CAHIERS DE
TOPOLOGIE ET GÉOMÉTRIE DIFFÉRENTIELLE
CATÉGORIQUES

KARLHEINZ BAUMGARTNER

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Cahiers de topologie et géométrie différentielle catégoriques, tome 16, nº 2 (1975), p. 169-213

<http://www.numdam.org/item?id=CTGDC_1975__16_2_169_0>

CAHIERS DE TOPOLOGIE
ET GEOMETRIE DIFFERENTIELLE

Vol. XVI-2 (1975)

STRUCTURE OF ADDITIVE CATEGORIES*

by Karlheinz BAUMGARTNER

INTRODUCTION

The first effort generalizing the theory of rings to additive categories was made by LEDUC [10], [11]. Using homological methods (small) additive categories have been investigated by MITCHELL [13], and by the french school (DARTOIS [2], HARARI [5], WEIDENFELD [17], [18], etc..).

The structure theory developed in this paper (being completely different from LEDUC's theory) depends on the concept of the centralizer of a set $\mathfrak{U} = \{ \mathfrak{U}_i : \mathfrak{B} \to \mathfrak{C} \mid i \in I \}$ of parallel functors and the corresponding evaluation E. In order to establish our theory we need the YONEDA-Lemma only. The key results (see 1 until 6 or 7) from which the whole theory flows are a generalized MORITA Theorem characterizing the generators in $\mathfrak{Ab}^{\mathfrak{B}}$ and a generalized BOURBAKI density theorem describing the evaluation of semisimple $\mathfrak{B}$-modules for a (small) category $\mathfrak{B}$. Afterwards we present a more detailed study of the theory which does not reflect all aspects in the special case of rings. The main reason of this is that the D.C.C. for left ideals does not imply the D.C.C. for ideals in general (see 8 until 14). Further developments of the theory including categories of quotients and also generalized GOLDIE theorems would be desirable. On the other hand let us remark that perhaps the investigations in the beginning of the paper are of more common interest. So only these things should be consulted by the reader who is not mainly interested in rings and generalizations or in the study of the structure of categories.

This presentation of the theory was last made possible by the discovery of the isomorphism theorem essentially due to my student L.

* Conférence donnée au Colloque d'Amiens 1973

SCHUMACHER. The whole theory might also be stated for so called K-categories over a commutative ring K (see MITCHELL [13]) using the category Mod_K of K-modules instead of the category $\mathfrak{Ab}$ of abelian groups as base-category. Expecting later on a consideration of the non-additive case and a generalization to closed base-categories, the restriction to this more concrete presentation which is probably improvable in its technic may be justified.

PRELIMINARIES

Let $\mathfrak{B}$ be an additive category. A *cosieve* or a *left ideal* in an object B of $\mathfrak{B}$ is a subfunctor of the additive *Hom*-functor $H^B : \mathfrak{B} \to \mathfrak{Ab}$. By an ideal in $\mathfrak{B}$ we mean a subfunctor of the additive *Hom*-bifunctor $Hom_{\mathfrak{B}}(-,.) : \mathfrak{B}^{op} \otimes \mathfrak{B} \to \mathfrak{Ab}$ (see also [13]). Let I be an ideal in $\mathfrak{B}$. Then

$$f \equiv 0 \quad \Longleftrightarrow \quad f \in I(B_1, B_2)$$

defines a congruence in the abelian group $Hom(B_1, B_2)$ for each B_1, $B_2 \in |\mathfrak{B}|$. In a very natural manner one gets the factor category $\mathfrak{B}/I$ with the canonical additive functor $S : \mathfrak{B} \to \mathfrak{B}/I$. Let us mention that for every additive functor $\mathfrak{U} : \mathfrak{B} \to \mathfrak{C}$ the ideal $Ker\,\mathfrak{U}$ (kernel of $\mathfrak{U}$) is defined by

$$Ker\,\mathfrak{U}(B_1, B_2) = \{ f \in Hom(B_1, B_2) \mid \mathfrak{U}f = 0 \}.$$

Moreover there is a unique faithful additive functor $\mathfrak{U}'$ such that the triangle

$$\begin{tikzcd} \mathfrak{B} \arrow[rr, "\mathfrak{U}"] \arrow[dr, "S"'] & & \mathfrak{C} \\ & \mathfrak{B}/Ker\,\mathfrak{U} \arrow[ur, "\mathfrak{U}'"'] & \end{tikzcd}$$

commutes.

Following the french terminology a $\mathfrak{B}$-module is an $\mathfrak{Ab}$-valued additive functor $\mathfrak{U} : \mathfrak{B} \to \mathfrak{Ab}$. Considering the category $\mathfrak{Ab}^{\mathfrak{B}}$ of $\mathfrak{B}$-modules ($\mathfrak{B}$ small), the usual forgetful functor $V : \mathfrak{Ab}^{\mathfrak{B}} \to Ens^{|\mathfrak{B}|}$ is monadic (see [2], [13]). Note also that $V(\mathfrak{S})_B$ is the underlying mapping of the $\mathfrak{Ab}$-homomorphism $\mathfrak{S}(B)$. The free $\mathfrak{B}$-module over $M = (\ldots, M_B, \ldots)$ is just

$$F(M) = \coprod_{B \in |\mathfrak{B}|} \; \coprod_{m_B \in M_B} H^B .$$

Finally let us recall that natural transformations between free $\mathfrak{B}$-modules can be described by row-finite matrices over the category $\mathfrak{B}$. Some further remarks on left ideals resp. ideals generated by a set of morphisms are needed and developed in 10.

I. BALANCED FUNCTORS

1. Centralizer, bicentralizer, evaluation.

Let $\mathfrak{B}$ be a small additive category and

$$\mathfrak{U} = \{ \mathcal{U}_i : \mathfrak{B} \to \mathcal{C} \mid i \in I \}$$

a set of parallel additive functors. Note that the functors $\mathcal{U}_i$ need not be different. Moreover let us consider the additive functor-category $\mathcal{C}^{\mathfrak{B}}$ of additive functors and the full (small) subcategory $\mathfrak{U}(\mathcal{C}^{\mathfrak{B}})$ of the $\mathcal{U}_i \in \mathfrak{U}$, with the inclusion $I_{\mathfrak{U}} : \mathfrak{U}(\mathcal{C}^{\mathfrak{B}}) \to \mathcal{C}^{\mathfrak{B}}$. Denoting now by $V_B : \mathcal{C}^{\mathfrak{B}} \to \mathcal{C}$ the forgetful functor (defined by $V_B(f) = f(B)$ with $B \in |\mathfrak{B}|$) the set

$$C(\mathfrak{U}) = \{ V_B \circ I_{\mathfrak{U}} \mid B \in |\mathfrak{B}| \}$$

of parallel additive functors $V_B \circ I_{\mathfrak{U}} : \mathfrak{U}(\mathcal{C}^{\mathfrak{B}}) \to \mathcal{C}$ is called the *centralizer of* $\mathfrak{U}$. The centralizer $CC(\mathfrak{U})$ of $C(\mathfrak{U})$ is called the *bicentralizer of* $\mathfrak{U}$. If $\mathfrak{B}$ is not small but the (large) set $\mathfrak{U}$ still isomorphic to a class, then $\mathfrak{U}(\mathcal{C}^{\mathfrak{B}})$ may also be isomorphic to an additive category. The centralizer $C(\mathfrak{U})$ is again isomorphic to a class and so $C(\mathfrak{U})(\mathcal{C}^{\mathfrak{U}(\mathcal{C}^{\mathfrak{B}})})$ may also be isomorphic to an additive category (see 7 until 13).

Now we consider the following fundamental diagram:

$$\begin{tikzcd}
 & \mathfrak{U}(\mathcal{C}^{\mathfrak{B}}) \arrow[dl, "I_{\mathfrak{U}}"'] \arrow[dd, shift left, "C(\mathfrak{U})"] \arrow[dd, shift right] & \\
\mathcal{C}^{\mathfrak{B}} \arrow[dr, "V_B"] & & \mathcal{C}^{\mathfrak{U}(\mathcal{C}^{\mathfrak{B}})} \arrow[dl, "V_{\mathcal{U}}"] \arrow[dl, shift left] \arrow[dl, shift right] \\
 & \mathcal{C} & \\
\mathfrak{B} \arrow[ur, shift left, "\mathfrak{U}"] \arrow[ur, shift right, "\mathcal{U}"'] \arrow[ur] \arrow[rr, "E"'] & & C(\mathfrak{U})(\mathcal{C}^{\mathfrak{U}(\mathcal{C}^{\mathfrak{B}})}) \arrow[ul, "CC(\mathfrak{U})"] \arrow[ul, shift left] \arrow[ul, shift right] \arrow[uu, "I_{C(\mathfrak{U})}"']
\end{tikzcd}$$

Then it is very easy to verify the following

THEOREM 1. *Let* $E:\mathfrak{B}\to C(\mathfrak{U})(\mathcal{C}^{\mathfrak{U}(\mathcal{C}^{\mathfrak{B}})})$ *be the additive (on objects surjective) functor defined by* $E(B)=V_B\circ I_{\mathfrak{U}}$ *and*

$$E(\beta)(\mathfrak{U})=\mathfrak{U}(\beta)\ for\ B,B'\in|\mathfrak{B}|,\ \beta:B\to B',\ \mathfrak{U}\in\mathfrak{U}.$$

Then $\mathfrak{U}=V_{\mathfrak{U}}\circ I_{C(\mathfrak{U})}\circ E$ *holds for every* $\mathfrak{U}\in\mathfrak{U}$.

REMARK 1. The functor

$$E:\mathfrak{B}\to dom\, C\, C(\mathfrak{U})=C(\mathfrak{U})(\mathcal{C}^{\mathfrak{U}(\mathcal{C}^{\mathfrak{B}})})$$

and the restricting evaluating functor $E^*:\mathfrak{B}\times\mathfrak{U}(\mathcal{C}^{\mathfrak{B}})\to\mathcal{C}$ (see SCHUBERT [15]) correspond by the equations

$$E^*(B,\mathfrak{U})=E(B)(\mathfrak{U})\ \text{ and }\ E^*(\beta,\mathfrak{S}_{\mathfrak{U},\mathfrak{U}'})=\mathfrak{S}_{\mathfrak{U},\mathfrak{U}'}(B')\circ E(\beta)(\mathfrak{U}).$$

So E will be called *the evaluation of* $\mathfrak{U}$.

In this paper we are mainly interested in the question under what conditions the evaluation E of the set $\mathfrak{U}$ is (almost) an isomorphism (see FAITH [3], LAMBEK [9], SUZUKI [16], etc...). Just this situation will be called the *bicentralizer-property of* $\mathfrak{U}$. In this case we also say that $\mathfrak{U}$ is *balanced.* If E is only full, $\mathfrak{U}$ is called *weakly balanced.* Futhermore $\mathfrak{U}$ is *faithful* if $\bigcap_{i\in I} Ker\,\mathfrak{U}_i=0$ holds.

A classical principle for structure theories is the following: Find axioms in order to characterize a «constructive» defined class of objects of the theory. So balanced functors seem to be an appropriate machinery developing a structure theory of additive categories. The «constructive» objects are full subcategories $dom\,C\,C(\mathfrak{U})$ of functor-categories. Axiomatic characterizations are obtained by finding conditions for the existence of a certain balanced set $\mathfrak{U}$ of functors.

2. A criterion for balanced $\mathfrak{B}$-modules.

We consider a small additive category $\mathfrak{B}$ and a set

$$\mathfrak{U}=\{\mathfrak{U}_i:\mathfrak{B}\to\mathfrak{Ab}\mid i\in I\}$$

of $\mathfrak{B}$-modules. Note that for every object $B\in|\mathfrak{B}|$ there is the natural

YONEDA-isomorphism

$$\Phi_B : H^{(H^B)} \xrightarrow{\sim} V_B \quad \text{defined by } \Phi_B(\mathfrak{U})(\xi) = \xi(B)(1_B) = X$$

for $\mathfrak{B}$-modules $\mathfrak{U}$ and natural transformations $\xi : H^B \to \mathfrak{U}$.

THEOREM 2. *The (faithful) set* $\mathfrak{U} = \{ \mathfrak{U}_i : \mathfrak{B} \to \mathfrak{Ab} \mid i \in I \}$ *of* $\mathfrak{B}$*-modules* $\mathfrak{U}_i$ *is weakly balanced if and only if for every* $B, B' \in |\mathfrak{B}|$ *every natural transformation between the restrictions* $H_{\mathfrak{U}}^{(H^B)}$ *and* $H_{\mathfrak{U}}^{(H^{B'})}$ *(on* $\mathfrak{U}(\mathfrak{Ab}^{\mathfrak{B}})$ *of* $H^{(H^B)}$ *and* $H^{(H^{B'})}$*) can be extended to a (unique) natural transformation between* $H^{(H^B)}$ *and* $H^{(H^{B'})}$. *Hence every* $\mathfrak{U}$ *containing all* $\mathfrak{B}$*-Hom-modules* $H^B : \mathfrak{B} \to \mathfrak{Ab}$ *(or, more generally, every* $\mathfrak{U}$ *defining a dense (see* [4], [15] *) subcategory in* $\mathfrak{Ab}^{\mathfrak{B}}$*) is balanced.*

PROOF. Let $f' : H^{(H^B)} \to H^{(H^{B'})}$ be a natural transformation and Φ'_B, resp. $\Phi'_{B'}$, the restriction on $\mathfrak{U}(\mathfrak{Ab}^{\mathfrak{B}})$ of Φ_B, resp. $\Phi_{B'}$. If $\mathfrak{U}$ is weakly balanced, then for

$$g' := \Phi'_{B'} \circ f' \circ \Phi'^{-1}_B : E(B) \to E(B')$$

there exists a $\beta : B \to B'$ such that $g' = E(\beta)$ holds. Then $g(\mathfrak{U}) = \mathfrak{U}(\beta)$ for any $\mathfrak{B}$-module $\mathfrak{U} : \mathfrak{B} \to \mathfrak{Ab}$ defines a natural transformation $g : V_B \to V_{B'}$, which extends g'. Hence $f := \Phi_{B'}^{-1} \circ g \circ \Phi_B$ is an extension of f'. Our condition is also sufficient. Since every $g' : E(B) \to E(B')$ is induced by a $g : V_B \to V_{B'}$, we must show the existence of a $\beta : B \to B'$ satisfying

$$g(\mathfrak{U}) = E(\beta)(\mathfrak{U}) = \mathfrak{U}(\beta) \quad \text{for every } \mathfrak{U} \in \mathfrak{U}$$

(i.e. with $g' = E(\beta)$). Let us now consider an element $x \in \mathfrak{U}(B)$ and the following abbreviations:

$$f := \Phi_{B'}^{-1} \circ g \circ \Phi_B, \quad \xi := \Phi_B^{-1}(\mathfrak{U})(x), \quad \mathfrak{S} := f(H^B)(1_{H^B}),$$
$$\beta := \mathfrak{S}(B')(1_{B'}) : B \to B'.$$

Then holds:

$$\begin{aligned} g(\mathfrak{U})(x) &= \Phi_{B'}(\mathfrak{U})(f(\mathfrak{U})(\xi)) = \Phi_{B'}(\mathfrak{U})(\xi \circ \mathfrak{S}) = \\ &= (\xi \circ \mathfrak{S})(B')(1_{B'}) = \xi(B')(\beta) = \mathfrak{U}(\beta)(\xi(B)(1_B)) = \\ &= \mathfrak{U}(\beta)(\Phi_B(\mathfrak{U})(\xi)) = \mathfrak{U}(\beta)(x) \end{aligned}$$

and hence $g(\mathcal{U})=\mathcal{U}(\beta)$ holds for every $\mathfrak{B}$-module $\mathcal{U}$. So an extension g, resp. f, of each g', resp. f', is unique if and only if $\mathfrak{U}$ is faithful. If now $\mathfrak{U}$ contains all $\mathfrak{B}$-*Hom*-modules $H^B:\mathfrak{B}\to\mathfrak{Ab}$ the evaluation E is full and $\mathfrak{U}$ faithful. Moreover the equations $\mathcal{U}=V_{\mathcal{U}}\circ I_{C(\mathcal{U})}\circ E$ for all $\mathcal{U}\in\mathfrak{U}$ show that E is faithful and (because of $\mathcal{U}=H^B$) injective (hence bijective) on objects. So E is an isomorphism. Using SCHUBERT [15] the last statement is routine. This completes the proof.

3. An Isomorphism Theorem.

We consider additive categories $\mathcal{C}$ with coproducts satisfying the following condition

(B) Let $\tau_j: C_j\to\coprod_{i\in I} C_i$ be the canonical coproduct morphisms, $\delta_{i,k}: C_i\to C_k$ the KRONECKER morphism and π_k defined by $\delta_{i,k}=\pi_k\circ\tau_i$ for $i\in I$. Then $\pi_k\circ f=\pi_k\circ g$ for all $k\in I$ and arbitrary parallel f,g implies always $f=g$.

Now let us state the following «Isomorphism Theorem»:

THEOREM 3. *Let $\mathfrak{B}$ be a small additive category and $\mathcal{C}$ an additive category with coproducts satisfying the condition* (B). *Moreover let us consider a set $\mathfrak{U}=\{\mathcal{U}_i:\mathfrak{B}\to\mathcal{C}\mid i\in I\}$ of parallel functors. Then there is an isomorphism T making the diagram*

$$\begin{tikzcd}
\mathfrak{B} \arrow[r, "E"] \arrow[dr, "E_{\coprod}"'] & dom\, C\, C(\mathfrak{U}) \\
& dom\, C\, C(\coprod_{i\in I}\mathcal{U}_i) \arrow[u, "T", "\wr"']
\end{tikzcd}$$

commutative.

PROOF: Let be $\delta_{i,k}:\mathcal{U}_i\to\mathcal{U}_k$ the KRONECKER morphism and π_k defined by $\delta_{i,k}=\pi_k\circ\tau_i$ for all $i\in I$. Denote $\mathcal{U}:=\coprod_{i\in I}\mathcal{U}_i$ and consider the following diagram:

$$\begin{tikzcd}
\mathfrak{U}_i(B) \arrow[rrr, "f_{i,j}"] \arrow[dr, "\tau_i(B)"] \arrow[ddd, dash, "\mathcal{S}_{i,k}(B)"'] & & & \mathfrak{U}_j(B') \arrow[dl, "\tau_j(B')"] \arrow[ddd, "\mathcal{S}_{j,l}(B)"] \\
& \mathfrak{U}(B) \arrow[r, "f"] \arrow[d, "\mathcal{S}(B)"] & \mathfrak{U}(B') \arrow[ur, "\pi_j(B')"] \arrow[d, "\mathcal{S}(B')"'] & \\
& \mathfrak{U}(B) \arrow[r] \arrow[dl, "\pi_k(B)"'] & \mathfrak{U}(B') \arrow[dr, "\pi_l(B')"] & \\
\mathfrak{U}_k(B) \arrow[ur, "\tau_k(B)"'] \arrow[rrr, "f_{k,l}"] & & & \mathfrak{U}_l(B')
\end{tikzcd}$$

Choosing $\mathcal{S}=\tau_i \circ \pi_i$ and $i=j=k\neq l$, one gets

$$\mathcal{S}_{i,l}:=\pi_l \circ \mathcal{S} \circ \tau_i = 0 \quad \text{and} \quad \mathcal{S}_{i,i}=1.$$

Hence for $f: E_{\amalg}(B) \to E_{\amalg}(B')$ we have

$$\begin{aligned}
f_{i,l} &:= \pi_l(B') \circ f \circ \tau_i(B) = \pi_l(B') \circ f \circ \mathcal{S}(B) \circ \tau_i(B) = \\
&= \pi_l(B') \circ \mathcal{S}(B') \circ f \circ \tau_i(B) = \pi_l(B') \circ \tau_i(B') \circ f_{i,i} = \\
&= \pi_l(B') \circ \mathcal{S}(B') \circ \tau_i(B') \circ f_{i,i} = \mathcal{S}_{i,l}(B') \circ f_{i,i} = 0.
\end{aligned}$$

Now $Tf(\mathfrak{U}_i)=f_{i,i}$ defines a morphism $Tf: E(B) \to E(B')$. By taking $\mathcal{S}=\tau_k \circ \mathcal{S}_{i,k} \circ \pi_i$ we have namely:

$$(\mathcal{S}) \qquad \pi_k(B') \circ \mathcal{S}(B') = \mathcal{S}_{i,k}(B') \circ \pi_i(B')$$

and (since $\mathfrak{U}(B)$ is a coproduct in $\mathcal{C}$ and $f_{i,l}=0$ holds for $i \neq l$)

$$(f) \qquad f_{k,k} \circ \pi_k(B) = \pi_k(B') \circ f.$$

Using $f \circ \mathcal{S}(B) = \mathcal{S}(B') \circ f$ and applying $(\mathcal{S})$ and (f) a «diagram-chasing» shows us what we wanted. Now let us define the functor

$$T: dom\, CC(\mathfrak{U}) \longrightarrow dom\, CC(\mathfrak{A})$$

in the following way. Each object of $dom\, CC(\mathfrak{U})$ has the form $E_{\amalg}(B)$ for $B \in |\mathfrak{B}|$. If $E_{\amalg}(B)=E_{\amalg}(B')$ holds, then $f=1$ shows $f_{i,i}=1$. Hence $T1=1$ and hence $T(E_{\amalg}(B))=E(B)$ is really a mapping of objects. Now for $f: E_{\amalg}(B) \to E_{\amalg}(B')$ define Tf as before. Then the condition (f) implies that T is an additive (on objects surjective) functor. We show that T is full and faithful. For an $f': E(B) \to E(B')$ let us write:

$$f_{i,i} = f'(\mathfrak{U}_i) \text{ and } f_{i,k} = 0 \quad (i \neq k).$$

Since $\mathfrak{S}_{i,l}(B') \circ f_{i,i} = f_{l,l} \circ \mathfrak{S}_{i,l}(B)$ and the condition (f) imply always

$$\pi_l(B') \circ \mathfrak{S}(B') \circ f \circ \tau_i(B) = \pi_l(B') \circ f \circ \mathfrak{S}(B) \circ \tau_i(B),$$

the morphism $f := \coprod_{i \in I} f_{i,i}$ belongs to $dom\, CC(\mathfrak{U})$ (trivially $f' = Tf$ holds). Then the condition (B) implies that T is full and obviously also faithful. Since $T^{-1}(1) = 1$ holds, the functor T is injective (hence bijective) on objects and so an isomorphism. This completes the proof.

REMARK 2. If $\mathcal{C} = \mathfrak{Ab}$ holds and if $\mathfrak{U}$ contains all $\mathfrak{B}$-Hom-modules $H^B : \mathfrak{B} \to \mathfrak{Ab}$, then $\mathfrak{U}$ and hence $\coprod \mathfrak{U}$ is balanced (Theorem 2 and Theorem 3). This immediately shows the existence of full embeddings of small additive categories $\mathfrak{B}$ into a category Mod_R of R-modules over a ring R (R is the ring of endomorphisms of $\coprod \mathfrak{U}$). Moreover it can be shown that $dom\, CC(\mathfrak{U})$ consists of cyclic, projective $[\mathfrak{U}, \mathfrak{U}]$-modules in the case of a free $\mathfrak{B}$-module $\mathfrak{U}$.

4. A generalized MORITA-Theorem.

A characterization of generators $\mathfrak{U}$ in $\mathfrak{Ab}^{\mathfrak{B}}$ ($\mathfrak{B}$ small) is given by

THEOREM 4 (see also FAITH [3]). *Let $\mathfrak{B}$ be a small additive category. Then the following conditions are equivalent:*

(1) $\mathfrak{U} : \mathfrak{B} \to \mathfrak{Ab}$ is a generator in $\mathfrak{Ab}^{\mathfrak{B}}$.

(2) $\mathfrak{U}$ is balanced and the centralizer $C(\mathfrak{U})$ consists of finitely generated projective $[\mathfrak{U}, \mathfrak{U}]$-modules.

PROOF: (1) $\Longrightarrow$ (2): Since $\{H^B \mid B \in |\mathfrak{B}|\}$ is balanced by Theorem 3, also $\coprod_{B \in |\mathfrak{B}|} H^B$ is balanced. Since $\mathfrak{U}$ is a generator there are epimorphisms $\varepsilon_B : \mathfrak{U}^{I(B)} \twoheadrightarrow H^B$. As is well known the morphisms $\varepsilon_B(B')$ are epimorphic (i.e. surjective) in $\mathfrak{Ab}$ for all $B, B' \in |\mathfrak{B}|$. Then the YONE-

DA-Lemma implies that always ε_B and hence

$$\varepsilon = \coprod_{B \in |\mathfrak{B}|} \varepsilon_B : \mathfrak{U}^I \rightleftarrows \coprod_{B \in |\mathfrak{B}|} H^B$$

is a retract. Let us also note that for each $B \in |\mathfrak{B}|$ the copower $I(B)$ can be choosen as a natural number (each element of an $\mathfrak{Ab}$-coproduct has only finitely many components unequal zero!). Now clearly

$$\mathfrak{U}^I \simeq \coprod_{B \in |\mathfrak{B}|} H^B \oplus W$$

holds, because this is true «pointwise». Since $\{W\} \cup \{H^B \mid B \in |\mathfrak{B}|\}$ is balanced by Theorem 3, we conclude that $\mathfrak{U}^I$ and hence that $\mathfrak{U}$ is balanced. Now let us consider a decomposition $\mathfrak{U}^n \simeq H^B \oplus W_B$ for a $B \in |\mathfrak{B}|$ and a natural number n. Denoting $S := [\mathfrak{U}, \mathfrak{U}]$ then in $\mathfrak{Ab}$ holds:

$$S^n = [\mathfrak{U}, \mathfrak{U}]^n \simeq [\mathfrak{U}^n, \mathfrak{U}] \simeq [H^B \oplus W_B, \mathfrak{U}] \simeq [H^B, \mathfrak{U}] \oplus [W_B, \mathfrak{U}] .$$

By left-composing with $\sigma \in S$ every such abelian group can be considered as an S-module and moreover all such $\mathfrak{Ab}$-isomorphisms can be considered as Mod_S-isomorphisms. By the YONEDA-Lemma we have the $\mathfrak{Ab}$-isomorphism

$$[H^B, \mathfrak{U}] \simeq \mathfrak{U}(B) \text{ with } f \mapsto f(1_B).$$

Again by left-composing with $\sigma \in S$ the groups $[H^B, \mathfrak{U}]$ and $\mathfrak{U}(B)$ can be considered as S-modules and the above $\mathfrak{Ab}$-isomorphism $f \mapsto f(1_B)$ can also be considered as Mod_S-isomorphism. Hence $[\mathfrak{U}(B)]_S \in C(\mathfrak{U})$ is finitely generated projective. Now we prove $(2) \Longrightarrow (1)$: Let be $[\mathfrak{U}(B)]_S \in C(\mathfrak{U})$ a finitely generated projective S-module. Then there is a natural number n and a Mod_S-isomorphism $S^n \simeq [\mathfrak{U}(B)]_S \oplus N_S$. In $\mathfrak{Ab}$ we have:

$$\begin{aligned}\mathfrak{U}^n(B') = [\mathfrak{U}(B')]^n &\simeq [S, [\mathfrak{U}(B')]_S]^n \simeq [S^n, [\mathfrak{U}(B')]_S] \\ &\simeq [[\mathfrak{U}(B)]_S, [\mathfrak{U}(B')]_S] \oplus [N_S, [\mathfrak{U}(B')]_S] .\end{aligned}$$

Obviously all $\mathfrak{Ab}$-isomorphisms are natural in B'. Since $\mathfrak{U}$ is balanced we have for each $B' \in |\mathfrak{B}|$ the $\mathfrak{Ab}$-isomorphism:

$$\mathfrak{U}^n(B') \simeq H^B(B') \oplus [N_S, [\mathfrak{U}(B')]_S]$$

which is natural in B'. Hence $\mathfrak{U}^n \simeq H^B \oplus W_B$ holds. Since every M in $\mathfrak{Ab}^{\mathfrak{B}}$ is an epimorph of a free $\mathfrak{B}$-module (i.e. a coproduct of certain $\mathfrak{B}$-*Hom*-modules $H^B, H^{B'}, \ldots$), we conclude that $\mathfrak{U}$ is a generator in $\mathfrak{Ab}^{\mathfrak{B}}$. This completes the proof.

REMARK 3. Of special interest are conditions such that $\mathfrak{U} = H^B : \mathfrak{B} \to \mathfrak{Ab}$ is balanced. GABRIEL-POPESCU's Theorem is an example for such a situation. Applying Theorem 4, we state further examples (simple categories $\mathfrak{B}$) later on, specially in 13. Note that for the full subcategory $\mathfrak{B}'$ (of an additive category $\mathfrak{B}$) consisting of the retracts, resp. coretracts, of an object $B \in |\mathfrak{B}|$ the functor H^B is a generator in $\mathfrak{Ab}^{\mathfrak{B}'}$, and hence balanced. So additive categories $\mathfrak{B}$ are always retract-colocally small, resp. coretract-locally small.

II. SEMISIMPLE $\mathfrak{B}$-MODULES.

5. Simple and semisimple objects.

An object $0 \not\simeq D_0$ of an additive category $\mathfrak{C}$ is called *simple* (or *irreducible*) if only the trivial subobjects (i.e. $0_{D_0}, 1_{D_0}$) exist. A coproduct $D = \coprod_{i \in I} D_i$ of simple objects D_i (with coproduct morphisms $d_i : D_i \to D$) is called *semisimple* (or completely reducible).

Let us first consider the case $\mathfrak{C} = \mathfrak{Ab}^{\mathfrak{B}}$ ($\mathfrak{B}$ small). If the $\mathfrak{B}$-*Hom*-module $H^B : \mathfrak{B} \to \mathfrak{Ab}$ is semisimple, then even $H^B \simeq \bigoplus_{i=1}^{n} Q_i$ holds for a natural number n and simple $\mathfrak{B}$-modules Q_i (because of the YONEDA-lemma and the $\mathfrak{Ab}$-coproducts !). Since (again the YONEDA-Lemma) a simple $\mathfrak{B}$-module Q with $Q(B) \neq 0$ is an epimorph of H^B, there is only a set of (non-isomorphic) simple $\mathfrak{B}$-modules. We shall call now an additive category $\mathfrak{B}$ *artinian* if every H^B $(B \in |\mathfrak{B}|)$ is an artinian object in $\mathfrak{Ab}^{\mathfrak{B}}$ (and not every $B \in |\mathfrak{B}|$ in $\mathfrak{B}$!).

Then ARTIN-WEDDERBURN's Theorem is true, namely:

THEOREM 5. *For a small additive category $\mathfrak{B}$ the following conditions are equivalent:*

(1) For each $B \in |\mathfrak{B}|$ the $\mathfrak{B}$-Hom-module $H^B : \mathfrak{B} \to \mathfrak{Ab}$ is (finitely) semisimple.

(2) The set $\{ Q_i \mid i \in I \}$ of non-isomorphic simple $\mathfrak{B}$-modules Q_i is balanced (and $\mathfrak{B}$ is artinian).

PROOF. Obviously (1) implies that $\mathfrak{B}$ is artinian. Since by (1) every H^B is a direct summand and hence epimorph of a copower of $\mathfrak{U} = \coprod_{i \in I} Q_i$ we conclude that $\mathfrak{U}$ is a generator. Hence by Theorem 4 and Theorem 3 the set $\{ Q_i \mid i \in I \}$ is balanced. Conversely by the so called SCHUR-Lemma (see also 7), the $[Q_i , Q_i]$ are divisionrings and for $i \neq j$ clearly $[Q_i , Q_j] = 0$ holds. Since by (2) the category

$$\mathfrak{B} \simeq dom\, C\, C \{ Q_i \mid i \in I \}$$

is obviously isomorphic to a full subcategory of a product category of vectorspace categories $Vec_{[Q_i, Q_i]}$, condition (1) follows immediately (see also 7, Lemma 2). This completes the proof.

Now let us consider an arbitrary additive category $\mathcal{C}$ having pullbacks (hence also finite biproducts!) and consider the diagram:

$$\begin{tikzcd}
B \cap C \arrow[rr, "g_C"] \arrow[dd, "g_B"'] \arrow[dr, "g"] & & C \arrow[dl, shift left, "\tau_C"] \arrow[dd, "\gamma"] \\
& B \oplus C \arrow[ur, shift left, "p_C"] \arrow[dl, shift left, "p_B"] \arrow[dr, "f"] & \\
B \arrow[ur, shift left, "\tau_B"] \arrow[rr, "\beta"'] & & D
\end{tikzcd}$$

Then obviously $f := \beta p_B + \gamma p_C$ is a monomorphism if and only if

$$Ker f = g := \tau_B g_B - \tau_C g_C = 0$$

(i.e. $B \cap C = 0$) holds.

Given now a semisimple object $D = \coprod_I D_i$ and a monomorphism $\beta : B \to D$, for $J \subseteq I$ let be $\delta_J : \coprod_J D_j \to D$ the canonical morphism induced by the $d_j : D_j \to D$ $(j \in J)$. If $J \subset I$ is maximal with the property that the canonical morphism

$$f : B \oplus \coprod_J D_j \to D$$

induced by β and δ_j is a monomorphism, then f factorizes over each $d_i = f \circ d_i'$. For $i \in J$ this is clear. For $i \in I - J$ the pullback of f and d_i must not be zero (because of the choice of J !), hence it must be isomorphic to D_i. So f is finally a monomorphic retract and so an isomorphism. Hence B is a direct (co-) factor of D.

Summarizing we have the following

THEOREM 6. *Let $\mathcal{C}$ be an additive category with pullbacks. Then for a semisimple object $D = \coprod_{i \in I} D_i$ being only a finite coproduct of simple objects D_i every subobject is a direct (co-) factor (since the existence of an above maximal $J \subset I$ is clear!). If the index set I is not finite one can get the same result using an «AB5-like» condition* (by ZORN's-Lemma again the existence of an above maximal $J \subset I$ can be established!) (see also [2], [17], etc..).

6. A generalized BOURBAKI density theorem.

Let us begin with the following

LEMMA 1. *Let $\mathfrak{B}$ be a small additive category and $\mathcal{C}$ an additive category with (finite) coproducts. Furthermore let be*

$$\overline{\mathfrak{B}} := dom\, CC(P \oplus Q) \quad and \quad \overline{P \oplus Q} \in CC(P \oplus Q)$$

for additive functors $P, Q : \mathfrak{B} \to \mathcal{C}$ and $\tau : P \rightarrowtail P \oplus Q$ resp. $\pi : P \oplus Q \twoheadrightarrow P$ the canonical injection, resp. projection. Denoting by $E : \mathfrak{B} \to \overline{\mathfrak{B}}$ the evaluation of $P \oplus Q$, then by:

$$P_0 EB := P(B) \overset{\tau_B}{\rightarrowtail} (P \oplus Q)(B) = \overline{P \oplus Q}\, EB$$

(as natural morphism), a unique subfunctor P_0 of $\overline{P \oplus Q} : \overline{\mathfrak{B}} \to \mathcal{C}$ is determined.

PROOF: For an $f : EB \to EB'$ belonging to $\overline{\mathfrak{B}}$ we have

$$\tau_{B'} \circ \pi_{B'} \circ \overline{P \oplus Q}(f) = \overline{P \oplus Q}(f) \circ \tau_B \circ \pi_B .$$

Then obviously by $P_0(f) := \pi_{B'} \circ P \oplus Q(f) \circ \tau_B$ a subfunctor P_0 is determined. Now consider the category Mod_R of R-modules. Each set $Hom_R(X, Y)$ becomes a topological space defining for $f: X \to Y$ and finitely many elements $x_1, x_2, \dots, x_n \in X$ a base-neighbourhood by

$$<f \mid x_1, x_2, \dots, x_n> = \{ g: X \to Y \mid g(x_i) = f(x_i), i = 1, \dots, n \}.$$

This so called «finite topology» makes Mod_R to a topological category (note using elements $x_1, \dots, x_n$ by a similar procedure arbitrary categories can be topologized!). Now a subcategory $\mathcal{C}$ of Mod_R is (topological-) dense, if always $Hom_{\mathcal{C}}(X, Y)$ is a dense subspace of $Hom_R(X, Y)$. Then we can state a generalized BOURBAKI density theorem namely:

THEOREM 7. *Let $\mathfrak{U} = \coprod \mathfrak{U}_i : \mathfrak{B} \to \mathfrak{Ab}$ be a semisimple (on objects injective) $\mathfrak{B}$-module. Then the additive category $E(\mathfrak{B})$ is dense in $domCC(\mathfrak{U})$ resp. $Mod_{[\mathfrak{U}, \mathfrak{U}]}$ (E is the evaluation of $\mathfrak{U}$!).*

PROOF. We must show that always $Hom_{E(\mathfrak{B})}(EB, EB')$ is dense in $Hom_{[\mathfrak{U}, \mathfrak{U}]}(EB, EB')$. Since the additive functor E is injective on objects, we see that $E(\mathfrak{B})$ is really an additive category.. Let us take now an $f: EB \to EB'$ belonging to $domCC(\mathfrak{U})$ and elements

$$x_1, \dots, x_n \in \mathfrak{U}(B) = \overline{\mathfrak{U}} EB \quad \text{with} \quad \overline{\mathfrak{U}} \in CC(\mathfrak{U}).$$

Consider furthermore the n-th «pointwise» copower $\mathfrak{U}^n$ of $\mathfrak{U}$. Then $(x_1, \dots, x_n) \in \mathfrak{U}^n(B)$ holds. Using the isomorphism T^{-1} of Theorem 3, we see that the n-th copower f^n of f belongs to $domCC(\mathfrak{U}^n)$. Obviously by

$$S(B'') = \{(\mathfrak{U}\beta x_1, \dots, \mathfrak{U}\beta x_n) \mid \beta \in Hom_{\mathfrak{B}}(B, B'')\}$$

a $\mathfrak{B}$-submodule S of $\mathfrak{U}^n$ is determined. Since $\mathfrak{U}$ is semisimple so is $\mathfrak{U}^n$, hence S is a direct summand of $\mathfrak{U}^n$ by Theorem 6. Now Lemma 1 implies that the $S(B'')$'s also determine a $domCC(\mathfrak{U}^n)$-submodule of $\overline{\mathfrak{U}}^n \in CC(\mathfrak{U}^n)$. Hence also

$$\overline{\mathfrak{U}}^n f^n(x_1, \dots, x_n) = (\overline{\mathfrak{U}} f x_1, \dots, \overline{\mathfrak{U}} f x_n) \in S(B').$$

So there must be a $\beta: B \to B'$ satisfying

$$\overline{\mathfrak{U}} E(\beta) x_i = \mathfrak{U}(\beta) x_i = \overline{\mathfrak{U}} f x_i \quad \text{for} \quad i = 1, \dots, n.$$

Hence $E(\mathfrak{B})$ is dense in $dom\, CC(\mathfrak{U})$ resp. $Mod_{[\mathfrak{U},\mathfrak{U}]}$ and the proof is complete.

7. Simple $\mathfrak{B}$-modules.

For a small additive category $\mathfrak{B}$ and a simple $\mathfrak{B}$-module $Q: \mathfrak{B} \to \mathfrak{Ab}$ the very well known SCHUR-Lemma states that $dom\, C(Q) = [Q, Q]$ is a divisionring. Since $C(Q)$ contains a (faithful) $[Q, Q]$-space which is not the zero-space, the injective ring homomorphism

$$[Q,Q] \to [Q(B), Q(B)] \quad (\text{for a certain } B \in |\mathfrak{B}|)$$

shows that $[Q, Q]$ is also a (small) divisionring even if $\mathfrak{B}$ is not small.

Now let us characterize simple $\mathfrak{B}$-modules as follows:

LEMMA 2. *Let $\mathfrak{B}$ be a (not necessarily small) additive category and $Q: \mathfrak{B} \to \mathfrak{Ab}$ a $\mathfrak{B}$-module. Then the following conditions are equivalent:*

(1) Q is simple.

(2) $Q \not\cong 0$ and for arbitrary $B, B' \in |\mathfrak{B}|$ and arbitrary elements $0 \neq n_B \in Q(B)$, $n_{B'} \in Q(B')$ there is a $\beta: B \to B'$ with $Q(\beta)(n_B) = n_{B'}$.

(3) $Q \not\cong 0$ and for each $B \in |\mathfrak{B}|$ with $Q(B) \not\cong 0$ there is a maximal $\mathfrak{B}$-submodule S^B of H^B such that the (on objects injective) simple factor $\mathfrak{B}$-module H^B/S^B is isomorphic to Q.

PROOF. By the additivity of Q obviously

$$\{ Q(\beta)(n_B) \mid \beta: B \to B' \}$$

is a subgroup of $Q(B')$. Since for $\beta': B' \to B''$ also

$$Q(\beta')(Q(\beta)(n_B)) = Q(\beta' \circ \beta)(n_B)$$

holds, we have

$$Q(\beta')(S(B')) \subset S(B'').$$

Hence S is a $\mathfrak{B}$-submodule of Q. Since $0 \neq n_B \in S(B)$, i.e. $S \not\cong 0$ holds, (1) implies (2). The converse is evident. Now let Q be a simple $\mathfrak{B}$-module with $Q(B) \not\cong 0$ for a certain $B \in |\mathfrak{B}|$. The YONEDA-Lemma assures us the existence of a natural transformation $0 \neq \tau: H^B \to Q$. Then it is clear that $S^B = Ker\,\tau$ is maximal and that $Q \simeq H^B/S^B$ holds. Conversely $Q = H^B/S^B$ is simple if S^B is maximal. Hence (1) $\Longleftrightarrow$ (3) is true. Since $H^B(B_1)$ and $H^B(B_2)$ are disjoint $(B_1 \neq B_2)$ every factor-$\mathfrak{B}$-module H^B/S^B must be injective on objects. This completes the proof of Lemma 2.

Now the condition (2) in Lemma 2 leads to two further lemmas (Lemma 3, Lemma 4), which are useful in order to investigate the bicentralizer-property of a simple $\mathfrak{B}$-module Q independently of Theorem 7. The following notations are used: For $B \in |\mathfrak{B}|$ suppose $\mu_B = V_B \circ I_Q$ in $C(Q)$ with the underlying group $Q(B)$. Moreover S^B is the underlying group of a $[Q,Q]$-subspace of μ_B. The finitely many elements n_B^1, ..., $n_B^i \in Q(B)$ are called $[Q,Q]$-linear independent modulo S^B if

$$\sum_{j=1}^{i} \lambda_j n_B^j \in S^B \quad \text{with} \quad \lambda_1, \ldots, \lambda_i \in [Q,Q]$$

always implies $\lambda_1 = \ldots = \lambda_i = 0$. For a $\mathfrak{B}$-subfunctor (cosieve) N^B of $H^B : \mathfrak{B} \to \mathfrak{Ab}$ we also consider

$$(0 : N^B) = \{ m \in Q(B) \mid Q(N^B)(m) = 0 \}$$

being obviously the underlying group of a $[Q,Q]$-subspace of μ_B.

LEMMA 3. *Let Q be a simple $\mathfrak{B}$-module and N^B a $\mathfrak{B}$-submodule (cosieve) of $H^B : \mathfrak{B} \to \mathfrak{Ab}$ for a $B \in |\mathfrak{B}|$. Furthermore let $n_B^1, \ldots, n_B^i \in Q(B)$ be $[Q,Q]$-linear independent modulo $(0 : N^B)$ and define the cosieve N_{i-1}^B by:*

$$N_{i-1}^B(B') = \{ \beta : B \to B' \mid Q(\beta)(n_B^1) = \ldots = Q(\beta)(n_B^{i-1}) = 0 \}.$$

Then for each $B' \in |\mathfrak{B}|$ and each $n_{B'} \in Q(B')$ there is a

$$\beta \in N_{i-1}^B(B') \cap N^B(B')$$

such that $Q(\beta)(n_B^i) = n_{B'}$ holds.

PROOF: We proceed by induction. For $i = 1$ we have $N_0^B = H^B$. Hence $N_0^B \cap N^B = N^B$ holds. Because of $n_B^1 \notin (0 : N^B)$ there is a $\bar{\beta} : B \to \bar{B}$ in $N^B(\bar{B})$ satisfying $Q(\bar{\beta})(n_B^1) \neq 0$. By Lemma 2 (2) we have a $\beta : B \to B'$ in $N^B(B')$ such that $Q(\beta)(n_B^1) = n_{B'}$ holds. So the case $i = 1$ is clear.

Using again Lemma 2 (2) for a $B' \in |\mathfrak{B}|$ with $Q(B') \neq 0$, we only must verify the existence of a

$$\beta \in N^B_{i-1}(B') \cap N^B(B') \quad \text{with} \quad Q(\beta)(n^i_B) \neq 0.$$

Assume the contrary. Then for each $B' \in |\mathfrak{B}|$ and each morphism

$$\beta \in N^B_{i-2}(B') \cap N^B(B') \quad \text{with} \quad Q(\beta)(n^{i-1}_B) = 0$$

always $Q(\beta)(n^i_B) = 0$ follows. By the induction hypothesis for each $n_{B'} \in Q(B')$ there is a

$$\beta_0 \in N^B_{i-2}(B') \cap N^B(B') \quad \text{with} \quad Q(\beta_0)(n^{i-1}_B) = n_{B'}.$$

So for each $B' \in |\mathfrak{B}|$ and each $\beta \in N^B_{i-2}(B') \cap N^B(B')$ the correspondence $Q(\beta)(n^{i-1}_B) \mapsto Q(\beta)(n^i_B)$ is an endomorphism

$$\tau(B') : Q(B') \longrightarrow Q(B')$$

of the abelian group $Q(B')$. Since obviously for each $\beta'' : B' \to B''$ the diagram:

$$\begin{tikzcd}
Q(B') \arrow[r, "\tau(B')"] \arrow[d, "Q(\beta'')"'] & Q(B') \arrow[d, "Q(\beta'')"] \\
Q(B'') \arrow[r, "\tau(B'')"'] & Q(B'')
\end{tikzcd}$$

commutes, $\tau \in [Q, Q]$ holds (i.e. τ is a natural transformation of Q !). So we have

$$Q(\beta)(n^i_B) = \tau(B')\,[Q(\beta)(n^{i-1}_B)] = Q(\beta)\,[\tau(B)(n^{i-1}_B)],$$

hence also

$$Q(\beta)\,[n^i_B - \tau(B)(n^{i-1}_B)] = 0$$

for each β in $N^B_{i-2}(B') \cap N^B(B')$ and each $B' \in |\mathfrak{B}|$. Since there is a $B' \in |\mathfrak{B}|$ with $Q(B') \neq 0$ (e.g. $Q(B) \neq 0$), by the induction hypothesis we have

$$\langle n^1_B, \ldots, n^{i-2}_B, (n^i_B - \tau(B)(n^{i-1}_B)) \rangle$$

and hence also $\langle n^1_B, \ldots, n^i_B \rangle$ must be $[Q, Q]$-linear dependent modulo $(0 : N^B)$. By this contradiction the proof is complete.

LEMMA 4. *Let Q be a simple $\mathfrak{B}$-module and N^B a $\mathfrak{B}$-submodule (cosieve) of $H^B: \mathfrak{B} \to \mathfrak{Ab}$ for a $B \in |\mathfrak{B}|$. Furthermore let $n_B^1, \ldots, n_B^i \in Q(B)$ be $[Q, Q]$-linear independent modulo $(0 : N^B)$. Then for each $B' \in |\mathfrak{B}|$ and arbitrary elements $q^1, \ldots, q^i \in Q(B')$ there is a $\beta : B \to B'$ in $N^B(B')$ such that $Q(\beta)(n_B^j) = q_j$ $(j = 1, 2, \ldots, i)$ holds.*

PROOF: Considering $N^B_{i-1,k}$ defined by:

$$N^B_{i-1,k}(B') = \{\beta : B \to B' \mid Q(\beta)(n_B^j) = 0; \ \ k \neq j \in \{1, 2, \ldots, i\}\},$$

Lemma 3 gives us a

$$\beta_k \in N^B_{i-1,k}(B') \cap N^B(B') \quad \text{with} \quad Q(\beta_k)(n_B^k) = q_k.$$

Hence

$$N^B(B') \ni \beta = \beta_1 + \ldots + \beta_i$$

has the desired property and the proof is complete.

REMARK 4. For the important case $N^B = H^B$ obviously our Lemma 3 and Lemma 4 follow by Theorem 7. In spite of that the above separate consideration seems to be useful.

III. STRUCTURE THEORY

8. Primitive categories.

A (not necessarily small) additive category $\mathfrak{B}$ is called *primitive* if there is a faithful simple $\mathfrak{B}$-module $Q : \mathfrak{B} \to \mathfrak{Ab}$. Let us mention that together with $\mathfrak{B}$ each equivalent category is primitive and that together with $Q : \mathfrak{B} \to \mathfrak{Ab}$ each isomorphic $\mathfrak{B}$-module is faithful resp. simple (but not necessarily injective on objects!). Moreover let us note that LEDUC's definition of the primitivity (see [10], [11]) is equivalent to our definition above.

Now we state the following density Theorem describing the structure of primitive categories, namely:

THEOREM 8. *Let $\mathfrak{B}$ be a (not necessarily small) additive category. Then the following conditions are equivalent:*

(1) $\mathfrak{B}$ is primitive.

(2) $\mathfrak{B}$ is isomorphic to a dense additive subcategory ($\neq 0$) of a vectorspace category Vec_K.

PROOF. (2) $\Longrightarrow$ (1): The restriction $Q:\mathfrak{B}\to\mathfrak{Ab}$ on $\mathfrak{B}$ of the forgetful functor $Vec_K\to\mathfrak{Ab}$ is a faithful $\mathfrak{B}$-module. Since $Q\not\simeq 0$ holds and since $\mathfrak{B}$ is a dense subcategory of Vec_K, we have for each $0\neq n_B\in Q(B)$, $n_{B'}\in Q(B')$ even a $\mathfrak{B}$-morphism $\beta: B\to B'$ with $Q(\beta)(n_B)=n_{B'}$. By Lemma 2 (2) then simplicity of Q follows. Hence $\mathfrak{B}$ is primitive.

(1) $\Longrightarrow$ (2): Now let $\mathfrak{B}$ be primitive and $Q:\mathfrak{B}\to\mathfrak{Ab}$ (without loss of generality!) an on objects injective, faithful, simple $\mathfrak{B}$-module (see also Lemma 2 (3)). Trivially $Q(\mathfrak{B})$ is an additive subcategory (isomorphic to $\mathfrak{B}$) of $\mathfrak{Ab}$. Hence $E(\mathfrak{B})$ is an additive subcategory (isomorphic to $\mathfrak{B}$) of $Vec_{[Q,Q]}$ (E is the evaluation of Q). Since Q is an embedding (i.e. faithful and injective on objects) so is E which is moreover by Theorem 1 surjective (hence bijective) on objects. We show that E is dense, i.e. that $E(\mathfrak{B})$ is a dense subcategory of $dom\,CC(Q)$ resp. $Vec_{[Q,Q]}$ (relative to the finite topology!). Suppose $f: EB\to EB'$ is a $[Q,Q]$-linear mapping and $\langle f\,|\,n_B^1,\dots,n_B^i\rangle$ a base-neighbourhood. Then by Lemma 4 (for $N^B=H^B$) or by Theorem 7 there is a $\beta: B\to B'$ satisfying

$$Q(\beta)(n_B^j)=f(n_B^j)\ \text{ for }\ j=1,2,\dots,i.$$

This completes the proof.

REMARK 5. Let $\mathfrak{B}$ be a primitive category. Then Lemma 2 (2) implies that each non-zero object $B\in|\mathfrak{B}|$ is a generator. By Theorem 8 one easily sees now that each non-zero object $B\in|\mathfrak{B}|$ is also a cogenerator. Each full subcategory $\mathfrak{B}'$ ($\neq 0$) of $\mathfrak{B}$ (hence also the ring $Hom_{\mathfrak{B}}(B,B)$ for each non-zero object $B\in|\mathfrak{B}|$) is again primitive by Lemma 2 (2).

Let us mention that for a simple $\mathfrak{B}$-module Q there is a GALOIS correspondence between the cosieves and the subspaces of a $[Q,Q]$-space $\mu_B=EB$ in $dom\,CC(Q)$. By the well known «annihilating principle» $S^B\subset Q(B)$ corresponds to

$$(0:S_B) = \{ f \in H^{EB} \mid f(S_B) = 0 \}$$

and $N^{EB} \subset H^{EB}$ corresponds to

$$(0:N^{EB}) = \{ m \in Q(B) \mid N^{EB}(m) = 0 \}.$$

The proof of the following result is quite similar as in [1] , p. 43 . We have namely:

THEOREM 9. *Let $Q: \mathfrak{B} \to \mathfrak{Ab}$ be a simple $\mathfrak{B}$-module and N^B a $\mathfrak{B}$-submodule (cosieve) of $H^B: \mathfrak{B} \to \mathfrak{Ab}$ for a $B \in |\mathfrak{B}|$. Then for each $B' \in |\mathfrak{B}|$ the closure (relative to the finite topology!) of*

$$Q(N^B(B')) = Q(N^B)[Q(B')]$$

is

$$(0:(0:Q(N^B)))[Q(B')].$$

REMARK 6. If $dom\,CC(Q)$ contains only finite-dimensional $[Q,Q]$-spaces, by Theorem 9 and Lemma 3 we conclude that the above GALOIS correspondence is strict, i.e. that

$$(0:(0:S_B)) = S_B \text{ and } (0:(0:N^{EB})) = N^{EB}$$

hold.

9. Primitive ARTIN-categories.

The previous Remark 6 also gives us a foundation for a categorical, i.e. «axiomatic» characterization of (primitive) additive categories being isomorphic to full subcategories ($\neq 0$) of finite-dimensional vectorspaces of Vec_K, i.e. to certain «constructive» defined categories (see also Corollary 6).

Now we can state the following

THEOREM 10. *For a (not necessarily small) additive category $\mathfrak{B}$ ($\neq 0$) the following conditions are equivalent:*

(1) $\mathfrak{B}$ is isomorphic to a full subcategory of finite-dimensional vectorspaces of Vec_K for a divisionring K.

(2) $\mathfrak{B}$ is simple (i.e. $\mathfrak{B}$ has only the trivial ideals) and artinian.

(3) $\mathfrak{B}$ is artinian and each non-zero object $B \in |\mathfrak{B}|$ is a generator

and a cogenerator. Moreover the endomorphism ring $Hom_{\mathfrak{B}}(B, B)$ *of a (hence of each) non-zero object* $B \in |\mathfrak{B}|$ *is simple (artinian).*

(4) $\mathfrak{B}$ *is a primitive* ARTIN-*category.*

So any primitive (simple) ARTIN-*category* $\mathfrak{B}$ *has a small (even countable) skeleton and the divisionring* K *determined by (1) is unique, up to isomorphisms.*

PROOF. $(1) \Longrightarrow (2)$: The canonical forgetful functor $Q: \mathfrak{B} \to \mathfrak{Ab}$ is faithful and simple. Hence $\mathfrak{B}$ is primitive. Now $[Q, Q]$ is a divisionring satisfying $K \subset [Q, Q]$. Considering a K-space $0 \neq B \in |\mathfrak{B}|$ by Remark 2 immediately $K = [Q, Q]$ follows. Hence $B = \mu_B$ holds for each B of $|\mathfrak{B}|$. So Q is balanced and $\mathfrak{B}$ artinian by Remark 6 (using the strict GALOIS correspondence!). Consider now an ideal $I \neq 0$ of $\mathfrak{B}$. Using the generator and cogenerator property of the non-zero objects $B \in |\mathfrak{B}|$, there is a $0 \neq \beta: B \to B$ in I. Since (as is well known) $Hom_{\mathfrak{B}}(B, B)$ is simple (artinian),

$$I(B, B) = Hom_{\mathfrak{B}}(B, B), \text{ hence } I = Hom_{\mathfrak{B}}(\cdot, \cdot)$$

follows. So $\mathfrak{B}$ is simple. Obvious is $(1) \Longrightarrow (3)$. Now let us prove $(2) \Longrightarrow (4)$. Consider a minimal cosieve $0 \not\subset N^B \subset H^B$ for a non-zero object $B \in |\mathfrak{B}|$. Clearly N^B is simple and injective on objects. By the simplicity of $\mathfrak{B}$ it is clear that N^B is faithful. Hence $\mathfrak{B}$ is primitive and (4) is true. $(3) \Longrightarrow (4)$: Let $B \in |\mathfrak{B}|$ be a non-zero object having a simple endomorphism ring $Hom_{\mathfrak{B}}(B, B)$ and consider a minimal cosieve $0 \not\subset N^B \subset H^B$. Clearly there is a $B' \in |\mathfrak{B}|$ with $N^B(B') \neq 0$. Since every non-zero object $B'' \in |\mathfrak{B}|$ is a cogenerator, we have immediately

$$N^B(B'') \neq 0 \text{ for every non-zero object } B'' \in |\mathfrak{B}|.$$

As before N^B is simple and even injective on objects. Now we shall show that N^B is also faithful. So let us consider a $0 \neq f: B_1 \to B_2$. We must assure the existence of a $\beta \in N^B(B_1)$ with $f \circ \beta \neq 0$. Since B_1 is a cogenerator, there is a $0 \neq g: B_2 \to B_1$ with $g \circ f \neq 0$. Since B is a cogenerator and a generator, there is a $0 \neq h: B_1 \to B$ with $h \circ g \circ f \neq 0$ and

$$0 \neq f': B \to B_1 \text{ with } 0 \neq x = h \circ g \circ f \circ f': B \to B.$$

Now $N^B(B)$ is a left ideal $(\neq 0)$ in $Hom_{\mathfrak{B}}(B, B)$. Assume

$$x \circ N^B(B) = 0.$$

Then the ideal I left-annihilating $N^B(B)$ is unequal zero. On the other hand (by simplicity of $Hom_{\mathfrak{B}}(B,B)$), $I = Hom_{\mathfrak{B}}(B,B)$ is impossible, since $1_B \circ N^B(B) \neq 0$ holds. Hence $x \circ N^B(B) \neq 0$ holds and so there is a $\beta' \in N^B(B)$ satisfying $x \circ \beta' \neq 0$. Then $\beta = f' \circ \beta' \in N^B(B_1)$ and $f \circ \beta \neq 0$ is true. Hence N^B is a simple embedding and $\mathfrak{B}$ is primitive. So (4) is true. (4) $\Longrightarrow$ (1): Let $Q: \mathfrak{B} \to \mathfrak{Ab}$ be a simple embedding. Consider the underlying group S_B of a finite-dimensional subspace of the $[Q,Q]$-space μ_B (with the underlying group $Q(B)$) for a non-zero object $B \in |\mathfrak{B}|$ and the equation

$$Q(N^B) = (0 : S_B) \cap Q(\mathfrak{B}).$$

Obviously N^B is a $\mathfrak{B}$-submodule of H^B. By Lemma 3 each proper ascending chain of subspaces resp. the underlying chain of abelian groups $S_B = S_B^1 \subset S_B^2 \subset \ldots$ corresponds with a proper descending chain of cosieves $N^B = N_1^B \supset N_2^B \supset \ldots$ which has only a finite length. So μ_B must be finite-dimensional and Q balanced by Theorem 8. Hence (1) is true. If an (arbitrary) primitive category $\mathfrak{B}$ has a minimal cosieve $L^{B_o} \not\simeq 0$ for a faithful simple $\mathfrak{B}$-module Q, we have $Q(B_o) \neq 0$ since Q is faithful and $B_o \neq 0$ holds. Then for a $0 \neq \lambda \in L^{B_o}(B')$ there is a $0 \neq u \in Q(B_o)$ satisfying $Q(\lambda)(u) \neq 0$, since Q is faithful. Hence

$$L^{B_o}(B) \ni \beta \mapsto Q(\beta)(u) \in Q(B)$$

causes a natural transformation $0 \neq \tau : L^{B_o} \to Q$. Hence τ is an isomorphism (see the beginning of 7) and $L^{B_o} \simeq Q$ holds. So all faithful simple $\mathfrak{B}$-modules are isomorphic and all minimal cosieves are faithful (and isomorphic). Hence all divisionrings $[Q,Q]$ must be isomorphic and so even more than the last statement is proved. This completes the proof.

REMARK 7. If $\mathfrak{B}$ is an ARTIN-category, then each faithful (on objects injective) simple $\mathfrak{B}$-module Q is balanced. This also can be proved more easily by Theorem 5 using a further result (Theorem 16). Finally let us mention that by ZORN's Lemma every (not necessarily artinian) simple additive category $\mathfrak{B}$ ($\neq 0$) is primitive (see also Theorem 20).

10. The radical.

An ideal I of a (not necessarily small) additive category $\mathfrak{B}$ is called *primitive* if $\mathfrak{B}/I$ is primitive. By Lemma 2 (2) only the kernels $Ker Q$ of simple $\mathfrak{B}$-modules Q are primitive ideals in $\mathfrak{B}$. As usual, the radical $rad\,\mathfrak{B}$ of the category $\mathfrak{B}$ is defined to be the intersection of all primitive ideals in $\mathfrak{B}$.

THEOREM 11 (see [13]). *Let $\mathfrak{B}$ be an additive category. Then for every non-zero object $B \in |\mathfrak{B}|$ the cosieve $rad(B,\cdot)$ is the intersection of all maximal cosieves $S^B \subset H^B$.*

PROOF. Let S^B be a maximal cosieve. Then $Ker(H^B/S^B)$ is a primitive ideal in $\mathfrak{B}$. Since $Ker(H^B/S^B)(B,\cdot) \subset S^B$ holds, by our definition of the radical,

$$rad\,\mathfrak{B}(B,\cdot) \subset \bigcap Ker(H^B/S^B)(B,\cdot) \subset \bigcap S^B$$

follows. Let now $Q: \mathfrak{B} \to \mathfrak{Ab}$ be a simple $\mathfrak{B}$-module. By the YONEDA-Lemma $1_B \mapsto 0 \neq x \in Q(B)$ induces always a surjective natural transformation $\delta: H^B \twoheadrightarrow Q$ with $\delta_{B'}(\beta: B \to B') = Q(\beta)(x)$. Clearly $Ker\,\delta = S^B$ is a maximal cosieve. So for $\beta \in \bigcap S^B$ always $Q(\beta) = 0$ holds. Hence

$$\bigcap S^B \subset rad\,\mathfrak{B}(B,\cdot)$$

is true. This completes the proof.

The following «internal» characterization of $rad\,\mathfrak{B}$ shows that the (not necessarily primitive!) ideal $rad\,\mathfrak{B}$ of $\mathfrak{B}$ coincides with KELLY's radical (see [7]).

THEOREM 12 (see [7], Lemma 6). *Let $\mathfrak{B}$ be an additive category. Then $f \in rad\,\mathfrak{B}(B_1, B_2)$ holds if and only if for each $g \in Hom_{\mathfrak{B}}(B_2, B_1)$ always $f_g = 1_{B_1} - g \circ f$ has a (multiplicative) left inverse f'_g in the ring $Hom_{\mathfrak{B}}(B_1, B_1)$ (i.e. $f'_g \circ f_g = 1_{B_1}$).*

PROOF: Let $f: B_1 \to B_2$ satisfy $f \notin rad\,\mathfrak{B}(B_1, B_2)$. Then there is a simple $\mathfrak{B}$-module $Q: \mathfrak{B} \to \mathfrak{Ab}$ with $Q(f) \neq 0$, i.e. there is a $0 \neq u \in Q(B_1)$ with $Q(f)(u) \neq 0$. By Lemma 2 (2) there is a $g: B_2 \to B_1$ satisfying:

$$Q(g \circ f)(u) = Q(g)(Q(f)(u)) = u = Q(1_{B_1})(u).$$

Hence $Q(1_{B_1} - g\circ f)(u)=0$ holds and so there is no

$$f'_g \in Hom_{\mathfrak{B}}(B_1, B_1) \text{ with } f'_g \circ (1_{B_1} - g\circ f) = 1_{B_1}.$$

Our condition is also necessary. For $f=0_{B_1,B_2}$ this is clear. Consider now $0\neq f \in rad\mathfrak{B}(B_1, B_2)$. Then for each morphism $g\in Hom_{\mathfrak{B}}(B_2, B_1)$ always $g\circ f \in rad\mathfrak{B}(B_1, B_1)$ holds. Since ($1_{B_1}\neq 0$!) obviously 1_{B_1} is not in $rad\mathfrak{B}(B_1, B_1)$ (see also Theorem 11), one has also

$$(1_B - g\circ f)\notin rad\mathfrak{B}(B_1, B_1).$$

Assume there is a cosieve $\bar{S}^{B_1}\neq H^{B_1}$ with $(1_{B_1} - g\circ f)\in \bar{S}^{B_1}$. Then there must be also a maximal cosieve S^{B_1} with $(1_{B_1} - g\circ f)\in S^{B_1}$. But by Theorem 11 we have $g\circ f\in S^{B_1}$. Hence $1_{B_1}\in S^{B_1}$ implies $S^{B_1}=H^{B_1}$, a contradiction. So (by $\bar{S}^{B_1}=H^{B_1}$) there is a f'_g with

$$f'_g \circ (1_{B_1} - g\circ f) = 1_{B_1}.$$

This completes the proof.

COROLLARY 1 (see [7]). *Let $\mathfrak{B}'$ be a full (additive) subcategory of an additive category $\mathfrak{B}$. Then*

$$rad\mathfrak{B}' = \mathfrak{B}' \cap rad\mathfrak{B}$$

(and specially $rad Hom_{\mathfrak{B}}(B_1, B_1) = rad\mathfrak{B}(B_1, B_1)$) holds.

COROLLARY 2 (see [7], Theorem 1). *The radical $rad\mathfrak{B}$ of an additive category $\mathfrak{B}$ is the largest ideal I in $\mathfrak{B}$ satisfying*

$$I(B,B)\subset rad Hom_{\mathfrak{B}}(B,B) \quad \text{for each} \quad B\in |\mathfrak{B}|.$$

PROOF: If $I(B,B)\subset rad Hom_{\mathfrak{B}}(B,B)$ holds, then also for $f\in I(B,B')$ and $g\in Hom_{\mathfrak{B}}(B',B)$ always $g\circ f\in I(B,B)\subset rad Hom_{\mathfrak{B}}(B,B)$ holds. Hence $1_B - g\circ f$ (by Theorem 12 for $Hom_{\mathfrak{B}}(B,B)$) has an inverse and hence (by Theorem 12 for $\mathfrak{B}$) finally $f\in rad\mathfrak{B}(B,B')$ is true. So

$$I\subset rad\mathfrak{B}$$

holds and the proof is complete.

COROLLARY 3. *If $\mathfrak{B}^{op}$ is the dual category of $\mathfrak{B}$, for objects B_1, $B_2\in |\mathfrak{B}| = |\mathfrak{B}^{op}|$, always: $rad\mathfrak{B}^{op}(B_1, B_2) = rad\mathfrak{B}(B_1, B_2)$ holds. Hence $rad\mathfrak{B}=0$ is equivalent to $rad\mathfrak{B}^{op}=0$.*

PROOF. First let us show that $rad\mathfrak{B}^{op}(B, B) = rad\mathfrak{B}(B, B)$ (by corollary 1 and the well known result for rings this is clear!). We need only verify $rad\mathfrak{B}^{op}(B, B) \subset rad\mathfrak{B}(B, B)$. So suppose $f \in rad\mathfrak{B}^{op}(B, B)$. Since for each $g \in Hom_{\mathfrak{B}^{op}}(B, B)$ always $f \cdot g \in rad\mathfrak{B}^{op}(B, B)$ holds, there is a f_g (Theorem 12) with $f_g \cdot (1_B - f \cdot g) = 1_B$. Hence $1_B - f_g$ (with $f_g \cdot f \cdot g$) belongs to $rad\mathfrak{B}^{op}(B, B)$. So there exists x satisfying

$$x \cdot (1_B - (1_B - f_g)) = 1_B = x \cdot f_g.$$

We have also $1_B - f \cdot g = x$ and hence $g \in Hom_{\mathfrak{B}^{op}}(B, B) = Hom_{\mathfrak{B}}(B, B)$ implies

$$(1_B - f \cdot g) \cdot f_g = 1_B = f_g \circ (1_B - g \circ f).$$

Then by our Theorem 12 it follows immediately $f \in rad\,\mathfrak{B}(B, B)$. Obviously writing

$$I(B_2, B_1) = rad\mathfrak{B}^{op}(B_1, B_2) \text{ resp. } J(B_1, B_2) = rad\mathfrak{B}(B_2, B_1),$$

we define an ideal I in $\mathfrak{B}$ resp. J in $\mathfrak{B}^{op}$. Since

$$rad\mathfrak{B}(B, B) = rad\mathfrak{B}^{op}(B, B),$$

Corollary 2 and Corollary 1 imply $I \subset rad\mathfrak{B}$, resp. $J \subset rad\mathfrak{B}^{op}$. Hence we have

$$rad\mathfrak{B}^{op}(B_1, B_2) \subset rad\mathfrak{B}(B_2, B_1) \subset rad\mathfrak{B}^{op}(B_1, B_2)$$

and the proof is complete.

REMARK 8. Corollary 3 generalizes results about the equivalence of «left and right semisimplicity» of (small artinian) additive categories (see [2], [13], [17]).

Theorem 12 implies also

COROLLARY 4. *If $\mathfrak{U}: \mathfrak{B} \to \mathfrak{C}$ is a full additive functor, then*

$$\mathfrak{U}(rad\mathfrak{B}(B_1, B_2)) \subset rad\mathfrak{C}(\mathfrak{U}B_1, \mathfrak{U}B_2).$$

The radical of ARTIN-categories has another interesting characterization, which will be obtained in the sequel after some preliminary investigations.

Let us start with classes M_i of morphisms of a category $\mathfrak{B}$. Consider now

$$\prod_{i=1}^{n} M_i = \{ f \mid \text{there is } f_1 \in M_1, \ldots, f_n \in M_n, \text{ with } f = f_1 \circ \ldots \circ f_n \}.$$

A class M of $\mathfrak{B}$-morphisms is said *nilpotent* if there is an n such that

$$M^n = \prod_{i=1}^{n} M_i \quad (\text{with } M_1 = \ldots = M_n = M)$$

is a subclass of the zero-ideal in $\mathfrak{B}$. Moreover M is said *discrete nilpotent* if for each (full) subcategory $\mathfrak{B}'$ (of $\mathfrak{B}$) having only a finite number of objects, $M \cap \mathfrak{B}'$ is always nilpotent. Finally let us denote by $\{ M \}_{\mathfrak{B}} = \{ M \}$ the ideal in $\mathfrak{B}$ generated by M (i.e. $\{ M \}$ is the smallest ideal containing M !).

LEMMA 5. *Let $\mathfrak{B}$ be an additive* ARTIN-*category. Then $rad\,\mathfrak{B}$ is nilpotent if and only if there is an n such that*

$$\{ rad^n \mathfrak{B} \} = \{ rad^{n+1} \mathfrak{B} \} = \ldots .$$

PROOF (see also [1], [6]): Trivially our condition is necessary. Assume that $rad\,\mathfrak{B}$ is not nilpotent. Then there are

$$f_1, f_2, \ldots, f_n, f_{n+1}, \ldots, f_{2n} \in rad\,\mathfrak{B} \text{ with } f_{2n} \circ \ldots \circ f_{n+1} \circ \ldots \circ f_1 \neq 0.$$

Denote $B = dom\, f_1$. Then the cosieve $L^B = \{ rad^n \mathfrak{B} \}(B, \cdot)$ shows that there are cosieves $L^B \subset H^B : \mathfrak{B} \to \mathfrak{Ab}$ satisfying

$$L^B \subset \{ rad^n \mathfrak{B} \} \text{ and } \{ rad^n \mathfrak{B} \} \circ L^B \neq 0$$

because

$$f_{2n} \circ \ldots \circ f_{n+1} \in \{ rad^n \mathfrak{B} \} \text{ and } f_n \circ \ldots \circ f_1 \in \{ rad^n \mathfrak{B} \}(B, \cdot).$$

Assume that L^B is already minimal. Then there is a $\beta : B \to B'$ in L^B, with $\{ rad^n \mathfrak{B} \} \circ \beta \neq 0$. Using

$$\{ rad^n \mathfrak{B} \} = \{ rad^{2n} \mathfrak{B} \}$$

there are finitely many $x_i, y_i \in rad^n \mathfrak{B}$ with

$$\left(\sum_{i=1}^{n} y_i \circ x_i \right) \circ \beta \neq 0.$$

So there are $x, y \in rad^n \mathfrak{B}$ with $x \circ y \circ \beta \neq 0$ and hence

$$\bar{L}^B = \{ rad^n \mathfrak{B} \} \circ \beta$$

is also a cosieve ($\neq 0$) with the above property. Since $\bar{L}^B \subset L^B$ holds, by the minimality of L^B it follows immediately $\bar{L}^B = L^B$. So there is an

$$f \in \{ rad^n \mathfrak{B} \} \subset \{ rad \mathfrak{B} \} = rad \mathfrak{B} \quad \text{with} \quad \beta = f \circ \beta .$$

By Theorem 12, we have an

$$f' \in Hom_{\mathfrak{B}}(B', B) \quad \text{with} \quad f' \circ (1_{B'} - f) = 1_{B'} .$$

Hence by the contradiction

$$0 = (\beta - f \circ \beta) - (1_{B'} - f') \circ (\beta - f \circ \beta) = f' \circ (1_{B'} - f) \circ \beta = \beta ,$$

the proof is complete.

REMARK 9. The ringoid $\mathfrak{B}$ of the ARTIN-rings (with a unit) shows that there are (not small) ARTIN-categories such that the radical $rad\mathfrak{B}$ of $\mathfrak{B}$ is not nilpotent.

In order to characterize now the radical of an ARTIN-category we need some further material which we have already announced at the end of our «Preliminaries».

Let $\mathfrak{B}'$ be an (additive) subcategory of an additive category $\mathfrak{B}$ and $M \subset H^{B'}$ a class of $\mathfrak{B}'$-morphisms with domain B'. Considering $H^{B'} : \mathfrak{B}' \to \mathfrak{Ab}$ resp. $H^{B'} : \mathfrak{B} \to \mathfrak{Ab}$ as additive functors, then

$$\{ M \}^{B'}_{\mathfrak{B}'} = \left\{ f \in H^{B'} : \mathfrak{B}' \to \mathfrak{Ab} \;\middle|\; \begin{array}{l} \text{there is } n \text{ such that for each } i \leq n \\ \text{there is } x_i \in M,\ y_i \in \mathfrak{B}' \text{ with} \\ \qquad \pm f = \sum_{i=1}^{n} y_i \circ x_i \end{array} \right\}$$

resp.

$$\{ M \}^{B'}_{\mathfrak{B}} = \left\{ f \in H^{B'} : \mathfrak{B} \to \mathfrak{Ab} \;\middle|\; \begin{array}{l} \text{there is } n \text{ such that for each } i \leq n \\ \text{there is } x_i \in M,\ y_i \in \mathfrak{B} \text{ with} \\ \qquad \pm f = \sum_{i=1}^{n} y_i \circ x_i \end{array} \right\}$$

is just the cosieve in $\mathfrak{B}'$ resp. $\mathfrak{B}$ generated by M (i.e. 1° $M \subset \{ M \}^{B'}_{\mathfrak{B}'}$; 2° a cosieve in $\mathfrak{B}'$ containing M also contains $\{ M \}^{B'}_{\mathfrak{B}'}$ and 3° $\{ M \}^{B'}_{\mathfrak{B}'}$

is a cosieve in $\mathfrak{B}'$). Similar results hold for ideals. Clearly

$$\{M\}^{B'}_{\mathfrak{B}'} \subset \mathfrak{B}' \cap \{M\}^{B'}_{\mathfrak{B}}$$

is always true. But one can verify immediately the following better result, namely:

LEMMA 6. *Let $\mathfrak{B}'$ be a full subcategory of an additive category $\mathfrak{B}$ and $M \subset H^{B'}$. Then we have even*

$$\{M\}^{B'}_{\mathfrak{B}'} = \mathfrak{B}' \cap \{M\}^{B'}_{\mathfrak{B}}.$$

Let us note that the same result holds for ideals. Now we have

LEMMA 7. *A full subcategory $\mathfrak{B}'$ of an* ARTIN*-category $\mathfrak{B}$ is again artinian. If $\mathfrak{B}'$ has only a finite number of objects, then the* D.C.C. *for ideals holds in $\mathfrak{B}'$.*

PROOF: Let $L_1^{B'} \supset L_2^{B'} \supset \dots$ be a descending chain of cosieves in $\mathfrak{B}'$ and $\{L_i\}^{B'}$ the cosieve in $\mathfrak{B}$ generated by $L_i^{B'}$. By Lemma 6, we have $L_i^{B'} = \mathfrak{B}' \cap \{L_i\}^{B'}$. But $\{L_1\}^{B'} \supset \{L_2\}^{B'} \supset \dots$ terminates after the n-th step. Hence the same holds for $L_1^{B'} \supset L_2^{B'} \supset \dots$. Furthermore let $I_1 \supset I_2 \supset \dots$ be a descending chain of ideals in $\mathfrak{B}'$. Since for each $B \in |\mathfrak{B}'|$ the chain of cosieves $I_1(B,\cdot) \supset I_2(B,\cdot) \supset \dots$ terminates after a finite number of steps, there must be an n such that

$$I_1(B,\cdot) \supset I_2(B,\cdot) \supset \dots$$

terminates after the n-th step for each $B \in |\mathfrak{B}'|$. This completes the proof.

Now we can state the following characterization of the radical of ARTIN-categories.

THEOREM 13. *The radical $rad\mathfrak{B}$ of an* ARTIN*-category $\mathfrak{B}$ is the largest discrete nilpotent ideal of $\mathfrak{B}$. If the* D.C.C. *for ideals holds in $\mathfrak{B}$, then $rad\mathfrak{B}$ is the largest nilpotent ideal and each discrete nilpotent ideal is nilpotent.*

PROOF: Let $\mathfrak{B}'$ be a full subcategory of $\mathfrak{B}$ with only finitely many objects. Since $\mathfrak{B}$ is artinian by Lemma 7, also $\mathfrak{B}'$ is artinian. Moreover the D.C.C. for ideals holds in $\mathfrak{B}'$. Then Lemma 5 implies that $rad\mathfrak{B}' =$

$\mathfrak{B}' \cap rad\mathfrak{B}$ is nilpotent (see also Corollary 1). Hence $rad\mathfrak{B}$ is discrete nilpotent. Now let I be a discrete nilpotent ideal. Since $I(B,B)$ is nilpotent by Theorem 12, we have

$$I(B,B) \subset rad\mathfrak{B}(B,B) = rad\,Hom_{\mathfrak{B}}(B,B)$$

since there is an n such that

$$[(g \circ z)^{n-1} + \dots + (g \circ z) + 1_B] \circ [1_B - (g \circ z)] = 1_B - (g \circ z)^n = 1_B$$

is true, for all $z \in I(B,B)$ and $g \in Hom_{\mathfrak{B}}(B,B)$. Hence Corollary 2 implies $I \subset rad\mathfrak{B}$. The last statement follows from the fact that there is an n such that $\{rad^n\mathfrak{B}\} = \{rad^{n+1}\mathfrak{B}\} = \dots$ holds. Hence by Lemma 5 $rad\mathfrak{B}$ is nilpotent. Since every discrete nilpotent (hence specially every nilpotent) ideal I is contained in $rad\mathfrak{B}$, we see that I is nilpotent and $rad\mathfrak{B}$ is the largest nilpotent ideal. This completes the proof.

11. Semiprimitive categories and subproducts.

A (not necessarily small) additive category $\mathfrak{B}$ with $rad\mathfrak{B} = 0$ is called *semiprimitive*. This is equivalent to the fact that there is a (large) faithful set $\mathfrak{U} = \{\mathfrak{U}_i \mid i \in I\}$ of simple $\mathfrak{B}$-modules. Without loss of generality we can assume that the $\mathfrak{U}_i$'s are not isomorphic (by pairs), and injective on objects. Furthermore each category which is equivalent to a semiprimitive category is again semiprimitive.

Now let Vec_{K_i} be topologized by the finite topology. Relative to the product topology the product category $\prod_I Vec_{K_i}$ is also topologized.

Then the following structure theorem characterizes semiprimitive categories.

THEOREM 14. *Let $\mathfrak{B}$ be a (not necessarily small) additive category. Then the following conditions are equivalent:*

(1) $\mathfrak{B}$ is semiprimitive.

(2) $\mathfrak{B}$ is isomorphic to a dense additive subcategory of a (large) product category $\prod_{i \in I} Vec_{K_i}$.

PROOF. If $\mathfrak{B}$ is semiprimitive (i.e. $rad\mathfrak{B} = 0$), there is a faithful (large) set $\mathfrak{U} = \{\mathfrak{U}_i \mid i \in I\}$ of non-isomorphic, on objects injective, simple $\mathfrak{B}$-

modules. Denote $K_i = [\mathfrak{U}_i, \mathfrak{U}_i]$, then $dom\, CC(\mathfrak{U})$ is the full subcategory of $\prod_{i \in I} Vec_{K_i}$ consisting of objects $E_{\mathfrak{U}}(B)$ for $B \in |\mathfrak{B}|$ having the object $E_{\mathfrak{U}_i}(B)$ as its i-th component. Then for $\mathfrak{U} := \coprod_{i \in I} \mathfrak{U}_i$, the isomorphism $T: dom\, CC(\mathfrak{U}) \to dom\, CC(\mathfrak{U})$ is obviously topological ($dom\, CC(\mathfrak{U})$ resp. $Mod_{[\mathfrak{U}, \mathfrak{U}]}$ are topologized by the finite topology!). Since by the BOURBAKI density Theorem (Theorem 7) $E_{\mathfrak{U}}(\mathfrak{B})$ is dense in $dom\, CC(\mathfrak{U})$, the category $E_{\mathfrak{U}}(\mathfrak{B}) = (T \circ E_{\mathfrak{U}})(B)$ (isomorphic to $\mathfrak{B}$) must also be dense in $dom\, CC(\mathfrak{U})$ resp. $\prod_{i \in I} Vec_{K_i}$. Conversely let $\mathfrak{B}$ be a dense additive subcategory of $\prod_{i \in I} Vec_{K_i}$ and $P_{\mathfrak{U}_i}: \mathfrak{B} \to Vec_{K_i}$ the restriction on $\mathfrak{B}$ of the $\mathfrak{U}_i$-th canonical product functor, and $V_i : Vec_{K_i} \to \mathfrak{Ab}$ the usual forgetful functor. By the density our Lemma 2 (2) implies the simplicity of $V_i \circ P_{\mathfrak{U}_i} : \mathfrak{B} \to \mathfrak{Ab}$. Clearly

$$\{ V_i \circ P_{\mathfrak{U}_i} \mid i \in I \}$$

is faithful. Hence $\mathfrak{B}$ is semiprimitive and the proof is complete.

Note that by Lemma 2 (2) a full subcategory of a (large) product category $\mathfrak{P} = \prod_{j \in J} P_j$ of primitive categories P_j is always semiprimitive (see also Corollary 1). A further characterization of semiprimitive categories by primitive categories is possible using so called subproducts of categories.

A *subproduct* $\pi_{j \in J} \mathfrak{A}_j$ of categories $\mathfrak{A}_j$ is a subcategory $\mathfrak{B}_\pi$ of the (large) product category $\mathfrak{P} = \prod_{j \in J} \mathfrak{A}_j$ such that the restriction $P_j' : \mathfrak{B}_\pi \to \mathfrak{A}_j$ of the canonical product functor $P_j : \mathfrak{P} \to \mathfrak{A}_j$ on $\mathfrak{B}_\pi$ is full.

It is easy to verify the following

LEMMA 8. *An additive category $\mathfrak{B}$ is isomorphic to a subproduct $\mathfrak{B}_\pi = \pi_{j \in J} \mathfrak{A}_j \simeq \mathfrak{B}$ if and only if there is a set (isomorphic to J) of full additive functors $R_j : \mathfrak{B} \to \mathfrak{A}_j$ with $\bigcap_{j \in J} Ker\, R_j = 0$ such that for $B_1, B_2 \in |\mathfrak{B}|$ the equations $R_j(B_1) = R_j(B_2)$ for $j \in J$ always imply $B_1 = B_2$.*

Call a subcategory $\mathfrak{B}_0$ of $\mathfrak{P} = \prod_{j \in J} \mathfrak{A}_j$ reduced if all $P_j' : \mathfrak{B}_0 \to \mathfrak{A}_j$

with $j \in J$ are non-isomorphic to zero. Then we have the following characterization of semiprimitive categories:

THEOREM 15 (see also LEDUC [11]). *An additive category $\mathfrak{B}$ $(\neq 0)$ is semiprimitive if and only if $\mathfrak{B}$ is isomorphic to a subproduct of primitive categories.*

PROOF: Suppose $rad\,\mathfrak{B} = 0$. Since there is a non-zero object, there is a non empty set of primitive ideals I_j of $\mathfrak{B}$ with $\bigcap_{j \in J} I_j = 0$. Then $R_j = S_j : \mathfrak{B} \to \mathfrak{B}/I_j$ satisfies the condition of Lemma 8, and so $\mathfrak{B}$ is isomorphic to an obviously reduced $(R_j \simeq 0)$ subproduct $\mathfrak{B}_\pi = \pi_{j \in J}(\mathfrak{B}/I_j)$ of the primitive categories $\mathfrak{B}/I_j$ $(j \in J)$. Now let $\mathfrak{B}_\pi = \pi_{j \in J} \mathfrak{P}_j$ be a reduced subproduct of primitive categories $\mathfrak{P}_j$. Then there is of course a non-zero object in $\mathfrak{B}_\pi$. Now we consider faithful simple $\mathfrak{P}_j$-modules $Q_j : \mathfrak{P}_j \to \mathfrak{Ab}$. Since $0 \not\simeq P_j' : \mathfrak{B}_\pi \to \mathfrak{P}_j$ is full and Q_j maps non-zero objects to non-zero objects by Lemma 2 (2), $Q_j \circ P_j' : \mathfrak{B}_\pi \to \mathfrak{Ab}$ is simple. Since for primitive ideals $Ker(Q_j \circ P_j')$ always

$$Ker(Q_j \circ P_j') = Ker\, P_j' \quad \text{and} \quad \bigcap_{j \in J} Ker\, P_j' = 0$$

hold, obviously $\mathfrak{B}_\pi$ is semiprimitive and the proof is complete.

As in JACOBSON [6] p. 15, we can prove:

LEMMA 9. *A subproduct $\mathfrak{B}_\pi = \pi_{j \in J} \mathfrak{A}_j$ is dense in the product category $\mathfrak{P} = \prod_J \mathfrak{A}_j$ (relative to the product-topology in $\mathfrak{P}$) if and only if for each finite subset $\{j_1, \dots, j_n\}$ of J:*

$$\mathfrak{B} = Ker\, P_{j_1}' + \bigcap_{\nu=2}^{n} Ker\, P_{j_\nu}' \quad (for \quad P_{j_\nu}' : \mathfrak{B}_\pi \to \mathfrak{A}_{j_\nu}).$$

12. Semiprimitive ARTIN-categories.

We start with the following

THEOREM 16 (see MITCHELL [13], p. 19-22). *A (not necessarily small) additive category $\mathfrak{B}$ is artinian and semiprimitive if and only if $H^B : \mathfrak{B} \to \mathfrak{Ab}$ is (finitely) semisimple for each non-zero object $B \in |\mathfrak{B}|$.*

PROOF: Suppose $H^B = \bigoplus_{i=1}^{n} T_i^B$ (T_i simple). Then $S_j^B = \bigoplus_{j+1=i}^{n} T_i^B$ is maximal and obviously $\bigcap_{i=1}^{n} S_i^B \simeq 0$ holds. Then by Theorem 11 it follows $rad\,\mathfrak{B} = 0$. Clearly each descending chain $H^B \supset R_1^B \supset R_2^B \supset \cdots$ has a finite length (at most length $n!$) and so $\mathfrak{B}$ is artinian. Conversely consider finite intersections of maximal cosieves S_ν^B (in H^B). Let $L^B = S_1^B \cap \cdots \cap S_n^B$ be a minimal cosieve with this property. Assume now $L^B \not\simeq 0$. By Theorem 11 we have

$$rad\,\mathfrak{B}(B,\cdot) = \bigcap_{max} S_\nu^B \simeq 0$$

and so there must be a maximal cosieve S_{n+1}^B such that $L^B \cap S_{n+1}^B$ be a proper subcosieve of L^B. By this contradiction,

$$S_1^B \cap \cdots \cap S_n^B \simeq 0$$

must be true. Without loss of generality we may assume that no S_i^B is superfluous in the above intersection. Denote by $T_j^B = \bigcap_{j \neq i=1}^{n} S_i^B$. Then $H^B = S_j^B \oplus T_j^B$ holds (T_j^B simple). Now if we take $Q_k^B = \bigcap_{i=1}^{k} S_i^B$, we can prove

$$H^B = T_1^B \oplus \ldots \oplus T_k^B \oplus Q_k^B \quad \text{for} \quad k = 1, 2, \ldots, n.$$

For $k = 1$ this is clear. By the noetherian isomorphism theorem

$$Q_k^B / Q_k^B \cap S_{k+j}^B \simeq Q_k^B + S_{k+j}^B / S_{k+j}^B$$

holds. Since also for $k < n$ in $\bigcap_{i=1}^{n-k} (Q_k^B \cap S_{k+j}^B) \simeq 0$ no $Q_k^B \cap S_{k+j}^B$ is superfluous, the $Q_k^B \cap S_{k+j}^B$ are not only maximal but also are unequal to Q_k^B. Hence again:

$$Q_k^B = (Q_k^B \cap S_{k+1}^B) \oplus \bigcap_{j=2}^{n-k} (Q_k^B \cap S_{k+j}^B) = Q_{k+1}^B \oplus T_{k+1}^B .$$

So together with $k < n$ also for $k+1$ our statement is true; hence also for n. Since $Q_n^B \simeq 0$, finally

$$H^B = T_1^B \oplus \ldots \oplus T_n^B$$

follows and the proof is complete.

REMARK 10. Consider a semiprimitive ARTIN-category $\mathfrak{B}$ and a $\mathfrak{B}$-submodule N^B of

$$M^B = \bigoplus_{i=1}^{m} T_i^B : \mathfrak{B} \to \mathfrak{Ab}$$

(T_i^B simple). Then there is a maximal (perhaps empty) subset

$$\{i_1,\dots,i_r\} \text{ of } \{1,2,\dots,n\} \text{ such that}$$
$$N^B \cap (T_{i_1}^B \oplus \dots \oplus T_{i_r}^B) \simeq 0.$$

By the maximality of $\{i_1,\dots,i_r\}$ we get immediately

$$M^B = N^B \oplus (T_{i_1}^B \oplus \dots \oplus T_{i_r}^B)$$

(see again Theorem 6). If now specially $N^B = T^B$ (simple), it follows $r = n-1$ and $T^B \simeq T_i^B$ (for a certain i). Hence by induction we see that

$$H^B = \bigoplus_{i=1}^{n} T_i^B = \bigoplus_{i=1}^{\bar{n}} \bar{T}_i^B$$

always implies $n = \bar{n}$ and $T_i^B \simeq \bar{T}_i^B$ (taking a certain choice of indices i).

REMARK 11. The structure of $Hom_{\mathfrak{B}}(B_1, B_2) \simeq Nat(H^{B_2}, H^{B_1})$ (YONEDA-Lemma) can be described more exactly using Theorem 16 and the SCHUR-Lemma. In this way (also very well known in ring theory!) MITCHELL (see [13], p.20) established the famous ARTIN-WEDDERBURN structure theorem for (small) additive categories (see also DARTOIS [2] p. 19-26). This theorem, already considered in 5 (Theorem 5), will be obtained again in a more concrete and detailed form by our general theory (see especially Theorem 17 and Theorem 22).

Let us consider a set $\{\mathfrak{B}_j \mid j \in J\}$ of ideals $\mathfrak{B}_j$ of an additive category $\mathfrak{B}$. If each $\beta \in \mathfrak{B}$ has a unique representation $\beta = \sum_{j \in J} \beta_j$ with $\beta_j \in \mathfrak{B}_j$ and only finitely many β_j's unequal to zero, we say that $\mathfrak{B}$ splits into a direct sum $\mathfrak{B} = \bigoplus_{j \in J} \mathfrak{B}_j$ of ideals $\mathfrak{B}_j$. In fact this is equivalent to say that $Hom_{\mathfrak{B}}(\cdot,.) : \mathfrak{B}^{op} \otimes \mathfrak{B} \to \mathfrak{Ab}$ is a coproduct of the ideals

$$\mathfrak{B}_j : \mathfrak{B}^{op} \otimes \mathfrak{B} \to \mathfrak{Ab} \text{ in } \mathfrak{Ab}^{\mathfrak{B}^{op} \otimes \mathfrak{B}}$$

(where $\mathfrak{B}_j$ is considered as a functor and as a class of morphisms simultaneously!). Obviously the $\mathfrak{B}_j$'s are additive categories having the same objects as $\mathfrak{B}$, but they are not subcategories of $\mathfrak{B}$ because the identity of $B \in |\mathfrak{B}_j|$ is $(1_B)_j$. Clearly $\mathfrak{B}_k \simeq \mathfrak{B} / \bigoplus_{k \neq j} \mathfrak{B}_j$ holds. Now let us call an object $B_0 \in |\mathfrak{B}_0|$ of a subcategory $\mathfrak{B}_0$ of $\mathfrak{P} = \prod_{j \in J} \mathfrak{A}_j$ *dis-*

crete if only finitely many of its projections

$$P'_{j_\nu}(B_0) \in |\mathfrak{A}_{j_\nu}| \quad (\nu = 1, 2, \ldots, n)$$

are non-zero objects. If $\mathfrak{B}_0$ has only discrete objects, we call $\mathfrak{B}_0$ discrete.

LEMMA 10. *An additive category $\mathfrak{B}$ is isomorphic to a full (reduced) discrete subcategory $\mathfrak{B}_\pi$ of $\mathfrak{P} = \prod_{j \in J} \mathfrak{A}_j$ if and only if $\mathfrak{B}$ splits into the direct sum $\mathfrak{B} = \bigoplus_{j \in J} \mathfrak{B}_j$ of ideals $\mathfrak{B}_j$ $(\neq 0)$ being (category) isomorphic to full subcategories of $\mathfrak{A}_j$.*

PROOF: Obviously

$$\mathfrak{A}'_k = \{ \alpha \in \mathfrak{P} \mid P_j(\alpha) = 0, k \neq j \in J \}$$

is an ideal in $\mathfrak{P}$ (category) isomorphic to $\mathfrak{A}_k$. Then $\mathfrak{A}''_k = \mathfrak{A}'_k \cap \mathfrak{B}_\pi$ is an ideal $(\neq 0)$ of $\mathfrak{B}_\pi$ which is also a full subcategory of $\mathfrak{A}'_k$, since $\mathfrak{B}_\pi$ is full. Since $\mathfrak{B}_\pi$ is discrete, $\mathfrak{B}_\pi = \bigoplus_{j \in J} \mathfrak{A}''_j$. Then $\mathfrak{B} \simeq \mathfrak{B}_\pi$ shows that our condition is necessary. Conversely consider the full (reduced) subcategory $\mathfrak{B}^o$ of $\mathfrak{P}^o = \prod_{j \in J} \mathfrak{B}_j$ consisting of objects A^o with $P^o_j(A^o) = B \in |\mathfrak{B}|$. Then $\mathfrak{B}^o$ is discrete, since $1_B = \sum_{j \in J} (1_B)_j$ holds. Now by the equations

$$P^o_j(R(\beta)) = \beta_j \quad \text{for } j \in J \quad \text{and} \quad \beta = \sum_{j \in J} \beta_j \in \mathfrak{B}$$

an isomorphism $R: \mathfrak{B} \to \mathfrak{B}^o$ will be defined. Since $\mathfrak{B}_j$ has a full embedding in $\mathfrak{A}_j$, there is a full embedding of $\mathfrak{P}^o$ in $\mathfrak{P} = \prod_{j \in J} \mathfrak{A}_j$ which transforms $\mathfrak{B}^o$ into a full (reduced) discrete subcategory $\mathfrak{B}_\pi$ of $\mathfrak{P}$ being isomorphic to $\mathfrak{B}^o$ resp. $\mathfrak{B}$. This completes the proof.

Using primitive (i.e. simple) ARTIN-categories, the structure of the semiprimitive ARTIN-categories $(\neq 0)$ can be described by the following

THEOREM 17. *Let $\mathfrak{B}$ be a (not necessarily small!) additive category. Then the following conditions are equivalent:*

(1) $\mathfrak{B}$ splits into the direct sum $\mathfrak{B} = \bigoplus_{j \in J} \mathfrak{B}_j$ of (as categories) primitive (i.e. simple) artinian ideals $\mathfrak{B}_j$ (i.e. $Hom_{\mathfrak{B}}(\cdot, .): \mathfrak{B}^{op} \otimes \mathfrak{B} \to \mathfrak{Ab}$

is semisimple and $\mathfrak{B}$ *artinian).*

(2) $\mathfrak{B}$ *is artinian semiprimitive and has a non-zero object.*

(3) $\mathfrak{B}$ *is isomorphic to a full reduced discrete subcategory* $\mathfrak{B}_\pi$ *of* $\mathfrak{P} = \prod_{j \in J} \mathfrak{P}_j$ *with primitive (i.e. simple)* ARTIN-*categories* $\mathfrak{P}_j$.

PROOF: (1) $\Longrightarrow$ (2). Clearly $\mathfrak{B}$ has a non-zero object. Since

$$\mathfrak{B}_k \simeq \mathfrak{B} / \bigoplus_{k \neq j} \mathfrak{B}_j$$

holds, $I_k = \bigoplus_{k \neq j} \mathfrak{B}$ is a primitive ideal in $\mathfrak{B}$. Since $\bigcap_{j \in J} I_j = 0$, it follows $rad\,\mathfrak{B} = 0$. A cosieve $T^B : \mathfrak{B} \to \mathfrak{Ab}$ induces now a cosieve

$$T_j^B = \{ \beta_j \mid \beta \in T^B \} : \mathfrak{B}_j \to \mathfrak{Ab}\,.$$

Obviously there are only finitely many $j_1, \dots, j_n \in J$ with $\beta_{j_\nu} \neq 0$ $(\nu = 1, 2, \dots, n)$ for each $\beta \in T^B$. Hence all cosieves T_j^B $(\neq T_{j_\nu}^B)$ are zero! Consider now a (proper) descending chain $T^B = T_1^B \supset T_2^B \supset \dots$. Then for each ν the (not necessarily proper) descending chain of cosieves $(T_1^B)_{j_\nu} \supset (T_2^B)_{j_\nu} \supset \dots$ in $\mathfrak{B}_{j_\nu}$ terminates at the r_ν-th step, since $\mathfrak{B}_j$ is artinian. Hence $T^B = T_1^B \supset T_2^B \supset \dots$ terminates at the $\max_{\nu = 1, \dots, n} r_\nu$-th step and so $\mathfrak{B}$ is artinian.

Using now Theorem 16, Theorem 5 and Theorem 10, the conclusion (2) $\Longrightarrow$ (3) is evident. But here we shall prove this in the following way. First there is a set of primitive ideals I_j with $\bigcap_{j \in J} I_j = 0$. Obviously the primitive factor categories $\mathfrak{P}_j = \mathfrak{B} / I_j$ are artinian (together with $\mathfrak{B}$!). By Lemma 8 (for

$$R_j = S_j : \mathfrak{B} \to \mathfrak{P}_j = \mathfrak{B} / I_j \;)$$

$\mathfrak{B}$ is isomorphic to a reduced subproduct $\mathfrak{B}_\pi = \pi_{j \in J} \mathfrak{P}_j$ of the primitive ARTIN-categories $\mathfrak{P}_j$. Let now $Q_j : \mathfrak{P}_j \to \mathfrak{Ab}$ be a faithful simple $\mathfrak{P}_j$-module. Then $Q_j \circ P_j'$ (for $P_j' : \mathfrak{B}_\pi \to \mathfrak{P}_j$) is simple and $Ker(Q_j \circ P_j') = Ker P_j'$ is true (see proof of Theorem 15 and Lemma 2 (2)). Since $P_j'(\mathfrak{B}_\pi) = \mathfrak{P}_j$ is primitive artinian (i.e. simple by Theorem 10), the ideal $Ker P_j'$ must be maximal.

Case 1: J is finite. Without loss of generality we can assume that no

I_j is superfluous. Then by Lemma 9, $\mathfrak{B}_\pi$ is dense in $\mathfrak{P} = \prod_{j \in J} \mathfrak{P}_j$ and hence $\mathfrak{B}_\pi$ is a full (reduced) and discrete subcategory of $\mathfrak{P}$.

Case 2: J is infinite. Then again by Lemma 9, $\mathfrak{B}_\pi$ is dense in $\mathfrak{P}$ and Remark 10 shows that $\mathfrak{B}_\pi$ is discrete. So $\mathfrak{B}_\pi$ is again a full (reduced, discrete) subcategory of $\mathfrak{P} = \prod_{i \in J} \mathfrak{P}_j$ and (3) is proved.

Finally (3) $\Longrightarrow$ (1) follows immediately by Lemma 10, Lemma 7 and Remark 5. This completes the proof.

REMARK 12. Theorem 17 is apparently a more detailed form of ARTIN-WEDDERBURN's theorem for additive categories (see again Theorem 5).

ARTIN-WEDDENBURN's theorem is also a consequence of:

COROLLARY 5. *Let $\mathfrak{B}$ ($\neq 0$) be a (semiprimitive) ARTIN-category. Then (there is) a faithful set $\mathfrak{U}$ of non-isomorphic on objects injective simple $\mathfrak{B}$-modules $\mathfrak{U}$ (which) is balanced.*

PROOF: Since $\mathfrak{U}$ is faithful, $E_{\mathfrak{U}}$ is an embedding which is bijective on objects (i.e. $\mathfrak{B} \simeq E_{\mathfrak{U}}(\mathfrak{B})$). Since (Lemma 4) $E_{\mathfrak{U}}(\mathfrak{B})$ is dense in $dom\, CC(\mathfrak{U})$ Lemma 3 (see also the proof of Theorem 10, (4) $\Longrightarrow$ (1)) implies that for for $\mathfrak{U} \in \mathfrak{U}$ always $E_{\mathfrak{U}}(\mathfrak{B}) = dom\, CC(\mathfrak{U})$ (the finite topology becomes the discrete one!). By Theorem 14, resp. Theorem 7, or by Lemma 9 and Theorem 10, $E(\mathfrak{B})$ is dense in $dom\, CC(\mathfrak{U})$. Then Theorem 16, resp. Remark 10, shows that $dom\, CC(\mathfrak{U})$ is a (legal!) discrete subcategory of a product of vectorspace categories. Hence $E_{\mathfrak{U}}(\mathfrak{B}) = dom\, CC(\mathfrak{U})$ holds and the proof is complete.

Since by Theorem 10 primitive and simple ARTIN-categories coincide, we shall prove, as an addition to Theorem 17, the following result (which is much stronger than the similar result mentioned in Remark 10).

THEOREM 18 (see [1], page 83). *Let*

$$\mathfrak{B} = \bigoplus_{j \in J} \mathfrak{B}_j = \bigoplus_{k \in K} \mathfrak{B}'_k$$

be two sum-decompositions of a (not necessarily small) additive category $\mathfrak{B}$ in (also as categories!) simple ideals $\mathfrak{B}_j$, resp. $\mathfrak{B}'_k$. Then the sets of the ideals $\mathfrak{B}_j$, resp. $\mathfrak{B}'_k$, coincide. Hence there is a bijection $f: J \to K$ such *that* $\mathfrak{B}'_{f(j)} = \mathfrak{B}_j$ *holds for all* $j \in J$. *The cardinal number*

$$dim\,\mathfrak{B} = |J| = |K|$$

is called the dimension of $\mathfrak{B}$.

PROOF : If I is an arbitrary ideal of $\mathfrak{B} = \underset{j \in J}{\oplus} \mathfrak{B}_j$ (where $\mathfrak{B}_j$ is not necessarily simple !), then $\mathfrak{B}_j \circ I = I \circ \mathfrak{B}_j$ coincides with the ideal

$$I_j = \{ \alpha_j \in \mathfrak{B}_j \mid \alpha \in I \}$$

of $\mathfrak{B}_j$. Consider now $\mathfrak{B} = \underset{j \in J}{\oplus} \mathfrak{B}_j = \underset{k \in K}{\oplus} \mathfrak{B}'_k$ for simple ideals ($\neq 0$) $\mathfrak{B}_j$, resp. $\mathfrak{B}'_k$. Since

$$\mathfrak{B}_j = \mathfrak{B}_j \circ \mathfrak{B} = \mathfrak{B}_j \circ \underset{k \in K}{\oplus} \mathfrak{B}'_k ,$$

$\mathfrak{B}_j \circ \mathfrak{B}'_k = 0$ cannot be true for all $k \in K$. Let us choose $k \in K$ with $\mathfrak{B}_j \circ \mathfrak{B}'_k$ not 0. Then $\mathfrak{B}_j \circ \mathfrak{B}'_k$ is an ideal ($\neq 0$) in $\mathfrak{B}_j$ and $\mathfrak{B}'_k$. By the simplicity of $\mathfrak{B}_j$ and $\mathfrak{B}'_k$, it follows immediately that we will have $\mathfrak{B}_j = \mathfrak{B}_j \circ \mathfrak{B}'_k = \mathfrak{B}'_k$. Hence each (of the different !) $\mathfrak{B}_j$ coincides with one (of the different !) $\mathfrak{B}'_k$. Since vice versa the same argument is true, both sets of ideals $\mathfrak{B}_j$, resp. $\mathfrak{B}'_k$, coincide. This completes the proof.

Now let us mention that $dim\,\mathfrak{B}$ and the classes of non-isomorphic divisionrings

$$K_j \quad (j \in J, \ |J| = dim\,\mathfrak{B})$$

determined by Theorem 10 are invariants for a semiprimitive ARTIN-category $\mathfrak{B}$. If $\mathbf{N}$ denotes the set of natural numbers and $f: J \to \mathbf{N}$ a discrete mapping (i.e. there are only finitely many $j \in J$ with $f(j) \neq 0$), then each $B \in |\mathfrak{B}|$ induces (Theorem 17 (3) or Theorem 5) a discrete mapping

$$f_B : J \to \mathbf{N} \quad \text{defined by} \quad f_B(j) = dim\,P_j E(B).$$

Again E denotes the evaluation of

$$\mathfrak{U} = \{ \text{simple } \mathfrak{U}_i \mid i \in J \}$$

and

$$P_j : \prod_{i \in J} Vec_{[\mathfrak{U}_i, \mathfrak{U}_i]} \to Vec_{[\mathfrak{U}_j, \mathfrak{U}_j]}$$

the j-th projection functor. So a semiprimitive ARTIN-category $\mathfrak{B}$ is uniquely determined (up to isomorphisms!) by the following system of invariants:

$$\begin{cases} |J| = dim\,\mathfrak{B}, \\ \{\text{divisionrings } K_j \mid j \in J\}, \\ \{\text{discrete } f_B : J \to \mathbf{N} \mid B \in |\mathfrak{B}|\}. \end{cases}$$

Finally let us also state:

THEOREM 19. *A (not necessarily small) semiprimitive* ARTIN *-category $\mathfrak{B}$ satisfies the* D.C.C. *for ideals if and only if $dim\,\mathfrak{B}$ is finite.*

PROOF: First, $dim\,\mathfrak{B} = 0$ means $\mathfrak{B} = 0$ and here the D.C.C. for ideals is true. If the D.C.C. for ideals is satisfied in $\mathfrak{B}$, then one sees by the proof of Theorem 17, case 1, that $dim\,\mathfrak{B}$ is finite. Now suppose finally

$$dim\,\mathfrak{B} = n \neq 0 \quad \text{finite.}$$

i.e. $\mathfrak{B} = \bigoplus_{\nu=1}^{n} \mathfrak{B}_\nu$, a direct sum-decomposition into simple (artinian) ideals. Then an ideal I of $\mathfrak{B}$ induces an ideal

$$I^\nu = \{\beta_\nu \in \mathfrak{B}_\nu \mid \beta \in I\}$$

of $\mathfrak{B}_\nu$. Now we consider a proper descending chain of ideals $I_1 \supset I_2 \supset \ldots$ in $\mathfrak{B}$. Since the $\mathfrak{B}_\nu$ are simple, $I_n = 0$ follows and the proof is complete.

REMARK 13. Theorem 19 shows that a semiprimitive ARTIN-category $\mathfrak{B}$ having only finitely many objects (e.g. a ring with unit!) has a finite dimension $dim\,\mathfrak{B}$. On the other hand $dim\,\mathfrak{B}$ may also be a (large) cardinal number representing a class. Contrary to Theorem 10, semiprimitive ARTIN-categories need not have a small skeleton (e.g. the corpoid-subcategory of the ringoid in Remark 9). One can easily verify that $\mathfrak{B}$ has a small skeleton if and only if $dim\,\mathfrak{B}$ is a small cardinal number. For this, consider a (full) equivalence from $\mathfrak{B}$ to a skeleton $\overline{\mathfrak{B}}$ and apply Theorem 17 (3) and Theorem 10. Especially (by Theorem 19) semiprimitive ARTIN-categories satisfying the D.C.C. condition for ideals have a small (even a countable) skeleton.

13. Simple categories.

Let $\mathfrak{B}$ be a (not necessarily small) additive category. If there are no non-trivial ideals in $\mathfrak{B}$ (i.e. $Hom_{\mathfrak{B}}(\cdot,.) : \mathfrak{B}^{op} \otimes \mathfrak{B} \to \mathfrak{Ab}$ is simple), we say that $\mathfrak{B}$ is *simple*. Generally let us consider a cosieve (left ideal):

$$0 \not\simeq S^B : \mathfrak{B} \longrightarrow \mathfrak{Ab}$$

and the ideal $\{S^B\}$ generated by S^B in $\mathfrak{B}$. Then

$$\{S^B\}(B_1, B_2) = \left\{ \begin{array}{l} x: B_1 \to B_2 \text{ such that:} \\ \text{there is } n \text{ such that for all } i \leqslant n \text{ there is} \\ t_i : B_1 \to B \text{ and } s_i \in S^B \text{ with } x = \sum_{i=1}^{n} s_i \circ t_i \end{array} \right\}$$

If $\mathfrak{B}$ is simple, by $\{S^B\} = \mathfrak{B}$ we have $1_{B_1} = \sum_{i=1}^{n} s_i \circ t_i$; hence

$$H^{B_1} = \sum_{i=1}^{n} S^B \circ t_i \quad \text{for some } t_i : B_1 \to B .$$

Let now $(S^B)^n$ be the n-th copower of S^B. Then by

$$(S^B)^n \ni (s_1, \ldots, s_n) \longmapsto \sum_{i=1}^{n} s_i \circ t_i$$

an epimorphic (surjective) natural transformation $\phi : (S^B)^n \twoheadrightarrow H^{B_1}$ is defined. This implies that S^B is a generator. Hence S^B is balanced (Theorem 4). It is easy to verify that each full subcategory $\mathfrak{B}'$ of a simple category $\mathfrak{B}$ (specially each ring $Hom_{\mathfrak{B}}(B, B)$) is again simple; one must only use Lemma 6 for ideals.

Now we can state the following structure theorem for (not necessarily small) simple additive categories, namely:

THEOREM 20. *Let $\mathfrak{B}$ be a (not necessarily small) additive category. Then the following conditions are equivalent:*

(1) $\mathfrak{B}$ is simple.

(2) $\mathfrak{B}$ is isomorphic to a full subcategory ($\neq 0$) of Mod_R consisting of finitely generated projective R-modules over a simple ring R. (Hence $\mathfrak{B}$ has always a small skeleton.)

PROOF: Let $\mathfrak{B}$ be simple. Since $H^B : \mathfrak{B} \to \mathfrak{Ab}$ is a generator and since: $[H^B, H^B] \simeq Hom_{\mathfrak{B}}(B, B)$ is simple, by Theorem 4 immediately (2) follows. Using FAITH [3], p. 209, each finitely generated projective module

$\mathfrak{U}$ over a simple ring R is a generator in Mod_R and moreover $Hom_R(\mathfrak{U}, \mathfrak{U})$ is a simple ring. If now (2) holds and $I \neq 0$ is an ideal in $\mathfrak{B}$, then there is $0 \neq \beta : B_1 \to B_2$ in I. For an arbitrary $0 \neq B \in |\mathfrak{B}|$ there is a cardinal number ν with $B^\nu \rightleftarrows B_2$. Hence

$$0 \neq \beta' : B_1 \xrightarrow{\beta} B_2 \rightarrowtail B^\nu \rightarrowtail \prod_\nu B .$$

By the product property of $\prod_\nu B$ there is also a $0 \neq \beta'' : B_1 \to B$ in I and a $0 \neq \beta''' : B \to B$ in I (B is a generator!). Since $Hom_{\mathfrak{B}}(B, B)$ is simple $I(B, B) = Hom_{\mathfrak{B}}(B, B)$ and hence also $I = Hom_{\mathfrak{B}}(\cdot, .)$ follows. So $\mathfrak{B}$ is simple and the proof is complete.

COROLLARY 6 (see also Theorem 10). *Let $\mathfrak{B}$ be a (not necessarily small) additive category. Then the following conditions are equivalent:*

(1) $\mathfrak{B}$ is simple with a minimal cosieve (left ideal) $S^B : \mathfrak{B} \to \mathfrak{Ab}$.

(2) $\mathfrak{B}$ is isomorphic to a full subcategory ($\neq 0$) of Vec_K consisting of finite dimensional K-spaces over a divisionring K. (Hence $\mathfrak{B}$ has always a countable skeleton.)

PROOF: Since S^B is a generator and since $[S^B, S^B]$ is a (small) divisionring (see again 7), (1) implies (2) by Theorem 4. Conversely by Theorem 20, clearly (2) implies (1). If now $0 \neq B \in |\mathfrak{B}|$ has dimension n, and if $R_{n-1} \subset B$ is a subspace of dimension n-1, there is a minimal cosieve

$$S^B : \mathfrak{B} \to \mathfrak{Ab} \text{ such that } S^B(x) = \{\beta : B \to X \mid Ker\,\beta \supset R_{n-1}\}.$$

14. Semisimple categories.

We call a (not necessarily small) additive category $\mathfrak{B}$ *semisimple* if $Hom_{\mathfrak{B}}(\cdot, .)$ is semisimple; i. e. $\mathfrak{B}$ splits into the direct sum of (also as categories!) simple ideals. Now we have:

THEOREM 21. *Let $\mathfrak{B}$ be a (not necessarily small) additive category. Then the following conditions are equivalent:*

(1) $\mathfrak{B}$ splits into a finite direct sum of (also as categories!) simple ideals $\mathfrak{B}_i$.

(2) $\mathfrak{B}$ satisfies the D.C.C. *for ideals and the zero-ideal is the intersection of maximal ideals.*

PROOF: Clearly (1) $\Longrightarrow$ (2) holds. Conversely we proceed similarly as in the proof of Theorem 16. So let us consider finite intersections of maximal ideals I_ν. Let $I = I_1 \cap \ldots \cap I_n$ be a minimal ideal with this property; then $I = 0$ must hold. Without loss of generality let us assume that no I_i is superfluous in $0 = I_1 \cap \ldots \cap I_n$. Define $\mathfrak{B}_j = \bigcap_{j \neq i=1}^{n} I_i$; then $I = I_j \oplus \mathfrak{B}_j$ with $\mathfrak{B}_j$ simple (also as category). Writing $\mathfrak{R}_k = \bigcap_{i=1}^{k} I_i$, we shall prove

$$\mathfrak{B} = \mathfrak{B}_1 \oplus \ldots \oplus \mathfrak{B}_k \oplus \mathfrak{R}_k \quad (k = 1, 2, \ldots, n).$$

For $k = 1$ this is clear. By the noetherian isomorphism theorem we have

$$\mathfrak{R}_k / \mathfrak{R}_k \cap I_{k+j} \simeq \mathfrak{R}_k + I_{k+j} / I_{k+j}.$$

Since for $k<n$ in $\bigcap_{j=1}^{n-k} (\mathfrak{R}_k \cap I_{k+j}) \simeq 0$ no $\mathfrak{R}_k \cap I_{k+j}$ is superfluous, likewise the $\mathfrak{R}_k \cap I_{k+j}$ are not only maximal in $\mathfrak{R}_k$ but also unequal to $\mathfrak{R}_k$. Hence again we have

$$\mathfrak{R}_k = (\mathfrak{R}_k \cap I_{k+1}) \oplus \bigcap_{j=2}^{n-k} (\mathfrak{R}_k \cap I_{k+j}) = \mathfrak{R}_{k+1} \oplus \mathfrak{B}_{k+1}.$$

So together with $k < n$ our statement holds also for $k+1$, hence for n. So $\mathfrak{R}_n = 0$ implies $\mathfrak{B} = \mathfrak{B}_1 \oplus \ldots \oplus \mathfrak{B}_n$ and the proof is complete.

Now, the following structure theorem characterizes (not necessarily small) semisimple categories, namely:

THEOREM 22. *Let $\mathfrak{B}$ be a (not necessarily small) additive category. Then the following conditions are equivalent:*

(1) $\mathfrak{B}$ is semisimple.

(2) The zero-ideal of $\mathfrak{B}$ is the intersection of maximal ideals and each subcategory $\mathfrak{B}_0$ of $\mathfrak{B}$ having only finitely many objects satisfies the D.C.C. *for ideals.*

(3) $\mathfrak{B}$ is isomorphic to a full and discrete (reduced) subcategory of a product category $\mathfrak{P} = \prod_{j \in J} \mathfrak{A}_j$ of simple categories $\mathfrak{A}_j$.

So a semisimple category $\mathfrak{B}$ is also semiprimitive.

PROOF: Assume (1). Then clearly the zero-ideal of $\mathfrak{B}$ is the intersection of maximal ideals. Let $\mathfrak{B}_0$ be a subcategory of $\mathfrak{B} = \bigoplus_{j \in I} \mathfrak{B}_j$ ($\mathfrak{B}_j$ simple) having only finitely many objects. Then obviously $\mathfrak{B}_0 = \bigoplus_{j\in I}(\mathfrak{B}_j \cap \mathfrak{B}_0)$. Since only finitely many $\mathfrak{B}_j \cap \mathfrak{B}_0$ are not the zero-ideal in $\mathfrak{B}_0$ and since these $\mathfrak{B}_j \cap \mathfrak{B}_0$ must be simple (by Lemma 6 for ideals!), immediately (2) follows (see also Theorem 21). Now let us assume (2). Then (2) holds also for each full subcategory $\mathfrak{B}'$ of $\mathfrak{B}$ (Lemma 6 for ideals!). Hence by Theorem 21 a $\mathfrak{B}_0$, resp. $\mathfrak{B}_0'$, from (2) splits into a finite direct sum of (also as categories!) simple ideals. Suppose $\mathfrak{B}_0' \supset \mathfrak{B}_0$ and let $\mathfrak{B}_{o,i}$ be a simple ideal in $\mathfrak{B}_0$. We shall show that the ideal $\{\mathfrak{B}_{o,i}\}_{\mathfrak{B}_0'}$ generated by $\mathfrak{B}_{o,i}$ in $\mathfrak{B}_0'$ is also simple. Consider a direct sum $\mathfrak{B}_0' = \bigoplus \mathfrak{B}'_{o,j}$ of simple ideals $\mathfrak{B}'_{o,j}$ in $\mathfrak{B}_0'$; then there is obviously $r < n$ with $\{\mathfrak{B}_{o,i}\}_{\mathfrak{B}_0'} = \bigoplus_{k=1}^{r} \mathfrak{B}_{o,j_k}$. Hence

$$\mathfrak{B}_{o,i} = \bigoplus_{k=1}^{r} (\mathfrak{B}_0 \cap \mathfrak{B}'_{o,j_k}).$$

Since $\mathfrak{B}_{o,i}$ is simple in $\mathfrak{B}_0$, there is an i with $\mathfrak{B}_{o,i} = \mathfrak{B}_0 \cap \mathfrak{B}'_{o,i}$. Since we have $\mathfrak{B}_{o,i} \subset \mathfrak{B}'_{o,i} \subset \{\mathfrak{B}_{o,i}\}_{\mathfrak{B}_0'}$, we conclude $\mathfrak{B}'_{o,i} = \{\mathfrak{B}_{o,i}\}_{\mathfrak{B}_0'}$. So $\{\mathfrak{B}_{o,i}\}_{\mathfrak{B}_0'}$ is really simple. Now let $\mathfrak{B}_{o,i}$ be again a simple ideal in a subcategory $\mathfrak{B}_0$ mentioned in (2). We consider the ideal $\mathfrak{B}_i = \{\mathfrak{B}_{o,i}\}_{\mathfrak{B}}$ generated by $\mathfrak{B}_{o,i}$ in $\mathfrak{B}$. Let $\mathfrak{B}_i'$ be an ideal ($\neq 0$) in $\mathfrak{B}$ with $\mathfrak{B}_i' \subset \mathfrak{B}_i$. Then there is obviously a subcategory $\mathfrak{B}_0' \supset \mathfrak{B}_0$ (like in (2)) satisfying $\mathfrak{B}_0' \cap \mathfrak{B}' \neq 0$ in $\mathfrak{B}_0'$. Clearly $\{\mathfrak{B}_{o,i}\}_{\mathfrak{B}_0'} \subset \mathfrak{B}_i \cap \mathfrak{B}_0'$. By

$$\{\{\mathfrak{B}_{o,i}\}_{\mathfrak{B}_0'}\}_{\mathfrak{B}} \supset \{\mathfrak{B}_{o,i}\}_{\mathfrak{B}} = \mathfrak{B}_i$$

(and Lemma 6 for ideals!), we have

$$\{\mathfrak{B}_{o,i}\}_{\mathfrak{B}_0'} = \{\{\mathfrak{B}_{o,i}\}_{\mathfrak{B}_0'}\}_{\mathfrak{B}} \cap \mathfrak{B}_0' \supset \mathfrak{B}_i \cap \mathfrak{B}_0'.$$

Hence

$$\{\mathfrak{B}_{o,i}\}_{\mathfrak{B}_0'} = \mathfrak{B}_i \cap \mathfrak{B}_0' \supset \mathfrak{B}_i' \cap \mathfrak{B}_0' \neq 0.$$

Since $\{\mathfrak{B}_{o,i}\}_{\mathfrak{B}_0'}$ is simple (see above), $\mathfrak{B}_i \cap \mathfrak{B}_0' = \mathfrak{B}_i' \cap \mathfrak{B}_0'$ and so

$$\mathfrak{B}_i' \cap \mathfrak{B}_0 = \mathfrak{B}_i \cap \mathfrak{B}_0 = \mathfrak{B}_{o,i}$$

(Lemma 6 for ideals!). By $\mathfrak{B}_{o,i} \subset \mathfrak{B}_i' \subset \mathfrak{B}_i$ immediately $\mathfrak{B}_i' = \mathfrak{B}_i$ follows. So $\mathfrak{B}_i = \{\mathfrak{B}_{o,i}\}_{\mathfrak{B}}$ is (together with $\mathfrak{B}_{o,i}$) simple. Hence $Hom_{\mathfrak{B}}(\cdot,.)$ is generated by its simple ideals and hence (by the usual arguments using the ZORN-Lemma) also semisimple. So (1) is true. By Lemma 10 obviously

(1) $\Longleftrightarrow$ (3). This completes the proof.

REMARK 14. By Theorem 18 we see that for a semisimple category $\mathfrak{B}$ the direct sum decomposition $\mathfrak{B} = \bigoplus_{j \in J} \mathfrak{B}_j$ into (also as categories!) simple ideals $\mathfrak{B}_j$ ($\neq 0$) is unique. The cardinality $|J|$ of J is an invariant and is called the dimension $dim\,\mathfrak{B}$ of $\mathfrak{B}$. As in Theorem 19, $dim\,\mathfrak{B}$ is finite if and only if the D.C.C. for ideals holds in $\mathfrak{B}$. As in Remark 13, $dim\,\mathfrak{B}$ is a small cardinal number if and only if $\mathfrak{B}$ has a small skeleton, etc.... Of course, Theorem 22 also implies Theorem 17. Moreover one notices that semisimple ARTIN-categories and semiprimitive ARTIN-categories coincide.

Abteilung für Mathematik
Ruhr Universität
BOCHUM. R. F. A.

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