On simply bireflective subcategories

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A description of an everyday-life concrete category follows often the following pattern: There is given

1. a construction producing structures on sets,
2. a mechanism of choice of well-behaved mappings among all mappings between structured sets,
3. a delimitation of the desired objects among all the ones obtained by the construction from (1) (the system of axioms on the structure in question).

In [5] there was proved that the approach over the categories $S(F)$ ([3], [4], [7], etc...; $F$ is a functor $\text{Set} \to \text{Set}$, the objects of $S(F)$ are the couples $(X, r)$ with $r \subseteq F(X)$, the morphisms $(X, r) \to (Y, s)$ are the triples $(r, f, s)$ with $f : X \to Y$ such that $F(f)(r) \subseteq s$) is of a fairly general validity for the tasks (1) and (2).

In the present paper we are going to discuss the tamest case of (3); namely, the delimitations leading to the subcategories that are both reflective and coreflective with, moreover, both the reflection and coreflection morphisms identity carried (following [4], we call them simply bireflective).

Consider the example of the symmetry axiom for binary relations. The category of sets with binary relations coincides with $S(Q)$ where $Q$ sends a set $X$ to $X \times X$ and a mapping $f$ to $f \times f$. We see that its subcategory of the symmetric relations is again of the type $S(G)$, namely with

$$G(X) = \{ A \mid A \subseteq X, 1 \leq |A| \leq 2 \}, \quad G(f)(A) = f(A).$$

and, moreover, its embedding into $S(Q)$ is naturally induced by the epi-transformation $Q \to G$ sending $(x, y)$ to $\{ x, y \}$. This observation led us for a moment to the conjecture that one might describe the simply bireflec-
tive subcategories of $S(F)$, generally, by means of epi-transformations $\varepsilon: F \to G$. One sees easily that this conjecture is false. There are simply bireflective subcategories which are not thus induced (e.g. in $S(Q)$ the subcategory of the $(X, r)$ such that $(x, x) \in r$ whenever $(x, y) \in r$ for a $y$). On the other hand, an epi-transformation always induces an embedding onto a simply reflective, but not necessarily onto a simply bireflective subcategory. If one, however, generalizes the definition of $S(F)$ to functors $F$ with values in the category of quasidiscrete spaces (see no 1), the situation is more satisfactory. Now, every simply bireflective subcategory is induced by an epi-transformation and one can give an explicit characterization of the epi-transformations which induce one. This is shown and discussed in no 4 and no 5 (the first three paragraphs are of a technical character). As an application, we give at the end of no 5 a complete list of « systems of axioms » on $A$-nary relations leading to simply bireflective subcategories of the category of all $A$-nary relations.

1. Quasidiscrete spaces.

1.1. A quasidiscrete space (abbr., QD-space) is a topological space in which the intersection of any system of open sets is open (see [1]).

In a QD-space we denote by $O_p A$ the smallest open subset containing $A$.

The category of all QD-spaces and their continuous mappings will be denoted by $QD Top$, its full subcategory generated by the $T_0$-spaces will be denoted by $QD Top_0$.

1.2. Let $(X, \tau)$ be a QD-space. Define a preorder $\preceq$ (more exactly, $\preceq_\tau$) on $X$ by:

$$x \preceq y \text{ iff } O_p \{x\} \subseteq O_p \{y\}.\$$

On the other hand, with a preorder $\preceq$ on $X$ we can associate a quasidiscrete topology declaring $A \subseteq X$ for open iff

$$x \preceq y \text{ and } y \in A \text{ imply } x \in A.$$

It is well-known (and very easy to check) that this construction yields an
isomorphism between \( \text{QD Top} \) and the category of preordered sets and monotone mappings. Since obviously a QD-space is \( T_0 \) iff
\[
\mathcal{O}_p \{ x \} = \mathcal{O}_p \{ y \} \text{ implies } x = y,
\]
\( \text{QD Top} \) corresponds under this isomorphism to the category of partially ordered sets.

1.3. Obviously, the continuity of a mapping \( f: X \to Y \) is characterized by the formula: \( f(\mathcal{O}_p \{ x \}) \subseteq \mathcal{O}_p \{ f( x) \} \).

1.4. **Conventions.** Whenever convenient, we will deal with the corresponding preorders instead of the topologies. If there is no danger of confusion, the preorders are indicated by \( \preceq \) simply without further specifications. We write
\[
x \sim y \text{ for } x \preceq y \& y \preceq x.
\]
Instead of \( \mathcal{O}_p \{ x \} \), we write \( \mathcal{O}_p x \).

The proofs of the following two lemmas are easy.

1.5. **Lemma.** Let \( f: X \to Y \) in \( \text{QD Top} \) be such that
\[
M, N \text{ open } \& f^{-1}(M) = f^{-1}(N) \text{ implies } M = N.
\]
Then for every \( y \in Y \) there is an \( x \in X \) such that \( f(x) \sim y \). Consequently, if moreover \( Y \) is \( T_0 \), \( f \) is onto.

1.6. **Lemma.** Let \( f, g: X \to Y \) in \( \text{QD Top} \) be such that
\[
f^{-1}(M) = g^{-1}(M) \text{ for every open } M.
\]
Then \( f(x) \sim g(x) \) for every \( x \in X \). Consequently, if moreover \( Y \) is \( T_0 \), \( f = g \).

1.7. If \( (X, \tau) \) is a QD-space, we denote by \( \mathcal{O}(X, \tau) \) the lattice of all the open subsets of \( (X, \tau) \). It is well-known (and very easy to see) that \( \mathcal{O}(X, \tau) \) is irreducibly generated, and the irreducible elements are exactly the sets of the form \( \mathcal{O}_p x \) with \( x \in X \). (\(^*\))

\(^*\) An irreducible element of a lattice \( \Lambda \) is a non-zero element \( a \in \Lambda \) such that \( a \sqcap b_i \) implies there is an \( i \) with \( a \preceq b_i \). \( \Lambda \) is said to be irreducibly generated if every \( x \in \Lambda \) is a union of irreducible elements.
1.8. The following fact is also well-known (and easy to prove):

An irreducibly generated lattice $A$ is a Boolean algebra iff its irreducible elements are disjoint. Consequently, $\mathcal{O}(X, \leq)$ is a Boolean algebra iff $\leq$ is symmetric (and, hence, an equivalence).

1.9. $QDTop_0$ is a reflective subcategory of $QDTop$. Let us denote by $J$ the embedding $QDTop_0 \subset QDTop$, by $L$ the reflection functor and by

$$\eta : 1_{QDTop} \rightarrow JL$$

the reflection transformation. (Suitable $L$ and $\eta$ may be obtained putting first

$$L'(X) = (\{ \mathcal{O}_p x \mid x \in X \}, \subset),$$

$$L'(f)(\mathcal{O}_p x) = \mathcal{O}_p f(x), \quad \eta_X'(x) = \mathcal{O}_p x, \quad \phi_X = \begin{cases} 1_{L(X)} & \text{for } X \notin QDTop_0 \\ \eta_X^{-1} & \text{for } X \in QDTop_0 \end{cases}$$

and then putting $L(f) = \phi_Y \circ L'(f) \circ \phi_X^{-1}$.)

1.10. The following property of the mappings $\eta_X$ is evident:

$M$ is open in $X$ iff $M = \eta_X^{-1}(N)$ for an open $N$ in $L(X)$.

2. The categories $S(F)$ with $F : Set \rightarrow QDTop$.

2.1. Let $F : Set \rightarrow QDTop$ be a functor (Set is the category of all sets and mappings). The category $S(F)$ is defined as follows: The objects are the couples $(X, r)$ with $r$ an open subset of $F(X)$; the morphisms from $(X, r)$ to $(Y, s)$ are the triples $(r, f, s)$ with $f : X \rightarrow Y$ such that $F(f)(r) \subset s$.

$S(F)$ will be regarded as a concrete category with the forgetful functor sending $(r, f, s)$ to $f$.

2.2. Thus defined categories $S(F)$ include the categories $S(F)$, with $F : Set \rightarrow Set$ introduced in [2] and studied in various papers. It suffices to regard a functor into Set as a functor into $QDTop$ with discrete values.

2.3. Let $\theta : F \rightarrow G$ be a transformation. Define a functor
2.4. **Observation.** \([\theta]\) is faithful and a right adjoint.

2.5. A transformation \(\theta : F \to G\) is said to be an *epi-transformation* if every \(\theta_X\) is a mapping onto.

(The epi-transformations are exactly the epimorphisms in the illegitimate category \([\text{Set}, \mathcal{QD Top}]\), which follows immediately by the co-continuousness of the evaluation functor and by the fact that the epimorphisms in \(\mathcal{QD Top}\) are onto.)

2.6. **Proposition.** If \(\varepsilon\) is an epi-transformation, then \([\varepsilon]\) is a full embedding.

**Proof.** (Quite analogous to the corresponding one for \(\text{Set}\)-valued functors - see [7]). Since \(\varepsilon_X\) are onto, \(\varepsilon_X^{-1}(r) = \varepsilon_X^{-1}(s)\) implies \(r = s\).

Thus, since \([\varepsilon]\) is faithful, it is one-to-one. If

\[F(f)(\varepsilon_X^{-1}(r)) \subseteq \varepsilon_Y^{-1}(s),\]

we have

\[G(f)(r) = G(f) \varepsilon_X \varepsilon_X^{-1}(r) = \varepsilon_Y F(f) \varepsilon_X^{-1}(r) \subseteq \varepsilon_Y \varepsilon_Y^{-1}(s) = s.\]

Thus \([\varepsilon]\) is also full.

2.7. **Remark.** If \([\varepsilon]\) is a full embedding, \(\varepsilon\) is not necessarily an epi-transformation, which is easily seen. It is necessarily an epi-transformation if the values of \(G\) are in \(\mathcal{QD Top}\) (see 1.5).

2.8. Evidently, we have:

**Proposition.** Let \(\varepsilon : F \to G\) be an epi-transformation. Then \([\varepsilon]\) is an isofunctor mapping \(S(G)\) onto \(S(F)\) iff the following condition holds:

For every \(X\), \(M\) is open in \(F(X)\) iff \(M = \varepsilon_X^{-1}(N)\) for an open set \(N\) in \(G(X)\).
3. The transformations $\eta F$.

3.1. In the notation of 1.9, for an $F: \text{Set} \to \mathcal{QD Top}$ put $F' = \mathcal{J} L F$. We have epi-transformations $\eta F: F \to F'$.

3.2. We have obviously $F'' = F'$ and $\eta F' = 1_{F'}$.

3.3. By 1.10 and 2.8 we obtain immediately:

**Proposition.** $[\eta F]$ is an isofunctor of $S(F')$ onto $S(F)$.

3.4. **Theorem.** Let $\phi: S(F) \to S(G)$ be an isofunctor such that $U' \phi = U$ for the natural forgetful functors $U$, $U'$. Then there is a natural equivalence $\kappa: G' \simeq F'$ such that the diagram

\[
\begin{array}{ccc}
S(F) & \xrightarrow{\phi} & S(G) \\
\downarrow{[\eta F]} & & \downarrow{[\eta G]} \\
S(F') & \xrightarrow{[\kappa]} & S(G')
\end{array}
\]

commutes.

**Proof.** Given $H, K: \text{Set} \to \mathcal{QD Top}_{\phi}$ and an isofunctor $\psi: S(H) \cong S(K)$ preserving the underlying mappings, the formula

$$\psi(X, r) = (X, \phi_X(r))$$

defines obviously lattice isomorphisms $\phi_X: \mathcal{O}(H(X)) \to \mathcal{O}(K(X))$. Since $\phi_X$ sends irreducibles to irreducibles and since $K(X)$ is a $T_0$-space, the formula

$$\check{\mathcal{O}}_p \lambda_X(x) = \phi_X(\check{\mathcal{O}}_p x)$$

defines homeomorphisms $\lambda_X: H(X) \cong K(X)$ (see 1.2) such that

$$\phi_X(r) = \lambda_X(r) \text{ for an open } r \subset H(X).$$

Let $f: X \to Y$ be a mapping, $x \in X$. Since $H(f)(\check{\mathcal{O}}_p x) \subset \check{\mathcal{O}}_p H(f)(x)$, we have

$$K(f)(\lambda_X(x)) \in K(f)(\check{\mathcal{O}}_p \lambda_X(x)) = K(f)(\phi_X(\check{\mathcal{O}}_p x)) \subset \phi_Y \check{\mathcal{O}}_p H(f)(x),$$

so that
Similarly, \( \mathcal{O}_p K(f) \lambda_X(x) \subset \mathcal{O}_p \lambda_Y H(f)(x) \).

Similarly, \( \mathcal{O}_p \lambda_Y H(f)(y) \subset \mathcal{O}_p \lambda_Y^* K(f)(y) \), and hence
\[
\mathcal{O}_p \lambda_Y H(f)(x) = \mathcal{O}_p \lambda_Y H(f)(y) = \phi_Y^* \mathcal{O}_p H(f)(y) \lambda_Y^*(y) \subset \mathcal{O}_p \lambda_Y K(f)(y) = \mathcal{O}_p K(f)(y) = \mathcal{O}_p K(f)(x).
\]

Thus
\[
\mathcal{O}_p K(f) \lambda_X(x) = \mathcal{O}_p \lambda_Y H(f)(x),
\]
and since the spaces are \( T_0 \), consequently, \( K(f) \lambda_X = \lambda_Y H(f) \), so that \( \lambda \) is a natural equivalence. Now, apply the just proved assertion to
\[
\psi = [\eta G]^* \phi [\eta F],
\]
and put \( \kappa = \lambda^* \).

3. REMARK. In particular, we see that \( S(F) \) and \( S(G) \) are equally carried (i.e. there is an isofunctor \( \phi \) with \( U' \phi = U \) ) iff \( F' \cong G' \). In fact, if \( S(F) \) and \( S(G) \) are isomorphic, they are equally carried necessarily (which, in essence, may be proved characterizing internally up to isomorphism the objects \( (1, \emptyset) \)). Thus, if \( S(F) \cong S(G) \), then \( F' \cong G' \).

4. Simply reflective subcategories.

4.1. Let \((\mathcal{K}, U)\) be a concrete category. A subcategory \( \mathcal{L} \) of \( \mathcal{K} \) is said to be simply reflective (resp. coreflective) in \((\mathcal{K}, U)\) if it is reflective (resp. coreflective) and if there is a reflection (resp. coreflection) transformation.

\[
\rho = (\rho_X: X \rightarrow X') \quad (\text{resp. } \rho = (\rho_X: X' \rightarrow X))
\]

such that
\[
\text{for every } X \in \text{obj} \mathcal{K}, \quad U(\rho_X) = 1_{U(X)} \quad (\text{cf. } [4]).
\]

4.2. PROPOSITION. Let \( \varepsilon: F \rightarrow G \) be an epi-transformation. Then \( [\varepsilon] \) maps \( S(G) \) onto an isomorphic simply reflective subcategory of \( S(F) \) such that, moreover,
\[
(* \text{ ) if } (X, a_i) \in \mathcal{K} \text{ for } i \in I \text{, then } (X, \bigcup_{i \in I} a_i) \in \mathcal{K}.
\]

PROOF follows immediately by 2.4, 2.6 and the fact that
\[
f^*(\bigcup_{i \in I} A_i) = \bigcup_{i \in I} f^*(A_i) \quad (\text{cf. also } [6]).
\]
4.3. Let $\mathcal{K}$ be a subcategory of an $S(F)$. Denote by $\mathcal{K}$ the full subcategory of $S(F)$ generated by all the objects of the form

$$(X, \bigcup_{i} a_{i})$$

with $(X, a_{i}) \in \text{obj} \mathcal{K}$ for every $i \in J$.

4.4. REMARK. Since $(X, \bigcup_{i} a_{i})$ may be expressed as a colimit of the diagram consisting of the identity carried morphisms $(X, \emptyset) \rightarrow (X, a_{i})$, and since the forgetful functor of $S(F)$ preserves colimits, we see that for a simply coreflective $\mathcal{K}$, we have $\mathcal{K} = \mathcal{K}$. On the other hand, one sees easily that the condition $\mathcal{K} = \mathcal{K}$ does not imply simple coreflectivity.

4.5. Let $\mathcal{K}$ be a simply reflective subcategory of $S(F)$. Denote by

$$\rho_{(X, a)}: (X, a) \rightarrow (X, \rho a)$$

the identity carried reflection. Define a functor $F_{\mathcal{K}}: \text{Set} \rightarrow \text{QD Top}$ as follows: The underlying set of $F_{\mathcal{K}}(X)$ coincides with that of $F(X)$ and the topology of $F_{\mathcal{K}}(X)$ is given by the preorder:

$$u \leq_{\mathcal{K}} v \text{ iff } \rho \mathcal{O}_{p} u \subseteq \rho \mathcal{O}_{p} v;$$

for a mapping $f$ put $F_{\mathcal{K}}(f)(u) = F(f)(u)$. (This is correct: if we have $\rho \mathcal{O}_{p} u \subseteq \rho \mathcal{O}_{p} v$, then by the basic property of reflections and by 1.3:

$$\rho \mathcal{O}_{p} F(f)(u) \subseteq \rho \mathcal{O}_{p} F(f)(v).$$

Since the preorder $\leq_{\mathcal{K}}$ is obviously stronger than the original one, we have an epi-transformation $\varepsilon_{\mathcal{K}}: F \rightarrow F_{\mathcal{K}}$ defined by $(\varepsilon_{\mathcal{K}})(u) = u$.

4.6. LEMMA. $M$ is open in $F_{\mathcal{K}}(X)$ iff $(X, M) \in  \mathcal{K}$.

PROOF. Let $M$ be open in $F_{\mathcal{K}}(X)$. Then for every $u \in M$, $\rho \mathcal{O}_{p} u \subseteq M$, so that $M = \bigcup_{u \in M} \rho \mathcal{O}_{p} u$. Thus, $(X, M) \in  \mathcal{K}$. On the other hand, suppose that $(X, M) \in  \mathcal{K}$ and $u \leq_{\mathcal{K}} v \in M$. Then, $u$ is in $\rho \mathcal{O}_{p} v \subseteq M$. Thus, $M$ is open.

4.7. THEOREM. Let $\mathcal{K}$ be a simply reflective subcategory of $S(F)$. Then $S(F_{\mathcal{K}}) =  \mathcal{K}$ and $[\varepsilon_{\mathcal{K}}]$ is the embedding $\mathcal{K} \subseteq S(F)$.

PROOF follows immediately by 2.6 and 4.6.
4.8. Remark. By 4.7 and 4.2, for a simply reflective \( \mathcal{K} \), the category \( \tilde{\mathcal{K}} \) is also simply reflective.

4.9. Theorem 4.7 is in a way converse to Proposition 4.2, stating that every simply reflective subcategory satisfying (*) is an image of an epi-transformation induced functor. By 3.3 we have another epi-transformation

\[ \varepsilon'_{\mathcal{K}} = \eta_{F_{\mathcal{K}}} \circ \varepsilon_{\mathcal{K}} : F \to F'_{\mathcal{K}} \]

inducing also an isomorphism of \( S(F'_{\mathcal{K}}) \) and \( \tilde{\mathcal{K}} \). We will show now that every epi-transformation \( \varepsilon : F \to G \) such that \( [\varepsilon] \) represents the embedding \( \mathcal{K} \subset S(F) \) lies in between \( \varepsilon_{\mathcal{K}} \) and \( \varepsilon'_{\mathcal{K}} \). We have

**Theorem.** Let \( \mathcal{K} \) be a simply reflective subcategory of \( S(F) \) such that \( \mathcal{K} = \tilde{\mathcal{K}} \). Let \( \varepsilon : F \to G \) be an epi-transformation and let there be an isofunctor \( \phi \) such that the diagram

\[
\begin{array}{ccc}
S(G) & \xrightarrow{[\varepsilon]} & S(F) \\
\phi \downarrow & & \downarrow \\
\mathcal{K} & \subset & S(F)
\end{array}
\]

commutes. Then there are epi-transformations

\[ F_{\mathcal{K}} \xrightarrow{\beta} G \xrightarrow{\gamma} F'_{\mathcal{K}} \]

such that the diagram

\[
\begin{array}{ccc}
F_{\mathcal{K}} & \xrightarrow{\varepsilon_{\mathcal{K}}} & F'_{\mathcal{K}} \\
\beta \downarrow & & \downarrow \\
F & \xrightarrow{\varepsilon} & G \\
\gamma \downarrow & & \downarrow \\
F_{\mathcal{K}} & \xrightarrow{\eta_{F_{\mathcal{K}}}} & F'_{\mathcal{K}}
\end{array}
\]

commutes.

**Proof.** By 3.4 there is a natural equivalence \( \kappa : G' \to F'_{\mathcal{K}} \) such that

\[
(3) \quad \phi^{-1} \left[ \eta_{F_{\mathcal{K}}} \right] = \left[ \eta_G \right] \left[ \kappa \right].
\]

Put \( \gamma = \kappa \circ \eta_{G} : G \to F'_{\mathcal{K}} \). Now, let \( M \) be open in \( G(X) \). Then \( (X, M) \) is in \( S(G) \) and hence \( (X, \varepsilon^{-1}_{X}(M)) = \phi(X, M) \varepsilon_{\mathcal{K}} \). Thus, by 4.6, \( \varepsilon^{-1}_{X}(M) \)
is open in $F\mathcal{K}(X)$. Consequently, we can define an epi-transformation $\beta: F\mathcal{K} \to G$ putting $\beta_X(u) = \varepsilon_X(u)$. Thus, we obtain immediately

(4) \hspace{1cm} \beta \circ \varepsilon\mathcal{K} = \varepsilon.

Further, we obtain (using (3), (4) and (1))

$$[\varepsilon\mathcal{K}] [\gamma \circ \beta] = [\gamma \circ \beta \circ \varepsilon\mathcal{K}] = [\beta \circ \varepsilon\mathcal{K}] [\gamma] =$$

$$= [\beta \circ \varepsilon\mathcal{K}] \phi^{-1} [\eta F\mathcal{K}] = [\varepsilon] \phi^{-1} [\eta F\mathcal{K}] = [\varepsilon \mathcal{K}] [\eta F\mathcal{K}].$$

Since $[\varepsilon\mathcal{K}]$ is an embedding, we have, hence, $[\gamma \circ \beta] = [\eta F\mathcal{K}]$. Since $(\gamma \circ \beta)_X, (\eta F\mathcal{K})_X: F\mathcal{K}(X) \to F'\mathcal{K}(X)$ and since $F'\mathcal{K}(X)$ is a $T_\circ$-space, we obtain by 1.6 finally $\eta F\mathcal{K} = \gamma \circ \beta$ . Thus, the diagram (2) commutes.

4.10. REMARK. On the other hand, if (2) in 4.9 commutes and if $\varepsilon$ is an epi-transformation, then it induces a full embedding onto $\mathcal{K}$. Really, $\beta$ is then necessarily an epi-transformation and we have $[\eta F\mathcal{K}] = [\beta] [\gamma]$ . Since $[\eta F\mathcal{K}]$ is an isofunctor and $[\beta]$ a full embedding, $[\beta]$ is an isofunctor.

5. Simply bireflective subcategories.

In this paragraph, we are going to characterize the epi-transformations $\varepsilon$ such that $[\varepsilon]$ is a full embedding onto a simply coreflective subcategory. By 2.4 and 2.7, every $[\varepsilon]$ is a full embedding onto a simply reflective subcategory, by 4.4 and 4.7 every simply bireflective (i.e. simply reflective and simply coreflective) subcategory of an $S(F)$ is represented by an embedding $[\varepsilon]$. Thus, we will obtain a characterization of all simply bireflective subcategories of $S(F)$.

5.1. LEMMA. Let $F, G: \text{Set} \to \mathcal{QDTop}$ be functors and $\tau: F \to G$ a transformation. Let $R: S(F) \to S(G)$ be a functor such that there is a natural equivalence

$$\kappa_{xy}: S(F)((\tau)(x), y) \cong S(G)(x, R(y))$$

such that
for $U$, $U'$ the natural forgetful functors. Then $(X, r') = R(X, r) = (X, C(r))$, where

$$C(r) = \cup \{ \mathcal{O}_p b \mid \tau_X^{-1}(\mathcal{O}_p b) \subset r \}.$$ 

**Proof.** Since $\tau_X^{-1}(C r) \subset r$, $I_X$ carries a morphism $[\tau](X, C r) \rightarrow (X, r)$ and hence it carries also a morphism $(X, C r) \rightarrow (X, r')$. Thus, $C r \subset r'$. On the other hand, if $b \in r'$, $I_X$ carries a morphism $(X, \mathcal{O}_p b) \rightarrow R(X, r)$. Consequently, $\tau_X^{-1}(\mathcal{O}_p b) \subset r$, and hence $b \in C r$.

5.2. **Remark.** For the $R$ from 5.1 we have $U' R = U$. The formula

$$U' R(\xi) = U(\xi)$$

is obvious immediately. Now consider a

$$\phi = (r, f, s) : (X, r) \rightarrow (Y, s).$$

We have

$$R(\phi) = S(G)(1, R(\phi)) \kappa_X \kappa_X^{-1}(1) = \kappa_X(S(F)(1, \phi)(\kappa_X^{-1}(1))) = \kappa_X(\phi \kappa_X^{-1}(1)).$$

Thus

$$U' R(\phi) = U' (\kappa_X(\phi \kappa_X^{-1}(1))) = U(\phi) U (\kappa_X^{-1}(1)) = U(\phi).$$

5.3. **Theorem.** Let $F, G : \text{Set} \rightarrow \text{QD Top}$ be functors, $\varepsilon : F \rightarrow G$ an epimorphism. $[\varepsilon]$ is a full embedding onto a simply coreflective (and, hence, simply bireflective) subcategory iff:

(A) for every $f : X \rightarrow Y$ and every $b \in G(X)$,

$$\mathcal{O}_p F(f) \varepsilon_X^{-1}(\mathcal{O}_p b) \in \varepsilon_Y^{-1}(\mathcal{O}_p G(f)(b)).$$

**Proof.** Let $[\varepsilon]$ be a full embedding onto a simply coreflective subcategory. Take an $f : X \rightarrow Y$ and a $b \in G(X)$. We have

$$\mathcal{O}_p F(f) \varepsilon_X^{-1}(\mathcal{O}_p b) \subset \varepsilon_Y^{-1}(\mathcal{O}_p G(f)(b)),$$

since, if

$$y \in F(f)(\varepsilon_X^{-1}(\mathcal{O}_p b)) \text{ and } z \in \varepsilon_X^{-1}(\mathcal{O}_p b)$$

is such that $F(f)(z) = y$, then
\[ \varepsilon_Y(y) = G(f) \varepsilon_X(z) \in G(f)(\mathcal{O}_p b) \subset \mathcal{O}_p G(f)(b). \]

Put \( r = \mathcal{O}_p F(f)(\varepsilon^*_X(\mathcal{O}_p b)) \). Then \( f \) carries a morphism
\[ [\varepsilon](X, \mathcal{O}_p b) \to (Y, r) \text{ in } S(F) \]
and hence also a morphism
\[ (X, \mathcal{O}_p b) \to (Y, r) = (Y, C r) \text{ in } S(G) \]
(\( C \text{ from 5.1} \)), so that in particular \( G(f)(b) \in C r \), i.e.
\[ \varepsilon^*_Y(\mathcal{O}_p G(f)(b)) \subset r = \mathcal{O}_p F(f) \varepsilon^*_X(\mathcal{O}_p b). \]

Now, let (A) hold. Define \( R : S(F) \to S(G) \) by \( R(r, f, s) = (C r, f, C s) \).
This is correct:
\[
G(f)(C r) = G(f)(\bigcup \{ \mathcal{O}_p b \mid \varepsilon^*_X(\mathcal{O}_p b) \subset r \}) = \\
= \bigcup \{ G(f)(\mathcal{O}_p b) \mid \varepsilon^*_X(\mathcal{O}_p b) \subset r \} \subset \bigcup \{ \mathcal{O}_p G(f)(b) \mid \varepsilon^*_X(\mathcal{O}_p b) \subset r \}
\]
\[
\subset \bigcup \{ \mathcal{O}_p G(f)(b) \mid \mathcal{O}_p F(f) \varepsilon^*_X(\mathcal{O}_p b) \subset s \} = \\
= \bigcup \{ \mathcal{O}_p G(f)(b) \mid \varepsilon^*_Y(\mathcal{O}_p G(f)(b)) \subset s \} \subset C s.
\]
We have \( F(f)(\varepsilon^*_X(r)) \subset s \) iff \( \mathcal{O}_p F(f) \varepsilon^*_X(r) \subset s \) iff
\[
\bigcup \{ \mathcal{O}_p F(f) \varepsilon^*_X(\mathcal{O}_p b) \mid b \in r \} \subset s
\]
iff
\[
\bigcup \{ \varepsilon^*_Y(\mathcal{O}_p G(f)(b)) \mid b \in r \} \subset s
\]
iff
\[ \varepsilon^*_Y(\mathcal{O}_p G(f)(r)) \subset s \text{ iff } \mathcal{O}_p G(f)(r) \subset C s. \]
Thus, \( f \) carries a morphism \([\varepsilon](x) \to y\) iff it carries a morphism \( x \to R(y)\).

5.4. REMARKS. 1° From the first part of the proof of 5.3 we see that the inclusion
\[ \mathcal{O}_p F(f) \varepsilon^*_X(\mathcal{O}_p b) \subset \varepsilon^*_Y(\mathcal{O}_p G(f)(b)) \]
holds for any \( \varepsilon \). Thus, the condition (A) is equivalent to the reverse inclusion
\[ \varepsilon^*_Y(\mathcal{O}_p G(f)(b)) \subset \mathcal{O}_p F(f) \varepsilon^*_X(\mathcal{O}_p b). \]
Rewriting this, we obtain the following condition on \( \varepsilon \) equivalent to (A):

(B) For every \( f : X \to Y \), every \( a \in F(Y) \) and every \( b \in G(X) \) such that
\[ \varepsilon_Y(a) \leq G(f)(b), \text{ there is a } c \in F(X) \text{ such that } \]
\[ a \leq F(f)(c) \text{ and } \varepsilon_X(c) \leq b. \]

2° In the case of an \( \varepsilon : F \to G \) such that \( F \) and \( G \) have discrete values, the condition (A) reduces to:

For every \( f : X \to Y \) and for every \( b \in G(X) \),

\[ F(f)(\varepsilon^Y_X(b)) = \varepsilon^Y_X(G(f)(b)). \]

5.5. Proposition. Let \( \varepsilon : F \to G \) satisfy (A). Then, whenever \( F(f) \) is an open mapping, \( G(f) \) is also an open mapping.

Proof. We have

\[ G(f)(\mathcal{O}_p b) = G(f)\varepsilon^Y_X(\mathcal{O}_p b) = \varepsilon_Y F(f) \varepsilon^Y_X(\mathcal{O}_p b). \]

If \( F(f) \) is an open mapping, we continue

\[ \cdots = \varepsilon_Y\mathcal{O}_p F(f) \varepsilon^Y_X(\mathcal{O}_p b) = \varepsilon_Y \varepsilon^Y_X \mathcal{O}_p G(f)(b) = \mathcal{O}_p G(f)(b). \]

5.6. Simply bireflective subcategories of the category of sets with \( A \)-nary relations: Let \( A \) be a set. An \( A \)-nary relation on a set \( X \) is a subset of \( X^A \); if \( r \) (resp. \( s \) ) is an \( A \)-nary relation on \( X \) (resp. \( Y \) ), a mapping \( f : X \to Y \) is an \( r s \)-homomorphism if

\[ a \in r \text{ implies } f \circ a \in s. \]

Thus, the category of sets with \( A \)-nary relations and their homomorphisms coincides with \( S(Q_A) \) where

\[ Q_A(X) = X^A, \quad Q_A(f)(a) = f \circ a, \]

the topology on \( Q_A(X) \) being discrete. By 4.7 and 5.4, we will obtain a complete list of bireflective subcategories of \( S(Q_A) \) if we list all the epi-transformations \( \varepsilon : Q_A \to G \) satisfying (A) and such that the underlying mappings of \( \varepsilon \) are the identities.

Consider such an epi-transformation. Since \( Q_A(X) \) are discrete, we have all \( Q_A(f) \) open and, hence, by 5.6, all

\[ G(f) : G(X) = (X^A, \leq) \to G(Y) = (Y^A, \leq) \]

are open. Thus, in particular, if \( a \leq \beta \) in \( G(X) \), then, since we have \( \beta = G(\beta)(1_A) \), there is a \( \phi \leq 1_A \) in \( G(A) \) such that
\[ \alpha = G(\beta)(\phi) = \beta \circ \phi. \]

Conversely, of course, if \( \phi \leq 1_A \), we have necessarily

\[ \beta \circ \phi = G(\beta)(\phi) \leq G(\beta)(1) = \beta. \]

Thus, in particular, if \( \phi' \leq 1_A \) and \( \psi \leq 1_A \), we have

\[ \phi \circ \psi \leq \phi \leq 1_A. \]

Hence, there is a submonoid \( M \) of \( A^A \) such that

\[ (1) \quad \alpha \leq \beta \text{ in } G(X) \iff \exists \phi \in M, \alpha = \beta \circ \phi. \]

On the other hand, let there be given a submonoid \( M \) of \( A^A \) and let us put

\[ G(X) = (X^A, \leq) \text{ with } \leq \text{ defined by the formula (1), } G(f)(\alpha) = f \circ \alpha. \]

(Obviously, thus defined \( \leq \) is transitive, and

\[ \alpha \leq \beta \text{ implies } G(f)(\alpha) \leq G(f)(\beta). \]

Define \( \varepsilon: Q_A \to G \) putting \( \varepsilon_X(\alpha) = \alpha \). If \( \varepsilon_Y(\alpha) \leq G(f)(\beta) \), i.e. if

\[ \alpha \leq f \circ \beta, \]

we have \( \alpha = f \circ \beta \circ \phi \) for a \( \phi \in M \). Put \( \gamma = \beta \circ \phi \). Then

\[ \alpha = G(f)(\gamma) \quad \text{and} \quad \varepsilon_X(\gamma) = \gamma \leq \beta. \]

Hence, the condition (B) is satisfied.

Thus, we conclude that the simply bireflective subcategories \( \mathcal{K} \) of \( S(Q_A) \) are exactly those obtained as follows: A submonoid \( M \) of \( A^A \) is given, and an object \( (X, r) \) of \( S(Q_A) \) is in \( \mathcal{K} \) iff

\[ \alpha \circ \phi \in r, \text{ for every } \alpha \in r \text{ and } \phi \in M. \]

Moreover, we see easily that, among them, the ones representable by an \( \varepsilon: Q_A \to G \) with discrete \( G(X) \) are exactly those where \( M \) is a group. (By 1.8, \( \phi \in M \), i.e. \( \phi \leq 1 \), implies \( 1 \leq \phi \); hence there exists a

\[ \psi \in M \text{ with } I = \phi \circ \psi. \]
REFERENCES