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HIGHER ORDER FRAMES AND LINEAR CONNECTIONS

by YUEN Ping Cheng

Introduction.

In the first part of this paper we develop some elementary properties of semi-holonomic $k$-frames parallel to those of holonomic $k$-frames. Our definition of a semi-holonomic $k$-frame is essentially equivalent to the one originally given by Ehresmann [1b]; our formulation, however, leads us easily to define a canonical 1-form $\theta_k$ on the principal fibre bundle $\widetilde{H}^k(V^n)$ of semi-holonomic $k$-frames on a differentiable manifold $V^n$. If we restrict $\theta_k$ to the principal sub-bundle $H^k(V^n)$ of holonomic $k$-frames on $V^n$, we obtain the canonical 1-form given by Kobayashi [3]. Our main result is the "Holonomy Theorem" where we give a geometrical interpretation of the holonomy conditions in terms of the canonical 1-form. This result will be useful for studying the integrability of higher order $G$-structures. These preliminary results served originally as an introductory part to a forthcoming paper which deals with the structure tensors of higher order regular $G$-structures and higher order geometric structures.

The second part of this paper deals with the higher order linear connections. Let $V^n$ be a differentiable manifold. A linear connection of order $k$ on $V^n$ is an infinitesimal connection on the principal fibre bundle $\widetilde{H}^k(V^n)$. Its torsion form is defined to be the exterior covariant derivative of $\theta_k$. There is a one-to-one correspondence between the set of linear connections of order $k$ (resp. quasi-holonomic linear connections of order $k$ without torsion) on $V^n$ and the set of invariant sections of the canonical projection $\widetilde{H}^{k+1}(V^n) \rightarrow H^1(V^n)$. We show further that a linear connection of order $k$ on $V^n$ is locally flat if and only if it can be obtained by successive prolongations of a first order linear connection without torsion and curvature. Some of these results have been summarized in [6] and are prepublished in French, in the first part of the author's thesis [7]. If $V^n$ is a differentiable manifold, $T_x(V^n)$ is the tangent vector space of $V^n$ at $x$. 

333
Part 1

HIGHER ORDER FRAMES

1. Semi-holonomic frames.

Let \( V_n \) be an \( n \)-dimensional \( C^\infty \)-differentiable manifold. A first order frame (or a 1-frame) of \( V_n \) at the point \( x \) is an invertible 1-jet of \( \mathbb{R}^n \) into \( V_n \) with source \( 0 \in \mathbb{R}^n \) and target \( x \in V_n \). The manifold of all 1-frames of \( V_n \), denoted by \( H^1(V_n) \), forms a principal fibre bundle over \( V_n \) with natural projection \( \pi^1_0 \) which assigns to each 1-jet its target, the structure group being \( \text{GL}(n, \mathbb{R}) \). The trivial bundle \( H^1(\mathbb{R}^n) = \mathbb{R}^n \times \text{GL}(n, \mathbb{R}) \) can be identified with the group of all affine transformations on \( \mathbb{R}^n \). There is a distinguished element in \( H^1(\mathbb{R}^n) \), namely the 1-frame \( e_1 \) of \( \mathbb{R}^n \) defined by the 1-jet of the identity mapping of \( \mathbb{R}^n \) onto \( \mathbb{R}^n \) with source \( 0 \).

Let \( b : H^1(V_n) \rightarrow H^1(V_n) \) be a local isomorphism. It induces a local diffeomorphism \( f \) of \( \mathbb{R}^n \) into \( V_n \) with \( f_0 \pi^1_0 = \pi^1_0 b \) (pseudo-products); we will denote all natural projections by the same symbol \( \pi \) with indices. We say that \( b \) is 1-admissible if the domain of \( b \) contains \( e_1 \) and \( b(e_1) = j^1_0 f \).

The manifold of 1-jets \( j^1_{e_1} b \), where \( b \) is a 1-admissible local isomorphism of \( H^1(\mathbb{R}^n) \) into \( H^1(V_n) \), will be denoted by \( H^2(V_n) \). There are two natural bundle structures on \( H^2(V_n) \):

i) \( H^2(V_n) \) forms a principal fibre bundle over \( H^1(V_n) \) with natural projection \( \pi^2_1 \) and structure group \( \text{M}^2_2 \) consisting of all 1-jets of 1-admissible local isomorphisms of \( H^1(\mathbb{R}^n) \) into \( H^1(\mathbb{R}^n) \) with source and target \( e_1 \). The structure group \( \text{M}^2_2 \) acts on \( H^2(V_n) \) on the right by the composition of jets. Moreover \( \pi^2_1(j^1_{e_1} b) = b(e_1) = j^1_0 f \).

ii) \( H^2(V_n) \) forms a principal fibre bundle over \( V_n \) with projection \( \pi^2_0 = \pi^1_0 \pi^2 \) and structure group \( \text{L}^2_2 \). Here \( \text{L}^2_2 \) is the fibre of \( H^2(\mathbb{R}^n) \) over the origin \( 0 \in \mathbb{R}^n \). The multiplication in \( \text{L}^2_2 \) is given by: if \( g_1 = j^1_{e_1} b_1 \in \text{L}^2_2 \) and \( g_2 = j^1_{e_1} b_2 \in \text{L}^2_2 \), then the pseudo-product \( b_1 \circ b_2 \) is a 1-admissible local isomorphism and \( g_1 \cdot g_2 = j^1_{e_1}(b_1 \circ b_2) \) depends only on \( j^1_{e_1} b_1 \) and \( j^1_{e_1} b_2 \). Notice there is again a distinguished element in \( H^2(\mathbb{R}^n) = \mathbb{R}^n \times \text{L}^2_2 \).
namely the element \( e_2 \) defined by the \( l \)-jet of the identity mapping of \( H^1(\mathbb{R}^n) \) onto \( H^1(\mathbb{R}^n) \) with source \( e_1 \). An element \( z \in \overline{H}^2(V_n) \) will be called a semi-holonomic 2-frame of \( V_n \) at the point \( x = \pi_0^2(z) \).

We define by recurrence the principal fibre bundle \( \overline{H}^k(V_n) \) of semi-holonomic \( k \)-frames of \( V_n \). Let us assume that we have defined the principal fibre bundle \( \overline{H}^{k-1}(V_n) \) of semi-holonomic \( (k-1) \)-frames of \( V_n \), with base space \( V_n \), structure group \( \overline{L}^k \) and projection \( \pi_{k-2} \) on \( \overline{H}^{k-2}(V_n) \). A local isomorphism \( u : \overline{H}^{k-1}(\mathbb{R}^n) \rightarrow \overline{H}^{k-1}(V_n) \) is said \((k-1)\)-admissible if:

1) \( u \) is \((k-2)\)-admissible, where \( v \) is the local isomorphism of \( \overline{H}^{k-2}(\mathbb{R}^n) \) into \( \overline{H}^{k-2}(V_n) \) induced by \( u \), such that \( v \circ \pi_{k-2} = \pi_{k-2} \circ u \).

2) \( u(e_{k-1}) = j_{e_{k-2}}^1 \), where \( e_{k-1} \) (resp. \( e_{k-2} \)) is the distinguished element in \( \overline{H}^{k-1}(\mathbb{R}^n) \) (resp. \( \overline{H}^{k-2}(\mathbb{R}^n) \)).

The set \( \overline{H}^k(V_n) \) of \( l \)-jets of the form \( j_{e_{k-1}}^1 u \), where \( u \) is a \((k-1)\)-admissible local isomorphism of \( \overline{H}^{k-1}(\mathbb{R}^n) \) into \( \overline{H}^{k-1}(V_n) \), forms a principal fibre bundle over \( V_n \) with structure group \( \overline{L}^k \); the underlying set of \( \overline{L}^k \) is just the fibre of \( \overline{H}^k(\mathbb{R}^n) \) over \( 0 \in \mathbb{R}^n \). The space \( \overline{H}^k(V_n) \) can also be regarded as a principal fibre bundle over \( \overline{H}^{k-1}(V_n) \) with structure group \( \overline{M}^k = \text{Ker}(\overline{L}^k - \overline{L}^{k-1}) \). An element \( z \) of \( \overline{H}^k(V_n) \) will be called a semi-holonomic \( k \)-frame of \( V_n \) at the point \( x \), where \( x \) is the projection of \( z \) into \( V_n \).

For \( m < k \), the natural projection \( \pi_m^k \) of \( \overline{H}^k(V_n) \) onto \( \overline{H}^m(V_n) \) is compatible with the surjective homomorphism of \( \overline{L}^k \) onto \( \overline{L}^m \). The distinguished element \( e_k \) in \( \overline{H}^k(\mathbb{R}^n) = \mathbb{R}^n \otimes \overline{L}^k \) is defined by the \( l \)-jet of the identity mapping of \( \overline{H}^{k-1}(\mathbb{R}^n) \) with source \( e_{k-1} \).

2. Canonical form on \( \overline{H}^k(V_n) \).

An element \( u \in \overline{H}^k(V_n) \) can be written as \( u = j_{e_{k-1}}^1 b \), where \( b \) is a \((k-1)\)-admissible local isomorphism of \( \overline{H}^{k-1}(\mathbb{R}^n) \) into \( \overline{H}^{k-1}(V_n) \); it determines a linear isomorphism \( \tilde{u} \) of \( \overline{E}^{k-1} = T_{e_{k-1}}(\overline{H}^{k-1}(\mathbb{R}^n)) \) onto \( T_u(\overline{H}^{k-1}(V_n)) \) with \( u' = \pi_{k-1}^k(\mu) \in \overline{H}^{k-1}(V_n) \). Since \( \overline{H}^{k-1}(\mathbb{R}^n) = \mathbb{R}^n \otimes \overline{L}^{k-1} \), we have a canonical decomposition \( \overline{E}^{k-1} = \mathbb{R}^n \otimes \overline{O}^{k-1} \), where \( \overline{O}^{k-1} \) is the Lie algebra of \( \overline{L}^{k-1} \). From now on, we will identify \( \mathbb{R}^n \) with a vector subspace of \( \overline{E}^{k-1} \) given by the canonical decomposition. Since \( \tilde{u} \) is a linear isomorphism, \( \tilde{u}(\mathbb{R}^n) \) is an \( n \)-dimensional vector subspace of \( T_u(\overline{H}^{k-1}(V_n)) \).
transversal to the fibres, called the horizontal $n$-plane associated to the $k'$-frame $u$.

Let $v$ be the projection of $u$ under $\pi^k_m$. The following diagram

$$
\begin{array}{ccc}
\tilde{E}^{k-1} & \xrightarrow{\tilde{u}} & T_u^* (\tilde{H}^{k-1}(V_n)) \\
\downarrow & & \downarrow \\
\tilde{E}^{m-1} & \xrightarrow{\tilde{v}} & T_v^* (\tilde{H}^{m-1}(V_n))
\end{array}
$$

is commutative, where $v'$ is the projection of $v$ under $\pi^m_{m-1}$ and where the vertical arrows are the natural projections.

Consider a vector $Z \in T_u(\tilde{H}^k(V_n))$. Its image $Z' = T \pi^k_{k-1}(Z)$ under the tangential map $T \pi^k_{k-1}$ is tangent to $\tilde{H}^{k-1}(V_n)$ at the point $u' = \pi^k_{k-1}(u)$.

The $\tilde{E}^{k-1}$-valued differential 1-form $\theta_k$ defined by

$$\theta_k(Z) = u^{-1}(T \pi^k_{k-1}(Z))$$

will be called the canonical form on $\tilde{H}^k(V_n)$. For $m < k$, we have the following commutative diagram

$$
\begin{array}{ccc}
T(\tilde{H}^k(V_n)) & \xrightarrow{\theta_k} & \tilde{E}^{k-1} \\
\downarrow & & \downarrow \\
T(\tilde{H}^m(V_n)) & \xrightarrow{\theta_m} & \tilde{E}^{m-1}
\end{array}
$$

where the vertical arrows are the natural projections.

The Lie group $\tilde{L}_n^k$ acts naturally on $\tilde{E}^{k-1}$ on the left. Each element $g$ of $\tilde{L}_n^k$ defines a linear isomorphism $\tilde{g}$ of $\tilde{E}^{k-1}$ onto $T_{g^*}(\tilde{H}^{k-1}(\mathbb{R}^n))$ with $g^* = \pi^k_{k-1}(g)$. The right translation $R_{g^*}^{-1} = R_g^{-1}(g^*)^{-1}$ determines a linear isomorphism $T R_{g^*}^{-1}$ of $T_{g^*}(\tilde{H}^{k-1}(\mathbb{R}^n))$ onto $\tilde{E}^{k-1}$. If we put $\rho(g) = T R_{g^*}^{-1}$, we obtain a linear representation $\rho$ of $\tilde{L}_n^k$ on the vector space $\tilde{E}^{k-1}$. For $m < k$,

$$
\begin{array}{ccc}
\tilde{E}^{k-1} & \xrightarrow{\rho(g)} & \tilde{E}^{k-1} \\
\downarrow & & \downarrow \\
\tilde{E}^{m-1} & \xrightarrow{\rho(\pi^k_m(g))} & \tilde{E}^{m-1}
\end{array}
$$
is a commutative diagram, where the vertical arrows are the natural projections.

**Proposition I.1.** The canonical form $\theta_k$ is a pseudo-tensorial 1-form on $\bar{H}^k(V_n)$ of type $(\rho, \bar{E}^{k-1})$, i.e.

$$\theta_k(TR_g(Z)) = \rho(g^{-1})\theta_k(Z)$$

for all $Z \in T(\bar{H}^k(V_n))$ and $g \in \Gamma_n^k$.

3. Holonomic Frames.

A diffeomorphism $f : V_n \to V'_n$ induces a principal fibre bundle isomorphism $f(k)$ of $\bar{H}^k(V_n)$ onto $\bar{H}^k(V'_n)$. This isomorphism $f(k)$ possesses the following properties:

- $i)$ $\pi^k_m \circ f(k) = f(m) \circ \pi^k_m$ for all $0 \leq m < k$;

- $ii)$ $f(k)$ is compatible with the canonical forms, i.e. $f(k)^*\theta_k = \theta_k$,

where $\theta_k$ (resp. $\theta_k'$) is the canonical form on $\bar{H}^k(V_n)$ (resp. $\bar{H}^k(V'_n)$).

**Theorem 1.2.** Let $\phi$ be a local diffeomorphism of $\bar{H}^k(V_n)$ into $\bar{H}^k(V'_n)$. Then locally $\phi = f(k)$ for some local diffeomorphism $f$ of $V_n$ into $V'_n$, if and only if $\phi$ is compatible with the canonical forms, i.e. $\phi^*\theta_k = \theta_k$.

It remains to show that the condition is sufficient. For this we will proceed by induction on $k$.

**Lemma I.3.** Let $\phi$ be a local diffeomorphism of $H^1(V_n)$ into $H^1(V'_n)$ with $\phi^*\theta_1 = \theta_1$. Then we can locally write $\phi = f(1)$ for some local diffeomorphism $f$ of $V_n$ into $V'_n$.

Consider a tangent vector $Z \in T_\xi(H^1(V_n))$ with $T\pi^1_0(Z) = 0$. The condition $\phi^*\theta_1 = \theta_1$ implies that $T\pi^1_0(T\phi(Z)) = 0$. Thus $\phi$ sends a tangent space to the fibre of $H^1(V_n)$ onto a tangent space to the fibre of $H^1(V'_n)$. This means that locally $\phi$ is a fibre map and induces a map $f$ of $V_n$ into $V'_n$ satisfying $f_0\pi^1_0 = \pi^1_0\phi$. We want to show that $\phi = f(1)$. Thus we want to show that for any $u$ with $\pi^1_0(u) = x$ we have $\phi(u) = j^1_x f_0 u$. Let $\xi \in \mathbb{R}^n$. Choose a vector $Z \in T_\xi(H^1(V_n))$ with $T\pi^1_0(Z) = \bar{u}(\xi)$. Then $(j^1_x f_0 Z)(\xi) = (T f_0 \bar{u})(\xi) = (T f_0 T\pi^1_0(Z)) = (T \pi^1_0 T\phi)(Z)$.

On the other hand, $(\phi(u))^{-1} T\pi^1_0 T\phi(Z) = (\bar{u}^{-1} T\pi^1_0)(Z) = \xi$. Thus
To prove the theorem for $k$ we may assume that it has been established for $k-1$. Let $Z \in T_u(\widetilde{H}^k(V_n))$ with $T\pi^k_{k-1}(Z)=0$. The condition $\phi^*\theta^*_k=\theta_k$ implies $(T\pi^k_{k-1} \circ T\phi)(Z)=0$. Thus $\phi$ is a local fibre map with respect to the fibrations $\widetilde{H}^k(V_n) \to \widetilde{H}^{k-1}(V_n)$ and $\widetilde{H}^k(V'_n) \to \widetilde{H}^{k-1}(V'_n)$.

There exists a local diffeomorphism $\psi$ of $\widetilde{H}^{k-1}(V_n)$ into $\widetilde{H}^{k-1}(V'_n)$ such that $\psi \circ \pi^k_{k-1} = \pi^k_{k-1} \circ \phi$. Since $\pi^k_{k-1} \circ \theta_{k-1} = T\pi^k_{k-2} \circ \theta_k$ (resp. $\pi^k_{k-1} \circ \theta'_{k-1} = T\pi^k_{k-2} \circ \theta'_{k}$), we have

$$
(\pi^k_{k-1} \circ \psi \circ \theta'_{k-1})(Z) = (\psi \circ T\pi^k_{k-1} \circ T\phi)(Z) = (T\pi^k_{k-1} \circ \theta'_{k-1} \circ T\phi)(Z) = (T\pi^k_{k-2} \circ \theta_k)(Z) = (\pi^k_{k-1} \circ \theta_{k-1})(Z)
$$

for all $Z \in T_u(\widetilde{H}^k(V_n))$. As $\pi^k_{k-1}$ is surjective, we deduce that $\psi \circ \theta'_{k-1} = \theta_{k-1}$. By the induction hypothesis, there exists a local diffeomorphism $f$ of $V_n$ into $V'_n$ such that locally $\psi = f^{(k-1)}$. We have thus $f^{(k-1)} \circ \pi^k_{k-1} = \pi^k_{k-1} \circ \phi$ locally. Now we are going to show that locally $\phi = f^{(k)}$. An element $u \in \widetilde{H}^k(V_n)$ determines a linear isomorphism $\widetilde{u} : \widetilde{E}^{k-1} \to T_u(\widetilde{H}^{k-1}(V_n))$ with $u' = \pi^k_{k-1}(u)$. Two elements $u$ and $v$ of $\widetilde{H}^k(V_n)$ are identical if and only if $\widetilde{u} = \widetilde{v}$. It suffices therefore to show that $\widetilde{\phi}(u) = f^{(k)}(u)$ for all $u \in \widetilde{H}^k(V_n)$. Let $\xi \in \widetilde{E}^{k-1}$. Choose a tangent vector $Z \in T_u(\widetilde{H}^k(V_n))$ with $\theta_k(Z) = \xi$. We have

$$
\theta_k(Z) = (\phi^*\theta^*_k)(Z) = (\phi^* \circ T\phi)(Z) = (\widetilde{\phi}(u)^{-1} \circ T\pi^k_{k-1} \circ T\phi)(Z).
$$

On the other hand, $\xi = \theta_k(Z) = (\widetilde{u}^{-1} \circ T\pi^k_{k-1})(Z)$. It follows that for all $\xi \in \widetilde{E}^{k-1}$,

$$
\widetilde{\phi}(u)(\xi) = (T\pi^k_{k-1} \circ T\phi)(Z) = (Tf^{(k-1)} \circ T\pi^k_{k-1})(Z) = (Tf^{(k-1)} \circ \widetilde{u})(\xi) = f^{(k)}(u)(\xi).
$$

We have therefore $\phi = f^{(k)}$ locally and our theorem is proved.
COROLLARY 1.4. Let \( \phi \) be a principal fibre bundle isomorphism of \( \widetilde{H}^k(V_n) \) onto \( \widetilde{H}^k(V'_n) \). Let \( f \) be the diffeomorphism of \( V_n \) onto \( V'_n \), induced by \( \phi \). Then \( \phi = f^{(k)} \) if and only if \( \phi_* \theta'_k = \theta_k \).

Consider a local diffeomorphism \( f \) of an open neighbourhood of \( 0 \in \mathbb{R}^n \) onto an open set of \( V_n \). It induces a \((k-1)\)-admissible local isomorphism \( f^{(k-1)} : \widetilde{H}^{k-1}(\mathbb{R}^n) \to \widetilde{H}^{k-1}(V_n) \). It follows that \( u = j^{1}_{e^{k-1}} f^{(k-1)} \) is an element of \( \widetilde{H}^{k}(V_n) \). We say that \( u \in \widetilde{H}^{k}(V_n) \) is a holonomic \( k \)-frame of \( V_n \) if \( u \) can be written as \( u = j^{1}_{e^{k-1}} f^{(k-1)} \) for some local diffeomorphism \( f \) of \( \mathbb{R}^n \) into \( V_n \). A \( k \)-frame \( u \) of \( V_n \) is holonomic if and only if one can find a representative for \( u \) compatible with the canonical forms. The set of holonomic \( k \)-frames of \( V_n \) forms a principal fibre subbundle \( H^{k}(V_n) \) of \( \widetilde{H}^{k}(V_n) \). Its structure group is the subgroup \( L_n^k \) of \( L_n^k \) consisting of holonomic elements. Notice there is a group isomorphism between \( L_n^k \) and the group of all invertible \( k \)-jets of \( \mathbb{R}^n \) into \( \mathbb{R}^n \) with source and target \( 0 \). The space \( H^{k}(V_n) \) can also be regarded as a principal fibre bundle over \( \widetilde{H}^{k-1}(V_n) \) with structure group \( M_n^k = M_n^k \cap L_n^k \), kernel of the surjective homomorphism \( L_n^k \to L_n^{k-1} \).

4. Relations between \( \widetilde{H}^{k}(V_n) \), \( \widetilde{P}^{k}(V_n) \) and \( \widetilde{J}^{k-1}(H^{1}(V_n)) \).

Let \( W \) and \( Y \) be two \( C^\infty \)-differentiable manifolds. We will denote by \( \tilde{J}^k(W, Y) \) the differentiable manifold of semi-holonomic \( k \)-jets of \( W \) into \( Y \). For the definition of semi-holonomic jets, see the works of Ehresmann. For \( m < k \), let \( p^k_m \) be the canonical projection of \( \tilde{J}^k(W, Y) \) onto \( \tilde{J}^m(W, Y) \). A jet \( X \in \tilde{J}^k(W, Y) \) is invertible if and only if \( p^k_1(X) \) is invertible. Let \( \tilde{J}^1(W, Y) \) denote the set of invertible jets in \( \tilde{J}^k(W, Y) \). This set is then the inverse image of \( \tilde{J}^1(W, Y) \) by the submersion \( p^k_1 \). Since \( \tilde{J}^1(W, Y) \) is an open submanifold of \( \tilde{J}^k(W, Y) \), it follows that \( \tilde{J}^k(W, Y) \) is an open submanifold of \( \tilde{J}^k(W, Y) \). Moreover, \( p^k_m : \tilde{J}^k(W, Y) \to \tilde{J}^m(W, Y) \) is a submersion.

A semi-holonomic \( k \)-frame (resp. holonomic \( k \)-frame) of \( V_n \) in the sense of Ehresmann is an invertible semi-holonomic \( k \)-jet (resp. invertible holonomic \( k \)-jet) of \( \mathbb{R}^n \) into \( V_n \) with source \( 0 \in \mathbb{R}^n \). The set \( \tilde{P}^k(V_n) \) (resp.
$p^k(V_n)$ of semi-holonomic $k$-frames (resp. holonomic $k$-frames) of $V_n$ in the sense of Ehresmann has a principal fibre bundle structure over $V_n$, the structure group being the group of all invertible semi-holonomic $k$-jets (resp. holonomic $k$-jets) of $\mathbb{R}^n$ into $\mathbb{R}^n$ with source and target $0 \in \mathbb{R}^n$. An element $u \in \bar{p}^k(V_n)$ can then be written as $u = j^1_0 f$, where $f$ is a differentiable mapping of $\mathbb{R}^n$ into $\bar{p}^{k-1}(V_n)$ satisfying the condition:

\[
 j^1_0(p^{k-1}_{k-2} \circ f) = f(0).
\]

Here we have also denoted by $p^{k-1}_{k-2}$ the canonical projection of $\bar{p}^{k-1}(V_n)$ onto $\bar{p}^{k-2}(V_n)$.

**Theorem 1.5.** There exists a canonical diffeomorphism $\nu_k$ of $\bar{H}^k(V_n)$ onto $\bar{p}^k(V_n)$ satisfying the properties:

1. $\nu_k$ is a fibre map, i.e. $p^k_0 \circ \nu_k = \pi^k_0$,
2. for $m < k$,

\[
 \begin{array}{ccc}
 \bar{H}^k(V_n) & \xrightarrow{\nu_k} & \bar{p}^k(V_n) \\
 \downarrow \pi^k_m & & \downarrow p^k_m \\
 \bar{H}^m(V_n) & \xrightarrow{\nu_m} & \bar{p}^m(V_n)
\end{array}
\]

is a commutative diagram;

3. $\nu_k$, restricted to $H^k(V_n)$, is a diffeomorphism of $H^k(V_n)$ onto $p^k(V_n)$.

We prove the theorem by induction on $k$. For $k = 1$, $H^1(V_n)$ is identical with $p^1(V_n)$ and $\nu_1$ is just the identity map. Let $u = j^1_1 h$ be an arbitrary element in $\bar{H}^2(V_n)$. If $\gamma_1$ denotes the zero section* of $H^1(\mathbb{R}^n)$ the mapping $u \rightarrow \nu_2(u) = j^1_0(\nu_1 \circ h \circ \gamma_1)$ defines a diffeomorphism of $\bar{H}^2(V_n)$ onto $\bar{p}^2(V_n)$, because the composition of jets is a differentiable map. Let us assume there exists $\nu_{k-1}$ such that, for all $z \in \bar{H}^{k-1}(V_n)$, $\nu_{k-1}(z) = (j^1_z, \nu_{k-1}) \circ z \circ (j^1_0 \eta_{k-2})$ where $z' = \eta_{k-2}^{k-1}(z)$ and $\eta_{k-2}$ is the zero section* of the trivial bundle $\bar{H}^{k-2}(\mathbb{R}^n) = \mathbb{R}^n \times \mathbb{R}^{k-2}$. Consider then an arbitrary element $y = j^1_1 g$ in $\bar{H}^k(V_n)$. If $\eta_{k-1}$ is the zero section* of $\bar{H}^{k-1}(\mathbb{R}^n) = \mathbb{R}^n \times \mathbb{R}^{k-1}$, $g' = \nu_{k-1} \circ g \circ \eta_{k-1}$ defines a local diffeomorphism of $\mathbb{R}^n$ to $\bar{p}^{k-1}(V_n)$. Since $j^1_0(p^{k-1}_{k-2} \circ g') = g'(0)$, the $i$-jet $j^1_0 g'$, which

*) corresponding to $\mathbb{R}^k \times \{e\}$, where $e$ is the unit element.

340

YUEN PING CHENG
is independent of the choice of g for y, is an element in $\overline{P}^k(V_n)$. The mapping $y \mapsto \nu_k(y) = j_0^1g'$ defines a diffeomorphism $\nu_k$ of $\overline{H}^k(V_n)$ onto $\overline{P}^k(V_n)$. It is easy to check that $\nu_k$ has the desired properties.

Consider the case where $V_n = \mathbb{R}^n$. Let us recall that the underlying set of $L^k_n$ is just the fibre of $\overline{H}^k(\mathbb{R}^n)$ over the origin 0. Since the multiplication in $L^k_n$ is given by the composition of jets, the restriction of $\nu_k$ to $L^k_n$ defines a group isomorphism of $L^k_n$ onto the group of all invertible semi-holonomic k-jets of $\mathbb{R}^n$ into $\mathbb{R}^n$ with source and target 0. It is easy to see that the diffeomorphism $\nu_k$ of the above theorem is compatible with this group isomorphism. We have therefore the following corollary:

**Corollary 1.6.** The principal fibre bundle $\overline{H}^k(V_n)$ (resp. $\overline{H}^k(V_n)$) is canonically isomorphic to $\overline{P}^k(V_n)$ (resp. $\overline{P}^k(V_n)$).

Let $E$ be a locally trivial fibre bundle over $V_n$. We will denote by $J^kE$ the differentiable manifold of k-jets of local sections of $E$. Let $\tilde{J}^2E = J^1(J^1E)$. The k-th non-holonomic prolongation of $E$ is defined by induction:

$$\tilde{J}^kE = J^1(\tilde{J}^{k-1}E).$$

We define also the semi-holonomic prolongation $\tilde{J}^kE$ by restricting ourselves to those local sections such that, for all $0 < m < k$, the local section $\sigma$ of $V_n$ into $\tilde{J}^mE$ satisfies the condition: $j_x^1(\pi^m_{m-1} \circ \sigma) = \sigma(x)$, where $\pi^m_{m-1}$ is the natural projection of $\tilde{J}^mE$ onto $\tilde{J}^{m-1}E$. We have

$$J^kE \subset \tilde{J}^kE \subset \tilde{J}^{k-1}(H^1(V_n)).$$

**Theorem 1.7.** There exists a canonical diffeomorphism $\mu_k$ of $\overline{H}^k(V_n)$ onto $\tilde{J}^{k-1}(H^1(V_n))$ satisfying the following properties:

1. For $k = 1$, $\mu_1$ is just the identity map of $H^1(V_n)$;
2. $\mu_k$ is a fibre map; more explicitly

$$
\begin{array}{ccc}
\overline{H}^k(V_n) & \xrightarrow{\mu_k} & \tilde{J}^{k-1}(H^1(V_n)) \\
\downarrow & & \downarrow \\
V_n & \xrightarrow{id} & V_n
\end{array}
$$

is a commutative diagram;
for $0 < m < k$, the following diagram commutes.

\[
\begin{array}{ccc}
\overline{H}^k(V_n) & \xrightarrow{\mu_k} & \overline{f}^{k-1}(H^1(V_n)) \\
\downarrow & & \downarrow \\
\overline{H}^m(V_n) & \xrightarrow{\mu_m} & \overline{f}^{m-1}(H^1(V_n))
\end{array}
\]

We prove the theorem by induction on $k$. For $k = 1$, $j^0_0(H^1(V_n)) = H^1(V_n)$ by definition and $\mu_1$ is just the identity map of $H^1(V_n)$. Let $u = j^1_1 b$ be an arbitrary element of $\overline{H}^2(V_n)$. Consider the local diffeomorphism $f$ of $\mathbb{R}^n$ into $V_n$ defined by the condition: $\pi^1_0 f = j^1_0$. If $\eta_1$ is the zero section of $H^1(\mathbb{R}^n) = \mathbb{R}^n \times L_1^1$, the mapping

\[x \mapsto \sigma(x) = b \circ \eta_1 \circ f^{-1}(x)\]

defines a local section $\sigma$ of $V_n$ into $H^1(V_n)$. If we put $\mu_2(u) = j^1_x \sigma$ with $x = \pi^1_0(u)$, the mapping $u \mapsto \mu_2(u)$ defines an injection of $\overline{H}^2(V_n)$ into $\overline{f}^1(H^1(V_n))$. This differentiable mapping $\mu_2$ is surjective. In fact let $\sigma$ be a local section of $V_n$ into $H^1(V_n)$ with $j^1_1 \sigma \in \overline{f}^1(H^1(V_n))$. The target $\sigma(x)$ can be written as $\sigma(x) = j^1_0 f$ for some local diffeomorphism $f$ of $\mathbb{R}^n$ into $V_n$. Let $b$ be the local isomorphism of $H^1(\mathbb{R}^n)$ into $H^1(V_n)$ defined by the conditions:

i) $\pi^1_0 f = j^1_0$,

ii) $\eta_1 = \sigma \circ f$.

It is easy to check that $b$ is $l$-admissible and $j^1_x \sigma = \mu_2(j^1_1 b)$. The mapping $\mu_2$ gives then a diffeomorphism of $\overline{H}^2(V_n)$ onto $\overline{f}^1(H^1(V_n))$ with the desired properties. Now, let us assume there exists $\mu_{k-1}$ and $\mu_{k-2}$ such that, for all $u \in \overline{H}^{k-1}(V_n)$, we have

\[\mu_{k-1}(u) = (j^1_u, \mu_{k-2}) \circ \omega \circ (j^1_0 \eta_{k-2}) \circ \omega^{-1}\]

with $u' = \pi^k_{k-2}(u)$, $\omega = \pi^k_1(u')$ and where $\eta_{k-2}$ is the zero section of $\overline{H}^{k-2}(\mathbb{R}^n) = \mathbb{R}^n \times L_1^{k-2}$. Let $z = j^1_0 b$ be an arbitrary element of $\overline{H}^k(V_n)$. Let $f$ be the local diffeomorphism of $\mathbb{R}^n$ into $V_n$ induced by $b$. If we denote by $\eta_{k-1}$ the zero section of $\overline{H}^{k-1}(\mathbb{R}^n) = \mathbb{R}^n \times L_1^{k-1}$, then
defines a local section of $V_n$ into $\widetilde{T}^{k-2}(H^1(V_n))$ and $\gamma_{k-1}'$ determines an element $\mu_k(x)$ of $\widetilde{T}^{k-1}(H^1(V_n))$ independent of the choice of the representative $b$ for $x$. It is easy to verify that $x \mapsto \mu_k(x)$ defines a diffeomorphism $\mu_k$ of $\widetilde{H}^k(V_n)$ onto $\widetilde{T}^{k-1}(H^1(V_n))$ satisfying the required conditions of the theorem.

**Corollary 1.8** [4c] $\overline{\mu}^k(V_n)$ and $\overline{T}^{k-1}(H^1(V_n))$ are canonically diffeomorphic.

### 5. Local coordinate systems in $\overline{H}^k(V_n)$.

Let $\{x^1, x^2, \ldots, x^n\}$ be the natural coordinate system in $\mathbb{R}^n$. Let $U$ be a coordinate neighbourhood in $V_n$ with a local coordinate system $\{y^1, y^2, \ldots, y^n\}$. Consider an element $u \in H^1(V_n)$ with projection

$$\pi^1_0(u) = y = (y^1, y^2, \ldots, y^n) \in U.$$ 

The 1-frame $u$ is completely determined by the linear isomorphism

$$\tilde{u} : T_0(\mathbb{R}^n) \longrightarrow T_y(V_n).$$

In terms of local coordinates, $\tilde{u}$ can be expressed by

$$\tilde{u} : v_i \longmapsto \sum_m y^m_i \tilde{v}_m \quad (1 \leq i \leq n, 1 \leq m \leq n),$$

where $v_i = \frac{\partial}{\partial x^i} \bigg|_0$, $\tilde{v}_m = \frac{\partial}{\partial y^m} \bigg|_y$ and $\det(y_i^m) \neq 0$.

The 1-frame $u$ is therefore completely determined by the set of local coordinates $(y^1, y^i_k)$ with $\det(y^i_k) \neq 0$. Thus we can take $\{y^i, y^i_k\}$ as a local coordinate system in $(\pi^1_0)^{-1}(U) \subset H^1(V_n)$. Similarly, we have a global coordinate system $\{x^1, x^i_k\}$ in $H^1(\mathbb{R}^n)$, with respect to which the distinguished element is given by $e_i = (0, \delta^i_k)$.

The $n + n^2$ vectors $\{s_i = (\frac{\partial}{\partial x^i})_1, s_k^i = (\frac{\partial}{\partial x^k})_j e_1\}$ form a basis for $E^1 = T_{e_1}(H^1(\mathbb{R}^n))$, and the $n + n^2$ local vector fields $\{\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^i_k}\}$ are linearly independent. Once again, any 2-frame $\nu$ is completely determined by the linear isomorphism $\tilde{\nu}$ associated to $\nu$. In terms of local coordinates, we have

$$\tilde{\nu} : E^1 \longrightarrow T_u(H^1(V_n)) \text{ with } \pi^2_1(\nu) = u = (y^i, y^i_k)$$
where $s_m = \left( \frac{\partial}{\partial y^m} \right) u$, $s_p = \left( \frac{\partial}{\partial y^p} \right) u$ and $T_u$ is the tangential map of $u$, $u$ being considered as a differentiable map. Thus $v$ is completely determined by the set of local coordinates $(y^i, y^i_1, y^i_{j_1} \ldots y^i_{j_k})$ with $\text{det}(y^i_{j_1} \ldots y^i_{j_k}) \neq 0$.

By iteration we have a coordinate neighbourhood $(\pi_k^k)^{-1}(U)$ in $\tilde{H}^k(V_n)$ with a local coordinate system $\{y^i, y^i_1, y^i_{j_1} \ldots y^i_{j_k}\}$ with $\text{det}(y^i_{j_1} \ldots y^i_{j_k}) \neq 0$. The natural projection of $\tilde{H}^k(V_n)$ onto $\tilde{H}^m(V_n)$ ($m < k$) is given by

$$(y^i, y^i_1, \ldots, y^i_{j_1} \ldots y^i_{j_k}) \rightarrow (y^i, y^i_{j_1}, \ldots, y^i_{j_1} \ldots y^i_{j_m}).$$

If $u = (a_i^1, a_i^1, \ldots, a_i^1 \ldots a_i^1) \in \tilde{H}^k(V_n)$, the associated linear isomorphism $\tilde{u}$ can be expressed by

$$t_j \rightarrow \sum (a_i^i t_i + \frac{1}{2!} a_i^i t_i^j + \ldots + \frac{1}{k!} a_i^i \ldots a_i^i \ldots a_i^i \ldots a_i^i t_i^1 \ldots t_i^j\ldots t_i^k - 1)$$

$$i_i \rightarrow T_u^t (i_i^1 \ldots i_i^k)$$

$$i_i^1 \ldots i_k - 1 \rightarrow T_u^t (i_i^1 \ldots i_k - 1)$$

where

$$i_i^1 \ldots i_m = \left( \frac{\partial}{\partial x^i \ldots x_{j_1} \ldots x_{j_m}} \right) e_{k - 1}$$

$$i_i^1 \ldots i_m = \left( \frac{\partial}{\partial y^i_1 \ldots y^i_{j_1} \ldots y^i_{j_m}} \right) u^*,$$

and $u^* = \pi_k^k(u)$. The local coordinates $a_i^j, \ldots, a_i^j$ are symmetrical with respect to the lower indices if and only if $u$ is a holonomic $k$-frame of $V_n$ [1c].

6. Holonomy Theorem.

Consider an arbitrary element $u$ in $\tilde{H}^k(V_n)$. In this paragraph we
give a necessary and sufficient condition for $u$ to be a holonomic $k$-frame. Let us recall that the horizontal $n$-plane defined by $u$ is just the image of the $R^n$-component of $\mathbb{E}^{k-1} = R^n \oplus \mathbb{O}^{k-1}_n$ under the linear isomorphism $\bar{u}$. It is tangent to $H^{k-1}(V_n)$ at the point $u' = \pi_{k-1}^k(u)$, if $u$ is holonomic.

For $k = 1$, there is no distinction between semi-holonomic frames and holonomic frames. For $k \geq 2$, $H^k(V_n) \subset \bar{H}^k(V_n)$.

**Proposition 1.9.** An element $u$ of $\bar{H}^2(V_n)$ is a holonomic 2-frame if and only if the 2-form $d\Theta_1$ vanishes on the horizontal $n$-plane associated to $u$.

Let $r_1, r_2, \ldots, r_n$ be a basis for $R^n$. The canonical form $\Theta_1$ on $H^1(V_n)$ can be expressed as follows:

$$\Theta_1 = \sum \Theta^i r_i.$$  

In terms of a local coordinate system $\{y^i, y^j\}$ in $H^1(V_n)$, the components $\Theta^i$ of $\Theta_1$ are given by

$$\Theta^i = \sum z^i_j dy^j,$$

where $(z^i_j)$ is the inverse matrix of $(y^i_j)$. By exterior differentiation, we get

$$d\Theta^i = \sum \frac{\partial z^i_j}{\partial y^m} dy^m \wedge dy^j.$$  

Let $u = (a^i, a^i_j, a^i_{jk}) \in \bar{H}^2(V_n)$. The horizontal $n$-plane $Q_u$ associated to $u$ is generated by the $n$ vectors

$$X_i = \sum (\frac{\partial}{\partial y^j})_u + \sum \frac{1}{2^i} a^i_{kj} (\frac{\partial}{\partial y^k})_u \quad (1 \leq i \leq n),$$

with $u' = \pi^2_1(u) = (a^i, a^i_j)$. The 2-form $d\Theta_1$ vanishes on $Q_u$ if and only if

$$d\Theta_1(X_j, X_k) = \sum (\frac{\partial z^i_j}{\partial y^m}, \frac{\partial z^i_k}{\partial y^m})_u^* \begin{vmatrix} 1 & a^p\delta_{ij} & a^m_j \\ \frac{1}{2^i} a^p q_j & a^m_j \\ \frac{1}{2^i} a^p q_k & a^m_k \end{vmatrix}$$

is zero for all $1 \leq i, j, k \leq n$. Since $(z^i_j)^{-1} = (y^i_j)^{-1}$, we have the relation $z^i_p y^p_k = \delta^i_k$. By differentiation, we get
where \( (b_i^j) = (a_j^i)^{-1} \). It follows that

\[
d \Theta^i(X_j, X_k) = - \frac{1}{2!} \sum \left( b_m^q b_p^j \left( a_k^m a_p^q - a_j^m a_q^p \right) \right)
\]

\[
= - \frac{1}{2!} \sum \left( b_p^j (a_k^p - a_j^p) \right)
\]

Since \( \text{det}(b_i^j) \neq 0 \), we conclude that \( d \Theta^i(X_j, X_k) = 0 \) for all \( 1 \leq i, j, k \leq n \) if and only if the \( a_j^i \) are symmetrical with respect to their lower indices.

Thus our proposition is proved.

For the general case where \( k > 2 \), we have the following «Holonomy Theorem»:

**Theorem 1.10.** An element \( u \in \widehat{H}^k(V_n) \) is a holonomic \( k \)-frame if and only if the following conditions are satisfied:

1. the horizontal \( n \)-plane \( Q_u \) associated to \( u \) is tangent to the submanifold \( H^{k-1}(V_n) \) of \( H_k(V_n) \);
2. the 2-form \( d \theta^{k-1} \) vanishes on \( Q_u \).

Let us assume that \( u \) is a holonomic \( k \)-frame. We can then write

\[ u = j^1_{e_{k-1}} f^{(k-1)} \]

for some local diffeomorphism \( f \) of \( \mathbb{R}^n \) into \( V_n \). If \( \theta_{k-1} \) and \( \hat{\theta}_{k-1} \) are respectively the canonical form on \( \widehat{H}^{k-1}(V_n) \) and \( H^{k-1}(\mathbb{R}^n) \), we have

\[ f^{(k-1)} \ast \theta_{k-1} = \hat{\theta}_{k-1} \].

It follows that \( f^{(k-1)} \ast d \theta_{k-1} = d \hat{\theta}_{k-1} \).

Now, the 2-form \( d \hat{\theta}_{k-1} \) vanishes on the \( \mathbb{R}^n \)-component of \( E^{k-1} = \mathbb{R}^n \oplus \mathbb{R}^{k-1} \). As a consequence, \( d \theta_{k-1} \) vanishes on \( Q_u \). The first condition is obviously necessary.

It remains to show that the conditions are sufficient. The first condition implies that \( u' = \pi_{k-1}^k(u) \) is a holonomic \((k-1)\)-frame, and that we can find a local coordinate system \( \{y^i, y^i_{j_1}, \ldots, y^i_{j_1 \ldots j_k}\} \) in \( \widehat{H}^k(V_n) \) such that \( u = (0, a^i_{j_1}, \ldots, a^i_{j_1 \ldots j_k}) \) where \( a^i_{j_1 \ldots j_m} \) are symmetrical with respect to their lower indices for \( 2 \leq m \leq k-1 \) and \( a^i_{j_1 \ldots j_k} \) is symmetrical with respect to the first \( k-1 \) lower indices. By a change of local coordinate systems, we can even suppose that \( a^i_j = \delta^i_j \) and \( a^i_{j_1 \ldots j_m} = 0 \) for
Let \( \{ x^1, x^2, \ldots, x^n \} \) be the natural coordinate system in \( \mathbb{R}^n \). By iteration, we define a global coordinate system \( \{ x^{i_1}, x^{i_2}, \ldots, x^{i_{k-1}} \} \) in \( \mathbb{H}^m(\mathbb{R}^n) \). Let \( z^a = x^{i_1} \cdots x^{i_p} \) with \( a = in^{p-1} + \cdots + i_1 \). The vectors

\[
  t_a = \left( \frac{\partial}{\partial z^a} \right) e^{k-2} \quad (1 \leq a \leq n^{k-1} + n^{k-2} + \cdots + n)
\]

form a basis for \( \mathbb{E}^{k-2} \) and we can write

\[
  \theta_{k-1} = \sum_a \theta^a t_a.
\]

An element \( v = (y^i, y^{i_1}, \ldots, y^{i_{k-1}}) \in \mathbb{H}^{k-1}(V_n) \) defines a linear isomorphism \( \tilde{v} \) of \( \mathbb{E}^{k-2} \) onto \( \mathbb{T}_v'(\mathbb{H}^{k-2}(V_n)) \) with \( v' = \pi^{k-2}_{k-2}(v) \). In terms of local coordinate systems, \( \tilde{v} \) is given by

\[
  \tilde{v} : t_a \mapsto \sum_{\beta} A_{\alpha}^{\beta} \tilde{t}_\beta
\]

where \( 1 \leq \alpha, \beta \leq n^{k-1} + n^{k-2} + \cdots + n \), \( t_a = \left( \frac{\partial}{\partial z^a} \right) v' \), with \( z^a = y^{i_1} \cdots y^{i_p} \). The matrix \( A = (A_{\alpha}^{\beta}) \) is of the form

\[
  A = \begin{pmatrix}
    A^{ij} & A^{i\omega} \\
    0 & J
  \end{pmatrix}
\]

where \( J \) is the matrix corresponding to the linear isomorphism \( \mathbb{T}_v' \). We have therefore

\[
  y^{i_1} \cdots y^{i_m} = A^{i_{j_1}}_{j_{m-1}} \quad \text{with} \quad \beta = in^{m-1} + \cdots + i_1 \].

Let \( B = (B_{\alpha}^{\beta}) \) be the inverse matrix of \( A = (A_{\alpha}^{\beta}) \). The components \( \theta^a \) of \( \theta_{k-1} \) can be expressed by

\[
  \theta^a = \sum_{\beta} B_{\alpha}^{\beta} d\tilde{z}^\beta.
\]

By exterior differentiation, we get

\[
  d\theta^a = \sum \left( \frac{\partial B_{\alpha}^{\beta}}{\partial z^\gamma} \right) d\tilde{z}^\gamma \wedge d\tilde{z}^\beta + \sum \left( \frac{\partial B_{\alpha}^{\beta}}{\partial y^{i_1} \cdots y^{i_{k-1}}} \right) dy^{i_1} \cdots dy^{i_{k-1}} \wedge d\tilde{z}^\beta.
\]

Since \( \sum B_{\mu}^\alpha A_{\mu}^{\nu} = \delta^\alpha_\nu \), we obtain by differentiation

\[
  \frac{\partial B_{\alpha}^{\beta}}{\partial z^\gamma} = \sum B_{\mu}^\alpha B_{\nu}^\beta \frac{\partial A_{\mu}^{\nu}}{\partial z^\gamma},
\]
\[
\frac{\partial B_\beta^\alpha}{\partial y_i^{j_1 \cdots j_{k-1}}} = -\sum B_\mu^\alpha B_\beta^\nu \left( \frac{\partial A_\mu^\nu}{\partial y_i^{j_1 \cdots j_{k-1}}} \right) ,
\]

hence

\[
d \theta^a = -\sum B_\mu^\alpha B_\beta^\nu \left( \frac{\partial A_\mu^\nu}{\partial \bar{z}_\beta} \right) d \bar{z}_\gamma \wedge d \bar{z}_\beta - \sum B_\mu^\alpha B_\beta^\nu \left( \frac{\partial A_\mu^\nu}{\partial y_i^{j_1 \cdots j_{k-1}}} \right) d y_i^{j_1 \cdots j_{k-1}} \wedge d \bar{z}_\beta .
\]

Let \( u = (0, \delta^i_1, 0, \ldots, a_{i_1 \cdots i_k}) \) and let \( Q_u \) be the horizontal \( n \)-plane of \( \mathcal{H}^{k-1}(V_n) \) associated to \( u \). \( Q_u \) is generated by the \( n \) vectors

\[
X_p = (\frac{\partial}{\partial y_p}) u' + \frac{1}{k!} \sum a_{i_1 \cdots i_{k-1}p} \left( \frac{\partial}{\partial y_{i_1 \cdots i_{k-1}}} \right) u' ,
\]

where \( u' = \pi_{k-1}^k(u) \) and \( 1 \leq p \leq n \).

The nullity of \( d \theta^a_{k-1} \) on \( Q_u \) implies that \( d \theta^a(X_p, X_q) = 0 \) for all \( 1 \leq p, q \leq n \) and \( 1 \leq a \leq n^{k-1} + n^{k-2} + \ldots + n \). We have then

\[
0 = d \theta^a(X_p, X_q)
= \sum B_\mu^a(u') B_\beta^\nu(u') \left( \frac{\partial A_\mu^\nu}{\partial y_{i_1 \cdots i_{k-1}}} \right) u' ,
\]

\[
= \frac{1}{k!} \sum a_{i_1 \cdots i_{k-1}p} \frac{\partial A_\beta^\nu}{\partial y_{i_1 \cdots i_{k-1}}} - a_{i_1 \cdots i_{k+2}pq} - a_{i_1 \cdots i_{k-2}pq} \]
with \( \beta = i n^{k-2} + \ldots + j_1 \). Since \( \text{det}(B_\beta^\nu(u')) \neq 0 \), we obtain

\[
a_{i_1 \cdots i_{k-2}pq} = a_{i_1 \cdots i_{k-2}pq} .
\]

It follows that the \( a_{i_1 \cdots i_{k-1}} \) are symmetrical with respect to their lower indices and thus \( u \) is a holonomic \( k \)-frame.

Let us call \( u \in \mathcal{H}^k(V_n) \) a \textit{quasi-holonomic} \( k \)-frame if the horizontal \( n \)-plane \( Q_u \) of \( \mathcal{H}^{k-1}(V_n) \) associated to \( u \) is tangent to the submanifold \( H^{k-1}(V_n) \). We will denote by \( \mathcal{H}^k(V_n) \) the set of quasi-holonomic \( k \)-frames. We have obviously \( H^k(V_n) \subset \mathcal{H}^k(V_n) \subset \mathcal{H}^k(V_n) \). From the above theorem a quasi-holonomic \( k \)-frame \( u \) is a holonomic one if and only if \( d \theta^a_{k-1} \) vanishes on the horizontal \( n \)-plane \( Q_u \) associated to \( u \).
7. Some remarks on $\widetilde{H}^k(\mathbb{R}^n)$.

In the preceding paragraphs, $\mathbb{R}^n \times \widetilde{L}_n^k$ has been identified with $\widetilde{H}^k(\mathbb{R}^n)$. In this identification, a couple $(x, g) \in \mathbb{R}^n \times \widetilde{L}_n^k$ is identified with the element $t_x^{(k)}(g) \in \widetilde{H}^k(\mathbb{R}^n)$, where $t_x$ denotes the translation in $\mathbb{R}^n$ sending the origin 0 to the point $x$. The tangent space $\widetilde{E}_k$ to $\widetilde{H}^k(\mathbb{R}^n)$ at the distinguished element $e_k$ has a canonical Lie algebra structure. Let us say a few words on this Lie algebra structure. Let $u = (x, g) \in \widetilde{H}^k(\mathbb{R}^n)$. The translation $t_x$ in $\mathbb{R}^n$ induces an automorphism $t_x^{(k)}$ of $\widetilde{H}^k(\mathbb{R}^n)$ which commutes with the right translations of $\widetilde{H}^k(\mathbb{R}^n)$ on itself, i.e.

$$t_x^{(k)} \circ R_b = R_b \circ t_x^{(k)}$$

for all $b \in \widetilde{L}_n^k$. In particular, $t_x^{(k)} \circ R_g = R_g \circ t_x^{(k)}$ gives a diffeomorphism of $\widetilde{H}^k(\mathbb{R}^n)$ onto itself that we will denote by $t_u$. We call a vector field on $\widetilde{H}^k(\mathbb{R}^n)$ invariant if it is invariant with respect to all diffeomorphisms of the form $t_u$, where $u$ is an arbitrary element of $\widetilde{H}^k(\mathbb{R}^n)$. There is a one-to-one correspondence between $\widetilde{E}_k$ and the set of invariant vector fields on $\widetilde{H}^k(\mathbb{R}^n)$. If $X, Y$ are two invariant vector fields on $\widetilde{H}^k(\mathbb{R}^n)$, so is the bracket $[X, Y]$. The vector space $\widetilde{E}_k$, endowed with this multiplication, becomes a Lie algebra over the field of real numbers. The Lie algebra $\widetilde{O}_n^k$ of $\widetilde{L}_n^k$ is a Lie subalgebra of $\widetilde{E}_k = \mathbb{R}^n \oplus \widetilde{O}_n^k$.

To every differentiable map $f$ of a differentiable manifold $W$ into $\widetilde{H}^k(\mathbb{R}^n)$, we can associate a differential 1-form $\omega_f = f^{-1} df$ with values in the Lie algebra $\widetilde{E}_k$ defined by $\omega_f(X) = (T f^{-1}_o T f)(X)$ for all $X$ in $T_x(W)$. In particular, if $W = \widetilde{H}^k(\mathbb{R}^n)$ and if $f$ is the identity map of $\widetilde{H}^k(\mathbb{R}^n)$, we get a differential 1-form $\omega$ on $\widetilde{H}^k(\mathbb{R}^n)$ with values in $\widetilde{E}_k$, called the invariant form on $\widetilde{H}^k(\mathbb{R}^n)$.

**Proposition 1.11.** The invariant form $\omega$ on $\widetilde{H}^k(\mathbb{R}^n)$ satisfies the equation

$$d\omega + [\omega, \omega] = 0.$$

We recall that the form $[\omega, \omega]$ is defined by $[\omega, \omega](X, Y) = [\omega(X), \omega(Y)]$ for all vector fields $X, Y$ on $\widetilde{H}^k(\mathbb{R}^n)$. Since the module of vector fields on $\widetilde{H}^k(\mathbb{R}^n)$ is generated by the invariant vector
fields, it suffices to prove the equation for two invariant vector fields $X$ and $Y$. We have
\[
d\omega(X, Y) = X\omega(Y) - Y\omega(X) - \omega([X, Y]) \]
\[= -\omega([X, Y]) = -[\omega(X), \omega(Y)]
\]
proving the proposition.

Remark: We have adopted the following convention for the exterior product:
\[(\alpha \wedge \beta)(X_1, X_2, \ldots, X_{p+q}) = \sum (-1)^{\varepsilon} \alpha_{i_1, \ldots, i_p} \beta_{i_{p+1}, \ldots, i_{p+q}},
\]
where the summation runs over all permutations $i_1, \ldots, i_p, i_{p+1}, \ldots, i_{p+q}$ of $\{1, 2, \ldots, p+q\}$ and where $\varepsilon$ denotes the signature of the corresponding permutation. With this convention, we have the following formula: if $\alpha$ is a $p$-form, then
\[d\alpha(X_1, \ldots, X_{p+1}) = \sum_{i=1}^{p+q} (-1)^{i+j} X_i \alpha(X_1, \ldots, \hat{X}_i, \ldots, X_{p+1}) + \sum_{i<j} (-1)^{i+j} \alpha([X_i, X_j], X_1, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_{p+1}).
\]

Kumpera pointed out to me that the above Lie algebra structure on $E^k$ comes from a canonical Lie group structure on $\tilde{H}^k(\mathbb{R}^n)$. Since $(x, g) \in \mathbb{R}^n \times \tilde{L}_n^k$ is identified with $t_x^{(k)}(g) = t_x^{(k)}(R_g(e_k)) = R_g \circ t_x^{(k)}(e_k)$, we have $(t_x^{(k)}(R_g) \circ t_x^{(k)}(R_g)) \circ t_x^{(k)}(R_g \circ R_g) = t_x^{(k)}(R_g \circ R_g)$. Let $L_n^k$ denote the underlying set of $\tilde{L}_n^k$ endowed with the following multiplication: $g \ast b = bg$ where $g \ast b$ denotes the product in $t\tilde{L}_n^k$ and $bg$ denotes the product in $\mathbb{R}^n \times t\tilde{L}_n^k$. With the identification $\tilde{H}^k(\mathbb{R}^n) = \mathbb{R}^n \times \tilde{L}_n^k$, $\tilde{H}^k(\mathbb{R}^n)$ becomes a Lie group isomorphic to $\mathbb{R}^n \times t\tilde{L}_n^k$. Moreover, if $u = (x, g), u' = (x', g')$, then
\[
u u' = (x + x', g' g) = t_{x+x'}^{(k)}(R_g, e_k)
\[= (t_{x'}^{(k)}(R_g) \circ t_{x'}^{(k)}(R_g))(e_k)
\[= (t_{x'}^{(k)}(R_g))(u') = t_u(u').
\]
where $t_u$ is the diffeomorphism defined in the opening paragraph of this section. In fact, $t_u$ is no other than the left translation defined by $u$ in
the Lie group $\overline{H}^k(\mathbb{R}^n)$. The Lie algebra structure on $\overline{E}^k$ defined above is precisely the Lie algebra of the Lie group $\overline{H}^k(\mathbb{R}^n)$. The invariant form $\omega$ is simply the Maurer-Cartan form of the Lie group $\overline{H}^k(\mathbb{R}^n)$.

Part II

HIGHER ORDER CONNECTIONS

1. Linear connections of order $k$.

An infinitesimal connection $\Gamma^k$ in the principal fibre bundle of semi-holonomic $k$-frames $\overline{H}^k(V_n)$ over $V_n$ will be called a linear connection of order $k$ of $V_n$. Let $\omega_k$ be its connection form. We will sometimes say that $\omega_k$ is a linear connection of order $k$ of $V_n$. If $D$ is the exterior covariant differentiation relative to $\omega_k$, the tensorial 2-form $\Theta_k = D \bar{\dot{\omega}}_k$ (resp. $\Omega_k = D \omega_k$) will be called the torsion form (resp. curvature form) of $\Gamma^k$ or $\omega_k$. For $Y, Z \in T(\overline{H}^k(V_n))$, $g \in \overline{E}^k_n$, we have

$$\Theta_k(TR_g(Y), TR_g(Z)) = \rho(g^{-1})\Theta_k(Y, Z).$$

where $\rho$ is the linear representation of $\overline{E}^k_n$ on $\overline{E}^{k-1}_n$ defined in Part I. If $Y$ or $Z$ is a vertical vector, then $\Theta_k(Y, Z) = 0$.

The linear representation $\rho$ induces a representation of $\overline{L}^k_n$ on $\overline{E}^{k-1}_n$: if $A \in \overline{L}^k_n$, $\xi \in \overline{E}^{k-1}_n$, we put

$$A \xi = \lim_{t \to 0} \frac{1}{t}(\rho(a_t)\xi - \xi)$$

where $a_t = \exp tA$ is the $t$-parameter group of transformations of $\overline{L}^k_n$ generated by $A$. In particular, if $\xi$ is vertical, i.e. $\xi \in \overline{E}^{k-1}_n$, we have

$$A \xi = -[T\pi_{k-1}^k(A), \xi].$$

Theorem II.1 (structure equations) Let $\omega_k$ be a linear connection of order $k$. Then

$$\Omega_k = d\omega_k + \omega_k \wedge \omega_k$$

$$\Theta_k = d\bar{\dot{\omega}}_k + \omega_k \wedge \bar{\dot{\omega}}_k + 3 \left[ T\pi_{k-1}^k \omega_k, T\pi_{k-1}^k \omega_k \right].$$
The first structure equation is well known. Let us show the second structure equation:

\[ \Theta_k(X, Y) = d \vartheta_k(X, Y) + \omega_k(X) \vartheta_k(Y) - \omega_k(Y) \vartheta_k(X) \]

\[ + 3 \left[ T \pi^{k}_{k-1} \circ \omega_k(X), T \pi^{k}_{k-1} \circ \omega_k(Y) \right] \]

for all vectors \( X \in T_u(\overline{H}^k(V_n)) \) and \( Y \in T_u(\overline{H}^k(V_n)) \). It is sufficient to verify the equality in the following three special cases:

i) \( X \) and \( Y \) are horizontal. In this case, \( \omega_k(X) = 0 \), \( \omega_k(Y) = 0 \) and the equation reduces to the definition of \( \Theta_k \).

ii) \( X \) and \( Y \) are vertical. Let \( X = A^* \) and \( Y = B^* \), where \( A^* \) and \( B^* \) are the fundamental vector fields on \( \overline{H}^k(V_n) \) corresponding to \( A = \omega_k(X) \) and \( B = \omega_k(Y) \) respectively. We have

\[ \Theta_k(X, Y) = 0; \]

\[ d \vartheta_k(X, Y) = X \vartheta_k(B^*) - Y \vartheta_k(A^*) - \vartheta_k(\left[ A^*, B^* \right]_u) \]

\[ = -\left[ T \pi^{k}_{k-1}(A), T \pi^{k}_{k-1}(B) \right] ; \]

\[ \omega_k(X) \vartheta_k(Y) = A \vartheta_k(B^*) \]

\[ = -\left[ T \pi^{k}_{k-1}(A), T \pi^{k}_{k-1}(B) \right] ; \]

\[ \omega_k(Y) \vartheta_k(X) = -\left[ T \pi^{k}_{k-1}(B), T \pi^{k}_{k-1}(A) \right] ; \]

and

\[ \left[ T \pi^{k}_{k-1} \circ \omega_k(X), T \pi^{k}_{k-1} \circ \omega_k(Y) \right] = \left[ T \pi^{k}_{k-1}(A), T \pi^{k}_{k-1}(B) \right] . \]

The equality holds.

iii) \( X \) is vertical and \( Y \) is horizontal. Let \( X = A^*_u \) with \( A = \omega_k(X) \in \overline{L}^k_n \). We can extend \( Y \) to an invariant horizontal vector field \( \tilde{Y} \) on \( \overline{H}^k(V_n) \). We have then

\[ d \vartheta_k(X, Y) = X \vartheta_k(\tilde{Y}) - Y \vartheta_k(A^*) - \vartheta_k(\left[ A^*, \tilde{Y} \right]_u) . \]

Since \( \vartheta_k(A^*) \) is constant, \( Y \vartheta_k(A^*) = 0 \). As \( \tilde{Y} \) is an invariant horizontal vector field, \( \left[ A^*, \tilde{Y} \right] = 0 \). Let \( a_t = \exp tA \) be the \( t \)-parameter group of transformations of \( \overline{L}^k_n \) generated by \( A \in \overline{L}^k_n \).

\[ d \vartheta_k(X, Y) = A^*_u \vartheta_k(\tilde{Y}) \]

\[ = \lim_{t \to 0} \frac{1}{t} (\rho(a_t^{-1}) \vartheta_k(\tilde{Y}) - \vartheta_k(\tilde{Y})) \]
Now, $\omega_k(Y) = 0$, $\Theta_k(X, Y) = 0$ and $\omega_k(X) \Theta_k(Y) = \Lambda \Theta_k(Y)$. The equality therefore holds.

The projection $\pi^k_m$ of $\overline{H}^k(V_n)$ onto $\overline{H}^m(V_n)$ being compatible with the natural surjection of $\overline{L}^k_n$ onto $\overline{L}^m_n$ ($m < k$), any linear connection $\omega_k$ (of order $k$) induces a linear connection $\omega_m$ of order $m$, given by

$$\pi^k_m \ast \omega_m = T \pi^k_m \circ \omega_k.$$

**Proposition II.2** Any linear connection $\omega_k$ of order $k$ induces canonically a linear connection $\omega_m$ of order $m < k$ given by

$$\pi^k_m \ast \omega_m = T \pi^k_m \circ \omega_k.$$

We have the relations:

$$\pi^k_m \ast \Omega_m = T \pi^k_m \circ \Omega_k,$$

$$\pi^k_m \ast \Theta_m = T \pi^{k-1}_{m-1} \circ \Theta_k.$$

Let us verify only the last formula. We know that

$$\pi^k_m \ast \theta_m = \theta_m \circ T \pi^k_m = T \pi^{k-1}_{m-1} \circ \theta_k.$$

As a consequence, $\pi^k_m \ast d \theta_m = T \pi^{k-1}_{m-1} \circ d \theta_k$. From the second structure equation, we obtain

$$\pi^k_m \ast \Theta_m = \pi^k_m \ast d \theta_m + \pi^k_m \ast \omega_m \wedge \pi^k_m \ast \theta_m$$

$$= T \pi^{k-1}_{m-1} \circ d \theta_k + T \pi^k_m \circ \omega_k \wedge T \pi^{k-1}_{m-1} \circ \theta_k$$

$$+ 3 \left[ T \pi^m_{m-1} \circ \pi^k_m \circ \omega_m, T \pi^m_{m-1} \circ \pi^k_m \ast \omega_m \right]$$

$$= T \pi^{k-1}_{m-1} \circ d \theta_k + T \pi^k_m \circ \omega_k \wedge T \pi^{k-1}_{m-1} \circ \theta_k$$

$$+ 3 \left[ T \pi^k_{m-1} \circ \omega_k, T \pi^k_{m-1} \circ \omega_k \right]$$

$$= T \pi^{k-1}_{m-1} \circ \Theta_k.$$

**Corollary II.3** If the torsion form (resp. the curvature form) of $\omega_k$ vanishes identically on $T(\overline{H}^k(V_n))$, the induced connection $\omega_m$ ($m < k$) is without torsion (resp. without curvature).

Let $\omega_k$ be a linear connection of $V_n$. We say that $\omega_k$ is quasi- ho-
loumonic if the connection form $\omega_k$, restricted to $T(H^k(V_n))$, defines a connection in the principal fibre bundle $H^k(V_n)$ over $V_n$. If $\omega_k$ is quasi-holonomic, all induced connections $\omega_m (m < k)$ are quasi-holonomic. The canonical connection in $\mathcal{H}^k(R^n) = R^n \times L_n^k$ is quasi-holonomic.

2. Second order linear connections.

Let $u$ be an element of $\mathcal{H}^2(V_n)$. Consider a coordinate neighbourhood $U$ of $a_0 = \pi_0^2(u)$ with a system of local coordinates $\{x^1, x^2, \ldots, x^n\}$. The 2-frame $u$ can be represented by a set of local coordinates $(x^i, x^{ij}, x^{ijk})$ with $\det(x^i) \neq 0$. Let $U'$ be another coordinate neighbourhood of $a_0$ with a system of local coordinates $\{y^1, y^2, \ldots, y^n\}$. The same $u$ is represented by $(y^i, y^{ij}, y^{ijk})$. The changes of local coordinates are given by

$$y^i = y^i(x)$$

$$y^j_i = \sum \left( \frac{\partial y^i}{\partial x^m} \right) y^m_j$$

$$y^{ijk} = \sum \left( \frac{\partial^2 y^m}{\partial x^i \partial x^k} \right) x^m_i + \sum \left( \frac{\partial y^p}{\partial x^i} \right) \left( \frac{\partial y^q}{\partial x^k} \right) x^i_p q.$$ 

An element $g \in L_2$ can be represented by $u = (a^i_j, a^{ijk})$ with $\det(a^i_j) \neq 0$. In terms of these coordinates, the multiplication in $L_2$ is given by

$$(a^i_j, a^{ijk}) \cdot (b^i_j, b^{ijk}) = (\Sigma a^i_m b^m_j + \Sigma a^i_j b^{jk} + \Sigma a^i_p q b^p b^q, a^{ijk} + \Sigma a^i_p q a^p a^q).$$

The action of $L_2$ on $\mathcal{H}^2(V_n)$ is given by

$$(x^i, x^{ij}, x^{ijk}) \cdot (a^i_j, a^{ijk}) = (x^i, \Sigma x^m_i a^m_j, \Sigma x^i a^{jk} + \Sigma x^i_p q a^p a^q).$$

Let $\alpha$ be the automorphism of $L_2$ defined by $\alpha(a^i_j, a^{ijk}) = (a^i_j, a^{ijk})$. It is evident that $\alpha$ leaves fixed every element in $L_2$. Moreover, $\alpha^2 = 1$.

**Proposition 2.1** There exists an involutive automorphism $\alpha$ of $L_2$ such that $L_2$ is the subgroup of all the fixed points of $\alpha$.

**Theorem 2.2** The homogeneous space $L_2^2 / L_2$ is weakly reductive: there exists a vector subspace $\mathfrak{m}$ of $L_2^2$ such that

$$L_2^2 = L_2^2 \oplus \mathfrak{m}$$

(identity sum),
LEMMA II.6 Let \( \alpha \) be an involutive automorphism of a Lie group \( \bar{G} \). The set of fixed points of \( \alpha \) forms a Lie subgroup \( G \) of \( \bar{G} \). Moreover, the homogeneous space \( \bar{G}/G \) is weakly reductive: there exists a vector subspace \( \mathfrak{M} \) of the Lie algebra \( \mathfrak{g} \) of \( \bar{G} \) such that
\[
\mathfrak{g} = \mathfrak{g} \oplus \mathfrak{M} \quad \text{(direct sum)}
\]
where \( \mathfrak{g} \) is the Lie algebra of \( G \). The vector space \( \mathfrak{M} \) can be given by
\[
\mathfrak{M} = \{ X \in \mathfrak{g} : T \alpha(X) = -X \}.
\]

Let \( \mathfrak{M} \) be the vector subspace of \( \mathfrak{Q}_n^2 \) defined by the above lemma. If \( X \in \mathfrak{M}, \ Y \in \mathfrak{M}, \ T \alpha(\ [X, Y]) = [T \alpha(X), T \alpha(Y)] = [-X, -Y] = [X, Y] \), showing that \( [X, Y] \in \mathfrak{Q}_n^2 \), i.e. \( [\mathfrak{M}, \mathfrak{M}] \subset \mathfrak{Q}_n^2 \). We have therefore the following result.

COROLLARY II.7 The homogeneous space \( \mathfrak{L}_n^2/\mathfrak{L}_n^2 \) is a symmetric space.

For the rest of this section, we fix once for all a decomposition \( \mathfrak{Q}_n^2 = \mathfrak{Q}_n^2 + \mathfrak{M} \), where \( \mathfrak{M} \) is the vector subspace defined in the theorem II.5. We denote by \( i \) the canonical injection of \( \mathfrak{H}^2(\mathfrak{V}_n) \) into \( \mathfrak{H}^2(\mathfrak{V}_n) \).

Let \( \bar{\omega}_2 \) be a connection form in \( \mathfrak{H}^2(\mathfrak{V}_n) \). We can write \( i^*\bar{\omega}_2 = \omega_2 + t \), where \( \omega_2 \) (resp. \( t \)) is the \( \mathfrak{Q}_n^2 \)-component (resp. \( \mathfrak{M} \)-component) of \( i^*\bar{\omega}_2 \). Since \( \text{ad}(\mathfrak{L}_n^2)\mathfrak{M} \subset \mathfrak{M} \), \( \omega_2 \) defines a connection in the principal fibre bundle \( \mathfrak{H}^2(\mathfrak{V}_n) \) over \( \mathfrak{V}_n \) and \( t \) is a \( \mathfrak{M} \)-valued tensorial 1-form on \( \mathfrak{H}^2(\mathfrak{V}_n) \), called the quasi-holonomic form of \( \bar{\omega}_2 \). Inversely, the couple \( (\omega_2, t) \) determines a connection \( \bar{\omega}_2 \) in \( \mathfrak{H}^2(\mathfrak{V}_n) \). In fact, if \( \xi \in T_u(\mathfrak{H}^2(\mathfrak{V}_n)) \) with \( u \in \mathfrak{H}^2(\mathfrak{V}_n) \), we can write \( \xi = \xi' + \xi'' \), where \( \xi' \) is a horizontal vector with respect to the connection \( \omega_2 \) and \( \xi'' \) is a vertical vector. Let us put \( \bar{\omega}_2(\xi) = t(\xi') + u^{-1}(\xi'') \). Now, if \( \bar{\xi} \in T_v(\mathfrak{H}^2(\mathfrak{V}_n)) \) where \( v \notin \mathfrak{H}^2(\mathfrak{V}_n) \), there exist \( u \in \mathfrak{H}^2(\mathfrak{V}_n) \) and \( g \in \mathfrak{L}_n^2 \) such that \( v = ug \) and \( \bar{\xi} = TR_g(\xi) \) for
some $\xi \in T_u(\mathbb{H}^2(V_n))$. It is easy to check that $\bar{w}_2(\xi) = \text{ad}(g^{-1})\bar{w}_2(\xi)$ does not depend on the choice of $u$ and $g$. The mapping $\xi \mapsto \bar{w}_2(\xi)$ gives the required connection form on $\mathbb{H}^2(V_n)$. Besides, $i^*\bar{w}_2 = \omega_2 + t$. We have thus established the following result.

**PROPOSITION II.8** There is a one-to-one correspondence between the set of all second order connections $\bar{w}_2$ of $V_n$ and the set of all couples $(\omega_2, t)$, where $\omega_2$ is a connection form in $\mathbb{H}^2(V_n)$ and $t$ is a $\mathbb{R}$-valued tensorial 1-form on $\mathbb{H}^2(V_n)$; the correspondence is given by

$$i^*\bar{w}_2 = \omega_2 + t.$$

**COROLLARY II.9** A linear connection $\bar{w}_2$ is quasi-holonomic if and only if its associated quasi-holonomic form $t$ vanishes identically on $\mathbb{H}^2(V_n)$.

Let $\phi$ be a tensorial form on $\mathbb{H}^2(V_n)$. From the structure equation

$$\bar{D}\phi = d\phi + \bar{w}_2 \wedge \phi$$

where $\bar{D}\phi$ is the exterior covariant derivative of $\phi$ with respect to $\bar{w}_2$, we deduce that

$$i^*(\bar{D}\phi) = i^*d\phi + i^*\bar{w}_2 \wedge i^*\phi$$

$$= d(i^*\phi) + i^*\bar{w}_2 \wedge i^*\phi$$

$$= d(i^*\phi) + \omega_2 \wedge i^*\phi + t \wedge i^*\phi.$$

The induced form $i^*\phi$ is a tensorial form on $\mathbb{H}^2(V_n)$. If $D$ is the exterior covariant differentiation with respect to $\omega_2$, we have

$$D(i^*\phi) = d(i^*\phi) + \omega_2 \wedge i^*\phi.$$

Thus

$$i^*(\bar{D}\phi) = D(i^*\phi) + t \wedge i^*\phi.$$

Let $\bar{\Omega}_2$ (resp. $\Omega_2$) be the curvature form of $\bar{w}_2$ (resp. $\omega_2$). From the structure equation

$$\bar{\Omega}_2 = d\bar{w}_2 + [\bar{w}_2, \bar{w}_2]$$

we have

$$i^*\bar{\Omega}_2 = i^*(d\bar{w}_2) + i^*[\bar{w}_2, \bar{w}_2]$$

$$= d(i^*\bar{w}_2) + [i^*\bar{w}_2, i^*\bar{w}_2]$$

856
The form $D t + [t, t]$ is a tensorial 2-form on $H^2(V_n)$. We may call it the quasi-holonomic curvature of $\omega_2$.

From the structure equation

$$\bar{\Theta}_2 = d\Theta_2 + \omega_2 \wedge \Theta_2 + 3 \left[ T\pi^2_{1 o} \omega_2, T\pi^2_{1 o} \omega_2 \right]$$

we have

$$i^* \bar{\Theta}_2 = i^* d\Theta_2 + i^* \omega_2 \wedge i^* \Theta_2 + 3 \left[ T\pi^2_{1 o} i^* \omega_2, T\pi^2_{1 o} i^* \omega_2 \right]$$

$$= \Theta_2 + t \wedge i^* \Theta_2 + 3 \left[ T\pi^2_{1 o} (\omega_2 + t), T\pi^2_{1 o} (\omega_2 + t) \right].$$

The form

$$T = t \wedge i^* \Theta_2 + 3 \left[ T\pi^2_{1 o} (\omega_2 + t), T\pi^2_{1 o} (\omega_2 + t) \right]$$

$$- 3 \left[ T\pi^2_{1 o} \omega_2, T\pi^2_{1 o} \omega_2 \right]$$

is a tensorial 2-form on $H^2(V_n)$, which may be called the quasi-holonomic torsion of $\omega_2$.

If $\omega_2$ is quasi-holonomic, its associated quasi-holonomic form $t$ vanishes identically on $H^2(V_n)$. Therefore, the quasi-holonomic curvature and the quasi-holonomic torsion of $\omega_2$ are zero.

3. $E$-connections.

Let $u$ be an arbitrary element of $L^1_{n}$. There exists a unique automorphism $f$ of the vector space $\mathbb{R}^n$ such that $u = i^1_0 f$. The induced map $f^{(k-1)}: \bar{H}^{k-1}(\mathbb{R}^n) \to \bar{H}^{k-1}(\mathbb{R}^n)$ is a $(k-1)$-admissible isomorphism, and $i^1_{e_{k-1}} f^{(k-1)} \in L^k$. The mapping $u \mapsto \iota^k(u) = i^1_{e_{k-1}} f^{(k-1)}$ gives a canonical identification of $L^1_{n}$ with a subgroup of $L^k_{n}$ (hence of $\bar{L}^k_{n}$). For $m < k$, $\iota^m = \pi^k_m \circ \iota^k$.

An invariant section of the fibration $\bar{H}^{k+1}(V_n) \to H^{k}(V_n)$, i.e. a lift $\phi_{k+1}$ of $H^{k}(V_n)$ into $\bar{H}^{k+1}(V_n)$ compatible with the canonical homomorphism $\iota^{k+1} : L^1_{n} \to \bar{L}^{k+1}_{n}$, will be called an $E$-connection of order $k$ of $V_n$. It is given by a reduction of the structure group of $\bar{H}^{k+1}(V_n)$ from $\bar{L}^{k+1}_{n}$ to $L^1_{n}$. There is a one-to-one correspondence between the set of all $E$-connections (of order $k$) of $V_n$ and the set of all semi-holonomic connections (of order $k$) defined in the sense of Ehresmann on the principal
We say that an E-connection \( \phi_{k+1} \) is symmetrical or holonomic (resp. quasi-holonomic) if
\[
\phi_{k+1}(H^1(V_n)) \subseteq H^{k+1}(V_n) \quad \text{(resp. } \phi_{k+1}(H^1(V_n)) \subseteq \tilde{H}^{k+1}(V_n))\).
\]
If \( \phi_{k+1} \) is symmetrical (resp. quasi-holonomic), all projections \( \pi_{m+1} \circ \phi_{k+1} \) of \( \phi_{k+1} \) are symmetrical.

Consider an open set \( U \) of \( V_n \) with a system of local coordinates \( \{x^1, x^2, \ldots, x^n\} \). In terms of the induced local coordinates, a lift \( \phi_{k+1} \) of \( H^1(V_n) \) into \( H^{k+1}(V_n) \) can be expressed by
\[
(x^i, x^j) \rightarrow (x^i, \ldots, x^i_{l_1l_2 \ldots l_{k+1}}).
\]
If \( \phi_{k+1} \) is invariant, the functions \( x^i_{l_1l_2 \ldots l_{k+1}} \) can be written in the form
\[
x^i_{l_1l_2} = -\sum \Gamma^i_{m_1m_2} x^m_1 x^m_2,
\]n
\[
x^i_{l_1l_2l_3} = -\sum \Gamma^i_{m_1m_2m_3} x^m_1 x^m_2 x^m_3,
\]
\[
\vdots
\]
\[
x^i_{l_1l_2 \ldots l_{k+1}} = -\sum \Gamma^i_{m_1m_2 \ldots m_{k+1}} x^m_1 x^m_2 \ldots x^m_{k+1}
\]
where \( \Gamma^i_{m_1m_2 \ldots m_{k+1}} \) are differentiable functions defined on \( U \). These are the Christoffel symbols of the \( \xi \)-connection \( \phi_{k+1} \). They are not entirely arbitrary; they have to satisfy certain conditions when we change the local coordinates system. It is clear that \( \phi_{k+1} \) is symmetrical if and only if all the Christoffel symbols are symmetrical with respect to their lower indices.

Let us consider some particular cases:

\textbf{case (i): } \( k = 1 \).

Let \( \Gamma^i_{rs} \) (resp. \( \tilde{\Gamma}^i_{rs} \)) be the Christoffel symbols of a first order \( \xi \)-connection \( \phi_2 \) relative to a coordinate neighbourhood \( U \) (resp. \( \tilde{U} \)) with a local coordinates system \( \{x^1, x^2, \ldots, x^n\} \) (resp. \( \{\tilde{x}^1, \tilde{x}^2, \ldots, \tilde{x}^n\} \)). If \( U \cap \tilde{U} \neq \emptyset \), we obtain easily the classical formula for the Christoffel symbols of a linear connection.
The quantities $\Gamma^i_{jk}$ define then a linear connection of $V_n$. On the other hand, if $u \in H^1(V_n)$, the lift $O_2(u)$ of $u$ determines a horizontal $n$-plane $Q\phi_2(u)$ of $H^1(V_n)$ at $u$. Since $\phi_2$ is compatible with $\iota^2 : l^1_n \rightarrow l^2_n$, it is easy to check that the distribution $u \rightarrow Q\phi_2(u)$ defines an infinitesimal connection on $H^1(V_n)$, thus a linear connection $\omega_1$ of $V_n$. The quantities $\Gamma^i_{jk}$ are simply the classical Christoffel symbols of the associated linear connection $\omega_1$. In fact, if $X_i = \sum x^i_j \frac{\partial}{\partial x^j}$ $(1 \leq i \leq n)$ is a basis for $T_x(V_n)$, with $x \in U$, the horizontal lift of $X_i$ at $u = (x^i, x^j) \in H^1(V_n)$ with respect to $\omega_1$, is given by

$$X^*_i = \sum x^i_j \frac{\partial}{\partial x^j} u + \sum x^i_j \frac{\partial}{\partial x^i} u$$

where $x^i_j = -\sum \Gamma^i_{rs} x^r x^s$. Let $y^i_j = (\frac{\partial}{\partial x^j}) e_1$ $(1 \leq i, j \leq n)$ be a basis for $\Omega^1_n$. The components of $\omega_1 = \sum \omega^i_j r^i_j$ can be expressed by

$$\omega^i_j = \sum y^i_k \left( dx^k + \sum c^k_{mp} x^p dx^m \right)$$

where $(y^i_j)$ is the inverse matrix of $(x^i_j)$ and $c^k_{mp}$ are the classical Christoffel symbols of the linear connection $\omega_1$. Consequently, $\omega^i_j (X^*_k) = 0$ for all indices $1 \leq i, j, k \leq n$. It follows that

$$x^i_j = -\sum \Gamma^i_{rs} x^r x^s = -\sum c^i_{rs} x^r x^s.$$ 

Since $det(x^i_j) \neq 0$, we have $\Gamma^i_{jk} = c^i_{jk}$.

**Proposition II.10 [4a]** (i) There is a one-to-one correspondence between the set of first order linear connections of $V_n$ and the set of invariant sections of $H^1(V_n)$ into $H^2(V_n)$.

(ii) Two linear connections of $V_n$ have the same torsion if and only if the images of $H^1(V_n)$ by the corresponding invariant sections are contained in a principal subbundle of $H^2(V_n)$ having the structure group $L^2_n$.

It remains to prove the second part of the proposition. Let $\phi_2, \bar{\phi}_2$
be two invariant sections of $H^1(V_n)$ into $\widetilde{H}^2(V_n)$. In terms of local coordinates, these $E$-connections are given by
\[
(x^i, x_j^i) \rightarrow \phi_2(x^i, x_j^i) = (x^i, x_j^i, -\Sigma_{rs}^{i} x^r_{j k}^s),
\]
\[
(x^i, x_j^i) \rightarrow \widetilde{\phi}_2(x^i, x_j^i) = (x^i, x_j^i, -\Sigma_{rs}^{i} x^r_{j k}^s).
\]
where $\Gamma_{jk}^{i}$, $\widetilde{\Gamma}_{jk}^{i}$ are the corresponding Christoffel symbols. As $\phi_2(x^i, x_j^i)$ and $\widetilde{\phi}_2(x^i, x_j^i)$ are on the same fibre of $\widetilde{H}^2(V_n)$, there exists an element $(\delta_j^{i}, g_{j k}) \in \tilde{M}^2_n = \text{Ker}(\tilde{L}_n^2 - \tilde{L}_n^1)$ such that
\[
(x^i, x_j^i, -\Sigma_{rs}^{i} x^r_{j k}^s) = (x^i, x_j^i, -\Sigma_{rs}^{i} x^r_{j k}^s)(\delta_j^{i}, g_{j k}).
\]
It follows that
\[
\Sigma_{rs}^{i} x^r_{j k}^s = \Sigma_{rs}^{i} x^r_{j k}^s - \Sigma_{m}^{i} x^m_{j k} g_{j k}.
\]
Consequently, we have
\[
(*) \quad \Sigma_{rs}^{i} (\Gamma_{rs}^{i} - \Gamma_{sr}^{i}) x^r_{j k} = \Sigma_{rs}^{i} (\Gamma_{rs}^{i} - \Gamma_{sr}^{i}) x^r_{j k} - \Sigma_{m}^{i} (g_{jk}^m - g_{jk}^m).
\]
If the two linear connections have the same torsion, that is if $\Gamma_{rs}^{i} - \Gamma_{sr}^{i} = \Gamma_{rs}^{i} - \Gamma_{sr}^{i}$, we have $\Sigma_{m}^{i} (g_{jk}^m - g_{jk}^m) = 0$. Since $\text{det}(x^i_m) \neq 0$, we get $g_{jk}^m = g_{jk}^m$, which shows that $(\delta_j^{i}, g_{j k}) \in M^2_n = \tilde{M}^2_n \cap L^2_n$. Hence the condition is necessary.

If $\phi_2$ and $\widetilde{\phi}_2$ map $H^1(V_n)$ into the same principal subbundle of $\widetilde{H}^2(V_n)$ having the structure group $L^2_n$, we still have the formula $(*)$ with $g_{jk}^m = g_{jk}^m$. Consequently,
\[
\Sigma_{rs}^{i} (\Gamma_{rs}^{i} - \Gamma_{sr}^{i}) x^r_{j k} = \Sigma_{rs}^{i} (\Gamma_{rs}^{i} - \Gamma_{sr}^{i}) x^r_{j k}.
\]
Since $\text{det}(x^i_j) \neq 0$, we get
\[
\Gamma_{rs}^{i} - \Gamma_{sr}^{i} = \Gamma_{rs}^{i} - \Gamma_{sr}^{i}.
\]
Hence the connections have the same torsion, proving that the condition is sufficient.

Case (ii): $k = 2$

An element of $L^3_n$ can be represented by a set of coordinates $(a^i_j, a^i_{jk}, a^i_{jkm})$ with $\text{det}(a^i_j) \neq 0$. The multiplication is given by
\[
(a^i_j, a^i_{jk}, a^i_{jkm})(b^i_j, b^i_{jk}, b^i_{jkm}) = (\delta_{r}^i b^r_{j}, \Sigma_{rs}^{i} b^r_{j} b^s_{j k} + \delta_{r}^i b^r_{j k}).
\]
If \( u = (x^i, x^j, x^i_{jk}, x^i_{jkm}) \in \mathcal{H}^3(V_n) \), the action of \( L_3^n \) on \( \mathcal{H}^3(V_n) \) can be expressed by

\[
\Sigma (a^i_{rst}, b^s_{jk}, a^i_{jkm}) + a^i_{rs} b^s_{jk} + a^i_{jkm} + a^i_{jkm} b^s_{jk} \).
\]

Consider an \( E \)-connection \( \phi_3 \) of order 2. In terms of local coordinates, \( cP \) is given by

\[
\sum (x^i, x^j, x^i_{jk}, x^i_{jkm})(a^i_{jk}, a^i_{jkm}) = (x^i, \sum x^i a^i_{jk}, \Sigma (x^i a^i_{jk} + x^i a^i_{jkm})),
\]

where \( \Gamma^i_{rs}, \Gamma^i_{rst} \) are the Christoffel symbols. If \( \bar{\Gamma}^i_{rs}, \bar{\Gamma}^i_{rst} \) are the Christoffel symbols of \( \phi_3 \) in an other local coordinates system, we have

\[
\bar{\Gamma}^i_{rst} = \sum (\frac{\partial \bar{x}^r}{\partial x^j})(\frac{\partial \bar{x}^s}{\partial x^k})(\frac{\partial \bar{x}^t}{\partial x^a}) + \sum (\frac{\partial^2 \bar{x}^r}{\partial x^j \partial x^k})(\frac{\partial x^a}{\partial x^r}).
\]

By direct computations, we have the following result:

**Proposition II.11** Let \( \Gamma^i_{jk}, \Gamma^i_{jkm} \) be the Christoffel symbols of a second order \( E \)-connection of \( V_n \). If the induced first order \( E \)-connection is symmetrical, then the following quantities

\[
A^i_{jkm} = \Gamma^i_{jkm} - \Gamma^i_{kjm},
\]

\[
B^i_{jkm} = \Gamma^i_{jkm} - \Gamma^i_{mkj},
\]

\[
C^i_{jkm} = \Gamma^i_{jkm} - \Gamma^i_{jmk}
\]

are respectively the components of a \( (1,3) \)-tensor on \( V_n \). The given \( E \)-connection is symmetrical if and only if these three tensors are zero.

4. Linear connections and \( E \)-connections.

The Lie group \( L_{n+1}^k \) (resp. \( L_{n+1}^k \)) acts linearly on \( E^k \) (resp. \( E^k = T_e \mathcal{H}^k(\mathbb{R}^n) \)) on the left. We denote by \( \bar{S}^k T \) (resp. \( S^k T \)) the associa-
ted vector bundle of $\widetilde{H}^{k+1}(V_n)$ (resp. $H^{k+1}(V_n)$) with standard fibre $\widetilde{E}^k$ (resp. $E^k$) and structure group $\widetilde{L}^{k+1}$ (resp. $L_n^{k+1}$). For $k = 0$, $S^0T = T(V_n)$.

**PROPOSITION 11.12** The vector bundle $\widetilde{S}^kT$ (resp. $S^kT$) is canonically isomorphic to the vector bundle $T(\widetilde{H}^k(V_n))/\widetilde{L}^k_n$ (resp. $T(H^k(V_n))/L^k_n$).

An element $u \in \widetilde{H}^{k+1}(V_n)$ determines a linear isomorphism $\tilde{u}$ of $\widetilde{E}^k$ onto $T_{u'}(\widetilde{H}^k(V_n))$ with $u' = \pi^{k+1}_0(u)$. On the other hand, $u$ can be considered as a linear isomorphism of $\widetilde{E}^k$ onto the fibre $(\widetilde{S}^kT)_x$ over $x$, where $x$ is the projection of $u$ on $V_n$. We have then a linear isomorphism $\tilde{u}_o^{-1}$ of $(\widetilde{S}^kT)_x$ onto $T_{u'}(\widetilde{H}^k(V_n))$. If $v$ is another element of $\widetilde{H}^{k+1}(V_n)$ with projection $x = \pi^{k+1}_0(v)$, we can write $v = u g$ for a unique $g \in \widetilde{E}^{k+1}$. Similarly, we have a linear isomorphism $\tilde{v}_o^{-1} : (\widetilde{S}^kT)_x \rightarrow T_{v'}(\widetilde{H}^k(V_n))$, where $v' = \pi^{k+1}_0(v)$. Now, $v = u_o \rho(g)$ and $\tilde{v} = T_{g} \tilde{u}_o \rho(g)$ with $g' = \pi^{k+1}_0(g) \in \widetilde{E}^k$. Consequently, $\tilde{v}_o^{-1} = T_{g'} \tilde{u}_o u^{-1}$. Since $\widetilde{H}^{k+1}(V_n) \rightarrow \widetilde{H}^k(V_n)$ is surjective, we get an isomorphism of $\widetilde{S}^kT$ onto $T(\widetilde{H}^k(V_n))/\widetilde{L}^k_n$. Similarly, one establishes an isomorphism of $S^kT$ onto $T(H^k(V_n))/L^k_n$.

P. Libermann showed that $T(\widetilde{H}^k(V_n))/\widetilde{L}^k_n$ (resp. $T(H^k(V_n))/L^k_n$) is canonically isomorphic to $\tilde{j}^kT$ (resp. $j^kT$), the $k$-th semi-holonomic (resp. holonomic) prolongation of the vector bundle $T(V_n)$. Thus, we have an isomorphism of $\widetilde{S}^kT$ (resp. $S^kT$) onto $\tilde{j}^kT$ (resp. $j^kT$).

$H^{k+1}(V_n)$ being a principal fibre subbundle of $\widetilde{H}^{k+1}(V_n)$ and the action of $L_n^{k+1}$ on $E^k$ being the restriction of that of $\widetilde{L}^{k+1}$ on $\widetilde{E}^k$, the vector bundle $S^kT$ can be considered as a vector subbundle of $\widetilde{S}^kT$.

The projection $\pi^{k+1}_m$ of $\widetilde{H}^{k+1}(V_n)$ onto $\widetilde{H}^{m+1}(V_n)$ induces a surjection $p_m^k$ of $\widetilde{S}^kT$ onto $\widetilde{S}^mT$. Moreover, the restriction of $p_m^k$ to each fibre of $\widetilde{S}^kT$ is linear. Similarly, we have a projection of $S^kT$ onto $S^mT$ for $m < k$.

An $\tilde{E}$-connection $\varphi_{k+1} : H^1(V_n) \rightarrow \widetilde{H}^{k+1}(V_n)$ induces a splitting of the following exact sequence of vector bundles

$$0 \rightarrow N^k \rightarrow \widetilde{S}^kT \rightarrow T(V_n) \rightarrow 0$$

where $N^k$ is the kernel of the projection $\widetilde{S}^kT \rightarrow T(V_n)$. More precisely, we have the following result:
THEOREM II.13 There exists a one-to-one correspondence between the set of E-connections of order k of \(V_n\) and the set of splittings of the exact sequence of vector bundles over \(V_n\):

\[
0 \rightarrow \overline{N}^k \rightarrow \overline{S}^k T \rightarrow T(V_n) \rightarrow 0.
\]

Let us first prove two lemmas:

LEMMA II.14 Let \(\overline{E}^k = R^n \oplus \overline{O}^k_n\) be the canonical decomposition of \(\overline{E}^k\) defined by the canonical connection in \(\overline{H}^k(R^n) = R^n \times \overline{L}^k_n\). For every other decomposition of \(\overline{E}^k\) of the form \(E^k = Q^k \oplus \overline{Q}^k_n\), there exists a unique \(g \in \overline{M}^{k+1} = \text{Ker } (T_n^{k+1} \rightarrow L_n^1)\) such that \(\rho(g)(R^n) = Q^k\).

We prove the lemma by induction on \(k\). For \(k = 1\), we have the canonical decomposition \(E^1 = R^n \oplus Q^1_n\). Let \(E^1 = Q^1 \oplus Q^1_n\) be another decomposition of \(E^1\). Consider a local section \(\sigma_1\) of \(H^1(R^n) \rightarrow R^n\) such that \(\sigma_1(0) = e_1\) and \(T \sigma_1(R^n) = Q^1\). Let \(f\) be the admissible local isomorphism of \(H^1(R^n)\) into \(H^1(R^n)\) defined by the condition: \(f \circ \eta_1 = \sigma_1\), where \(\eta_1\) is the «zero section» of \(H^1(R^n) = R^n \times L_n^1 \rightarrow R^n\). The 1-jet \(j^1_{e_1} f = g\) defines an element \(g \in \overline{M}^2_n = \text{Ker } (T_n^2 \rightarrow L_n^1)\) satisfying the property: \(\rho(g)(R^n) = Q^1\). Uniqueness follows from the fact that the neutral element is the only element of \(\overline{M}^2_n\) leaving stable the two components of \(E^1 = R^n \oplus Q^1_n\).

Let us assume that the lemma is proved for \(m \leq k - 1\). If \(\overline{E}^k = Q^k \oplus \overline{Q}^k_n\) is a decomposition of \(\overline{E}^k\), we may consider a local section \(\sigma_k\) of \(\overline{H}^k(R^n) \rightarrow R^n\) satisfying the conditions: \(\sigma_k(0) = e_k\) and \(T \sigma_k(T_0(R^n)) = Q^k\). Now,

\[
\overline{E}^{k-1} = T \pi_n^{k-1}(\overline{E}^k) = T \pi_n^{k-1}(Q^k) \oplus T \pi_n^{k-1}(\overline{Q}^k_n) = T \pi_n^{k-1}(Q^k) \oplus \overline{O}^{k-1}_n.
\]

From the induction hypothesis, there is a unique \(g' \in \overline{M}^k = \text{Ker } (T_n^k \rightarrow L_n^1)\) such that \(\rho(g')(R^n) = T \pi_n^{k-1}(Q^k)\). Let \(h\) be the admissible local isomorphism of \(\overline{H}^k(R^n)\) into \(\overline{H}^k(R^n)\) defined by the condition: \(h \circ \eta_k = R_{g'} \circ \sigma_k\), where \(\eta_k\) is the «zero section» of \(\overline{H}^k(R^n) = R^n \times \overline{L}^k_n \rightarrow R^n\). The 1-jet \(j^1_{e_k} h\) defines an element \(g\) of \(\overline{M}^{k+1} = \text{Ker } (T_n^{k+1} \rightarrow L_n^1)\) such that \(\rho(g)(R^n) = Q^k\). Suppose that there is another \(\tilde{g} \in \overline{M}^{k+1}\) satisfying the condition: \(\rho(\tilde{g})(R^n) = Q^k\). We have then \(\rho(\pi_n^{k+1}(\tilde{g}))(R^n) = T \pi_n^{k-1}(Q^k)\). Consequently, \(g' = \pi_n^{k+1}(\tilde{g})\). We can write \(\tilde{g} = g m_0\) where \(m_0\) is an ele-
ment of $\text{Ker}(\overline{L}_n^{k+1} \rightarrow \overline{L}_n^k)$. Since the neutral element is the only element of $\text{Ker}(\overline{L}_n^{k+1} \rightarrow \overline{L}_n^k)$ leaving stable the two components of $\overline{E}^k = \mathbb{R}^n \oplus \overline{L}_n^k$, we conclude that $\overline{g} = g$ proving the uniqueness of $g$.

**Lemma II.15** The Lie group $\iota^{k+1}(L^1_n)$ is the largest subgroup of $\overline{L}_n^{k+1}$ which leaves invariant the two direct summands of $\overline{E}^k = \mathbb{R}^n \oplus \overline{L}_n^k$.

It is easy to check that $\iota^{k+1}(L^1_n)$ leaves invariant the two direct summands of $\overline{E}^k = \mathbb{R}^n \oplus \overline{L}_n^k$. Now, consider an element $g \in \overline{L}_n^{k+1}$ such that $\rho(g)(\mathbb{R}^n) = \mathbb{R}^n$. Let $g_0 = \iota^{k+1}(g)$; the action of $\iota^{k+1}(g_0), g^{-1}$ on $\mathbb{R}^n \subset \overline{E}^k$ is trivial. Consequently, we have $g = \iota^{k+1}(g_0) \in \iota^{k+1}(L^1_n)$ in virtue of the preceding lemma.

Let us go back to the proof of the theorem. We have seen that there is a mapping $F$ of the set of $\mathcal{E}$-connections of order $k$ of $V_n$ into the set of splittings of the exact sequence of vector bundles over $V_n$:

$$0 \rightarrow \overline{N}^k \rightarrow \overline{S}^k T \rightarrow T(V_n) \rightarrow 0.$$  

This mapping $F$ is injective. Let us consider two $\mathcal{E}$-connections $\phi_{k+1}$ and $\psi_{k+1}$ which induce the same splitting

$$F(\phi_{k+1}) = F(\psi_{k+1}) : T(V_n) \rightarrow \overline{S}^k T.$$  

If $y \in T(V_n)$, we can write $y = q_1(u, \xi)$, where $u \in H^1(V_n)$, $\xi \in \mathbb{R}^n$ and $q_1$ is the natural projection of $H^1(V_n) \times \mathbb{R}^n$ onto $T(V_n)$. The condition $F(\phi_{k+1})(y) = F(\psi_{k+1})(y)$ implies that

$$q_{k+1}(\phi_{k+1}(u), \xi) = q_{k+1}(\psi_{k+1}(u), \xi),$$

where we have denoted by $q_{k+1}$ the natural projection of $\overline{H}^{k+1}(V_n) \times \overline{E}^k$ onto $\overline{S}^k T$. From the above lemma, we deduce that $\phi_{k+1}(u) = \psi_{k+1}(u)$ for all $u \in H^1(V_n)$. Let us show that $F$ is surjective. Consider a splitting of the exact sequence

$$0 \rightarrow \overline{N}^k \rightarrow \overline{S}^k T \rightarrow T(V_n) \rightarrow 0$$

given by the lift $\sigma : T(V_n) \rightarrow \overline{S}^k T$. Let $x$ be an arbitrary element of $V_n$. An element $u$ of the fibre of $\overline{H}^{k+1}(V_n)$ over $x$ determines a linear isomorphism of $\overline{E}^k$ onto $(\overline{S}^k T)_x$. The image $u^{-1}(\sigma(T_x(V_n)))_x$ is a vector subspace of $\overline{E}^k$. More exactly, we have $\overline{E}^k = u^{-1}(\sigma(T_x(V_n)))_x \oplus \overline{L}_n^k$. From
the lemma II.14, there exists a $g \in \overline{M}^{k+1} = \text{Ker}(\overline{L}^{k+1}_n \to \overline{L}^{1}_n)$ such that $\rho(g)(R^n) = u^{-1}(\sigma(T_x(V_n)))$. The element $v = ug \in \overline{H}^{k+1}(V_n)$ defines therefore a linear isomorphism of $\overline{k} = R^n \oplus \overline{S}_n^k$ onto $(\overline{k}T)_x$, mapping $R^n$ onto $\sigma(T_x(V_n))$. Every element of $\overline{H}^{k+1}(V_n)$ lying on the fibre over $x$ and having the same property is of the form $v g_0$ with $g_0 \in \iota^{k+1}(L^{1}_n)$. Since $x$ is arbitrary, we obtain in this way a principal subbundle of $\overline{H}^{k+1}(V_n)$ with structure group $\iota^{k+1}(L^{1}_n)$, hence the $E$-connection that we are looking for.

The vector bundle $T(\overline{H}^{k}(V_n))/L^{k}_n$ is isomorphic to $\overline{k}T$. We have therefore a one-to-one correspondence between the set of linear connections of order $k$ of $V_n$ and the set of splittings of the exact sequence of vector bundles

$$0 \to \overline{H}^{k} \to \overline{k}T \to T(V_n) \to 0.$$ 

From the preceding result, we have

**Theorem II.16** There is a one-to-one correspondence between the set of linear connections of order $k$ and the set of $E$-connections of the same order.

Consider an $E$-connection $\phi_{k+1} : H^1(V_n) \to \overline{H}^{k+1}(V_n)$. Let $\phi_k = \eta^{k+1}_k \circ \phi_{k+1}$. If $u \in H^1(V_n)$, $\phi_{k+1}(u)$ determines a horizontal $n$-plane of $\overline{H}^k(V_n)$ at $\phi_k(u) \in \overline{H}^k(V_n)$. We obtain thus a field of $n$-planes of $\overline{H}^k(V_n)$ defined on $\phi_k(H^1(V_n))$. It is easy to check that this local field is invariant with respect to the right translations defined by the elements of $\iota^k(L^{1}_n)$ on $\overline{H}^k(V_n)$. Consequently, we can extend it to a global field of $n$-planes of $\overline{H}^k(V_n)$ invariant with respect to the right translations of $L^{k}_n$ on $\overline{H}^k(V_n)$. We obtain thus a linear connection $\omega_k$ of order $k$ of $V_n$. This correspondence $\phi_{k+1} \to \omega_k$ is exactly the one we have established in the above theorem. For $k = 1$, we have a one-to-one correspondence between the set of symmetrical linear connections of $V_n$ and the set of invariant sections of $H^1(V_n)$ into $H^2(V_n)$ (cf. Prop. I.9 and Prop. II.10). Let us assume that there is a one-to-one correspondence between the set of symmetrical $E$-connections of order $m$ ($m \leq k - 1$) and the set of quasi-holonomic linear connections of the same order having zero torsion. If $\phi_{k+1}$ is a
symmetrical \( \mathcal{E} \)-connection of order \( k \), the corresponding linear connection \( \omega_k \) is quasi-holonomic and without torsion (cf. Theorem I.10). Inversely let \( \omega_k \) be a quasi-holonomic linear connection having zero torsion and let \( \phi_{k+1} \) be the corresponding \( \mathcal{E} \)-connection established in the above theorem. The connection projection \( \omega_{k-1} \) (of order \( k-1 \)) of \( \omega_k \) is a quasi-holonomic connection without torsion. From the induction hypothesis, the corresponding \( \mathcal{E} \)-connection \( \phi_k \) is symmetrical. It is easy to check that \( \phi_k = \pi_k^{k+1} \circ \phi_{k+1} \). Hence \( \phi_{k+1}(\Lambda^1(V_n)) \subset H^{k+1}(V_n) \) from the Holonomy Theorem. We have thus established the following result:

**Corollary II.17** There is a one-to-one correspondence between the set of symmetrical \( \mathcal{E} \)-connections and the set of quasi-holonomic linear connections without torsion.

5. **Pseudo-connections and multi-connections.**

A pseudo-connection of order \( k \) of \( V_n \) is a couple \((\psi_{k+1}, \Psi_{k+1})\), where \( \Psi_{k+1} \) is a homomorphism of \( \overline{L}_k \) into \( \overline{L}_{k+1} \) and \( \psi_{k+1} \) is a differentiable lift of \( \overline{H}^k(V_n) \) into \( \overline{H}^{k+1}(V_n) \) such that

\[
\psi_{k+1}(ug) = \psi_{k+1}(u) \Psi_{k+1}(g)
\]

for all \( u \in \overline{H}^k(V_n) \) and \( g \in \overline{L}_n \). It follows that \( \Psi_{k+1} \) is a lift of \( \overline{L}_k \) into \( \overline{L}_{k+1} \). The condition of compatibility implies that an invariant vector field of \( \overline{H}^k(V_n) \) can be lifted to an invariant vector field of \( \overline{H}^{k+1}(V_n) \). We obtain thus an infinitesimal connection in the principal fibre bundle \( E^{k+1} \rightarrow E^k(V_n) \), or equivalently, a splitting of the exact sequence of vector bundles over \( V_n \)

\[
0 \rightarrow \overline{N}_k^{k+1} \rightarrow \overline{S}^{k+1} \rightarrow \overline{S}^k \rightarrow 0
\]

where \( \overline{N}_k^{k+1} \) is the kernel of \( \overline{S}^{k+1} \rightarrow \overline{S}^k \).

Consider a pseudo-connection \((\psi_{k+1}, \Psi_{k+1})\) of \( V_n \). The lift \( \psi_{k+1} \) of \( \overline{H}^k(V_n) \) into \( \overline{H}^{k+1}(V_n) \) defines an absolute parallelism on \( \overline{H}^k(V_n) \). If \( Z \in T_u(\overline{H}^k(V_n)) \), we put \( \alpha(Z) = \overline{\psi}_{k+1}^{-1}(u) \). The mapping \( Z \mapsto \alpha(Z) \) defines a differentiable 1-form \( \alpha \) on \( \overline{H}^k(V_n) \) with values in \( \overline{E}^k \). There is an induced linear representation of \( \overline{L}_k \) on \( \overline{E}^k \) given by

\[
\sigma = \rho \circ \Psi_{k+1}.
\]
where we have denoted by $\rho$ the linear representation of $L_n$ on $E^k$.

If $Z \in T(\overline{H}^k(V_n))$, we have $\alpha(TR_g(Z)) = \sigma(g^{-1})\alpha(Z)$, i.e. $\alpha$ is a pseudotensorial 1-form on $\overline{H}^k(V_n)$, called the pseudo-connection form of $(\psi_{k+1}, \Psi_{k+1})$.

A multi-connection of order $k$ of $V_n$ is given by a sequence of pseudo-connections $(\psi_{m+1}, \Psi_{m+1})$, $m = 1, 2, \ldots, k$ such that $\Psi_{m+1} \circ \iota^m = \iota^{m+1}$. The composite map $\phi_{k+1} = \psi_{k+1} \circ \psi_{k} \circ \ldots \circ \psi_2$ defines an $\mathcal{E}$-connection of $V_n$. Inversely, given a sequence of homomorphisms $\Psi_{m+1}: L_n^m \rightarrow L_n^{m+1}$ such that $\Psi_{m+1} \circ \iota^m = \iota^{m+1}$ ($m = 1, 2, \ldots, k$), an $\mathcal{E}$-connection $\phi_{k+1}: H^k(V_n) \rightarrow \overline{H}^{k+1}(V_n)$ determines a multi-connection of order $k$ of $V_n$.

We are going to define a natural sequence of group homomorphisms

\[ L_n^1 \xrightarrow{\Lambda_2} L_n^2 \xrightarrow{\Lambda_3} \ldots \xrightarrow{\Lambda_{k+1}} L_n^k \xrightarrow{\iota^1} L_n^{k+1} \xrightarrow{\iota^2} \ldots \]

satisfying the conditions: $\pi_{k+1}^{k+1} \circ \Lambda_{k+1} = \text{identity}$, $\Lambda_{k+1} \circ \iota^k = \iota^{k+1}$ for $k = 2, 3, \ldots$. We put $\Lambda_2 = \iota^2$, the canonical injection of $L_n^1$ into $L_n^2$. It induces a lift of $H^1(\mathbb{R}^n) = \mathbb{R}^n \times L_n^1$ into $H^2(\mathbb{R}^n) = \mathbb{R}^n \times L_n^2$. We will denote this lift by the same symbol $\Lambda_2$. Let $u = j_{e_1}f \in L_n^2$, where $f$ is an admissible local isomorphism of $H^1(\mathbb{R}^n)$ into $H^1(\mathbb{R}^n)$. Consider the local isomorphism $b$ of $\overline{H}^2(\mathbb{R}^n)$ into $\overline{H}^2(\mathbb{R}^n)$ defined by the condition:

\[ b \circ \eta_2 = R_u \circ \Lambda_2 \circ R_u^{-1} \circ f \circ \eta_1, \]

where $u' = \pi_2^1(u)$ and $\eta_i$ ($i = 1, 2$) are the «zero sections». The 1-jet $j_{e_2}^1 b$ depends uniquely on $u$ and the mapping $u \rightarrow \Lambda_3(u) = j_{e_2}^1 b$ defines a group homomorphism of $L_n^2$ into $L_n^3$ satisfying the required conditions. Let us assume that we have defined homomorphisms $\Lambda_2, \Lambda_3, \ldots, \Lambda_k$ satisfying the required conditions. Let $v = j_{e_{k-1}}^1 b \in L_n^k$, where $b$ is an admissible local isomorphism of $\overline{H}^{k-1}(\mathbb{R}^n)$ into $\overline{H}^{k-1}(\mathbb{R}^n)$. Consider the admissible local isomorphism $g$ of $\overline{H}^k(\mathbb{R}^n)$ into $\overline{H}^k(\mathbb{R}^n)$ defined by the condition:

\[ g \circ \eta_k = R_v \circ \Lambda_k \circ R_v^{-1} \circ b \circ \eta_{k-1}, \]

with $v' = \pi_{k-1}^{k-1}(v)$ and $\eta_i$ ($i = k-1, k$) are the «zero sections». It is easy to check that the mapping $v \rightarrow \Lambda_{k+1}(v) = j_{e_k}^1 g$ defines a group homomorphism of $L_n^k$ into $L_n^{k+1}$ with the desired properties. We obtain thus a natural sequence of group homomorphisms.
PROPOSITION II.18 There is a one-to-one correspondence between the set of $\mathcal{E}$-connections of order $k$ of $V_n$ and the set of multi-connections of the form $\{(\lambda_m, \Lambda_m)\}_{2 \leq m \leq k}$, where the $\Lambda_m$ are the homomorphisms of the natural sequence.

6. Prolongations of linear connections.

We have seen that a linear connection of order 1 of $V_n$ can be given by an invariant section $\phi_2$ of $H^1(V_n)$ into $H^2(V_n)$. We are going to construct a lift of $\phi_2(H^1(V_n))$ into $H^3(V_n)$. Let $u = j_{e_1}^1 f \in \phi_2(H^1(V_n))$, where $f$ is an admissible local isomorphism of $H^1(R^n)$ into $H^1(V_n)$. Let $h$ be the admissible local isomorphism of $H^2(R^n)$ into $H^2(V_n)$ defined by: $b \circ \eta_2 = \phi_2 \circ f \circ \eta_1$. The mapping $u \mapsto \phi_3^3(u) = j_{e_1}^2 h$ defines a lift of $\phi_2(H^1(V_n))$ into $H^3(V_n)$. The composite mapping $\phi_3 = \phi_3^3 \circ \phi_2$ defines an invariant section of $H^1(V_n)$ into $H^3(V_n)$. The $\mathcal{E}$-connection $\phi_3$ obtained by this way or the corresponding linear connection of order 2 will be called the first prolongation of $\phi_2$. The principal subbundle $\phi_3(H^1(V_n))$ of $H^3(V_n)$, possesses the following property: for every $v \in \phi_3(H^1(V_n))$, there exists an admissible local isomorphism $g$ of $H^2(R^n)$ into $H^2(V_n)$ such that $v = j_{e_2}^1 g$ and that $g$ maps the (local) zero section of $H^2(R^n)$ into $\phi_2(H^1(V_n))$. By means of this property, we can construct a lift $\phi_4^3$ of $\phi_3(H^1(V_n))$ into $H^4(V_n)$ and the composite mapping $\phi_4 = \phi_4^3 \circ \phi_3$ defines an $\mathcal{E}$-connection of order 3, called the second prolongation of $\phi_2$. Notice that the projections of $\phi_4$ are respectively $\phi_3$ and $\phi_2$. By iterations, we construct the $k$-th prolongation of $\phi_2$.

If we consider only the prolongations of linear connections of order 1 of $V_n$, we do not obtain all the linear connections of higher order of $V_n$. A linear connection of order $k$ is called simple if it is the $(k-1)$-th prolongation of a first order linear connection of $V_n$.

Let $\omega_k$ (resp. $\omega_k'$) be a linear connection of order $k$ of $V_n$ (resp. $V'_n$). We will say that $\omega_k$ is equivalent to $\omega_k'$ if there exists a diffeomorphism $f$ of $V_n$ onto $V'_n$ such that $f^{(k)} \ast \omega_k' = \omega_k$.
A linear connection $\omega_k$ is called locally flat if it is locally equivalent to the canonical connection in the trivial bundle $H^k(\mathbb{R}^n) = \mathbb{R}^n \times L^k$.

**Theorem II.19** A linear connection of order $k$ is locally flat if and only if it is simple, without torsion and without curvature.

It is well known that a first order connection is locally flat if and only if its torsion and curvature are zero. For $k > 1$, the conditions are obviously necessary, because the canonical connection in $H^k(\mathbb{R}^n)$ is simple, without torsion and without curvature. Let us show that the conditions are sufficient. Consider such a linear connection $\omega_k$. The connection projection $\omega_1$ of order 1 of $\omega_k$ is locally flat, because its torsion and its curvature are both zero. Since $\omega_k$ is simple, we can obtain $\omega_k$ by taking the successive prolongations of $\omega_1$. Let $\phi_{k+1}$ be the invariant section of $H^1(V_n)$ into $\widetilde{H}^{k+1}(V_n)$ corresponding to $\omega_k$. We put $\phi_k = \eta_k \circ \phi_{k+1}$.

For all $y \in H^1(V_n)$, the horizontal $n$-plane of $\widetilde{H}^k(V_n)$ associated to the $(k+1)$-frame $\phi_{k+1}(y)$ is tangent to $\phi_k(H^1(V_n))$, because $\omega_k$ is simple. From the «Holonomy Theorem», we have $\phi_{k+1}(H^1(V_n)) \subset H^{k+1}(V_n)$.

On the other hand, the nullity of the curvature form of $\omega_k$ implies that the distribution of $n$-planes of $\widetilde{H}^k(V_n)$ defined by $\omega_k$ is involutive. Let $W$ be the maximal integral submanifold passing through $u \in \phi_k(H^1(V_n))$. The canonical form $\theta_k$ (resp. $\bar{\theta}_k$) of $\widetilde{H}^k(V_n)$ (resp. $\widetilde{H}^k(\mathbb{R}^n)$), restricted to $W$ (resp. $Q = \eta_k(\mathbb{R}^n)$), will be denoted by $\theta_W$ (resp. $\bar{\theta}_Q$). These forms $\theta_W$ and $\bar{\theta}_Q$ have their values in $\mathbb{R}^n \subset E^{k-1}$. Consider the 1-form $\beta = p_1^* \theta_W - p_2^* \bar{\theta}_Q$ on the product manifold $W \times Q$, where $p_i$ ($i = 1, 2$) are the projections on $W$ and $Q$ respectively. In terms of a basis $\{a^1, a^2, ..., a^n\}$ for $\mathbb{R}^n$, the components $\beta_i$ of $\beta$ are linearly independent. Consider now the module $\mathfrak{M}$ of vector fields $X$ on $W \times Q$ such that $\beta_i(X) = 0$ for $i = 1, 2, ..., n$. If $X \in \mathfrak{M}$, $Y \in \mathfrak{M}$, we have

$$d\beta(X, Y) = X \beta(Y) - Y \beta(X) - \beta([X, Y]) = -\beta([X, Y]).$$

On the other hand, $d\beta(X, Y) = 0$. Consequently, $[X, Y] \in \mathfrak{M}$ showing that $\mathfrak{M}$ is involutive. Therefore, there exists a maximal integral submanifold $M$ of dimension $n$ passing through $(u, e_k) \in W \times Q$. For any non-zero vector $Z$ tangent to $p_2^{-1}(e_k)$, $\beta(Z) \neq 0$. We can find an open neighbourhood $U$ of $e_k$ in $Q$ and a differentiable section $\lambda$ of $U$ into $W \times Q$ such that we
have $\lambda(U) \subset M$. Let $b = p_1 \circ \lambda$. The form $\beta$ vanishes identically on $M$, we have $\lambda^* \beta = 0$, showing that $\hat{\theta}_h = b^* \tilde{\theta}_w$. We can now extend $b$ to a local isomorphism $\tilde{b}$ of $\overline{H}^k(\mathbb{R}^n)$ into $\overline{H}^k(V_n)$ satisfying $\hat{\theta}_h = \tilde{b}^* \tilde{\theta}_w$. In virtue of theorem 1.2, we can find an open neighbourhood $N$ (resp. $N'$) of $0 \in \mathbb{R}^n$ (resp. $x = \pi^k_0(u) \in V_n$) and a diffeomorphism $f$ of $N$ onto $N'$ such that locally $\tilde{b} = f(x)$. Consequently, $\omega_k$ is locally flat.

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