 \downarrow & & \downarrow & & \downarrow \\
 E(k) & \xrightarrow{x} & F(i) & \xrightarrow{y} & G(j)
 \end{array}$$

commutes and  $N$  contains an initial subcategory  $N_0$  consisting of those pairs  $(x, y)$  such that  $yx = 0$ . There is an initial functor

$$\phi: N_0 \rightarrow M: (x, y) \rightarrow y,$$

and so we can obtain a diagram in  $C^{N_0}$ :

$$\begin{array}{ccccc}
 Ker \bar{f} \approx (Ker \bar{f}) \phi & \longrightarrow & \bar{F} \phi & \xrightarrow{\bar{f} \phi} & \bar{G} \phi \\
 & & \uparrow g' & & \\
 & & E \psi & & 
 \end{array}$$

where  $\psi: N_0 \rightarrow K$  sends  $(x, y)$  to  $k$ , the index of the domain of  $x$ . Clear-

ly  $g'$  factors through the kernel of  $\bar{f}\phi$  as required.

A similar dual discussion describes the cokernel of  $f$ .

In the above discussion we have made use of the fact that a morphism  $f: F \rightarrow G$  is the zero morphism iff for any index  $j$  of  $G$  and  $i$  of  $F$  such that  $f_j^i: F(i) \rightarrow G(j)$  represents  $f$ , there is some  $i_0 \rightarrow i$  so that the composite map

$$F(i_0) \xrightarrow{p_{i_0}^i} F(i) \xrightarrow{f_j^i} G(j)$$

is zero. This is immediate from the definition of maps in  $pro(C)$  as it is merely the statement that the elements

$$f_j^i, 0 \in Hom_C(F(i), G(j))$$

have the same image in  $\varinjlim_I Hom_C(F(i), G(j))$ .

As a consequence of this applied to the identity map on  $F$ , we find that  $F = 0$  iff, for any index  $i$  of  $F$ , there is an index  $i_0 \rightarrow i$  such that  $p_{i_0}^i = 0$ .

As was mentioned before, none of this material is new. It has been collected here together for the convenience of the reader since the methods of proof and the results are scattered around in the literature. For further information on procategories, we refer the reader to the following sources:

Artin and Mazur [1], Appendix; Deligne in Hartshorne [3], Appendix; Duskin [6]; and Grothendieck [13].

From now on we will assume that  $C$  is a Grothendieck category.

## 2. ESSENTIAL EQUIVALENCE OF PROJECTIVE SYSTEMS.

At the start of Section 1, we introduced the terminology «of type  $C'$ » where  $C'$  was a full subcategory of  $C$ . We also saw in Section 1 that if  $C' = \{0\}$  was the full subcategory consisting of the zero object, then a pro-object  $F$  could be isomorphic to  $0$  without having any of its «objects» zero: For example let  $I$  be any index category and  $X$  any object

of  $C$ , define a pro-object  $X_0: I \rightarrow C$  by:

$$\begin{aligned} X_0(i) &= X \text{ for each } i \text{ in } I, \\ p_{X,j}^i &= 0 \text{ for each } i \rightarrow j \text{ in } I. \end{aligned}$$

$X$  could be chosen to be non-zero, but  $X_0 \approx 0$ . Thus in this case  $X_0$  is isomorphic to something in  $pro(C')$  without being in fact in  $pro(C')$  itself. To enable a sensible classification of pro-objects to be carried out, one has to allow for this. This is handled in the following definition:

Suppose  $C_1$  is a full subcategory of  $C$ , then  $X$  in  $pro(C)$  is said to be *essentially of type  $C_1$*  if it is isomorphic to some object of  $pro(C_1)$ .

This idea of «essential» properties of pro-objects has been studied by Verdier [40] and Laudal [23] (for a fuller account of [40], see Duskin's Strasbourg Notes [6]). The internal description of when  $X \approx 0$  in  $pro(C)$  is mirrored by internal descriptions of when  $X$  is essentially of type  $C_1$ . Such an internal description was given by Laudal [23] (cf. Duskin's Notes [6]). Although we will not make very extensive use of this result, it is useful as a means to check whether or not a given pro-object is essentially of a certain type, we therefore include it and will prove it, as proofs are difficult to come by.

PROPOSITION 2.1. *In order that a pro-object be essentially of type  $C_1$ , it is necessary and sufficient that its defining system  $F: I \rightarrow C$ , say, have the property:*

(\*) *for each  $i$  in  $I$  there is a  $j$  and  $j \rightarrow i$  such that the transition  $p_{F,i}^j$  factors through an object  $M_i^j$  of  $C_1$ .*

PROOF. Let  $F$  have the property mentioned above. We form a new indexing category  $C(I)$  as follows:

- the ordered pair  $(j, i)$  is in  $C(I)$  if  $i, j$  are in  $I$  and there is a factorisation

$$p_{F,i}^j: F(j) \rightarrow M_i^j \rightarrow F(i)$$

with  $M_i^j$  in  $C_1$ . (We will write

$$s_i^j: F(j) \rightarrow M_i^j \text{ and } t_i^j: M_i^j \rightarrow F(i) .)$$

- There is a single morphism in  $C(I)$  between  $(j', i')$  and  $(j, i)$  precisely when there are morphisms  $j' \rightarrow j$  and  $i' \rightarrow i$  in  $I$ .

There are two projections :

$$\begin{aligned} p_1: C(I) &\rightarrow I, & p_1(j, i) &= j, \\ p_2: C(I) &\rightarrow I, & p_2(j, i) &= i, \end{aligned}$$

and the property (\*) ensures that both are initial, so the pro-objects  $F$ ,  $Fp_1$  and  $Fp_2$  are isomorphic. We refine  $C(I)$  by looking at an initial subcategory  $D(I)$  of  $C(I)$  with the same objects but with a single morphism from  $(j', i')$  to  $(j, i)$  precisely when a map  $i' \rightarrow j$  exists. Thus

$$F_1 = Fp_1 a \quad \text{and} \quad F_2 = Fp_2 a$$

are both isomorphic to  $F$ , where  $a: D(I) \rightarrow C(I)$  is the inclusion.

We now define  $M$  by:

$$M: D(I) \rightarrow C, \quad M(j, i) = M_i^j,$$

and

$$p_{M, (j, i)}^{(j', i')} = M((j', i') \rightarrow (j, i)) = s_i^j p_{F, j}^{i'} t_{i'}^{j'}.$$

Since  $t_i^{j'}, s_i^{j'} = p_{F, i'}^{j'}$ , it is easily checked that this does in fact define a pro-object in  $C$ ; of course it is of type  $C_1$  since each  $M(j, i)$  is in  $C_1$ .

If  $(j', i') \rightarrow (j, i)$  in  $D(I)$ , there are diagrams

$$\begin{array}{ccc} F_1(j, i) & \xrightarrow{s(j, i)} & M(j, i) \\ \downarrow & & \downarrow \\ F_1(j', i') & \xrightarrow{s(j', i')} & M(j', i') \end{array}$$

and

$$\begin{array}{ccc} M(j, i) & \xrightarrow{t(j, i)} & F_2(j, i) \\ \downarrow & & \downarrow \\ M(j', i') & \xrightarrow{t(j', i')} & F_2(j', i') \end{array}$$

both commuting, where

$$s(j, i) = s_i^j \quad \text{and} \quad t(j, i) = t_i^j$$

modulo the identifications of  $F_1(j, i)$  with  $Fp_1 a(j, i) = F(j)$  and of

$F_2(j, i)$  with  $Fp_2a(j, i) = F(i)$ . These diagrams express the fact that the  $\{s_i^j\}$  and  $\{t_i^j\}$  give morphisms

$$s: F_1 \rightarrow M \quad \text{and} \quad t: M \rightarrow F_2.$$

Thus it is obvious that the composed maps

$$F(j) = F_1(j, i) \xrightarrow{s(j, i)} M(j, i) \xrightarrow{t(j, i)} F_2(j, i) = F(i)$$

are precisely the transitions of  $F$  and hence that  $ts$  is the natural isomorphism  $x: F_1 \rightarrow F_2$  given by the initiality condition. Now if we write  $p = s x^{-1} t$  we have that  $p$  is an idempotent and we can form a projective system

$$\dots \rightarrow M \xrightarrow{p} M \xrightarrow{p} M \rightarrow \dots$$

within  $pro(C)$ . Since  $pro(C)$  is left complete, we can use this system to define a limit of itself as in Section 1, we call this «interlaced» limiting system  $\bar{M}$ ; since  $M$  is in  $pro(C_1)$ , so is  $\bar{M}$ . We have now merely to show that  $F$  is isomorphic to  $\bar{M}$ . The above projective system in  $pro(C)$  is clearly isomorphic to the system

$$\dots \rightarrow M \xrightarrow{t} F_2 \xrightarrow{x^{-1}} F_1 \xrightarrow{s} M \xrightarrow{t} F_2 \xrightarrow{x^{-1}} F_1 \xrightarrow{s} M \rightarrow \dots$$

obtained by splitting up each  $p = s x^{-1} t$  into its constituent parts. This is, in turn, isomorphic to

$$\dots \rightarrow F_2 \xrightarrow{x^{-1}} F_1 \xrightarrow{x} F_2 \xrightarrow{x^{-1}} F_1 \xrightarrow{x} F_2 \rightarrow \dots$$

which collapses to give the constant system on  $F_1$  or  $F_2$  depending on the collapse used. Hence  $\bar{M}$  and  $F_1$  must be isomorphic, i. e.,  $F$  is isomorphic to a pro-object of type  $C_1$  as required.

The converse is easier. Suppose given an isomorphism  $f: F \rightarrow M$  with inverse  $g: M \rightarrow F$ , with  $M$  in  $pro(C_1)$ ; we must check condition (\*), so we are given some  $i$  in the index category of  $F$ . By our previous work, if  $M: K \rightarrow C$ , there is a set map  $\phi: I \rightarrow K$  and a map

$$g(i): M(\phi(i)) \rightarrow F(i) \quad \text{in } C$$

representing  $g$ . Similarly there is a set map  $\psi: K \rightarrow I$  and for each  $k$  in  $K$ , a map

$$f(k): F(\psi(k)) \rightarrow M(k) \text{ in } C$$

representing  $f$ . Taking  $k = \phi(i)$  we get a map

$$F(\psi\phi(i)) \xrightarrow{f(\phi(i))g(i)} F(i)$$

representing the composite  $fg = id_F$ . Since these two maps are equal, there is some  $j$  in  $I$  such that  $j \rightarrow \psi\phi(i)$  and

$$F(j) \xrightarrow{p_i^j} F(i) = F(j) \xrightarrow{p_{\psi\phi(i)}^j} F(\psi\phi(i)) \xrightarrow{f(\phi(i))} M(\phi(i)) \begin{matrix} \searrow g(i) \\ F(i) \end{matrix}$$

i.e., the transition  $p_i^j$  factors through an object in  $C_1$ , so  $F$  satisfies condition (\*).

The next theorem is of central importance in what follows. It is stated incorrectly in both Verdier [40] and Duskin [6], but the mistake is not that serious.

THEOREM 2.2. *Suppose  $C_1$  is closed under extensions, i. e., if  $A_1, A_3$  are in  $C_1$  and there is a short exact sequence*

$$0 \rightarrow A_1 \rightarrow A_2 \rightarrow A_3 \rightarrow 0$$

*in  $C$ , then  $A_2$  is also in  $C_1$ . If*

$$(2.2.1) \quad 0 \rightarrow F_1 \xrightarrow{u} F_2 \xrightarrow{v} F_3 \rightarrow 0$$

*is an exact sequence in  $pro(C)$  and  $F_1, F_3$  are essentially of type  $C_1$ , then  $F_2$  is essentially of type  $C_1$ .*

PROOF. First we reindex so as to be able to replace (2.2.1) by an isomorphic exact sequence indexed by a single indexing category, i. e., so as to represent (2.2.1) by a pro-object in the category of exact sequences in  $C$ . To this end, we consider the category of all pairs  $(u, v)$ :

$$F_1(i) \xrightarrow{u_1} F_2(j) \xrightarrow{v_1} F_3(k)$$

such that  $v_1 u_1 = 0$  and  $v_1$  represents  $v$  and  $u_1$  represents  $u$  in the sense introduced in Section 1. For each such pair  $(u_1, v_1)$  we associate the short exact sequence

$$0 \rightarrow \text{Ker } v_1 \rightarrow F_2(j) \rightarrow \text{Im } v_1 \rightarrow 0$$

and if  $\alpha: (u_1, v_1) \rightarrow (u_2, v_2)$  is a morphism in this category (i.e., a diagram

$$\begin{array}{ccccc} F_1(i) & \xrightarrow{u_1} & F_2(j) & \xrightarrow{v_1} & F_3(k) \\ p_i^{i'} \uparrow & & \uparrow p_j^{j'} & & \uparrow p_k^{k'} \\ F_1(i') & \xrightarrow{u_2} & F_2(j') & \xrightarrow{v_2} & F_3(k') \end{array}$$

representing a map of pairs), then we associate to  $\alpha$  the corresponding map of exact sequences

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{Ker } v_1 & \longrightarrow & F_2(j) & \longrightarrow & \text{Im } v_1 \longrightarrow 0 \\ & & \uparrow & & \uparrow p_j^{j'} & & \uparrow \\ 0 & \longrightarrow & \text{Ker } v_2 & \longrightarrow & F_2(j') & \longrightarrow & \text{Im } v_2 \longrightarrow 0 \end{array}$$

This assignment gives therefore a «pro-exact sequence» in  $C$  and there are, in the limit, isomorphisms

$$F_1 \approx \langle \text{Ker } v_1 \rangle \quad \text{and} \quad F_3 \approx \langle \text{Im } v_1 \rangle$$

induced by the natural morphisms

$$F_1(i) \rightarrow \text{Ker } v_1 \quad \text{and} \quad \text{Im } v_1 \rightarrow F_3(k).$$

We thus will assume that (2.2.1) is indexed by some category  $I$ . We next assume that  $F_1$  and  $F_3$  satisfy condition (\*) of 2.1, and we will check that  $F_2$  also satisfies this condition.

Given  $i$  in the indexing category  $I$  we can find some  $j$  in  $I$  and a factorisation of  ${}_1p_i^j: F_1(j) \rightarrow F_1(i)$  as

$$F_1(j) \xrightarrow{{}_1s_i^j} {}_1M_i^j \xrightarrow{{}_1t_i^j} F_1(i).$$

Also by refining  $j$  further if necessary, we can assume that, for this  $j$ , the morphism  ${}_3p_i^j: F_3(j) \rightarrow F_3(i)$  also factorises as

$$F_3(j) \xrightarrow{{}_3s_i^j} {}_3M_i^j \xrightarrow{{}_3t_i^j} F_3(i),$$

where of course  ${}_1M_i^j$  and  ${}_3M_i^j$  are in  $C_I$ .

We repeat the procedure starting with  $j$  to find a  $k$  sufficiently

«fine» so that  ${}_1P_j^k$  and  ${}_3P_j^k$  factor through  ${}_1M_j^k$  and  ${}_3M_j^k$  respectively.

Thus we have a large commutative diagram

$$(2.2.2) \quad \begin{array}{ccccccc} 0 & \longrightarrow & F_1(i) & \longrightarrow & F_2(i) & \longrightarrow & F_3(i) \longrightarrow 0 \\ & & \uparrow M_i^j & & \uparrow 2P_i^j & & \uparrow 3M_j^i \\ 0 & \longrightarrow & F_1(j) & \longrightarrow & F_2(j) & \longrightarrow & F_3(j) \longrightarrow 0 \\ & & \uparrow M_j^k & & \uparrow 2P_j^k & & \uparrow 3M_j^k \\ 0 & \longrightarrow & F_1(k) & \longrightarrow & F_2(k) & \longrightarrow & F_3(k) \longrightarrow 0 \end{array}$$

where all the maps are the obvious ones.

Working with the bottom rectangle first we take the pushout of

$$\begin{array}{ccc} {}_1M_j^k & \dashrightarrow & B_j^k \\ \uparrow {}_1S_j^k & & \uparrow \\ F_1(k) & \xrightarrow{u(k)} & F_2(k) \end{array}$$

and form the induced exact sequence

$$\begin{array}{ccccccc} 0 & \longrightarrow & {}_1M_j^k & \longrightarrow & B_j^k & \longrightarrow & F_3(k) \longrightarrow 0 \\ & & \uparrow & \text{p.o.} & \uparrow \sigma_j^k & & \uparrow \\ 0 & \longrightarrow & F_1(k) & \longrightarrow & F_2(k) & \longrightarrow & F_3(k) \longrightarrow 0 \end{array}$$

(as in the Yoneda description of the extension group). By the universal property of pushouts,  ${}_2P_j^k$  factors through  $\sigma_j^k$  as, say,  ${}_2P_j^k = \tau_j^k \sigma_j^k$ . We have thus replaced (2.2.2) by the following diagram :

$$(2.2.3) \quad \begin{array}{ccccccc} 0 & \longrightarrow & F_1(i) & \xrightarrow{u(i)} & F_2(i) & \xrightarrow{v(i)} & F_3(i) \longrightarrow 0 \\ & & \uparrow {}_1P_i^j t_j^k & & \uparrow {}_2P_i^j \tau_j^k & & \uparrow {}_3t_i^j \\ & & & & & & \uparrow {}_3M_i^j \\ & & & & & & \uparrow {}_3S_i^j P_j^k \\ 0 & \longrightarrow & {}_1M_j^k & \longrightarrow & B_j^k & \longrightarrow & F_3(k) \longrightarrow 0 \\ & & \uparrow {}_1S_j^k & & \uparrow \sigma_j^k & & \uparrow \\ 0 & \longrightarrow & F_1(k) & \xrightarrow{u(k)} & F_2(k) & \xrightarrow{v(k)} & F_3(k) \longrightarrow 0 \end{array}$$

We next take the induced exact sequence given by the diagram

$$\begin{array}{ccccccc}
 0 & \longrightarrow & {}_1M_j^k & \longrightarrow & {}_2M_i^k & \longrightarrow & {}_3M_i^j \longrightarrow 0 \\
 & & \uparrow & & \uparrow & \text{p. b.} & \uparrow {}_3s_i^j \\
 0 & \longrightarrow & {}_1M_j^k & \longrightarrow & B_j^k & \longrightarrow & F_3(k) \longrightarrow 0
 \end{array}$$

where the right hand square is a pullback. We thus can replace (2.2.3) finally by (2.2.4) (by using the universal property of pullbacks):

$$(2.2.4) \quad \begin{array}{ccccccc}
 0 & \longrightarrow & F_1(i) & \xrightarrow{u(i)} & F_2(i) & \xrightarrow{v(i)} & F_3(i) \longrightarrow 0 \\
 & & \uparrow {}_1p_i^j t_j^k & & \uparrow {}_2t_i^k & & \uparrow {}_3t_i^j \\
 0 & \longrightarrow & {}_1M_j^k & \longrightarrow & {}_2M_i^k & \longrightarrow & {}_3M_i^j \longrightarrow 0 \\
 & & \uparrow {}_1s_j^k & & \uparrow {}_2s_j^k & & \uparrow {}_3s_i^j p_j^k \\
 0 & \longrightarrow & F_1(k) & \xrightarrow{u(k)} & F_2(k) & \xrightarrow{v(k)} & F_3(k) \longrightarrow 0
 \end{array}$$

where the vertical centre maps factorise  ${}_2p_i^k$ .

Since  $C_1$  is closed under extensions, it follows that  ${}_2M_i^k$  is in  $C_1$ , so condition (\*) is satisfied by  $F_2$ .

If we combine this proof with that of Proposition 2.1 applied to the subcategory of the category of exact sequences in  $C$  consisting of exact sequences in  $C_1$ , we obtain the following useful result.

PROPOSITION 2.3. *If*

$$0 \rightarrow F_1 \rightarrow F_2 \rightarrow F_3 \rightarrow 0$$

*is an exact sequence in  $\text{pro}(C)$ ,  $C_1$  is a full subcategory of  $C$  which is closed under extensions and  $F_1$  and  $F_3$  are essentially of type  $C_1$ , then there is a short exact sequence in  $\text{pro}(C_1)$  isomorphic to the given sequence, i. e., there is*

$$\begin{array}{ccccccc}
 0 & \rightarrow & M_1 & \rightarrow & M_2 & \rightarrow & M_3 \rightarrow 0 \\
 & & \uparrow \approx & & \uparrow \approx & & \uparrow \approx \\
 0 & \rightarrow & F_1 & \rightarrow & F_2 & \rightarrow & F_3 \rightarrow 0 ;
 \end{array}$$

*moreover the first and last isomorphisms can be specified to start with.*

PRO-OBJECTS IN GROTHENDIECK CATEGORIES

One of the most important consequences of the proof of 2.2 is the following:

PROPOSITION 2.4. *Given any short exact sequence*

$$0 \longrightarrow F_1 \longrightarrow F_2 \longrightarrow F_3 \longrightarrow 0$$

*in  $\text{pro}(C)$ , there is an index category  $I$  and a short exact sequence*

$$0 \longrightarrow E_1 \longrightarrow E_2 \longrightarrow E_3 \longrightarrow 0$$

*in  $C^I$  (considered as a part of  $\text{pro}(C)$ ) such that*

$$\begin{array}{ccccccc} 0 & \longrightarrow & E_1 & \longrightarrow & E_2 & \longrightarrow & E_3 & \longrightarrow & 0 \\ & & \uparrow \cong & & \uparrow \cong & & \uparrow \cong & & \\ 0 & \longrightarrow & F_1 & \longrightarrow & F_2 & \longrightarrow & F_3 & \longrightarrow & 0 \end{array}$$

*commutes, where the vertical isomorphisms are from the various initiality relations; i. e., any short exact sequence in  $\text{pro}(C)$  is isomorphic to a «pro-short exact sequence in  $C$ ».*

COROLLARY 2.5. *Suppose  $F_1$  is a subobject of  $F_2$ , i. e., there is a monomorphism  $0 \rightarrow F_1 \xrightarrow{f} F_2$ ; then  $f$  can be represented up to isomorphism by a pro-monomorphism, i. e., a promap all of whose components are monomorphisms.*

PROOF. Form

$$0 \longrightarrow F_1 \xrightarrow{f} F_2 \longrightarrow \text{Coker } f \longrightarrow 0$$

and use 2.4.

COROLLARY 2.6. *If  $F_3$  is a quotient object of  $F_2$  then the quotient epimorphism  $F_2 \xrightarrow{g} F_3 \rightarrow 0$  can be represented up to isomorphism by a pro-epimorphism.*

PROOF. Form

$$0 \longrightarrow \text{Ker } g \longrightarrow F_2 \xrightarrow{g} F_3 \longrightarrow 0$$

and use 2.4.

COROLLARY 2.7. *If  $C_1$  is closed under subobjects and  $F_2$  is essentially of type  $C_1$ , then any subobject of  $F_2$  is also essentially of type  $C_1$ .*

PROOF. Take  $0 \rightarrow F_1 \rightarrow F_2$  ; replacing  $F_2$  by  $M_2$  in  $pro(C_1)$  we still get  $0 \rightarrow F_1 \rightarrow M_2$  ; now use 2.5 to find a pro-monomorphism  $0 \rightarrow E_1 \rightarrow E_2$  . The method outlined in the proof of 2.2 shows that this can be done in such a way that  $E_2$  is in  $pro(C_1)$  and hence  $E_1$  must also be in  $pro(C_1)$  , i. e.,  $F_1$  is essentially of type  $C_1$  .

Similarly and dually one obtains :

COROLLARY 2.8. *If  $C_1$  is closed under quotients, then any quotient of a pro-object  $F_2$  which is essentially of type  $C_1$  is also essentially of type  $C_1$  .*

In order to sum up these results in a way relevant to the use which will be made of them later, we will introduce some notation and a well-known definition.

If  $C_1$  is a full subcategory of  $C$  , then we will denote by  $E(C_1)$  the full subcategory of  $pro(C)$  consisting of the pro-objects which are essentially of type  $C_1$  .

A subcategory  $C_1$  of  $C$  is called *thick* (épaisse) if, for each short exact sequence in  $C$  :

$$0 \rightarrow A \rightarrow B \rightarrow D \rightarrow 0 ,$$

$B$  is in  $C_1$  iff both  $A$  and  $D$  are in  $C_1$  . (The translation of épaisse varies according to the author. Demers who translated Gabriel and Zisman [9] uses «thick». Popescu [26] uses «dense». We will use «thick» since it seems to be nearer the original. The terminology «Serre subcategory» is also sometimes used but seems somewhat clumsy even though it assigns the origin of the idea correctly. )

PROPOSITION 2.9. *If  $C_1$  is a thick subcategory of  $C$ , then  $E(C_1)$  is a thick subcategory of  $pro(C)$ .*

The converse is also true but we will not need it and hence will not prove it. It is non-trivial.

Following Gabriel [8] we could in this situation form the quotient category  $pro(C)/E(C_1)$  and the obvious thing to expect would be some

natural connection with  $pro(C/C_1)$ . In fact there is such a natural connection and, although we will not be studying it in detail here, it is worth noting how it arises.

There are canonical functors

$$T: C \rightarrow C/C_1 \quad \text{and} \quad \bar{T}: pro(C) \rightarrow pro(C)/E(C_1)$$

such that the usual universality properties hold. For instance, if  $X$  is an object of  $C_1$ ,  $T(X) \approx 0$  and, for any additive functor  $S: C \rightarrow D$  such that  $S(X) \approx 0$  for all  $X$  in  $Ob C_1$ , there is a unique functor  $S': C/C_1 \rightarrow D$  so that  $S = S'T$ ; similarly for  $\bar{T}$ .

If we extend  $T$  «pointwise» to a functor

$$pro T: pro(C) \rightarrow pro(C/C_1),$$

we find that, for any pro-object  $X$  which is of type  $C_1$ ,  $pro T(X) \approx 0$ . Of course this is also true for any  $X$  which is essentially of type  $C_1$  (to see this, note that  $X$  is isomorphic to something in  $pro(C_1)$  or that, by 2.1, the transitions of  $pro T(X)$  will eventually factor through  $0$ , and hence  $pro T(X) \approx 0$ ). Thus for each  $X$  in  $E(C_1)$ ,  $pro T(X) \approx 0$ , which implies that there is a unique functor

$$T': pro(C)/E(C_1) \rightarrow pro(C/C_1).$$

One might expect  $T'$  to be an equivalence, but at present it is not known in general it is or it is not. The only information available is that, if  $C_1$  is a localising subcategory, then  $T'$  is an equivalence, but in the analogous non-additive homotopy theory problem,  $T'$  is most certainly not an equivalence.

### 3. EXTENDING TORSION THEORIES AND LOCALISATIONS.

At this point in the development of the ideas of this paper there are two directions in which we may go. The one which we will not yet take is to consider what happens if  $C_1$  is a localising subcategory, and hence the significance of the existence of a section functor  $S$ , which would be a right adjoint to  $T$ . This route leads speedily to a useful conclusion, but omits much of the structure of this localisation situation. It seems

better to proceed at a more leisurely pace and to look at the algebraic details of the various structures involved in the main result instead of smashing the problem to pieces with an instant categorical sledgehammer.

We therefore will need the details of how to attack localisation from as many different directions as possible, and for this we refer the reader to Gabriel [8], Popescu [26] or Hacque [16, 17].

We first assume as before that  $C_I$  is a thick subcategory of  $C$  and recall the following result (cf. Hacque [16], page 25, or Popescu [26], page 174, 4.4, Lemma 4.1).

For any object  $M$  in  $C$  the following conditions are equivalent :

a) For each morphism  $u : P \rightarrow Q$  with  $Ker u$  and  $Coker u$  in  $C_I$ ,

$$Hom_C(u, M) : Hom_C(Q, M) \rightarrow Hom_C(P, M)$$

is a bijection.

b) Each subobject of  $M$  appearing in  $C_I$  is null and any short exact sequence

$$0 \longrightarrow M \longrightarrow N \longrightarrow P \longrightarrow 0$$

with  $P$  in  $C_I$  splits.

c) For any  $P$  in  $C$ ,

$$T(P, M) : Hom_C(P, M) \rightarrow Hom_{C/C_I}(T(P), T(M))$$

is a bijection.

As usual we say that  $M$  is  $C_I$ -closed if it satisfies these equivalent conditions.

We need to know the connection, if any, between the  $E(C_I)$ -closed pro-objects in  $pro(C)$  and the essentially  $C_I$ -closed pro-objects in  $pro(C)$ .

PROPOSITION 3.1. *If  $M : I \rightarrow C$  is essentially  $C_I$ -closed, then it is also  $E(C_I)$ -closed.*

PROOF. Examination of the three equivalent conditions cited above should convince the reader that conditions b and c will be difficult to verify. For b, one can easily show that the sequence is «locally» split, but to show that the various «splittings» fit together to make a promap will be

difficult; and for  $c$  we do not know enough about  $pro(C)/E(C_1)$  to be able to describe  $\text{Hom}$  in that category. (In fact, it is precisely condition  $c$  which provides a description of  $\text{Hom}$  in  $pro(C)/E(C_1)$  and enables us to describe  $pro(C)/E(C_1)$ .) Thus we are left with condition  $a$ .

We break the proof in two parts by splitting  $u$  as a monomorphism composed with an epimorphism. Firstly the epimorphism  $P \rightarrow \text{Im } u$ . This we will relabel to be  $u$  itself as this will cause no confusion; so we consider an epimorphism  $u: P \rightarrow Q$  with  $\text{Ker } u$  in  $E(C_1)$ . We need to replace  $u$  by an isomorphic promap by reindexing in such a way that  $u$  is still epimorphic and  $\text{Ker } u$  is of type  $C_1$ ; this we do as follows. By the proof of 2.2 we can replace the short exact sequence

$$0 \longrightarrow \text{Ker } u \longrightarrow P \xrightarrow{u} Q \longrightarrow 0$$

by a pro-short exact sequence - we assume this has been done. Now  $\text{Ker } u$  is in  $E(C_1)$ , so by 2.1 given any index  $i$  there is a  $j$  with  $j \rightarrow i$  and a factorisation of this transition

$$K(j) \xrightarrow{s_i^j} M_i^j \xrightarrow{t_i^j} K(i),$$

with  $M_i^j$  in  $C_1$  (writing  $(\text{Ker } u)(j)$  as  $K(j)$  for simplicity). We thus obtain the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & K(i) & \longrightarrow & P(i) & \longrightarrow & Q(i) \longrightarrow 0 \\ & & \uparrow & & \uparrow & & \uparrow \\ & & M_i^j & & & & \\ & & \uparrow & & & & \\ 0 & \longrightarrow & K(j) & \longrightarrow & P(j) & \longrightarrow & Q(j) \longrightarrow 0. \end{array}$$

As in similar situations before, we take the pushout of

$$M_i^j \longleftarrow K(j) \longrightarrow P(j)$$

to obtain a new diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & K(i) & \longrightarrow & P(i) & \longrightarrow & Q(i) \longrightarrow 0 \\ & & \uparrow & & \uparrow & & \uparrow \\ 0 & \longrightarrow & M_i^j & \longrightarrow & P_i^j & \longrightarrow & Q(j) \longrightarrow 0 \\ & & \uparrow & \text{p.o.} & \uparrow & & \parallel \\ 0 & \longrightarrow & K(j) & \longrightarrow & P(j) & \longrightarrow & Q(j) \longrightarrow 0 \end{array}$$

where two vertical maps in the middle column compose to give the appropriate transition. We thus get a new pro-short exact sequence, indexed by  $D(I)$  (cf. Proof of 2.1)

$$0 \longrightarrow M \longrightarrow \bar{P} \xrightarrow{\bar{u}} \bar{Q} \longrightarrow 0,$$

where

$$M(j, i) = M_i^j, \quad \bar{P}(j, i) = P_i^j \quad \text{and} \quad \bar{Q}(j, i) = Q(j).$$

It is fairly easily seen that  $\bar{u}: \bar{P} \rightarrow \bar{Q}$  is isomorphic to  $u: P \rightarrow Q$ , since both are initial subsystems of the promap represented by the two right hand columns of the above diagram and the maps between columns. Clearly  $M$  is  $\text{Ker} \bar{u}$  so we can replace  $u$  by a map in which  $\text{Ker} u$  is actually of type  $C_I$  and not just essentially of type  $C_I$ . Now we assume this done and can thus consider an epimorphism  $u: P \rightarrow Q$  which is a promap, indexed by  $I$ , say, such that  $\text{Ker} u(i)$  is in  $C_I$  for each  $i$  in  $I$ .

We are given some  $M$  in  $\text{pro}(C)$  which is essentially  $C_I$ -closed, since we have to examine the natural function

$$\text{Hom}(u, M): \text{Hom}(Q, M) \rightarrow \text{Hom}(P, M);$$

we can replace  $M$  by an isomorphic pro-object actually of «type  $C_I$ -closed», just as we could replace  $u$  by an isomorphic promap with special properties. Thus we assume:  $M: J \rightarrow C$  satisfies  $M(j)$  is  $C_I$ -closed for each  $j$  in  $J$ . Thus for each  $i$  in  $I$ ,  $j$  in  $J$  we have  $u(i): P(i) \rightarrow Q(i)$  has  $\text{Ker} u(i)$  in  $C_I$  and  $M(j)$  is  $C_I$ -closed, so that the induced natural map

$$\text{Hom}_C(u(i), M(j)): \text{Hom}_C(Q(i), M(j)) \rightarrow \text{Hom}_C(P(i), M(j))$$

is a bijection. Moreover given  $i' \rightarrow i$  and  $j' \rightarrow j$  the obvious diagrams commute and all the horizontal maps are bijections, thus

$$\begin{aligned} \lim_{\overleftarrow{J}} (\lim_{\overrightarrow{I}} \text{Hom}_C(u(i), M(j))) : \\ \lim_{\overleftarrow{J}} (\lim_{\overrightarrow{I}} \text{Hom}_C(Q(i), M(j))) \rightarrow \lim_{\overleftarrow{J}} (\lim_{\overrightarrow{I}} \text{Hom}_C(P(i), M(j))) \end{aligned}$$

must be a bijection, as required.

The proof for  $u$  a monomorphism follows obviously a similar, if partially dual, path and hence will be omitted.

We next turn our attention to torsion theories. Recall from Dickson [5] (cf. Popescu [26], Section 4.8) that a *torsion theory* for the category  $C$  consists of a pair  $(T, F)$  of full subcategories of  $C$  satisfying the following axioms:

- (i)  $T \cap F = 0$ ,
- (ii)  $T$  is closed under quotients,
- (iii)  $F$  is closed under subobjects,
- (iv) for each  $X$  in  $C$ , there is a short exact sequence

$$0 \longrightarrow X' \longrightarrow X \longrightarrow X'' \longrightarrow 0$$

with  $X'$  in  $T$  and  $X''$  in  $F$ .

An alternative but equivalent set of properties involves (iv) above and in addition the two axioms:

- (v)  $T$  and  $F$  contain complete isomorphism classes,
- (vi) if  $X$  is in  $T$  and  $Y$  in  $F$ , then  $Hom_C(X, Y) = 0$ .

Suppose now that  $(T, F)$  is a torsion theory in the Grothendieck category  $C$ . By 2.7 and 2.8,  $E(T)$  is closed under quotients and  $E(F)$  under subobjects. Moreover (vi) is clearly satisfied for  $X$  in  $E(T)$  and  $Y$  in  $E(F)$  as is (v). Thus an obvious question to ask is:

Is  $(E(T), E(F))$  a torsion theory in  $pro(C)$ ?

We have to verify (iv). We first check (i) as this is quite simple.

LEMMA 3.2.  $E(T) \cap E(F) = 0$ .

PROOF. Suppose  $X$  is in both  $E(T)$  and  $E(F)$ . We will if necessary replace  $X$  by an isomorphic pro-object of type  $T$ , but only essentially of type  $F$ . Thus given any index  $i$  there is a  $j$  with  $j \rightarrow i$  such that the relevant transition factors as

$$X(j) \xrightarrow{s_i^j} M_i^j \xrightarrow{t_i^j} X(i),$$

where  $M_i^j$  is in  $F$  and of course  $X(j)$  is in  $T$ . By (vi) for  $(T, F)$ , we have  $s_i^j = 0$ , so the transition  $p_i^j$  is zero, i.e.,  $X \approx 0$ .

As is well-known (Popescu [26], Section 8, page 200, Stenström [39] or Lambek [22]), any hereditary torsion theory  $(T, F)$  (that is one

in which  $T$  is closed under subobjects) is completely determined by a «subfunctor of the identity»  $t$  satisfying:

- (i)  $t^2 = t$ ,
- (ii)  $t(X/t(X)) = 0$  for all  $X$  in  $C$ .

$t$  is the torsion radical associated with  $(T, F)$ .  $t(X)$  is the maximal subobject of  $X$  which is in  $T$  and the exact sequence of axiom (iv) can in this case be written

$$0 \longrightarrow t(X) \longrightarrow X \longrightarrow X/t(X) \longrightarrow 0.$$

If  $(E(T), E(F))$  is a hereditary torsion theory, the obvious candidate for the torsion radical  $\bar{t}$  will be the extension to  $pro(C)$  of the torsion radical of  $(T, F)$ . (Note  $(T, F)$  is bound, in this case, to be hereditary.) Now  $E(T)$  is closed under subobjects if  $T$  is, by 2.7, so we wish to examine it to see if  $\bar{t}$  is a torsion radical, or alternatively use it to show that  $(E(T), E(F))$  is a hereditary torsion theory. So, suppose  $X$  is any pro-object in  $C$

LEMMA 3.3.  $\bar{t}(X)$  is the maximal subobject of  $X$  appearing in  $E(T)$ .

PROOF. If  $X: I \rightarrow C$ , then  $\bar{t}(X)(i) = t(X(i))$ . Now suppose  $Y$  is in  $E(T)$  and  $Y$  is a subobject of  $X$ . By 2.5 we may assume  $Y$  is indexed by  $I$  and the monomorphism  $u: Y \rightarrow X$  satisfies:  $u(i)$  is a monomorphism for each  $i$ . Applying  $\bar{t}$  to  $u$  gives an inclusion

$$t(u(i)): t(Y(i)) \rightarrow t(X(i)) \text{ for each } i.$$

By assumption  $Y$  is essentially of type  $T$ , so by 2.1 there is, for each such  $i$ , a  $j \rightarrow i$  such that the transition  $\gamma_i^j$  factors:

$$Y(j) \xrightarrow{s_i^j} T_i^j \xrightarrow{t_i^j} Y(i).$$

Applying  $t$  gives

$$\begin{array}{ccccc} Y(j) & \longrightarrow & T_i^j & \longrightarrow & Y(i) \\ \uparrow & & \uparrow & & \uparrow \\ t(Y(j)) & \longrightarrow & T_i^j & \longrightarrow & t(Y(i)) \end{array}$$

the vertical maps being natural inclusions from property (ii) of  $t$ . Thus the inclusion map  $\bar{t}(Y) \rightarrow Y$  has cokernel a pro-object such that the tran-

sition corresponding to  $j \rightarrow i$  factors through  $0$ , i.e., this inclusion is an isomorphism in  $pro(C)$ , so  $\bar{t}(Y) \approx Y$  and the diagram

$$\begin{array}{ccc} \bar{t}(Y) & \xrightarrow{\bar{t}(u)} & \bar{t}(X) \\ \approx \downarrow & & \downarrow \\ Y & \xrightarrow{u} & X \end{array}$$

shows that  $u$  factors through  $\bar{t}(X)$ , so  $\bar{t}(X)$  is the maximal subobject of  $X$  which appears in  $E(T)$ .

PROPOSITION 3.4. *If  $(T, F)$  is a hereditary torsion theory on  $C$ , then  $(E(T), E(F))$  is a hereditary torsion theory in  $pro(C)$ .*

PROOF. It remains only to check (iv), so suppose  $X$  is in  $pro(C)$ . There is a short exact sequence

$$0 \rightarrow \bar{t}(X) \rightarrow X \rightarrow X/\bar{t}(X) \rightarrow 0,$$

$\bar{t}(X)$  is in  $E(T)$  and  $X/\bar{t}(X)$  is represented by the pro-object with

$$(X/\bar{t}(X))(i) = X(i)/\bar{t}(X(i)),$$

so  $X/\bar{t}(X)$  is of type  $F$  and hence is in  $E(F)$ .

COROLLARY 3.5. *The associated torsion radical of  $(E(T), E(F))$  is the extension to  $pro(C)$  of the torsion radical of  $(T, F)$ .*

We now can look at the case when  $C'$  is a localising subcategory.  $C'$  is localising if it is thick and the projection functor  $T: C \rightarrow C/C'$  has a right adjoint  $S$ . The characterisations of localising subcategories are many (see for instance Hacque [16, 17]) but the most useful, for our purposes, shows that any localising subcategory  $C'$  must form the torsion class of a hereditary torsion theory on  $C$  and moreover, if  $t$  is the associated torsion radical, each  $X/t(X)$  is embeddable in a  $C'$ -closed object (cf. Popescu [26], 4.4.5, page 177, or Hacque [16], 4.2.8, page 30). This condition is both necessary and sufficient.

PROPOSITION 3.6. *If  $C'$  is a localising subcategory of  $C$ , then  $E(C')$  is a localising subcategory of  $pro(C)$ .*

PROOF. Since  $C'$  is a hereditary torsion class, we can deduce from 3.4 that  $E(C')$  is so as well.  $C'$  being localising implies that for any  $X$  in  $C$ ,  $X/t(X)$  can be embedded in a  $C'$ -closed object of  $C$ . In fact writing  $L = ST$ , there is a natural transformation  $\psi: I \rightarrow L$ ,  $L(X/t(X))$  is  $C'$ -closed for each  $X$  and  $\psi(X/t(X))$  is a monomorphism (see Hacque [16 or 17]). Thus if we use the extension  $\bar{L}$  of  $L$  to  $pro(C)$  and invoke 3.1 and 3.5, we find that  $\bar{L}(X/\bar{t}(X))$  is  $E(C')$ -closed for each  $X$  in  $pro(C)$  and that

$$\bar{\psi}(X/\bar{t}(X)): X/\bar{t}(X) \rightarrow \bar{L}(X/\bar{t}(X))$$

is the desired embedding in a  $E(C')$ -closed pro-object. Thus  $E(C')$  is localising.

In our investigation of this localisation, we have now the following information: we know the quotient functor  $\bar{T}$  and the associated subcategory  $Ker \bar{T} = E(C')$ ; we know that  $\bar{T}$  has a section  $\bar{S}$ , but we as yet have no description of  $\bar{S}$  other than as being right adjoint to  $\bar{T}$ ; we also know that the image of  $\bar{S}$ , that is the  $E(C')$ -closed objects, contains the essentially  $C'$ -closed objects, however we do not know if these form all the  $E(C')$ -closed objects. In order to increase our knowledge in these directions, we adopt an approach and some terminology from Hacque [17].

Let  $A$  be any category; a *localising system* in  $A$  is a pair  $(L, \psi)$  where  $L: A \rightarrow A$  is a functor which commutes with finite projective limits and  $\psi: Id_A \rightarrow L$  is a natural transformation for which the associated natural transformations

$$\psi L: L \rightarrow L^2 \quad \text{and} \quad L\psi: L \rightarrow L^2$$

are equal isomorphisms.

An object  $M$  of  $A$  is an *invariant* of  $(L, \psi)$  if the morphism:  $\psi(M): M \rightarrow L(M)$  is an isomorphism.

Two localising systems  $(L, \psi)$  and  $(L', \psi')$  are *equivalent* if there is a natural isomorphism  $\epsilon: L \rightarrow L'$  such that  $\psi' = \epsilon \circ \psi$ .

With these definitions, a *local system* in  $A$  is an «equivalence class»  $[L, \psi]$  of localising systems  $(L, \psi)$  in  $A$ .

Hacque [17] shows there is a one-one correspondence between local systems in  $A$  and «local subcategories» of  $A$ . By a *local subcategory*, he means a subcategory  $L$  of  $A$  for which the inclusion  $S': L \rightarrow A$  has a left adjoint  $T'$  which commutes with finite projective limits. In fact, if  $A$  is abelian, then  $L$  is also abelian,  $T'$  is exact and commutes with inductive limits whilst  $S'$  commutes with all projective limits (Lemme 1.2 of [17]). Later in that paper Hacque shows (1.6) any localisation in  $A$  can be determined uniquely by any one of the following:

- a) a localising subcategory  $C$  of  $A$ ,
- b) a local subcategory  $L$  of  $A$ ,
- c) a local system  $[L, \psi]$  in  $A$ .

The method of passage between these is simple:

- a)  $C = \text{Ker } T = \text{Ker } T' = \text{Ker } L$ ,
- b)  $L =$  full subcategory of  $C$ -closed objects  $=$  full subcategory of invariants of  $(L, \psi)$ ,
- c)  $[L, \psi]$  is given by  $L = ST$  or  $L = S'T'$ , with, in either case:  
 $\psi: Id_A \rightarrow L$  the unit of the adjunction.

In our position we know the localising subcategory  $E(C')$  of  $pro(C)$ ; we need to know  $L$  and  $[L, \psi]$ . In order to find these we compare the given localisation with another.

We have in  $C$  the following data: a localising subcategory  $C'$ , a quotient functor  $T$  with section functor  $S$ , a localising system  $[L, \psi]$  with  $L = ST$  and a local subcategory  $L$  consisting of the  $C'$ -closed objects. Extending all this structure «pointwise» to  $pro(C)$  we have a localising subcategory  $E(C')$ , a functor

$$pro T: pro(C) \rightarrow pro(C/C')$$

with a right adjoint

$$pro S: pro(C/C') \rightarrow pro(C)$$

(thus  $(pro L, pro \psi)$  is a localising system in  $pro(C)$ ). The subcategory  $E(L)$  is a local subcategory of  $pro(C)$  since it is the full completion of  $pro(L)$  and there is a left adjoint

$$\text{pro } T': \text{pro}(C) \rightarrow \text{pro}(L)$$

to the inclusion

$$\text{pro } S: \text{pro}(L) \rightarrow \text{pro}(C) ;$$

it thus remains to check that  $\text{pro } T'$  preserves finite projective limits but this is clear from the reindexing results, since we can calculate limits pointwise in some  $L^I$  and then use  $T'^I: L^I \rightarrow C^I$  before passing back into  $\text{pro}(C)$ .

$$E(C') = \text{Ker}(\text{pro } T) = \text{Ker}(\text{pro } T') = \text{Ker}(\text{pro } L),$$

so by Hacque's characterizations we get the following result:

**THEOREM 3.7.** *If  $C'$  is a localising subcategory of  $C$  determining a local subcategory  $L$  of  $C$  and a local system  $[L, \psi]$  in  $C$ , then  $E(C')$  is a localising subcategory of  $\text{pro}(C)$  whose associated local subcategory of  $E(C')$ -closed objects is precisely  $E(L)$  and whose associated local system is precisely  $[\text{pro } L, \text{pro } \psi]$ .*

Thus to all intents and purposes one can extend localisations from  $C$  to  $\text{pro}(C)$  merely by using the extended localising functors and an adequate use of the word «essentially».

**REMARKS.** (i) This result compares favorably with those obtained on the localisation in [31]; there the localisation existed only if  $C$  was semiartinian and even then the description of the localisation was beyond the tools available. The questions raised at the end of that paper are trivial in this case as the answers form an integral part of our result above.

(ii) It is worth noting that the change-of-rings situation considered in [30] corresponds exactly to a localisation extended from  $\text{Mod-}A$  to  $\text{pro}(\text{Mod-}A)$ .

(iii) A comparison between the above situation and the situation in the non-abelian case as considered in pro-homotopy theory [27] suggests that the reason for the simple solution here is the existence of a right adjoint for  $T$ . This cannot exist in the homotopy situation since one cannot «realise» homotopy theory within any of the usual categories used in homotopy.

4. TRIPLES AND LOCALISATION IN  $pro(C)$ .

This section is essentially an aside and is not necessary for the main flow of the work. However it answers, in particular cases, one of problems asked in the Introduction.

The connection between triples and localisation is fairly well-known. When the base category  $C$  is abelian, Heinicke [18] shows the following results :

Let  $T = \langle G, \eta, \mu \rangle$  be a triple on  $C$  ; thus  $G: C \rightarrow C$  is a functor,

$$\eta: Id_C \rightarrow G \quad \text{and} \quad \mu: G^2 \rightarrow G$$

are natural transformations for which the diagrams

$$\begin{array}{ccc} G & \xrightarrow{G\eta} & G^2 \\ \eta G \downarrow & & \downarrow \mu \\ G^2 & \xrightarrow{\mu} & G \end{array} \quad \text{and} \quad \begin{array}{ccc} G^3 & \xrightarrow{G\mu} & G^2 \\ \downarrow \mu G & & \downarrow \mu \\ G^2 & \xrightarrow{\mu} & G \end{array}$$

commute.

Let  $C^T$  denote the Eilenberg-Moore category of  $T$  ; thus an object in  $C^T$  is a pair  $(X, \phi)$  where  $X$  is an object of  $C$  and  $\phi: G(X) \rightarrow X$  is a morphism in  $C$  for which

$$\begin{array}{ccc} X & \xrightarrow{\eta(X)} & G(X) \\ & \searrow I_X & \downarrow \phi \\ & & X \end{array} \quad \text{and} \quad \begin{array}{ccc} G^2(X) & \xrightarrow{G\phi} & G(X) \\ \downarrow \mu(X) & & \downarrow \phi \\ G(X) & \xrightarrow{\phi} & X \end{array}$$

commute.

A morphism  $f: (X, \phi) \rightarrow (X', \phi')$  in  $C^T$  is a morphism  $f: X \rightarrow X'$  in  $C$  for which

$$\begin{array}{ccc} G(X) & \xrightarrow{G(f)} & G(X') \\ \phi \downarrow & & \downarrow \phi' \\ X & \xrightarrow{f} & X' \end{array}$$

commutes.

Heinicke [18] uses the term «localising triple» for a triple in which  $G$  is left exact and  $\mu$  is an equivalence. He proves, amongst other things,

that in this case  $(G, \eta)$  is a localising system in the sense (due to Hacque [17]) introduced above.

However of most interest to us here is his proof of the identification of  $C^T$  (when  $T = \langle G, \eta, \mu \rangle$  is localising) with the full subcategory of  $C$  determined by those objects  $X$  for which  $\eta(X)$  is an isomorphism. (Note his results are stated only for a category of modules, but they do not depend on this fact.)

Linking up Heinicke [18] and Hacque [17] shows that, for a localising triple  $T$ ,  $C^T$  is the same as the local subcategory determined by  $(G, \eta)$ . We can now use this to reinterpret Theorem 3.7 above in terms of localising triples.

**THEOREM 4.1.** *Let  $T$  be a localising triple on a (locally small) abelian category  $C$  and let  $\bar{T}$  be its extension to  $pro(C)$ , i. e.,*

$$\bar{T} = (pro\ G, pro(\eta), pro(\mu));$$

*then there is an isomorphism of categories between  $pro(C)^{\bar{T}}$  and  $E(C^T)$ .*

The strength - isomorphism of categories rather than mere equivalence - of this result is due mainly to the power of the results of Heinicke [18] and thus to the particular hypothesis that  $T$  is localising. Without this hypothesis one might conjecture that  $pro(C)^T$  and  $E(C^T)$  would be equivalent, but «isomorphic» would seem at first sight to be too strong.

Another reason for 4.1 is that we had at our disposal the powerful characterisations of localisations in (locally small) abelian categories. Removal of «abelian» raises more problems. First and foremost is that of determining what one means by localisation in this context. It is hoped to make an examination of this the subject of another paper.

## 5. KRULL - GABRIEL DIMENSION.

(From now on  $C$  will denote a Grothendieck category.)

In this section we recall the essentials of Gabriel's generalisation of Krull dimension. We will also have to introduce some new definitions for later use. The principal references will be :

Gabriel [8], Popescu [26], Gordon and Robson [12].

Recall that an object  $M$  in  $C$  is a *finite length* if there is a filtration

$$0 = M_{-1} \subset M_0 \subset \dots \subset M_n = M$$

of  $M$  by subobjects  $M_i$  such that each quotient  $M_i/M_{i-1}$  is simple.

We define the *Krull-Gabriel filtration* of  $C$  to be the ordinal indexed collection  $\{C_\alpha\}$  of localising subcategories of  $C$  defined as follows:

$C_{-1} = \{0\}$ , the zero category,

$C_0$  is the smallest localising subcategory of  $C$  containing all objects of finite length. Let  $T_0: C \rightarrow C/C_0$  be the quotient functor and  $S_0$  the associated section functor.

Assuming that  $\alpha = \beta + 1$ , that  $C_\beta$  is defined and is a localising subcategory of  $C$ , we denote by  $T_\beta: C \rightarrow C/C_\beta$  the quotient functor and by  $S_\beta$  the corresponding section functor. An object  $M$  is in  $C_\alpha$  iff  $T_\beta(M)$  is in  $(C/C_\beta)_0$ .

If  $\alpha$  is a limit ordinal, then  $C_\alpha$  is the smallest localising subcategory of  $C$  containing  $\bigcup_{\beta < \alpha} C_\beta$ .

We will be interested only in the  $C_\alpha$  for  $\alpha$  finite and we will write

$$KG\text{-dim} M = n \quad \text{if } M \text{ is in } C_n \text{ but not in } C_{n-1}.$$

REMARK. A few words need to be said as to why we have chosen to use this, the earlier version of Gabriel rather than his later version introduced on his joint work with Rentschler [10]. This latter seems to have attracted more attention amongst algebraists than his original version; see for instance the memoir of Gordon and Robson and the bibliography there [11]. However our methods depend heavily on the use of localising subcategories, whilst the Gabriel-Rentschler version only uses thick subcategories (for this see Gordon and Robson [12]). This alone would mitigate against use of the later version here, however the connection between the two versions, as indicated in [12] is such that those objects of  $C$  with a given Krull-Gabriel dimension form a larger class than those with the

later version of that dimension. In fact, if  $M$  has a Krull-Gabriel dimension, then it has a Krull dimension (à la Gabriel-Rentschler) iff each homomorphic image of  $M$  has finite uniform dimension (i. e., each homomorphic image of  $M$  contains no infinite direct sum of non-zero subobjects). Thus in proving results related to KG-dimension, we combine ease of method with the benefits of considering a larger class at any one time.

Returning to the localising subcategories  $C_n$  of the Krull-Gabriel filtration, each is determined by an idempotent torsion radical  $\tau^n$  defined by:  $\tau^n(M)$  is the maximal subobject of  $M$  which is in  $C_n$ . It will be necessary to consider how we may construct  $\tau^n$  in each case.

For  $n = 0$ , it is relatively straightforward and is, of course, well known. We start by defining a left exact subfunctor of the identity as follows:

$$\tau_1^0(M) = \text{soc}(M) = \bigoplus \{ S \mid S \subset M, S \text{ is simple} \}.$$

Using this and transfinite recursion one defines an ordinal sequence of subfunctors of the identity:

If  $\alpha = \beta + 1$  and  $\tau_\beta^0$  is defined, then  $\tau_\alpha^0$  is given by

$$\frac{\tau_\alpha^0(M)}{\tau_\beta^0(M)} \approx \tau_1^0\left(\frac{M}{\tau_\beta^0(M)}\right).$$

If  $\alpha$  is a limit ordinal,  $\tau_\alpha^0(M) = \bigcup_{\beta < \alpha} \tau_\beta^0(M)$ .

Finally one checks that  $\tau_\alpha^0(M) = \bigcup_{\alpha} \tau_\alpha^0(M)$ .

Thus  $M$  is in  $C_0$  iff  $\tau_\alpha^0(M) = M$ . In this case  $\tau_\alpha^0(M) = M$  for some ordinal  $\alpha$ , and the minimal such  $\alpha$  is called the *0-length* (or length) of  $M$ . Note that « $M$  is of finite length» has an unambiguous meaning although the «value» of the length will in general be different in the two cases.

To define  $\tau^n$  intrinsically for  $n > 0$  is harder. We first assume that  $\tau^{n-1}$  has been defined and say that  $M$  is *n-simple* if

- (i)  $\tau^{n-1}(M) = 0$ , and
- (ii)  $T_{n-1}(M)$  is simple.

Thus if  $M$  is  $n$ -simple and  $N \subset M$ ,  $N$  cannot be in  $C_{n-1}$  and the quotient  $M/N$  is in  $C_{n-1}$ . The simplest example of this is, of course, the case :

$$n = 1 \text{ for } C = Ab \text{ with } M = Z.$$

$Z$  is torsion free (i.e.,  $\tau^0(Z) = 0$ ) and if  $N \subset Z$ , then  $N = nZ$  for some  $n$  so  $Z/N$  is torsion, hence in  $C_0$ .

We shall assume for simplicity that  $\tau^{n-1}(M) = 0$  to start with — the adjustment to the general case will follow. Let

$$\tau_1^n(M) = \Sigma \{ S \mid S \subset M, S \text{ is } n\text{-simple} \}.$$

In general, this sum is not direct, since for example

$$\tau_1^1(Z) = \Sigma \{ mZ \mid m \in Z \} = Z$$

is certainly not direct! However it is always an essential extension of a direct sum of  $n$ -simples, since  $n$ -simples are coirreducible (i.e., are essential extensions of all subobjects) and hence there is a KRSG-decomposition by  $n$ -simples (see Popescu [26], Chapter 5). (We will use  $\Sigma$  for «sum» and  $\oplus$  for «direct sum».)

The relationship between  $\tau_1^n$  and  $\tau_1^0$  is:

$$T_{n-1} \tau_1^n = \text{soc}(T_{n-1}) = \tau_1^0 T_{n-1}.$$

This relation will be fundamental in the sequel.

Defining  $\tau^n(M)$  as before by recursion, we obtain  $\tau^n$  and the concept of  $n$ -length (cf. Porter [34]). We can easily lift the restriction  $\tau^{n-1}(M) = 0$  by defining, for a general  $M$ ,  $\tau^n(M)$  by:

$$\frac{\tau^n(M)}{\tau^{n-1}(M)} = \tau_\alpha^n\left(\frac{M}{\tau^{n-1}(M)}\right);$$

thus  $\tau_\alpha^n(M)$  will be the maximal subobject of  $M$  satisfying

$$T_{n-1} \tau_\alpha^n(M) = T_{n-1} \tau_\alpha^n\left(\frac{M}{\tau^{n-1}(M)}\right).$$

We next need a generalisation of the notion of finitely generated. The usual concept is not the right one for use in this context - it would, for instance, impose finite  $n$ -length on all objects being considered. We therefore introduce a weaker form which, for want of a better term, will

be called «pseudo-finitely generated» (or «p.f.g.» for short). We define the term solely on objects with finite Krull-Gabriel dimension as it will be defined recursively.

If  $n = -1$ , all objects in  $C_{-1}$  are p.f.g.

Now assume the term is defined as far as  $C_{n-1}$  is concerned and suppose  $M$  is a sum of  $n$ -simple objects; then  $M$  is p.f.g. if it satisfies the two conditions:

(i)  $T_{n-1}(M)$  is a direct sum of finitely many simple objects of  $C/C_{n-1}$  and

(ii) If  $N \subset M$  is such that  $M/N$  is in  $C_{n-1}$ , then  $M/N$  is p.f.g.

In general  $M$  in  $C_n$  is p.f.g. if

(iii)  $\tau^{n-1}(M)$  is p.f.g. and

(iv) for each  $\alpha$ , writing  $\bar{M} = M/\tau^{n-1}(M)$ , one has  $\tau_1^n(M/\tau_\alpha^n(\bar{M}))$  is p.f.g. in the above sense.

PROPOSITION 5.1. *If  $M$  is a Noetherian object of finite Krull-Gabriel dimension, then  $M$  is p.f.g.*

PROOF. If  $KG\text{-dim } M = -1$ , the result is trivial, so suppose the result holds for all Noetherian objects with KG-dimension less than  $n$ .

First we consider the case that  $M$  is a sum of  $n$ -simples. By Popescu ([26], page 372),  $T_{n-1}(M)$  is Noetherian and so has only finitely many simple direct summands; hence (i) is satisfied. If  $N \subset M$  is such that  $M/N$  is in  $C_{n-1}$ , then  $M/N$  is Noetherian and hence by the induction hypothesis is p.f.g.; hence (ii) is satisfied.

In general if  $M$  is in  $C_n$  and is Noetherian, then  $\tau^{n-1}(M)$  is in  $C_{n-1}$  and is Noetherian and hence is p.f.g., whilst

$$M/\tau_\alpha^n(M) \approx \bar{M}/\tau_\alpha^n(\bar{M})$$

is a factor object of  $M$ , hence Noetherian. Since it is a sum of  $n$ -simples, it is p.f.g. by the first case considered above.

COROLLARY 5.2. *For a right Noetherian ring  $A$  and any finitely generated right  $A$ -module  $M$  of finite Krull-Gabriel dimension,  $M$  is p.f.g.*

This Corollary will be crucial in showing that the result of Gruson

and Jensen mentioned in the Introduction is in fact a special case of the main result of the next section.

**6. KRULL-GABRIEL DIMENSION AND THE VANISHING OF  $\varprojlim^{(i)}$ .**

In [36] Roos proved the following result :

Let  $A$  be a regular ring and  $M$  a projective system of  $A$ -modules of finite type, then

$$\varprojlim^{(i)} M = 0 \quad \text{for all } i > \dim A.$$

Another result of the same type is due to Jensen [20] :

Let  $M$  be a projective system of Artinian modules over a commutative ring  $A$ , then

$$\varprojlim^{(i)} M = 0 \quad \text{for all } i > 0$$

if either  $A$  is Noetherian or each  $M(i)$  is Noetherian.

Further developments of the same theme include Jensen's removal of «commutative» and «Noetherian» from the above result and also his extensions of the result of Roos to quotients of finite dimensional Gorenstein rings. Finally he proved :

If  $A$  is a local Noetherian ring of Krull dimension  $l$ , then

$$\varprojlim^{(i)} M = 0 \quad \text{for all } i \geq 2,$$

for all projective systems of  $A$ -modules of finite type.

(These latter results are all in Jensen's Note [21] : the first in Chapter 7 and the last two in Chapter 9.)

Jensen also conjectured ([21], page 82) that a result similar to these was true for arbitrary Noetherian rings and recently he and Gruson (details unpublished as yet, cf. [14]) have proved this to be true. Specifically they have proved :

Let  $A$  be a right Noetherian ring and  $M$  a projective system of finitely generated right  $A$ -modules. If

$$(\text{Krull dimension of } M(\alpha)) < n$$

for all  $\alpha$ , then  $\varprojlim^{(i)} M = 0$  for  $i > n$ .

The fact that for Artinian modules, no condition on the ring is necessary suggests that all these results should follow as special cases of some result such as the following:

If  $M$  is a projective system of Noetherian modules and

$$KG\text{-dim } M(a) \leq n \text{ for all } a,$$

then  $\varprojlim^{(i)} M = 0$  for  $i > n$ .

In this Section we will prove this theorem by the obvious method of recursion on  $n$ . In fact, we will prove a more general result and then use 5.1 to deduce this result.

As in previous work, we introduce the classes  $L^{(k)}$ ,

$$L^{(k)} = \{ M \mid M \text{ in } \text{pro}(C), \varprojlim^{(i)} M = 0 \text{ for } i > k \},$$

and we recall various of the properties of  $L^{(k)}$  - for convenience we repeat the proofs.

LEMMA 6.1. For each  $k$ ,  $L^{(k)}$  is closed under extensions.

PROOF. Given

$$0 \longrightarrow M' \longrightarrow M \longrightarrow N \longrightarrow 0$$

in  $\text{pro}(C)$  with  $M', N \in L^{(k)}$ , then the long exact sequence for the  $\varprojlim^{(i)}$  immediately gives  $M \in L^{(k)}$ .

From now on, we shall assume  $C$  has exact products. Usually direct limits are not exact in  $\text{pro}(C)$ , however given an object  $X: I \rightarrow C$  and a family of subobjects  $\{X_\alpha\}$  of  $X$  in such a way as the resulting diagram can be considered as belonging to  $C^I$ , the direct limit of the  $X_\alpha$  in  $\text{pro}(C)$  can be taken within  $C^I$  to give a subobject of  $X$  (in  $\text{pro}(C)$ ). Of course this sort of limit is exact since  $C^I$  is AB5. We will call such limits *special direct limits*. (I would like to thank L. Gruson and S. Zdravkowska for pointing out the necessity of working with special direct limits in the following theory.)

LEMMA 6.2. For each  $k$ ,  $L^{(k)}$  is closed under (filtered) special direct limits.

PROOF. Suppose  $\{M_\alpha\}$  is a special direct system in  $D$  and, for each  $\alpha$ ,

$\varprojlim^{(j)} M_\alpha = 0$  if  $j > k$ . There is a double complex defined as follows:

Firstly for any pro-object  $N$  in  $C$ , let  $\Pi(N)$  be the cochain complex defined by:

$$\Pi(N)^k = \prod_{i_0 \leq \dots \leq i_k} N(i_0, \dots, i_k),$$

where  $N(i_0, \dots, i_k) = N(i_0)$  and  $\delta^m: \Pi(N)^m \rightarrow \Pi(N)^{m+1}$  is given by:

$$\delta^m(n)(i_0, \dots, i_{m+1}) = p_{i_0}^{i_1} n(i_1, \dots, i_{m+1}) + \sum_{j=1}^{m+1} (-1)^j n(i_0, \dots, \hat{i}_j, \dots, i_{m+1}).$$

Note that  $H^p(\Pi(N)^*) = \varprojlim^{(p)} N$  (cf. Roos [36] or Jensen [21], Chapter 4). Dually, if  $N: I^{op} \rightarrow C$  is an inductive or direct system in  $C$ , we set

$$\Sigma(N)_k = \bigoplus_{i_0 \leq \dots \leq i_k} N(i_0, \dots, i_k),$$

where  $N(i_0, \dots, i_k) = N(i_0)$  and the differential  $\partial^k$  is given by

$$\partial^k(j(i_0, \dots, i_k)n) = j(i_1, \dots, i_k) p_{i_0}^{i_1} n + \sum_{i=1}^{k+1} (-1)^i j(i_0, \dots, \hat{i}_i, \dots, i_k)n \text{ for } n \in N(i_0),$$

where  $j(i_0, \dots, i_k)$  is the natural monomorphism from  $N(i_0, \dots, i_k)$  into  $\Sigma(N)_k$  as the  $(i_0, \dots, i_k)$ -th summand. Again we get

$$H_0(\Sigma(N)_*) = \varinjlim N,$$

and since  $\varinjlim$  is exact,  $H_k(\Sigma(N)_*) = 0$  for  $k \neq 0$ .

Since we have a special direct system  $\{M_\alpha\}$  of projective systems, we can form up the double complex  $\Sigma(\Pi M)$ , and study its spectral sequences. Adopting the indexing system suggested by Hilton and Stambach [19] rather than the classical one, we obtain

$${}_1E_1^{p,q} = \begin{cases} \varprojlim^{(-p)} \varinjlim^{(q+p)} M & \text{for } p \leq 0, q \geq -p, \\ 0 & \text{otherwise} \end{cases}$$

and

$${}_2E_1^{p,q} = \begin{cases} \varinjlim^{(-p)} \varprojlim^{(q+p)} M & \text{for } p \leq 0, q \geq -p, \\ 0 & \text{otherwise} \end{cases}$$

and which, if they converge, will converge to the same limit.

Now generally 4-th quadrant spectral sequences cannot be expected to converge, but since  $C$  is an AB5 category,  $\varprojlim$  is exact and by assumption

$$\varprojlim^{(j)} M_\alpha = 0 \quad \text{for } j > k,$$

many of the terms are zero. In fact, we get

$${}_1E_1^{p,q} = \begin{cases} 0 & \text{if } q \neq -p, \\ \varprojlim^{(-p)} \varinjlim M_\alpha & \text{if } q = -p, \end{cases}$$

and

$${}_2E_1^{p,q} = \begin{cases} 0 & \text{if } p \neq 0 \text{ or } q > k, \\ \varinjlim \varprojlim^{(q)} M_\alpha & \text{if } p = 0 \text{ and } q \leq k. \end{cases}$$

Putting  $B = \text{Tot}(\Sigma \Pi(M))$ , the  ${}_2E$ -sequence clearly gives

$$H^q(B) = \begin{cases} \varinjlim(\varprojlim^{(q)} M_\alpha) & \text{for } 0 \leq q \leq k, \\ 0 & \text{otherwise.} \end{cases}$$

Feeding this back into  ${}_1E$  gives

$$\varprojlim^{(q)}(\varinjlim M_\alpha) = 0 \quad \text{if } q > k.$$

Thus  $\varinjlim M_\alpha$  is in  $L^{(k)}$  as promised.

LEMMA 6.3. *If  $f: M \rightarrow N$  is a morphism in  $\text{pro}(C)$  such that  $\text{Ker} f$  and  $\text{Coker} f$  are in  $L^{(k)}$ , then the induced limiting morphisms*

$$\varprojlim^{(i)} f: \varprojlim^{(i)} M \rightarrow \varprojlim^{(i)} N$$

*are isomorphisms for  $i > k + 1$ .*

PROOF. Look at the long exact sequence corresponding to

$$0 \rightarrow \text{Ker} f \rightarrow M \rightarrow \text{Im} f \rightarrow 0;$$

it gives

$$\begin{aligned} \dots \rightarrow \varprojlim^{(k)} \text{Im} f \rightarrow \varprojlim^{(k+1)} \text{Ker} f \rightarrow \varprojlim^{(k+1)} M \rightarrow \\ \rightarrow \varprojlim^{(k+1)} \text{Im} f \rightarrow \varprojlim^{(k+2)} \text{Ker} f \rightarrow \dots \end{aligned}$$

and since

$$\varprojlim^{(k+i)} \text{Ker} f = 0 \quad \text{for } i > 0,$$

$\varprojlim^{(q)} M \rightarrow \varprojlim^{(q)} \text{Im} f$  is an isomorphism for  $q \geq k$ .

Similarly consider the long exact sequence corresponding to

$$0 \rightarrow \text{Im } f \rightarrow N \rightarrow \text{Coker } f \rightarrow 0;$$

clearly  $\varprojlim^{(k+i)} \text{Im } f \rightarrow \varprojlim^{(k+i)} N$  is an isomorphism for  $i > 1$ .

Thus putting these two isomorphisms together to get the induced morphism  $\varprojlim^{(q)} f$  for  $q > k+1$  completes the proof.

We next need a result on projective systems of semi-simple objects. (This result appears in a slightly different form in [30, 32].)

LEMMA 6.4. *Suppose  $M$  is such that each  $M(i)$  is a direct sum of at most  $n$  simple objects of  $C$ , then  $M$  is essentially constant and the canonical morphism  $h(\varprojlim M) \rightarrow M$  is an isomorphism in  $\text{pro}(C)$ , in particular*

$$\varprojlim^{(i)} M = 0 \quad \text{for } i > 0.$$

Moreover  $\varprojlim M$  is a finite direct sum of simple objects of  $C$ .

PROOF. Since, if  $S$  and  $S'$  are two non-isomorphic simple objects,

$$\text{Hom}_C(S, S') = 0,$$

we can apply the  $S$ -socle functor, i. e., the functor  $\tau_S^0$  where  $S$  is a simple object of  $C$  and

$$\tau_S^0(M) = \bigoplus \{ N \mid N \subset M, N \approx S \},$$

and split  $M$  into a possibly infinite direct sum  $M \approx \bigoplus_S \tau_S(M)$ , indexed by isomorphism types of simple objects, where  $\tau_S(M(i))$  consists of finite direct sums of copies of  $S$ . In fact, since each  $M(i)$  involves only finitely many simples, this sum must in fact be finite and hence we can restrict attention to the case where all the summands in  $M(i)$  are isomorphic to one single  $S$ , i. e.,  $\tau_S(M) = M$ .

We first suppose the index  $i$  is fixed; then the family  $\{p_i^j(M(j))\}$  is a family of subobjects of the Artinian object  $M(i)$ . Hence there is some index  $f(i)$  such that  $p_i^{f(i)}(M(f(i)))$  is a minimal element of this family, and so

$$p_i^j M(j) = p_i^{f(i)} M(f(i)) \quad \text{for all } j > f(i).$$

Now we form the category  $E(I)$  with objects ordered pairs  $(j, i)$  with

$j \geq i$  in  $I$ , and a map

$$(j', i') \rightarrow (j, i) \text{ if } j' \geq j \text{ and } i' \geq i.$$

Let  $M_E: E(I) \rightarrow C$  be defined by

$$M_E(j, i) = p_i^j(M(j)),$$

with maps  $M_E(j', i') \rightarrow M_E(j, i)$  given by  $p_i^{i'}$  restricted to  $p_i^{j'}M(j')$ .

The diagonal functor

$$\Delta: I \rightarrow E(I), \quad \Delta(i) = (i, i),$$

is initial and  $M = M_E \Delta$ , so  $M_E \approx M$ ; similarly the functor

$$\Delta_f: I \rightarrow E(I), \quad \Delta_f(i) = (f(i), i),$$

is cofinal and

$$M_E \Delta_f(i) = p_i^{f(i)}M(f(i))$$

is a projective system for which all transitions are epimorphic. Thus  $M$  is isomorphic to a projective system for which all the transitions are epimorphic; moreover by the method shown above this new system also satisfies the hypothesis of the theorem. We may thus replace  $M$ , if necessary, by this new pro-object.

Now consider, for fixed  $i$ , the number of summands  $k(j, i)$  in the kernel of  $p_i^j$  for each  $j > i$ . Since each  $p_j^k$  is onto,  $k(j, i)$  is increasing with  $j$  and, since it is bounded, it must achieve a maximum, i. e., there is some  $j_0$  such that

$$k(j_0, i) \geq k(j, i) \text{ for all } j > i.$$

However, if  $j_1 \rightarrow j_2$ , then  $k(j_1, i) \geq k(j_2, i)$ , so

$$k(j, i) = k(j_0, i) \text{ for all } j \geq j_0.$$

Again using the fact that transitions are onto, we obtain that  $p_{j_0}^j$  is a monomorphism and hence is an isomorphism for all  $j > j_0$ , i. e.,  $M$  is a pro-object for which the transitions are cofinally isomorphisms. The result follows.

**COROLLARY 6.5.** *Suppose  $M$  is such that  $M(i)$  is a finite direct sum of simple objects of  $C$ ; then  $M$  is in  $L^{(0)}$ , i. e.,  $\varprojlim^{(i)} M = 0$  for  $i > 0$ .*

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PROOF.  $M$  is the special direct limit of subobjects of the type satisfying the conditions of 6.4, thus the result follows from 6.2 and 6.4.

COROLLARY 6.6. *If  $M$  is p.f.g. and  $KG\text{-dim } M(i) \leq 0$  for all indices  $i$ , then  $\lim^{(j)} M = 0$  for all  $j > 0$ , i. e.  $M \in L^{(0)}$ .*

PROOF. Since  $M$  is p.f.g., each  $r_1^0(M/r_\alpha^0(M))$  satisfies the conditions of 6.5. Using 6.1 for non-limit ordinals and 6.2 for limit ordinals shows that each  $r_\alpha^0(M)$  is in  $L^{(0)}$  and hence  $r^0(M) = \bigcup_\alpha r_\alpha^0(M)$  is in  $L^{(0)}$ . Since  $M$  is in  $pro(C_0)$ ,  $r^0(M) = M$ , so we are finished with the proof.

If we denote the subcategory of  $C_n$  consisting of p.f.g. objects by  $C_{n,p.f.g.}$ , we obtain the first (or rather zero-th) case of the general theorem:

COROLLARY 6.7. *If  $M$  is in  $E(C_{0,p.f.g.})$ , then  $M$  is in  $L^{(0)}$ .*

Clearly any Artinian object in  $C$  is in  $C_{0,p.f.g.}$ , so we reobtain Jensen's result [21], page 57, 7.2 (cf. also the discrete case of Oberst [25], page 512, 5.20 and Demazure and Gabriel [4], V, 2.2, page 563).

COROLLARY 6.8. *If  $M$  is essentially Artinian, then  $M$  is in  $L^{(0)}$ .*

Although the proof of the main theorem would have produced these corollaries by itself, the proof of these results indicates the general plan of attack to be taken in the proof of the theorem. It is important to note that although the only part of 6.4 used in these corollaries is the conclusion that

$$\lim_{\leftarrow}^{(i)} M = 0 \quad \text{for } i > 0,$$

in fact the more important conclusions from the point of view of the main theorem are that the natural map  $h(\lim_{\leftarrow} M) \rightarrow M$  is an isomorphism and the description of  $\lim_{\leftarrow} M$ .

THEOREM 6.9. *Let  $M$  be in  $E(C_{n,p.f.g.})$ ; then  $M \in L^{(n)}$ .*

PROOF. The case  $n = -1$  is more or less trivial as  $E(C_{-1})$  contains only the essentially zero objects and, by the functionality of  $\lim_{\leftarrow}$  on  $pro(C)$ , these have zero limits and zero derived limits.

The case  $n = 0$  has already been dealt with but would anyway have been given as an especially easy special case of the general inductive step.

We will make the assumption that all objects in  $E(C_{n-1}, p.f.g.)$  are also in  $L^{(n-1)}$  and we will consider first an object  $M$  in  $E(C_{n-1}, p.f.g.)$  satisfying:

(i)  $\tau^{n-1}(M) = 0,$

(ii)  $\bar{T}_{n-1}(M)$  satisfies the conditions of Lemma 6.4 as an object of

$$pro(C)/E(C_{n-1}) \approx pro(C/C_{n-1}).$$

Thus  $\bar{T}_{n-1}(M)$  is isomorphic in  $pro(C/C_{n-1})$  to a finite direct sum of simples  $h(\bigoplus_{i=1}^m S_i)$ . Each simple  $S_i$  is equal to  $T_{n-1}(N_i)$  for some p.f.g.  $n$ -simple  $N_i$  and since  $T_{n-1}$  is exact and the sum is finite,

$$h(\bigoplus_{i=1}^m S_i) \approx h(T_{n-1}(\bigoplus_{i=1}^m N_i))$$

and by the construction of  $\bar{T}_{n-1}$ , this is the same as  $\bar{T}_{n-1}h(\bigoplus_{i=1}^m N_i)$ . Collecting up these isomorphisms, we obtain an isomorphism

$$\bar{T}_{n-1}(M) \approx \bar{T}_{n-1}h(\bigoplus_{i=1}^m N_i).$$

We will write  $N = \bigoplus_{i=1}^m N_i$  for short and note that  $\tau^{n-1}(N) = 0$ . Now from localisation theory one obtains the following result (cf. Popescu [26], page 172, 4.3.9):

Any morphism of  $C/A$  can be written as  $T(s_2)^{-1}T(f)T(s_1)^{-1}$  where  $s_1$  and  $s_2$  are invertible modulo  $A$  (i.e., each has its kernel and cokernel in the localising subcategory  $A$ ),  $s_1$  is a monomorphism and  $s_2$  is an epimorphism.

Interpreting this in our situation we obtain a representation of the isomorphism  $T_{n-1}(M) \rightarrow T_{n-1}h(M)$  in  $pro(C)/E(C_{n-1})$  as the image of morphisms

$$M \xleftarrow{s_1} M' \xrightarrow{f} N' \xleftarrow{s_2} h(N),$$

and since the result is an isomorphism,  $f$  must also have  $Ker f$  and  $Coker f$

in  $E(C_{n-1})$ .  $s_1$  is a monomorphism, so  $M'$  satisfies  $\tau^{n-1}(M') = 0$  and is in  $E(C_{n,p.f.g.})$ .  $s_2$  is an epimorphism, but  $\text{Ker } s_2 \in E(C_{n-1})$  and, since  $\tau^{n-1}(h(N)) = 0$ ,  $\text{Ker } s_2 = 0$  and  $s_2$  is itself an isomorphism. We thus have a diagram

$$M \xleftarrow{s_1} M' \xrightarrow{f} h(N)$$

or at least we may replace our original diagram by this one. We will show that  $s_1$  and  $f$  satisfy the conditions of Lemma 6.3 for  $k = n-1$  and hence induce isomorphisms of  $\varprojlim^{(i)} M$  with  $\varprojlim^{(i)} M'$ , and of the latter with  $\varprojlim^{(i)} h(N)$  for all  $i \geq n+1$ . Since

$$\varprojlim^{(i)} h(N) = 0 \quad \text{for all } i,$$

this will complete the proof for this special case.

Since each  $\text{Cokers}_1(i)$  is a quotient of the corresponding  $M(i)$  which is p.f.g. in the first sense (i.e.,  $T_{n-1}(M)$  has finitely many simple summands and each quotient of  $M(i)$  in  $C_{n-1}$  is p.f.g.), we have that  $\text{Cokers}_1$  is in  $E(C_{n-1,p.f.g.}) \subset L^{(n-1)}$ , by the inductive assumption.  $\text{Ker } f \subset M' \subset M$  and hence must be zero (recall  $\tau^{n-1}(M') = 0$ ). Finally,  $h(N)$  is p.f.g. and so  $\text{Coker } f$  is in  $E(C_{n-1,p.f.g.})$  and thus in  $L^{(n-1)}$ . Applying 6.3 as indicated shows that  $M$  is in  $L^{(n)}$ .

Now, if we suppose, more generally, that  $M$  is such that each  $T_{n-1}(M(i))$  is a direct sum of finitely many simples, then  $M$  is the special direct limit of objects of the kind already considered, so an application of 6.2 will do the trick.

If  $M$  is in  $E(C_{n,p.f.g.})$  and  $\tau^{n-1}(M) = 0$ , then each  $\tau_1^n(M/\tau_\alpha^n(M))$  is as above and hence is in  $L^{(n)}$  and the use of 6.1 for non-limit ordinals and 6.2 for limit ordinals completes this case to show  $M$  is in  $L^{(n)}$ .

Finally, if  $M$  is anything in  $E(C_{n,p.f.g.})$ , then  $\tau^{n-1}(M)$  is in  $E(C_{n-1,p.f.g.})$  and hence in  $L^{(n-1)}$ . A long exact sequence argument corresponding to the short exact sequence

$$0 \longrightarrow \tau^{n-1}(M) \longrightarrow M \longrightarrow M/\tau^{n-1}(M) \longrightarrow 0$$

shows

$$\varprojlim^{(i)} M \approx \varprojlim^{(i)} M / \tau^{n-1}(M) \text{ for } i > n$$

and as objects of the form  $M / \tau^{n-1}(M)$  have already been handled,  $M$  is in  $L^{(n)}$  and the proof is complete.

COROLLARY 6.10. *Let  $M$  be a projective system of Noetherian objects of  $C$  such that  $\text{KG-dim } M(i) < n$  for all  $i$ , then*

$$\varprojlim^{(j)} M = 0 \text{ for all } j > n.$$

In particular if  $C = \text{Mod-}A$  where  $A$  is a right Noetherian ring, then any projective system  $M$  of finitely generated right  $A$ -modules of Krull dimension not greater than  $n$  satisfies  $\varprojlim^{(i)} M = 0$  for  $i > n$ .

APPLICATIONS.

There are many uses in the literature of the main result in the case where  $n = 0$  so all the modules are Artinian. It thus seems likely that there will be a rich play off when the possibility of extending these results has been investigated; this seems by no means easy, as often, unfortunately for thoughts of applications for  $n > 0$ , the Artinian condition occurs naturally from the start. Much work remains to be done in this area. Given this we will limit ourselves to one «canonical» application. This particular application has received the same use many times as it requires little preparatory work.

COROLLARY 6.11. *Let  $A$  be a commutative ring, then  $\text{Ext}_A^i(M, N) = 0$  for all flat  $A$ -modules  $M$ , all  $A$ -modules  $N$  which are p.f.g. of Krull dimension  $\leq n$  and for all  $i > n$ .*

PROOF. One can represent  $M$  as a direct limit of finitely generated free modules  $M = \varinjlim F_\alpha$ ; now use the spectral sequence with

$$E_1^{p,q} = \varprojlim^{(p)} \text{Ext}_A^{(q-p)}(F_\alpha, N) \Rightarrow \text{Ext}^n(\varinjlim F_\alpha, N)$$

(cf. Jensen [21], 4.2, page 35).

$$\text{Ext}^{(q-0)}(F_\alpha, N) = 0 \text{ if } q \neq p$$

and is a finite direct sum of copies of  $N$  if  $q = p$ , hence  $E_1^{p,q} = 0$  unless  $p = q$  and, by 6.9, unless  $p \leq n$ . The result follows by the usual

sort of spectral sequence collapse.

REMARK. For  $n = 0$ , Jensen fits this sort of result into a set of equivalent conditions for  $M$  to be a finite product of complete local rings (Jensen [21], page 68, Theorem 8.1). This, of course, raises the interesting possibility of classifying Noetherian commutative rings  $A$  via the condition:

$$\varprojlim^{(i)} M = 0 \text{ for all projective systems of } A\text{-modules } M = \{M_\alpha\} \text{ of finite type and for all } i > n.$$

Clearly if  $KG\text{-dim } A \leq n$ , then this follows from 6.10, but what if  $KG\text{-dim } A \geq n + 1$ ? For  $n = 0$ , this problem is tackled by Jensen's Theorem. Some of his implications seem to generalise, but others do not. Above all one needs to know what sort of condition to put in place of «finite product of complete local rings».

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