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GERMS OF QUASI-CONTINUOUS FUNCTIONS

by YUH-CHING CHEN

Introduction.

The notion of quasi-topological spaces was first introduced by Kowalsky [12] under the German name «Limesräume». Since then it has been applied to various branches of mathematics such as differential Geometry [1], [8], functional analysis [1], [2], [4], [6], [7], theory of differentiations [1], [3], [10], [14], [15], and algebraic topology [16]. It was Bastiani [1] who first applied this notion to differentiable manifolds and introduced the French term *-quasi-topologie-* which is not related to the quasi-topology defined by Spanier [18]. Since this work is inspired by some works of Ehresmann's school [8], [14], [15], [16], the term *quasi-topology* here is a translation of the French «quasi-topologie».

In this paper, we try to generalize the notions of germs of functions and sheaves in topological sense to that of π -germs of functions and π -sheaves in quasi-topological sense and to study the relations between these notions. We begin with a brief review of some basic definitions and properties on quasi-topologies and the introduction of the notion of germs and π -germs of functions using inductive limits. Then we generalize the notions of pre-sheaves and sheaves over a topological space to that of π -presheaves and π -sheaves over a quasi-topological space and show that every π -sheaf E is reflected by the sheaf of germs of quasi-continuous local sections of E . In fact the category of π -sheaves over a quasi-topological space (X, π) contains a reflective full subcategory isomorphic to the category of abelian sheaves over the underlying topological space (X, T_π) of (X, π) . Finally, we show that the canonical injective structure of this reflective subcategory determines an effacement structure [13] on the category of π -sheaves. The homological Algebra of this effacement structure appears more complicated than the relative

homological Algebra of Eilenberg-Moore [9] and Maranda [17]. We shall defer this pending further investigations.

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1. Germs and π -germs of functions.

π will always stand for a quasi-topology on a set X . It is a function that associates to each $x \in X$ a family $\pi(x)$ of filters of subsets of X satisfying the conditions:

- (1) $F_1, F_2 \in \pi(x)$ implies $F_1 \cap F_2 \in \pi(x)$,
- (2) $F_1 \in \pi(x)$ and $F_2 \supset F_1$ implies $F_2 \in \pi(x)$,
- (3) the filter x^ϵ of all subsets of X containing x is in $\pi(x)$.

If $F \in \pi(x)$, we say that F converges to x in π . The pair (X, π) is called a *quasi-topological space*.

Let (X, π) and (E, τ) be quasi-topological spaces. A function $f: (X, \pi) \rightarrow (E, \tau)$, often written $f: X \rightarrow E$, is (π, τ) -continuous (called *quasi-continuous* in [1], [15]) if, for every $x \in X$ and every $F \in \pi(x)$, the images of the sets in F under f generate a filter $f(F) \in \tau(f(x))$. f is *quasi-open* if for every $x \in X$ and every $G \in \tau(f(x))$, there is $F \in \pi(x)$ with $f(F) \subset G$. If A is a subset of X , the quasi-topology induced on A by π is denoted $\pi|A$ and we say that $(A, \pi|A)$ is a (quasi-topological) *subspace* of (X, π) .

A topology T on X is identified with the quasi-topology π_T on X in which the filter of neighborhoods of $x \in X$ is the smallest filter in $\pi_T(x)$. The category $\mathcal{T}$ of topological spaces and continuous functions is identified with a full subcategory of the category $\mathcal{Q}\mathcal{T}$ of quasi-topological spaces and quasi-continuous functions (see e.g. [15]).

The *underlying topology* T_π of π is defined as follows: A set $U \subset X$ is open (in T_π) if and only if, for every $x \in U$, $F \in \pi(x)$ implies

$U \in F$. Thus every (π, τ) -continuous function $f: X \rightarrow E$ is continuous in the underlying topologies T_π and T_τ . Note that $1: (X, \pi) \rightarrow (X, T_\pi)$ is (π, T_π) -continuous. In fact, the underlying topology functor $T: \mathcal{QJ} \rightarrow \mathcal{J}$ is left adjoint to the inclusion functor $\mathcal{J} \subset \mathcal{QJ}$, i.e., $\mathcal{J}$ is identified with a coreflective subcategory of $\mathcal{QJ}$ (in French term, T is a projector functor). For further definitions and properties concerning quasi-topologies we refer the readers to [1], [16].

Convention. Let A be a subset of X . A (π, τ) -continuous function $f: A \rightarrow E$ is the restriction to A of a $(\pi|_U, \tau)$ -continuous function from an open neighborhood U of A to E . Thus if α is the directed set (by inclusion) of open neighborhoods of A and if $\mathcal{QJ}(U, E)$ denotes the set of (π, τ) -continuous functions from $U \in \alpha$ to E , then the *restriction map*

$$(1.1) \quad r: \lim_{\rightarrow \alpha} \mathcal{QJ}(U, E) \rightarrow \mathcal{QJ}(A, E)$$

is a surjection, where $\{\mathcal{QJ}(U, E) \mid U \in \alpha\}$ forms a direct system of sets of (π, τ) -continuous functions with genuine restriction maps.

Let $Q(x) = \bigcap \{F_i \mid F_i \in \pi(x)\}$ be the filter that is the intersection of all filters F_i in $\pi(x)$. Then each $A_x \in Q(x)$ is the union of a family of subsets of X one from each filter $F_i \in \pi(x)$. A set $A_x \in Q(x)$ is called a π -neighborhood of x (thus every neighborhood is a π -neighborhood). Notice that: (1) each $A_x \in Q(x)$ contains x , but $Q(x)$ may not converge to x in π , and (2) A_x may not be a π -neighborhood of another point $y \in A_x$.

We proceed now to define germs and π -germs of functions using inductive limits. Let $O(x)$ be the set of open neighborhoods of x (open in the underlying topology T_π of π). Then $O(x) \subset Q(x)$. Order both sets $O(x)$ and $Q(x)$ by inclusion. Then $O(x)$ is a directed subset of $Q(x)$. The inductive limit

$$\lim_{\rightarrow} \mathcal{QJ}(A, E), \quad A \in Q(x) \quad (\text{resp. } \lim_{\rightarrow} \mathcal{QJ}(U, E), \quad U \in O(x)),$$

is the set of π -germs (resp. germs) of (π, τ) -continuous functions at x . Each $f: A \rightarrow E$ in $\mathcal{QJ}(A, E)$ (resp. $f: U \rightarrow E$ in $\mathcal{QJ}(U, E)$) determines

a π -germ (resp. germ) f_x of a (π, τ) -continuous function at x . Often, we simply call f_x the *limit of f at x* . Since $O(x) \subset Q(x)$ as directed sets, there is a map

$$(1.2) \quad \gamma_x: \lim_{\rightarrow} \mathcal{QJ}(U, E) \rightarrow \lim_{\rightarrow} \mathcal{QJ}(A, E)$$

that associates to each germ f_x at x a π -germ $f'_x = \gamma_x(f_x)$ at x . It follows from (1.1) that γ_x is surjective. If $O(x)$ is cofinal in $Q(x)$, then the notions of germs and π -germs coincide. In particular, this is the case when π is topological.

2. π -sheaves.

A map $p: E \rightarrow X$ of quasi-topological spaces (E, τ) and (X, π) defines a π -sheaf E if the following conditions are satisfied (see [5]):

(S1) For every point $f_x \in E$ with $p(f_x) = x$, there exists a subset $U_f \subset E$ containing f_x such that the map $p|_{U_f}$ is a (τ, π) -homeomorphism of $(U_f, \tau|_{U_f})$ onto an open neighborhood U_x of x ;

(S2) τ is the final quasi-topology determined by all $\tau|_{U_f}$ via inclusion maps;

(S3) For every $x \in X$, the stalk $E_x = p^{-1}(x)$ is an abelian group and the group operations are quasi-continuous in τ .

In particular, if all U_f in (S1) can be chosen open in the underlying topology T_τ of τ , then we say that p *spreads* E over X and E is a π -*spreading space*. It is easy to see that:

PROPOSITION 2.1. *If E is a π -sheaf, then p is (τ, π) -continuous and quasi-open (cf. proposition 1.2.16 of [15]).*

COROLLARY 2.2. *If (E, τ) is a π -spreading space, then (E, T_τ) is an abelian sheaf over the topological space (X, T_π) . (We assume that the readers are familiar with the general theory of abelian sheaves).*

Let $p: (E, \tau) \rightarrow (X, \pi)$ be a π -sheaf. A *section of E over a subset A of X* is a function $s: A \rightarrow E$ which is the restriction to A of a section s' of E over an open neighborhood U of A (i.e. $s': U \rightarrow E$ is a $(\pi|_U, \tau)$ -continuous function such that $p s'$ is the identity of U); in particular, s is (π, τ) -continuous on A . The set $\Gamma(A, E)$ of sections of E

over A is an abelian group (the addition is pointwise). It follows that the restriction map

$$(2.1) \quad r: \lim_{\rightarrow \alpha} \Gamma(U, E) \rightarrow \Gamma(A, E)$$

is an epimorphism for every π -neighborhood A of $x \in X$. This will be referred to as property (PS) in the definition of π -presheaves in the next section.

A map $\phi: E \rightarrow F$ of π -sheaves (E, τ) and (F, σ) is called a π -homomorphism if: (1) ϕ is (τ, σ) -continuous, and (2) for every $x \in X$ the map $\phi_x = \phi|_{E_x}$ is a group homomorphism of E_x into F_x . We write $\phi = \{\phi_x | x \in X\}$. The classes of π -sheaves and π -homomorphisms form a category $\mathcal{L}_\pi$ called the *category of π -sheaves*.

PROPOSITION 2.3. *The class of π -spreading spaces form a full subcategory $\mathcal{L}_X$ of $\mathcal{L}_\pi$. If $\mathcal{L}_T$ denotes the category of abelian sheaves over the topological space (X, T_π) , then the underlying topology functor $T: \mathcal{L}_\pi \rightarrow \mathcal{T}$ induces an isomorphism $T_X: \mathcal{L}_X \rightarrow \mathcal{L}_T$ of categories.*

Indeed, T_X sends a π -spreading space $p: (E, \tau) \rightarrow (X, \pi)$ to an abelian sheaf $p: (E, T_\tau) \rightarrow (X, T_\pi)$. The inverse of T_X is defined as follows. Let $\pi: (E, \mathcal{U}) \rightarrow (X, T_\pi)$ be an abelian sheaf, where $\mathcal{U}$ is a topology on E . Then, by definition, for every point $f_x \in E$ with $p(f_x) = x$, there is a $U_f \in \mathcal{U}$ such that $p|_{U_f}$ is a homeomorphism of U_f onto an open neighborhood U_x of x . Endow each U_f with a quasi-topology τ_f that makes $p|_{U_f}$ a (τ, π) -homeomorphism and let τ be the final quasi-topology on E determined by all τ_f via inclusion maps. Then $p: (E, \tau) \rightarrow (X, \pi)$ is a π -spreading space. T_X^{-1} carries $p: (E, \mathcal{U}) \rightarrow (X, T_\pi)$ to $p: (E, \tau) \rightarrow (X, \pi)$.

3. Construction of π -sheaves.

Let $\mathcal{Q}_\pi$ be the category whose class of objects is the set

$$\{\emptyset\} \cup \bigcup_{x \in X} \mathcal{Q}(x) = \{A_x \in \mathcal{Q}(X) | x \in X\} \cup \{\emptyset\}$$

of π -neighborhoods of points of X and whose morphisms are inclusion maps, and let Ab be the category of abelian groups and homomorphisms. A π -presheave is a contravariant functor $P: \mathcal{Q}_\pi \rightarrow Ab$ satisfying the condition:

(PS) For every $A \in Q(x)$, the restriction map $r : \lim_{\rightarrow \alpha} P(U) \rightarrow P(A)$ is an epimorphism, where α is the set of all open neighborhoods U of A directed by inclusion. A *homomorphism* of π -presheaves is a natural transformation of functors. π -presheaves and their homomorphisms form a category $\mathcal{P}_\pi$ of functors.

A typical example of a π -presheaf is the π -presheaf ΓE of local sections of a π -sheaf E defined as follows. For every inclusion map $i: B \rightarrow A$ in Q_π , the map $(\Gamma E)(i): \Gamma(A, E) \rightarrow \Gamma(B, E)$ is the restriction map of sections of E over A to that of E over B . The property (PS) is verified by (2.1). In fact, there is a functor $\Gamma: \mathcal{Q}_\pi \rightarrow \mathcal{P}_\pi$ called a *local section functor*.

Let P be a π -presheaf. We shall construct the *associated π -sheaf* SP of P as follows. For every $x \in X$ let

$$(3.1) \quad (SP)_x = \lim_{\rightarrow} P(A_x), \quad A_x \in Q(x)$$

be the set of limits f_x of $f \in P(A_x)$ at x , and let

$$(3.2) \quad SP = \bigcup_{x \in X} (SP)_x.$$

We shall endow SP with a quasi-topology τ so that the projection $p: SP \rightarrow X$ defined by $p(f_x) = x$ is a π -sheaf: For each open set U of X and for each $f \in P(U)$ let

$$(3.3) \quad U_f = \{f_x \in (SP)_x \mid x \in U \text{ and } f_x = \text{limit of } f \text{ at } x\}$$

be the set of points of SP which are the limits of f at points of U . Endow each U_f with a quasi-topology that makes $p|_{U_f}: U_f \rightarrow U$ a (τ_f, π) -homeomorphism. Then we have

$$SP = \bigcup \{U_f \mid U \text{ open in } X, f \in P(U)\},$$

and τ_f and τ_g agree on $U_f \cap V_g$ for any two sets U_f and V_g defined by (3.3). τ is the final quasi-topology on SP determined by all τ_f via inclusion maps. Then

PROPOSITION 3.1. $p: (SP, \tau) \rightarrow (X, \pi)$ is a π -sheaf called the *associated π -sheaf* of the π -presheaf P .

In practice, most of π -sheaves are constructed in this way from π -

presheaves of π -germs of functions satisfying some prescribed properties such as quasi-continuous, quasi-holomorphic [15], etc... In fact the introduction of the notion of π -sheaves is motivated by this sort of examples. We should point out that in the construction above the sets U_f are not open in T_τ in general. Since the limit f_x of $f \in P(U)$ is taken on the directed set $Q(x)$, not on $O(x)$, there may exist $f, g \in P(U)$ with $U_f \cap U_g$ not open in U_f or U_g .

Finally, we shall see that there is a functor $S: \mathcal{P}_\pi \rightarrow \mathcal{Q}_\pi$ that sends a homomorphism $\rho: P \rightarrow P'$ of π -presheaves to a π -homomorphism $\phi: SP \rightarrow SP'$ defined as follows. Since ρ is a natural transformation of functors, it consists of a family of group homomorphisms $\rho_A: P(A) \rightarrow P'(A)$ indexed by the objects A of Q_π . For a point $f_x \in SP$, that is the limit of $f \in P(A)$ at x , let $\phi(f_x)$ be the limit of $\rho_A(f) \in P'(A)$ at x , i.e.

$$(3.4) \quad \phi(f_x) = g_x, \text{ where } g = \rho_A(f) \in P'(A).$$

It is obvious that ϕ is a π -homomorphism and that S is a functor.

4. The functors S and Γ .

THEOREM 4.1. *The functor $S: \mathcal{P}_\pi \rightarrow \mathcal{Q}_\pi$ is left adjoint to the functor $\Gamma: \mathcal{Q}_\pi \rightarrow \mathcal{P}_\pi$.*

The proof will follow two lemmas.

LEMMA 1. *There is a natural transformation from the composite functor $S\Gamma$ of S and Γ to the identity functor of $\mathcal{Q}_\pi$.*

PROOF. Let E be a π -sheaf. Then for every $x \in X$,

$$(4.1) \quad (S\Gamma E)_x = \lim_{\rightarrow} (\Gamma E)(A) = \lim_{\rightarrow} \Gamma(A, E), \quad A \in Q(x).$$

That is, $(S\Gamma E)_x$ is the group of π -germs at x of local sections of E ; a point in $(S\Gamma E)_x$ is the π -germ s_x at x represented by a section $s \in \Gamma(A, E)$. Let $\theta(s_x) = s(x)$. Then $\theta: S\Gamma E \rightarrow E$ so defined is a π -homomorphism. The class of θ (indexed by the objects E of $\mathcal{Q}_\pi$) form a natural transformation from $S\Gamma$ to the identity functor of $\mathcal{Q}_\pi$. Moreover, it is easily shown that:

COROLLARY. *The quasi-topology of E is the final quasi-topology deter-*

mined by that of $S\Gamma E$ via θ , i.e., θ is a π -epimorphism.

LEMMA 2. There is a natural transformation from the identity functor of $\mathcal{P}_\pi$ to the composite functor ΓS of Γ and S .

PROOF. Let P be a π -presheaf. Then $(\Gamma SP)(A) = \Gamma(A, SP)$ for every $A \in |Q_\pi|$. Define a map $b_A: P(A) \rightarrow \Gamma(A, SP)$ by $b_A(f) = s$ with $s(x) = f_x$ for every $x \in A$. This is well defined since, for every $f \in P(A)$, the family $\{f_x | x \in A\}$ do define a section s of SP over A . It is easily checked that b_A is a homomorphism and that

$$(4.2) \quad b = \{b_A: P(A) \rightarrow (\Gamma SP)(A) | A \in |Q_\pi|\}$$

is a π -homomorphism from P to ΓSP . The family of such b (indexed by the objects P of $\mathcal{P}_\pi$) form a natural transformation from the identity functor of $\mathcal{P}_\pi$ to ΓS .

PROOF OF THEOREM 4.1. We want to show that there is a natural equivalence

$$(4.2) \quad j: \mathcal{Q}_\pi(SP, E) \rightarrow \mathcal{P}_\pi(P, \Gamma E)$$

of the set of π -homomorphisms from SP to E to the set of homomorphisms from P to ΓE . For any $\phi \in \mathcal{Q}_\pi(SP, E)$, define $j(\phi) = \Gamma(\phi)b$ as in the diagram

$$\begin{array}{ccc} P & \xrightarrow{j(\phi)} & \Gamma E \\ b \downarrow & \searrow \Gamma(\phi) & \\ \Gamma SP & \xrightarrow{\quad} & \Gamma E \end{array} .$$

Then $j(\phi)$ consists of a set of homomorphisms $j(\phi)_A: P(A) \rightarrow (\Gamma E)(A)$ defined by

$$(4.4) \quad j(\phi)_A(f) = (\Gamma(\phi)_A b_A)(f) = \Gamma(\phi)_A(s) = \phi s,$$

where $s \in \Gamma(A, E)$ is the section $s(x) = f_x$. We claim that j is a bijection with inverse k defined by $k(\rho) = \theta S(\rho)$ for every $\rho \in \mathcal{P}_\pi(P, \Gamma E)$. Indeed,

$$(4.5) \quad \begin{aligned} (jk(\rho))_A(f)(x) &= \theta S(\rho)(s(x)) = \theta S(\rho)(f_x) = \\ &= \theta(\rho_A(f))_x = \rho_A(f)(x) \end{aligned}$$

for every $f \in P(A)$ shows that $j k(\rho) = \rho$. On the other hand,

$$(4.6) \quad \begin{aligned} k j(\phi)(f_x) &= \theta S(j(\phi))(f_x) = \theta(j(\phi)_A(f))_x = \\ &= (j(\phi)_A(f))(x) = \phi(f_x) \end{aligned}$$

shows that $k j(\phi) = \phi$. Since j and k are defined by functors and natural transformations, the bijection j is natural.

REMARK. (1) $\mathcal{P}_\pi$ and $\mathcal{L}_\pi$ are additive categories and j is indeed a natural isomorphism of groups.

(2) $\mathcal{P}_\pi$ and $\mathcal{L}_\pi$ are not abelian categories. For example, $\mathcal{P}_\pi$ is not closed under the formation of kernels since the property (PS) which is defined by colimits is not preserved by kernels.

5. The subcategory $\mathcal{L}_X$ of $\mathcal{L}_\pi$.

Recall that T_π is the underlying topology of π . Regard T_π as a category with morphisms inclusion maps; then it is a full subcategory of $\mathcal{Q}_\pi$. Let $\mathcal{P}_X$ be the category of presheaves over (X, T_π) , i.e., the category of contravariant functors from T_π to Ab . Then there is a functor $R': \mathcal{P}_\pi \rightarrow \mathcal{P}_X$ defined by $R'(P) = P|_{T_\pi}$. On the other hand, we define a functor $J': \mathcal{P}_X \rightarrow \mathcal{P}_\pi$ as follows. For every presheaf $G: T_\pi \rightarrow Ab$, let $J'G$ be a mapping on $\mathcal{Q}_\pi$ to Ab with

$$(5.1) \quad (J'G)(A) = \lim_{\rightarrow \alpha} G(U), \quad \forall A \in |\mathcal{Q}_\pi|,$$

where α is the set of open neighborhoods of A directed by inclusion. Then $J'G$ verifies the property (PS) and thus defines a π -presheaf. By a routine limit argument in category theory, one shows that the correspondence $G \rightarrow J'G$ defines a functor J' from $\mathcal{P}_X$ to $\mathcal{P}_\pi$ and that

PROPOSITION 5.1. *J' is left adjoint to R' . Moreover, the composite functor $R'J'$ of R' and J' is naturally equivalent to the identity functor of $\mathcal{P}_X$ (and therefore J' is a full embedding).*

Recall that $\mathcal{L}_X$ is a full subcategory of $\mathcal{L}_\pi$ (see proposition 2.3). $\Gamma|_{\mathcal{L}_X}$ defines a functor Γ' from $\mathcal{L}_X$ to $\mathcal{P}_X$ that can be identified with the composite functor $R'\Gamma J$, where J is the inclusion functor of $\mathcal{L}_X$ in $\mathcal{L}_\pi$. On the other hand, a functor $S': \mathcal{P}_X \rightarrow \mathcal{L}_X$ with

$$(5.2) \quad S'G = \bigcup_{x \in X} (S'G)_x, \text{ where } (S'G)_x = \varinjlim G(U), U \in O(x),$$

can be defined by replacing Q_π by T_π in the construction of S in section 3. Notice that here the limit f_x of f at x is taken on $O(x)$ instead of $Q(x)$; contrary to the remark of section 3, $U_f \cap V_g$ is always open in τ_f and τ_g . Therefore, the quasi-topology τ on $S'G$ is the only quasi-topology that renders each U_f an open subset of $S'G$ (cf. [16], p.28). In fact, the family of all subsets U_f form a basis for the underlying topology T_τ on $S'G$. Similar to theorem 4.1 we have

PROPOSITION 5.2. *S' is left adjoint to Γ' . Moreover, the composite functor $S'\Gamma'$ is naturally equivalent to the identity functor of $\mathcal{L}_X$.*

If $\mathcal{L}_X$ is identified with the category $\mathcal{L}_T$ of abelian sheaves over (X, T_π) by the functor T_X of proposition 2.3, then the functors S' and Γ' are identified with the associated sheaf functor and the local section functor, respectively, of the theory of sheaves.

Like $\mathcal{L}_T$, $\mathcal{L}_X$ is an abelian category with enough injectives; it is AB5 (see [11]). The injective structure on $\mathcal{L}_X$ is called *the canonical injective structure on $\mathcal{L}_X$* .

6. Germs of local sections of a π -sheaf.

In the diagram

$$\begin{array}{ccc} \mathcal{L}_X & \begin{array}{c} \xrightarrow{J} \\ \xleftarrow{R} \end{array} & \mathcal{L}_\pi \\ \Gamma' \downarrow \uparrow S' & & \Gamma \downarrow \uparrow S \\ \mathcal{P}_X & \begin{array}{c} \xleftarrow{R'} \\ \xrightarrow{J'} \end{array} & \mathcal{P}_\pi \end{array}$$

of categories and functors, let $R = S'R'\Gamma$. Then for any π -sheaf E ,

$$(6.1) \quad (RE)_x = (S'R'\Gamma E)_x = \varinjlim \Gamma(U, E), U \in O(x),$$

since $(R'\Gamma E)(U) = \Gamma(U, E)$ for every U in T_π . Thus $RE = \bigcup_{x \in X} (RE)_x$

is the π -sheaf (indeed a sheaf) of germs of local sections of E . Since

$$(6.2) \quad S\Gamma E = \bigcup_{x \in X} \varinjlim \Gamma(A, E), \quad A \in Q(x)$$

is the π -sheaf of π -germs of local sections of E , formula (1.2) and property (PS) show that there is a surjective π -homomorphism $\zeta : RE \rightarrow S\Gamma E$ defined by the set of group epimorphisms:

$$(6.3) \quad \zeta_x : \varinjlim \Gamma(U, E) \rightarrow \varinjlim \Gamma(A, E), \quad x \in X.$$

We claim that the quasi-topology σ' on $S\Gamma E$ is the final quasi-topology determined by the quasi-topology σ on RE via ζ and therefore ζ is a π -epimorphism. Indeed, since σ (resp. σ') is the final quasi-topology determined by the family $\sigma_f = \sigma|_{U_f}$ (resp. $\sigma'_f = \sigma'|_{U_f}$) via inclusion maps $U_f \subset RE$ (resp. $U_f \subset S\Gamma E$), every $p|_{U_f}$ is a (σ, π) -homeomorphism (resp. (σ', π) -homeomorphism) of U_f onto U . Now, let (Y, σ'') be a quasi-topological space and let $\phi : S\Gamma E \rightarrow Y$ be a map such that $\phi\zeta$ is (σ, σ'') -continuous. Then, since each $\phi|_{U_f} = (\phi\zeta|_{U_f})(\zeta|_{U_f})^{-1}$ is (σ', σ'') -continuous, so is ϕ (cf. [16]). This shows that σ' is the final quasi-topology determined by σ via ζ .

Recall (corollary of lemma 1 of section 4) that $\theta : S\Gamma E \rightarrow E$ is a π -epimorphism; so is

$$(6.4) \quad \psi : RE \rightarrow E, \quad \psi = \theta\zeta.$$

We identify $\mathcal{L}_X$ with $\mathcal{L}_T$ and see that every π -sheaf E is a quotient of the sheaf of germs of local sections of E . More generally, we shall prove that $\mathcal{L}_X$ is a reflective subcategory of $\mathcal{L}_\pi$ and that ψ of (6.4) is a reflection. Thus every π -sheaf is *reflected* by the sheaf of germs of its local sections.

THEOREM 6.1. $R : \mathcal{L}_\pi \rightarrow \mathcal{L}_X$ is right adjoint to the inclusion functor $J : \mathcal{L}_X \rightarrow \mathcal{L}_\pi$, i.e., R is a reflector.

PROOF. We want to show that for E' in $\mathcal{L}_X$ and E in $\mathcal{L}_\pi$, there is a natural bijection

$$(6.5) \quad \mathcal{L}_\pi(JE', E) \rightarrow \mathcal{L}_X(E', RE).$$

We observe that $J = SJ'\Gamma'$. Indeed, since

$$(J'\Gamma'E')(A) = \varinjlim \Gamma(U, E'), \quad \forall A \in |Q_\pi|,$$

we have

$$(S J' \Gamma' E')_x = \lim_{\rightarrow} (J' \Gamma' E')(A) = \lim_{\rightarrow} \Gamma(U, E') = E'_x.$$

Therefore $S J' \Gamma' E' = E'$. Now, by theorem 4.1 and proposition 5.1,

$$\mathcal{L}_\pi(J E', E) = \mathcal{L}_\pi(S J' \Gamma' E', E) \approx \mathcal{P}_\pi(J' \Gamma' E', \Gamma E) \approx \mathcal{P}_X(\Gamma' E', R' \Gamma E).$$

Since $S' | \Gamma'(\mathcal{L}_X)$ is a full embedding, proposition 5.2 shows that

$$\mathcal{P}_X(\Gamma' E', R' \Gamma E) \approx \mathcal{L}_X(S' \Gamma' E', S' R' \Gamma E) \approx \mathcal{L}_X(E', R E).$$

Thus, $\mathcal{L}_\pi(J E', E)$ is naturally isomorphic to $\mathcal{L}_X(E', R E)$.

COROLLARY 6.2. $R | \mathcal{L}_X$ is the identity functor of $\mathcal{L}_X$.

7. An effacement structure on $\mathcal{L}_\pi$.

Let the canonical injective structure of $\mathcal{L}_X$ be denoted $(\mathbb{M}, \varepsilon)$ and identify $\mathcal{L}_X$ with $\mathcal{L}_T$. Then $\mathbb{M}$ is the class of sheaf monomorphisms and ε is the class of injective sheaves [11]. $(\mathbb{M}, \varepsilon)$ induces an effacement structure $(\mathcal{F}', \mathcal{H}')$ on $\mathcal{L}_X$, where $\mathcal{F}' = \mathbb{M}$ and $\mathcal{H}'$ is the class of π -homomorphisms that factor through an injective object of $\mathcal{L}_X$ (see proposition 2.13 of [13]).

The notion of an *effacement structure* on a category C was first defined by Zimmermann [19] under the German name "Erweiterungspaare". It consists of a pair $(\mathcal{F}, \mathcal{H})$ of two classes of morphisms of C satisfying the following three conditions:

- (1) $\mathcal{H}$ is the class of all morphisms $b: A \rightarrow B$ such that, for every $f: X \rightarrow Y$ in $\mathcal{F}$ and for every given $u: X \rightarrow A$ in C , there is a morphism $v: Y \rightarrow B$ such that $vf = bu$;
- (2) $\mathcal{F}$ is the class of all morphisms $f: X \rightarrow Y$ such that, for every $b: A \rightarrow B$ in $\mathcal{H}$ and for every given $u: X \rightarrow A$ in C , there is a morphism $v: Y \rightarrow B$ such that $vf = bu$;
- (3) for every object A in C , there is a morphism $f \in \mathcal{F} \cap \mathcal{H}$ with domain A .

We need also the following definition from [13]. $\mathcal{F}' \subset \mathcal{F}$ is a *sub-basis* for $\mathcal{F}$ if $\mathcal{F}$ is the class of morphisms $f: X \rightarrow Y$ such that there exists a push-out

$$\begin{array}{ccc}
 X' & \xrightarrow{f'} & Y' \\
 \downarrow & & \downarrow \\
 X & \xrightarrow{\phi} & Z
 \end{array}$$

with $f' \in \mathcal{F}'$ and a morphism $k: Y \rightarrow Z$ with $k f = \phi$.

Let $(\mathcal{F}', \mathcal{H}')$ be the effacement structure on $\mathcal{L}_X$ induced by $(\mathcal{M}, \varepsilon)$ as mentioned in the first paragraph. We have

THEOREM 7.1. *There is an effacement structure $(\mathcal{F}, \mathcal{H})$ on $\mathcal{L}_\pi$ in which $\mathcal{H} = R^{-1}(\mathcal{H}')$ and $\mathcal{F}$ has $\mathcal{F}'$ as a subbasis.*

In view of theorem 4.6 of [13], $(\mathcal{F}, \mathcal{H})$ is the inverse transfer of $(\mathcal{F}', \mathcal{H}')$ by the pair of adjoint functors J and R provided that push-outs exist in $\mathcal{L}_\pi$.

Given π -homomorphisms $\phi: E \rightarrow F$ and $\psi: E \rightarrow G$, we shall construct the push-out K of ϕ by ψ . For every $x \in X$, let K_x be the push-out of $\phi_x: E_x \rightarrow F_x$ by $\psi_x: E_x \rightarrow G_x$ (since push-outs exist in Ab) and let $K = \bigcup_{x \in X} K_x$. Then we have a diagram

$$\begin{array}{ccc}
 E & \xrightarrow{\phi} & F \\
 \psi \downarrow & & \downarrow i \\
 G & \xrightarrow{j} & K
 \end{array}$$

Endow K with the final quasi-topology τ determined by the quasi-topologies of F and G via i and j . Then (K, τ) is a π -sheaf which is the push-out of ϕ by ψ .

Finally, we shall generalize the notion of injective resolutions in sheaf theory to that of π -injective resolutions. Recall that for every sheaf E' (e.g. $E' = R E$ of a π -sheaf E) there is an injective resolution

$$0 \longrightarrow E' \longrightarrow Q'_*$$

defined as follows: Choose a map $i': E' \rightarrow Q'_0$ in $\mathcal{F}' \cap \mathcal{H}'$ (i.e. i' is a monomorphism with Q'_0 injective) and let G_0 be the cokernel of i' . Then we obtain an exact sequence

$$0 \longrightarrow E' \xrightarrow{i'} Q'_0 \xrightarrow{j'} G_0 \longrightarrow 0$$

Choose a map $k': G_0 \rightarrow Q'_1$ in $\mathcal{F} \cap \mathcal{H}'$ and let G_1 be the cokernel of k' . Then an exact sequence

$$0 \longrightarrow E' \xrightarrow{i'} Q'_0 \xrightarrow{d'_0} Q'_1 \longrightarrow G_1 \longrightarrow 0, \quad d'_0 = k'j',$$

is obtained. Continuing in this way, we get an injective resolution

$$0 \longrightarrow E' \xrightarrow{i'} Q'_0 \xrightarrow{d'_0} Q'_1 \xrightarrow{d'_1} Q'_2 \longrightarrow \dots$$

We shall generalize this construction to one for a π -sheaf E . Let Q_0 be the push-out of $i': RE \rightarrow Q'_0$ by the reflection $\psi: RE \rightarrow E$

$$\begin{array}{ccc} RE & \xrightarrow{i'} & Q'_0 \\ \psi \downarrow & & \downarrow \psi_0 \\ E & \xrightarrow{i} & Q_0 \end{array}.$$

Then it is easy to show that, since $i' \in \mathcal{F} \cap \mathcal{H}'$, $i \in \mathcal{F} \cap \mathcal{H}$. Moreover, i is a π -monomorphism. In the diagram

$$(7.1) \quad \begin{array}{ccccccc} 0 & \longrightarrow & RE & \xrightarrow{i'} & Q'_0 & \xrightarrow{d'_0} & Q'_1 & \xrightarrow{d'_1} & Q'_2 & \longrightarrow & \dots \\ & & \downarrow \psi & & \downarrow j' & \nearrow k' & \downarrow & \nearrow & \downarrow & & \\ 0 & \longrightarrow & E & \xrightarrow{i} & Q_0 & \xrightarrow{h} & Q_1 & \xrightarrow{g} & Q_2 & \longrightarrow & \dots \\ & & & & \downarrow j & \nearrow k & \downarrow & \nearrow & \downarrow & & \\ & & & & F_0 & & F_1 & & & & \end{array}$$

let F_0 be the cokernel of i , Q_1 be the push-out of k' by h , F_1 be the cokernel of $d'_0 = kj$ and Q_2 be a push-out again. Then by repeating this construction we obtain an exact sequence

$$(7.2) \quad 0 \xrightarrow{i} E \xrightarrow{d_0} Q_0 \xrightarrow{d_1} Q_1 \longrightarrow Q_2 \longrightarrow \dots$$

called a π -injective resolution of E in $\mathcal{L}_\pi$. We remark that, in general, nei-

ther Q_i is injective in $\mathcal{L}_\pi$, nor is RQ_i in $\mathcal{L}_X$. We call (7.2) a « π -injective» resolution just because it is constructed out of an *injective* resolution of RE . Nevertheless, when Q'_* is replaced by another injective resolution of RE , the corresponding π -injective resolution of E is chain homotopic to (7.2). Therefore, there is a cohomology theory on $\mathcal{L}_\pi$ defined by resolutions that generalizes the sheaf cohomology. We shall study this further under a more general topic.

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