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COMPLETELY REGULAR RELATIONAL ALGEBRAS

by R. J. PERRY (*)

ABSTRACT

If (K, τ) is a completely regular space, the quotient space (K', τ') obtained by identifying points whose neighborhood systems coincide is a Tychonoff space and, as such, is a subspace of a compact Hausdorff space. We generalize this to subcategories of the relational algebras of an arbitrary triple on sets.

1. Introduction.

Barr introduced relational algebras in [1] by weakening the structure map λ in the T-algebra (K, λ) from a function to a relation. Manes [5] extended the notion to triples on certain categories besides sets. The basic weakeness has been a dearth of examples.

We introduce subcategories of the relational algebras of an arbitrary triple on sets which can be regarded as generalizing completely regular spaces and Tychonoff spaces. We hope that these intermediate steps between T-algebras and relational algebras will provide an arena in which more interesting examples may be found.

We begin with basic definitions. Let $\mathbf{T} = (T, \eta, \mu)$ be a given triple on sets.

DEFINITION. (1) The objects of the category $P(\mathbf{T})$, called **T**-*prealgebras* are pairs (K, λ), where

$$K \text{ is a set and } \lambda \subset T(K) \times K.$$

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A morphism $f : (K, \lambda) \to (A, \alpha)$ is a function $f : K \to A$ for which

$$(x, y) \in \lambda \quad \text{implies} \quad (T(f)(x), f(y)) \in \alpha .$$

(2) A $\mathbf{T}$-prealgebra (K, λ) is said to satisfy the *reflexive law* if

$$(\eta(K)(y), y) \in \lambda \text{ for all } y \in K .$$

The full subcategory $Ref(\mathbf{T})$ of reflexive $\mathbf{T}$-prealgebras consists of those prealgebras satisfying the reflexive law.

(3) If $\lambda \subset T(K) \times K$, let

$$p_1 : \lambda \to T(K) \quad \text{and} \quad p_2 : \lambda \to K$$

be the projections. We define $\hat{T}(\lambda) \subset T^2(K) \times T(K)$ as follows:

$(x, y) \in \hat{T}(\lambda)$ if and only if there exists $w \in T(\lambda)$ with
$T(p_1)(w) = x$ and $T(p_2)(w) = y$.

A prealgebra (K, λ) is said to satisfy the *transitive law* if

$$(x, y) \in \lambda \hat{T}(\lambda) \quad \text{implies} \quad (\mu(K)(x), y) \in \lambda .$$

The full subcategory $R(\mathbf{T})$ of relational $\mathbf{T}$-algebras consists of those prealgebras satisfying both the reflexive and the transitive laws.

(4) Let $f : (K, \lambda) \to (A, \alpha)$ in $P(\mathbf{T})$. f is said to be *algebraic* (Manes [5]) provided

$$(T(f)(x), f(y)) \in \alpha \quad \text{implies} \quad (x, y) \in \lambda .$$

Moreover we say that

(i) (K, λ) is an *algebraic subobject of* (A, α) when f is monic;

(ii) (A, α) is an *algebraic quotient object of* (K, λ) when f is epi. //

Let $\mathbf{B}$ the ultrafilter triple. Barr [1] has shown that

$$R(\mathbf{B}) = \text{topological spaces},$$

while Manes [4] has shown that

$$Ens^{\mathbf{B}} = \text{compact Hausdorff spaces}.$$

We produce subcategories $CR(\mathbf{T})$ and $CRH(\mathbf{T})$ of $R(\mathbf{T})$ with

$$CR(\mathbf{B}) = \text{completely regular spaces},$$

$$CRH(\mathbf{B}) = \text{Tychonoff spaces}.$$

If (K, τ) is a completely regular space, the quotient space (K', τ') obtained by identifying points whose neighborhood systems coincide is a Tychonoff space and, as such, is a subspace of a compact Hausdorff space. We show that an object $(K, \lambda) \in CR(\mathbf{T})$ possesses an appropriate algebraic quotient object $(K_\sim, \lambda_\sim) \in CRH(\mathbf{T})$ with $(K_\sim, \lambda_\sim)$ an algebraic subobject of a T-algebra.

We also observe that $CR(\mathbf{T})$ introduces some structure into the amorphous category $P(\mathbf{T})$ by providing a right bicategory structure (I, P), from which it follows easily that $Ref(\mathbf{T})$, $R(\mathbf{T})$ and $CR(\mathbf{T})$ are P-reflective subcategories.

To simplify the exposition, the uniqueness of both objects and morphisms is only to within equivalence.

2. $CR(\mathbf{T})$ and $CRH(\mathbf{T})$.

In [5], Manes shows that $Ens^{\mathbf{T}}$ is a reflective subcategory of $Ref(\mathbf{T})$. We list those details necessary to our work.

Let $(K, \lambda) \in Ref(\mathbf{T})$. Set

$$L = \{((x, y), (z, y)) \mid (x, y) \text{ and } (z, y) \in \lambda\}.$$

Define functions f and $g: L \to T(K)$ by $f(w) = x$ and $g(w) = z$, where $w = ((x, y), (z, y))$; then

$$\mu(K) T(f) \quad \text{and} \quad \mu(K) T(g): (T(L), \mu(L)) \to (T(K), \mu(K))$$

are $\mathbf{T}$-homomorphisms. Let $q: (T(K), \mu(K)) \to (Q, \theta)$ be the coequalizer of these morphisms in $Ens^{\mathbf{T}}$; then $q\eta(K): (K, \lambda) \to (Q, \theta)$ is the reflection map.

DEFINITION. (1) If $(K, \lambda) \in Ref(\mathbf{T})$, set

$$\overline{\lambda} = \{(x, y) \mid q(x) = q\eta(K)(y)\}.$$

(2) The full subcategory $CR(\mathbf{T})$ of *completely regular relational* $\mathbf{T}$-*algebras* consists of those $(K, \lambda) \in Ref(\mathbf{T})$ with $\overline{\lambda} = \lambda$.

(3) The full subcategory $CRH(\mathbf{T})$ of *Tychonoff relational* $\mathbf{T}$-*algebras* consists of those $(K, \lambda) \in CR(\mathbf{T})$ with λ a partial function. //

LEMMA 1. *(1)* $\lambda \subset \overline{\lambda}$ *;*

(2) $(K, \overline{\lambda}) \in CR(\mathbf{T})$ *;*

(3) $CR(\mathbf{T}) \subset R(\mathbf{T})$.

PROOF. (1) If $(x, y) \in \lambda$, $w' = ((x, y), (\eta(K)(y), y)) \in L$ implies

$$q(x) = qf(w') = qg(w') = q\eta(K)(y).$$

(2) By (1), $\overline{\lambda}$ is reflexive. We use the above notation with $\overline{\lambda}$ in place of λ and stars on all appropriate symbols. If $((x, y), (z, y)) \in L^*$, then

$$q(x) = q\eta(K)(y) = q(z)$$

implies

$$q\mu(K)T(f^*) = q\mu(K)T(g^*) .$$

Hence there exists a unique $\mathbf{T}$-homomorphism t with $tq^* = q$. Thus $\overline{\lambda} = \lambda$, since $(a, b) \in \overline{\overline{\lambda}}$ implies

$$q(a) = tq^*(a) = tq^*\eta(K)(b) = q\eta(K)(b).$$

(3) It suffices to show that $\overline{\lambda}$ is transitive. Now

$$q\mu(K)T(p_1) = qT(p_2),$$

since they are $\mathbf{T}$-homomorphisms equalized by $\eta(\lambda)$. Let $(x, y) \in \overline{\lambda}\,\hat{T}(\overline{\lambda})$; then there exist $w \in T(\overline{\lambda})$ and $z \in T(K)$ with

$$T(p_1)(w) = x, \quad T(p_2)(w) = z \quad \text{and} \quad (z, y) \in \overline{\lambda}.$$

Hence

$$q\mu(K)(x) = q\mu(K)T(p_1)(w) = qT(p_2)(w) = q(z) = q\eta(K)(y)$$

and $(\mu(K)(x), y) \in \overline{\lambda}$. //

LEMMA 2. *If* $f : (K_1, \lambda_1) \to (K_2, \lambda_2)$ *in* $Ref(\mathbf{T})$, *then*

$$f : (K_1, \overline{\lambda}_1) \to (K_2, \overline{\lambda}_2).$$

PROOF. Let $w = ((x, y), (z, y)) \in L_1$; then

$$w^* = ((T(f)(x), f(y)), (T(f)(z), f(y))) \in L_2 .$$

Thus

$$q_2T(f)f_1(w) = q_2T(f)(x) = q_2f_2(w^*) =$$

$$q_2g_2(w^*) = q_2T(f)(z) = q_2T(f)g_1(w) ;$$

it follows that

$$q_2 T(f)\mu(K_1)T(f_1) = q_2 T(f)\mu(K_1)T(g_1).$$

By definition of q_1, there exists

$$t:(Q_1,\theta_1)\to(Q_2,\theta_2) \quad \text{with} \quad tq_1 = q_2 T(f).$$

If $(a,b)\in\overline{\lambda}_1$, then

$$q_2 T(f)(a) = tq_1(a) = tq_1\eta(K_1)(b) =$$
$$= q_2 T(f)\eta(K_1)(b) = q_2\eta(K_2)f(b),$$

showing $(T(f)(a), f(b))\in\overline{\lambda}_2$. //

LEMMA 3. *Let $f:(K,\lambda)\to(A,\alpha)$ in $Ref(\mathbf{T})$ with f algebraic.*

(1) $(A,\alpha)\in CR(\mathbf{T})$ implies $(K,\lambda)\in CR(\mathbf{T})$;

(2) If f is epi, $(K,\lambda)\in CR(\mathbf{T})$ implies $(A,\alpha)\in CR(\mathbf{T})$.

PROOF. (1) is immediate from Lemma 2.

(2) There exists $g: A\to K$ with $fg = 1(A)$. g is algebraic and (1) applies. //

DEFINITION. (1) Let $(K,\lambda)\in Ref(\mathbf{T})$. Set

$$x\sim y \quad \text{if} \quad (\eta(K)(x), y)\in\lambda.$$

λ is said to be *complete* if $\sim$ is an equivalence relation on K.

(2) Let $(K,\lambda)\in Ref(\mathbf{T})$ with λ complete. Let $K_\sim$ be the set of equivalence classes and $p: K\to K_\sim$ the natural map. Set

$$\lambda_\sim = \{(a,b) \mid a = T(p)(x) \ \text{and}\ b = p(y) \ \text{for some}\ (x,y)\in\lambda\}. \quad //$$

Since $(K_\sim,\lambda_\sim)\in Ref(\mathbf{T})$, let $q'\eta(K_\sim):(K_\sim,\lambda_\sim)\to(Q',\theta')$ be the reflection map into $Ens^{\mathbf{T}}$.

THEOREM. *$(K,\lambda)\in CR(\mathbf{T})$ if and only if:*

(1) $(K_\sim,\lambda_\sim)$ is an algebraic quotient object of (K,λ);

(2) $(K_\sim,\lambda_\sim)\in CRH(\mathbf{T})$;

(3) $(K_\sim,\lambda_\sim)$ is an algebraic subobject of a T-algebra.

PROOF. Let $(K,\lambda)\in CR(\mathbf{T})$. If $(\eta(K)(x), y)\in\lambda$, then

$$q\eta(K)(x) = q\eta(K)(y),$$

from which it follows easily that λ is complete.

$h : K_\sim \to Q$ with $h p(a) = q\eta(K)(a)$ is well-defined. Suppose

$$T(p)(x) = T(p)(x') \quad \text{and} \quad p(y) = p(y') \quad \text{with} \quad (x', y') \in \lambda ;$$

then $y \sim y'$ implies $q\eta(K)(y) = q\eta(K)(y')$. To show p is algebraic we need $(x, y) \in \lambda$. But

$$q(x) = \theta T q\eta(K)(x) = \theta T(h) T(p)(x) = \theta T(h) T(p)(x') =$$

$$= \theta T q\eta(K)(x') = q(x') = q\eta(K)(y') = q\eta(K)(y).$$

Thus $(x, y) \in \overline{\lambda} = \lambda$ and (1) holds via p.

By Lemma 3 (2), $(K_\sim, \lambda_\sim) \in CR(\mathbf{T})$. If

$$(T q' \eta(K_\sim)(x),\ q' \eta(K_\sim)(y)) \in \theta',$$

then

$$q'(x) = \theta' T q' \eta(K_\sim)(x) = q' \eta(K_\sim)(y).$$

Whence $(x, y) \in \overline{\lambda_\sim} = \lambda_\sim$ and $q' \eta(K_\sim)$ is algebraic.

If $(T(p)(x), p(y))$ and $(T(p)(x), p(z)) \in \lambda_\sim$, then (x, y) and $(x, z) \in \lambda$, since p is algebraic. Thus

$$q\eta(K)(y) = q(x) = q\eta(K)(z),$$

$$(\eta(K)(y), z) \in \overline{\lambda} = \lambda, \quad p(y) = p(z),$$

and $\lambda_\sim$ is a partial function. Whence $(K_\sim, \lambda_\sim) \in CRH(\mathbf{T})$.

Suppose $q' \eta(K_\sim)(a) = q' \eta(K_\sim)(b)$; then

$$(\eta(K_\sim)(a),\ b) \in \overline{\lambda_\sim} = \lambda_\sim .$$

Since $\lambda_\sim$ is a partial function and $(K_\sim, \lambda_\sim) \in Ref(\mathbf{T})$, it follows that $a = b$ and $q' \eta(K_\sim)$ is monic. Thus $(K_\sim, \lambda_\sim)$ is an algebraic subobject of the T-algebra (Q', θ') via $q' \eta(K_\sim)$. //

3. Examples.

Kamnitzer [2] lists the following examples of relational algebras:

$R(\mathbf{I})$ = reflexive and transitive relations [1],

where $\mathbf{I}$ is the identity triple;

$R(\mathbf{B})$ = topological spaces [1],

where $\mathbf{B}$ is the ultrafilter triple;

$R(\mathbf{P})$ = generalized sup-semilattices [6],
where $\mathbf{P}$ is the power set triple;

$R(\mathbf{T}_1)$ = generalized semigroups [6],
where $\mathbf{T}_1$ is the semigroup triple;

$R(\mathbf{T}_2)$ = generalized R-modules,
where $\mathbf{T}_2$ is the left R-module triple.

In addition Barr [1] lists

$R(\mathbf{I}') = \{((K,\lambda),K_0) \mid \lambda$ is reflexive and transitive, K_0 is a ray$\}$,
where $\mathbf{I}' = (T,\eta,\mu)$ with $T(K) = K+1$ (coproduct);

$R(\mathbf{B}') = \{(K,K_0) \mid K$ is a topological space, K_0 a closed subspace$\}$,
where $\mathbf{B}' = (T,\eta,\mu)$ with $T(K) =$ the set of all ultrafilters on $K+1$.

Finally we let

$\mathbf{T}_3$ = the abelian idempotent semigroup triple,

$\mathbf{T}_4$ = the group triple.

It is known that $Ens^{\mathbf{I}} = Ens$, $Ens^{\mathbf{B}}$ = compact Hausdorff spaces [4], $Ens^{\mathbf{P}}$ = complete lattices, $Ens^{\mathbf{T}_1}$ = semigroups, $Ens^{\mathbf{T}_2}$ = left R-modules, $Ens^{\mathbf{I}'}$ = pointed sets, $Ens^{\mathbf{B}'}$ = pointed compact Hausdorff spaces, $Ens^{\mathbf{T}_3}$ = semilattices = abelian idempotent semigroups, $Ens^{\mathbf{T}_4}$ = groups.

Applying our theorem we see that:

$CRH(\mathbf{I}) = Ens$,

$CR(\mathbf{I})$ = equivalence relations,

$CRH(\mathbf{B})$ = Tychonoff spaces,

$CR(\mathbf{B})$ = completely regular spaces,

$CRH(\mathbf{P})$ = partially ordered sets,

$CR(\mathbf{P})$ = reflexive and transitive relations,

$CRH(\mathbf{T}_1)$ = partial semigroups,

$CR(\mathbf{T}_1) = \{(K,\lambda_1,\lambda_2) \mid \lambda_1$ is an equivalence relation on K, and $\lambda_2 \subset (K\times K)\times K$ with

(1) $((x,y),z)\in\lambda_2$ implies $((x',y'),z')\in\lambda_2$, whenever $(x,x'), (y,y'), (z,z')\in\lambda_1$; and

(2) $((x,y),z)\in\lambda_2$ and $((x,y),t)\in\lambda_2$ imply that $(z,t)\in\lambda_1$; the associative law holds when applicable$\}$.

$CRH(\mathbf{T}_2)$ = partial left R-modules,

$CR(\mathbf{T}_2)$ = an extension of the two operations to equivalence classes as in $CR(\mathbf{T}_1)$,

$CRH(\mathbf{I}')$ = pointed sets,

$CR(\mathbf{I}') = \{((K,\lambda), K_o) \mid \lambda$ is an equivalence relation, K_o is a ray $\}$,

$CRH(\mathbf{B}') = \{(K, K_o) \mid K$ is a Tychonoff space, K_o a closed subspace$\}$,

$CR(\mathbf{B}') = \{(K, K_o) \mid K$ is a completely regular space, K_o is a closed subspace $\}$,

$CRH(\mathbf{T}_3)$ = partially ordered sets,

$CR(\mathbf{T}_3)$ = reflexive and transitive relations,

$CRH(\mathbf{T}_4)$ = partial semigroups with cancellation laws,

$CR(\mathbf{T}_4)$ = those objects of $CR(\mathbf{T}_1)$ with the following property: Given $((a,b),c) \in \lambda_2$ and $((x,y),z) \in \lambda_2$ with $(c,z) \in \lambda_1$ then: (1) $(a,x) \in \lambda_1$ implies $(b,y) \in \lambda_1$,
(2) $(b,y) \in \lambda_1$ implies $(a,x) \in \lambda_1$.

4. Structure in $P(\mathbf{T})$.

DEFINITION. (1) Let $(K,\lambda) \in P(\mathbf{T})$. Set

$$\lambda^* = \lambda \cup \{(\eta(K)(y), y) \mid y \in K\};$$

(2) Set $P = \{1(K) : (K,\lambda) \to (K,\rho) \mid \rho \subset \overline{\lambda}^*\}$;

(3) Set $I = \{g : (K,\rho) \to (A,\alpha) \mid$ if $g : (K,\rho') \to (A,\alpha)$ with $\rho \subset \rho' \subset \overline{\rho}^*$, then $\rho' = \rho\}$. //

Application of Proposition 1.3 of Kennison [3] yields:

PROPOSITION 1. *(I, P) is a right bicategory structure on $P(\mathbf{T})$.*

The definition of I together with the transitive law yields:

PROPOSITION 2. *If $g : (K,\rho) \to (A,\alpha) \in I$, then:*

(1) (A,α) reflexive implies (K,ρ) reflexive;

(2) (A,α) transitive implies (K,ρ) transitive;

(3) $(A,\alpha) \in CR(\mathbf{T})$ implies $(K,\rho) \in CR(\mathbf{T})$.

By Theorem 1.2 of [3] it follows that $Ref(\mathbf{T})$, $R(\mathbf{T})$ and $CR(\mathbf{T})$ are all P-reflective subcategories of $P(\mathbf{T})$.

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