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Costacks - The simplicial parallel of sheaf theory

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Parallel to sheaf theory, costack theory is concerned with the study of the homology theory of simplicial sets with general coefficient systems. A coefficient system on a simplicial set $K$ with values in an abelian category $\mathcal{A}$ is a functor from $K$ to $\mathcal{A}$ ($K$ is a category of simplexes); it is called a precostack; it is a simplicial parallel of the notion of a presheaf on a topological space $X$. A costack on $K$ is a «normalized» precostack, it is realized as a simplicial set over $K$; it is a simplicial «espace étalé».

The theory developed here is functorial; it implies that «all» homology theories are derived functors. Although the treatment is completely independent of Topology, it is however almost completely parallel to the usual sheaf theory, and many of the same theorems will be found in it though the proofs are usually quite different.

In Chapter I we define the direct image functor $f_*$ and the inverse image functor $f^*$ of a simplicial map $f$ and show how this pair of adjoint functors supply projectives and injectives to a category of precostacks. It turns out that, if the coefficient category $\mathcal{A}$ has enough projectives, then there are enough projective precostacks on $K$. Thus the usual homological algebra applies to categories of precostacks.

Chapter II begins with the definition of the homology of $K$ with a general system of coefficients; it is defined in the same way as singular homology is defined. The simplicial parallel of the Vietoris-Begle theorem follows immediately from the definition. The homology theory so defined is proved to be the left derived functor of the 0-th homology functor $H_0$ (and so $H_0$ is called a cosection functor). This enables us to give a con-
ceptual proof of a theorem of Moore and Puppe on the equivalence of homology and homotopy theories for simplicial abelian groups. It also follows that the chain homology functor, denoted $H^C$, is the left derived functor of $H^C_0$. Thus any homology theory that is defined by chain homology of complexes of modules is a derived functor. In section 7, spectral sequences of a simplicial map are constructed. A theorem which is the simplicial parallel of the Leray spectral sequence of a map is proved.

Chapter III starts with functorial relations between precostacks and costacks (see Theorem 8.2). It is proved that a projective costack is projective as a precostack and that an exact sequence of costacks is exact as a sequence of precostacks. It also proves that the associated costack of a projective precostack is projective. Thus, when homology is concerned, we choose to regard costacks just as precostacks and make no attempt to set up a separated homology theory for costacks. Finally, we define relative costacks and precostacks for computing relative homology.
0. Categories, functors, and modules.

Let \( \mathcal{C} \) be a category and let \( A, B \) be objects of \( \mathcal{C} \); the set of morphisms in \( \mathcal{C} \) with source \( A \) and target \( B \) is denoted by \( \mathcal{C}(A, B) \). If the class of objects of \( \mathcal{C} \) is a set, then we say that \( \mathcal{C} \) is a small category.

The following symbols are fixed throughout this paper:
- \( \mathcal{A} \), an abelian category;
- \( \mathcal{Ab} \), the category of abelian groups;
- \( \Lambda \), a commutative ring with unit;
- \( \mathcal{M} \), the category of unitary \( \Lambda \)-modules;
- \( \mathcal{C} \), the category of chain complexes in \( \mathcal{A} \) of the form
  \[ ... \to X_n \to X_{n-1} \to ... \to X_0 \to A \to 0, \text{ or } X_* \to A. \]

Let \( \mathcal{C} \) be a small category and let \( \mathcal{C}^\mathcal{C} \) be the category of functors (covariant) from \( \mathcal{C} \) to \( \mathcal{A} \); if morphisms are natural transformations of functors, then \( \mathcal{C}^\mathcal{C} \) is an abelian category; exactness in \( \mathcal{C}^\mathcal{C} \) is fiberwise, i.e. a sequence of functors

\[ \ldots \to F_{n+1} \to F_n \to F_{n-1} \to \ldots \]

is exact if and only if, for each object \( x \) in \( \mathcal{C} \), the sequence

\[ \ldots \to F_{n+1}(x) \to F_n(x) \to F_{n-1}(x) \to \ldots \]

is exact in \( \mathcal{A} \). Products, coproducts (sums), and limits in \( \mathcal{C}^\mathcal{C} \) are obtained from that of \( \mathcal{A} \) by fiberwise constructions, e.g.

\[ (\prod F_i)(x) = \prod F_i(x), \quad x \in \mathcal{C}. \]

By [6] the category \( \mathcal{A} \) is said to be \( AB(3) \) (resp. \( AB^*(3) \)) if it has arbitrary coproducts (resp. products). If the coproduct (resp. product) functor is exact on \( \mathcal{A} \), then \( \mathcal{A} \) is said to be \( AB(4) \) (resp. \( AB^*(4) \)). An \( AB(3) \) category \( \mathcal{A} \) is said to be \( AB(5) \) if the direct limit of any directed system of objects of \( \mathcal{A} \) exists and if the direct limit functor, \( \text{lim} \), is exact on \( \mathcal{A} \). For example the category of modules \( \mathcal{M} \) is \( AB(3), AB^*(3), AB(4), AB^*(4) \) and \( AB(5) \). Since products, coproducts, and limits of
the functor category $\mathcal{C}$ are fiberwise, we have

**Lemma 0.1.** $\mathcal{C}$ inherits the following properties of $\mathcal{C}: AB(3)$, $AB^*(3)$, $AB(4)$, $AB^*(4)$, $AB(5)$.

An object $P$ is a projective (object) of $\mathcal{C}$ if the functor $\mathcal{C}(P,-) : \mathcal{C} \to \mathcal{C}$ is exact; $P$ is small if, for any coproduct $\Sigma_i P$ of copies of $P$,

$$\mathcal{C}(\Sigma_i P, A) \cong \Sigma_i \mathcal{C}(P, A).$$

A generator of $\mathcal{C}$ is an object $U$ of $\mathcal{C}$ such that $\mathcal{C}(U, A) \not= 0$ for any $A \not= 0$ in $\mathcal{C}$. Injectives and cogenerators are defined dually. For example $\Lambda$ is a projective generator of $\mathbb{M}$; $\mathbb{Q}/\mathbb{Z}$ (rationals mod 1) is an injective cogenerator of $\mathcal{A}$. $\mathcal{C}$ has enough projectives (resp. injectives) if, for each object $A$, there is a projective $P$ (resp. injective $E$) and an exact sequence $P \to A \to 0$ (resp. $0 \to A \to E$).

**Lemma 0.2** [6, Theorem 1.10.1]. An $AB(5)$ abelian category with a generator has enough injectives.

A generator has enough injectives. For example $\mathbb{M}$ has enough injectives. More precisely, for each module $A$, there is an injective envelope $E$ of $A$ [8, p. 103]. It is well known that $\mathbb{M}$ has enough projectives: let $P$ be the free module generated by the set $A$; then one gets an exact sequence $P \to A \to 0$.

The notion of adjoint functors plays an important role in this paper; here are a few preliminary remarks. Let $\mathcal{C}$ and $\mathcal{C}'$ be categories, a functor $F : \mathcal{C} \to \mathcal{C}'$ is a left adjoint functor of the functor $G : \mathcal{C}' \to \mathcal{C}$ if $\mathcal{C}'(Fx, x')$ is naturally equivalent to $\mathcal{C}(x, Gx')$, $x \in \mathcal{C}$, $x' \in \mathcal{C}'$. The left adjoint of $G$ is unique up to an equivalence of functors; we shall say that $F$ is the left adjoint of $G$ and $G$ is the right adjoint of $F$ and write $F \dashv G : (\mathcal{C}, \mathcal{C}')$.

Left (resp. right) adjoint functors preserve epimorphisms, cokernels, and right limits (resp. monomorphisms, kernels, and left limits).

**Lemma 0.3.** Let $\mathcal{C}$ and $\mathcal{C}'$ be abelian categories, and $F : \mathcal{C} \to \mathcal{C}'$. If $F$ is exact (resp. $F$ is exact) then $F$ preserves projectives (resp. $G$ preserves injectives).

**Proof.** Let $P$ be a projective in $\mathcal{C}$; then $\mathcal{C}(P,-)$ is exact. Since $\mathcal{C}'(FP,-) \cong \mathcal{C}(P,G-)$ and $G$ is exact, $\mathcal{C}'(FP,-)$ is exact and $FP$
is projective.

For $F \rightarrow G : (C, C')$, consider the natural equivalences

$$\mathcal{C}(Fx, Fy) \simeq \mathcal{C}(x, G(Fx)) \quad \text{and} \quad \mathcal{C}(Gx', Gx') \simeq \mathcal{C}'(F(Gx'), x'),$$

the morphisms $x \rightarrow GFx$ and $FGx' \rightarrow x'$ corresponding to $1 : Fx$ and $1 : Gx'$ give natural transformations

$$(0.1) \quad \theta : 1 \rightarrow GF \quad \text{and} \quad \rho : FG \rightarrow 1.$$ 

Let $C'$ be a faithful subcategory of $C$, i.e. $C'(x'_1, x'_2) = C(x'_1, x'_2)$, and $J : C' \rightarrow C$ be the inclusion functor of $C' \subseteq C$. If $J$ has a left (resp. right) adjoint functor $R$, then $C'$ is coreflective (resp. reflective) in $C$; $R$ is the coreflector (resp. reflector) and $R(x)$ is the coreflection (resp. reflection) of $x$ in $C'$. Thus, for any $x'$ in $C'$ and morphism $f : x \rightarrow x'$ (resp. $f' : x' \rightarrow x$), there is a unique morphism $g : R(x) \rightarrow x'$ (resp. $g' : x' \rightarrow R(x)$) such that $g \theta_x = f$ (resp. $\rho_x g' = f'$). For example, the category of abelian sheaves on a space $X$ is coreflective in the category of abelian presheaves on $X$; the associated sheaf functor is the coreflector. For other informations on coreflective and reflective subcategories, see Freyd [5]; notice that our coreflections are the reflections of [5].
CHAPTER I

PRECOSTACKS

1. Definition and examples of precostacks.

Let $K = \bigcup_{n \geq 0} K_n$ be a simplicial set (a semi-simplicial complex); $K_n$ is the set of $n$-simplexes of $K$, the face operators are $d^i : K_n \rightarrow K_{n-1}$, $0 \leq i \leq n$, the degeneracy operators are $s^j : K_n \rightarrow K_{n+1}$, $0 \leq j \leq n$. $K$ is a small category: objects are simplexes; morphisms are determined by $d^i$ and $s^j$ in the covariant manner, e.g. if the $i$-th face of $x_n \in K_n$ is $x_{n-1} \in K_{n-1}$, then there is a morphism $d^i : x_n \rightarrow x_{n-1}$. A precostack over $K$, with values in $\mathcal{C}$, is a functor (covariant) $\Lambda : K \rightarrow \mathcal{C}$. The category of precostacks of modules over $K$ is the functor category $\mathcal{M}^K$. By lemma 0.2 we have

**Proposition 1.1.** $\mathcal{M}^K$ is abelian, $AB(3)$, $AB^*(3)$, $AB(4)$, $AB^*(4)$, and $AB(5)$.

Let $\mathcal{C}_K$ be the category of simplicial sets over $K$: objects are simplicial maps $L \rightarrow K$ with target $K$; morphisms are commutative diagrams

\[
\begin{array}{ccc}
L & \rightarrow & L' \\
\downarrow & & \downarrow \\
K & \rightarrow & K
\end{array}
\]

If $\mathcal{M}_K$ denotes the subcategory of $\Lambda$-module objects of $\mathcal{C}_K$, then $\mathcal{M}_K \simeq \mathcal{M}^K$, see [2]. The "geometrical realization" of a precostack of modules over $K$ is a simplicial set over $K$ with fibers over simplexes of $\Lambda$-modules; face operators and degeneracy operators are $\Lambda$-homomorphisms on fibers.

**Example 1.1.** Let $<0>$ be a simplicial point: a collapsed simplicial set with one simplex in each dimension. Then a simplicial set is the geometrical realization of a precostack of sets over $<0>$. The category of simplicial $\Lambda$-modules is identified with the category of precostacks $\mathcal{M}<0>$.

**Example 1.2.** Let $X$ be a topological space and let $\Delta(X)$ be the total singular complex of $X$ (a simplicial set), then a local system of coefficients on $X$ gives rise to a precostack over $\Delta(X)$.

**Example 1.3.** Let $E \rightarrow X$ be a sheaf of modules over $X$; then $S(E) \rightarrow S(X)$
is (the geometrical realization of) a precostack of modules.

2. Direct and inverse images.

Let \( f : K \to L \) be a simplicial map; \( f \) is a functor when \( K \) and \( L \) are regarded as categories of simplexes. Let \( B \) be a precostack of modules over \( L \), i.e. \( B \in \mathcal{M}^L \); then the composite functor \( Bf \) is a precostack of modules over \( K \). \( Bf \) is called the inverse image of \( B \) under \( f \) and is denoted by \( f^\# B \). \( f \) induces a functor \( f^* : \mathcal{M}^L \to \mathcal{M}^K \) on precostacks. Also, \( f \) induces a functor \( f_* : \mathcal{M}^K \to \mathcal{M}^L \) in the following manner. For \( A \) in \( \mathcal{M}^K \), let \( f_* A \) in \( \mathcal{M}^L \) be the precostack over \( L \) that assigns to each \( y \in L \) the module \( \Sigma_{f(x)} = y A(x) \); the values of \( f_* A \) acting on morphisms of \( L \) are obtained in the natural way by the universal property of coproducts. For example for a morphism \( d : y \to dy \) of \( L \) given by a face operator \( d \), \( (f_* A)(d) \) is the unique homomorphism that renders the diagram

\[
\begin{array}{ccc}
A(x) & \xrightarrow{A(d)} & A(dx) \\
\downarrow{\text{inj}} & & \downarrow{\text{inj}} \\
\Sigma_{f(x)} = y A(x) & \xrightarrow{(f_* A)(d)} & \Sigma_{f(x)} = dy A(x)
\end{array}
\]

commutative. \( f_* A \) is called the direct image of \( A \) under \( f \).

For simplicial maps \( f : K \to L \) and \( g : L \to M \) we have \( (gf)_* = g_* f_* \) and \( (gf)^\# = f^* g^\# \).

**Lemma 2.1.** \( f_* \) and \( f^* \) are exact functors, i.e. they take exact sequences to exact sequences.

This follows from the definition of \( f_* \) and \( f^* \) and the fact that categories of precostacks of modules are \( AB(4) \).

**Lemma 2.2.** \( f_* \) is the left adjoint of \( f^* \), i.e. there is a natural isomorphism

\[
\mathcal{M}^K(A, f^* B) \to \mathcal{M}^L(f_* A, B)
\]

of abelian groups.

**Proof.** An element of \( \mathcal{M}^K(A, f^* B) \) is a natural transformation \( \varphi = \{ \varphi_x : x \in K \} : A \to f^* B = Bf \)
of functors. For each such $\varphi$ there corresponds a natural transformation $\Psi = \{ \Psi_y : y \in L \} : f_\# A \to B$ defined by the universal mapping diagram of $\Sigma f(x) = y A(x)$:

\[
\begin{array}{c}
\begin{array}{ccc}
A(x) & \xrightarrow{i_x} & \Sigma f(x) = y A(x) = (f_\# A)(y) \\
\varphi_x & \downarrow & \downarrow \Psi_y \\
(f^\# B)(x) & = & B(y),
\end{array}
\end{array}
\]

$\varphi_x = \Psi_y i_x$. This correspondence gives rise to an isomorphism.

**Proposition 2.3.** $f_\#$ preserves projectives, $f^\#$ preserves injectives.

This follows from Lemmas 0.3, 2.1, and 2.2.

3. Projectives and injectives of $\mathbb{M}^K$.

Let $<n>$ be the simplicial set of the standard $n$-simplex and let $\delta$ be its only non-degenerate $n$-simplex; $<0>$ is a simplicial point as indicated before; the simplicial standard 1-simplex $<1>$ plays in simplicial homotopy theory a similar role as the unit interval $I$ plays in the homotopy theory of topological spaces. Given a simplicial set $K = \bigcup_{n \geq 0} K_n$, for each $x \in K_n$ the correspondence $\delta \mapsto x$ determines uniquely a simplicial map $x^\delta : <n> \to K$ [8, p. 237]. This shows that, if $\Lambda^n$ is the constant precostack over $<n>$ with value $\Lambda$, then for any $A$ in $\mathbb{M}^{<n>}$

\[
\mathbb{M}^{<n>}(\Lambda^n, A) \simeq \mathbb{M}(\Lambda, A(\delta)) \simeq A(\delta).
\]

**Theorem 3.1.** $\mathbb{M}^K$ has enough projectives and injectives.

**Proof.** Let $U = \Sigma_{x \in K} (x^\delta \Lambda^n)$, $n = \dim x$, then $U$ is a projective generator of $\mathbb{M}^K$. As exactness in $\mathbb{M}^K$ is fiberwise, it follows from (3.1) that $\mathbb{M}^{<n>}(\Lambda^n, -)$ is exact and so $\Lambda^n$ is projective in $\mathbb{M}^{<n>}$; $x^\delta \Lambda^n$ is projective in $\mathbb{M}^K$ since $x^\delta$ preserves projectives. Now $U$ as a coproduct of projectives is itself projective. $U$ is a generator since, for $A \neq 0$ in $\mathbb{M}^K$,

\[
\mathbb{M}^K(U, A) = \mathbb{M}^K(\Sigma x^\delta \Lambda^n, A) \simeq \prod \mathbb{M}^K(x^\delta \Lambda^n, A) \\
\simeq \prod \mathbb{M}^{<n>}(\Lambda^n, x^\delta A) \simeq \prod (x^\delta A)(\delta) \\
\simeq \prod A(x^\delta(\delta)) = \prod_{x \in K} A(x) \neq 0.
\]

$\mathbb{M}^K$ has enough projectives since each $A$ in $\mathbb{M}^K$ is a quotient of $\Sigma_i U$. 
the index set is the set \( \mathcal{M}^K(U, A) \cong \prod_{x \in K} A(x) \). Finally, by Lemmas 0.1 and 0.2, \( \mathcal{M}^K \) has enough injectives since it is \( AB(5) \) and has a generator \( U \).

This theorem tells us that there is a homology theory on \( \mathcal{M}^K \) defined by derived functors and computed by resolutions. In particular, when \( K = = \langle 0 \rangle \) is a simplicial point, \( 0: \langle n \rangle \to \langle 0 \rangle \) is the only simplicial map. 

\( 0 : \mathcal{M}^{<n>} \to \mathcal{M}^{<0>} \) maps \( \Lambda^n \) onto the \( n \)-dimensional model simplicial module also denoted by \( \Lambda^n \) (we adopt the term "model" from \([8, p. 237]\)). Thus in \( \mathcal{M}^{<0>} \) the coproduct of model simplicial modules of all dimensions, \( U = \sum_{n \geq 0} \Lambda^n \), is a projective generator. We shall show that the classical homology theory of simplicial modules defined by chain modules is a derived functor.
CHAPTER II
HOMOLOGY


For each $A$ in $\mathbb{R}^K$, let $\partial A$ be the chain module of $A$: the module of $n$-chains is $\Sigma x A(x)$, where $x$ ranges over all $n$-simplexes of $K$, the boundary operator $\partial_n$ is the alternating sum of the face operators $d^i_n$. The homology of $K$ with coefficients in $A$ is by definition the chain homology of $\partial A$ and is denoted by $H(K; A)$, or sometimes by $\text{HA}$. Since the correspondence $A \mapsto \partial A$ defines an (additive) exact functor, $H = \{ H_n : n \geq 0 \}$ is a homological functor (a connected sequence of functors in the sense of [1, p. 43]).

Let $f : K \rightarrow L$ be a simplicial map, it is easily seen that the chain modules

$$\partial A = \partial(f_* A), \quad A \text{ in } \mathbb{R}^K.$$ 

We have the dual version of the Vietoris-Begle Theorem:

**Lemme 4.1.** $H(K; A) \cong H(L; f_* A)$.

We shall now prove the following main theorem on the homology of precostacks.

**Theorem 4.2.** $H = H(K; -)$ is «the» (unique up to natural equivalence of functors) left derived functor of $H_o(K; -)$, i.e. $H(K; A)$ or $HA$ is the chain homology of the chain module obtained by applying the functor $H_o(K; -)$ to a projective resolution of $A$ in $\mathbb{R}^K$.

**Proof.** Since $H_o(K; -)$ is right exact, by the isomorphism criterion of Cartan and Eilenberg [1, p. 87] it suffices to show that $H_q(K; P) = 0$ for $q > 0$ and $P$ projective. Since a projective $P$ is a summand of a coproduct $\Sigma U$ of copies of $U$ and since $H(K; -)$ preserves coproducts, it suffices to show that $H_q(K; U) = 0$ for $q > 0$. Now

$$\partial U = \partial(\Sigma x \in K x^5 \Lambda^n) = \Sigma x \in K \partial(x^5 \Lambda^n) = \Sigma_{n \geq 0} \partial \Lambda^n$$

by Lemma 4.1; $H_q(K; U)$ is isomorphic to the coproduct of the $q$-th homo-
logy group of $\partial \Lambda^n$, $n = 0, 1, 2, \ldots$. Since the $\partial \Lambda^n$ are acyclic \( [8, \text{p.} 238] \), $H_q(K; U) = 0$ for $q > 0$ and the theorem is proved.

$H_0(K; A)$ is the cosection of $A$ over $K$ (dual to the notion of section in sheaf theory). Since $H_0$ is right exact and preserves coproducts, it is a «tensor product» functor; its left derived functor $H$ is therefore a «torsion product» functor. Sometimes we write $H_n(K; A) = \text{Tor}_n^K(A)$ for $K 
eq <0>$, and $H_n(<0>; A) = \text{Tor}_n A$ (c.f. [10]).

Applying this theorem to the examples in Section 1, we see that the classical homology theory of simplicial modules, the singular theory, and the homology with local coefficients are all derived functors; they can be computed by resolutions.

Finally let $f : K \to L$ be a simplicial map and let $B$ be a precostack in $\mathbb{M}^L$; $f$ induces a homomorphism

\[(4.3) \quad f_\#: H(K; f^\# B) \to H(L; B)\]

in the following way. Let $P_\#$ be a projective resolution of $f^\# B$; then $f_\# P_\#$ is a projective resolution of $f_\# f^\# B$, since $f_\#$ is exact and preserves projectives. By «the comparison theorem» the map $\rho : f_\# f^\# B \to B$ of (0.2) lifts uniquely (up to homotopy) to a chain map from $f_\# P_\#$ to a projective resolution of $B$. This chain map induces the homomorphism $f_\#$ of (4.3). $f_\#$ can also be obtained from a naturally defined chain map from the chain module $\partial(f^\# B)$ to the chain module $\partial B$; we leave the details to the reader.

5. A remark on simplicial homotopy.

A simplicial set $K = \bigcup_{n \geq 0} K_n$ is a Kan complex if, for each set of $n$ \((n-1)\)-simplexes

$$\sigma^0, \sigma^1, \ldots, \sigma^{k-1}, \sigma^{k+1}, \ldots, \sigma^n, \quad 0 \leq k \leq n,$$

with

$$d_i^{i-1} \sigma^i = d_i \sigma^j \quad \text{for} \quad l \leq j, \quad i \neq k \neq j,$$

there is a $n$-simplex $\sigma$ with $d_i \sigma = \sigma^i$ for $i \neq k$. A Kan complex with a fixed 0-simplex $e$ as its «base point» is called a pointed Kan complex. For example, a simplicial group is a pointed Kan complex with base point the
identity element of the 0-dimensional group \([0]\). Since we are interested primarily in simplicial abelian groups, we shall restrict the general homotopy theory on pointed Kan complexes to the category of simplicial abelian groups \(\text{Ab} < 0\).

Let \(\pi = \{\pi_q; q \geq 0\}\) be a homotopy theory on \(\text{Ab} < 0\) (\(\pi\) is characterized by axioms analogue to that of the homology theory of Eilenberg and Steenrod, see [7, Chap. II]); then each \(\pi_q\) is an additive functor from \(\text{Ab} < 0\) to \(\text{Ab}\). The \(q\)-th homotopy group \(\pi_q(K, e)\) is defined, according to J.C. Moore, as the quotient group

\[
\bigcap_{i \leq q} \text{Kernel } d_i^q / d_{q+1}^q \cap \bigcap_{q+1} \text{Kernel } d_{q+1}^q,
\]

see [7, Chap. IV]. We shall give a conceptual proof of the following theorem of Moore and Puppe.

**Proposition 5.1.** The homotopy groups and homology groups of a simplicial abelian group are naturally isomorphic.

**Proof.** Let \(A\) be a simplicial abelian group. Since \(H(<0>; A)\) is a left derived functor of \(H_o(<0>, A)\) and since \(H_o(<0>; A)\) is equivalent to \(\pi_o(A)\), we shall show that \(\pi\) is a left derived functor of \(\pi_o\) on \(\text{Ab} < 0\). Since every short exact sequence of simplicial abelian groups is a fiber sequence of Kan complexes, there is associated to it a homotopy sequence of homotopy groups connected by connecting homomorphisms. The correspondence is functorial and so \(\pi\) is a homological functor. To complete the proof we shall show that \(\pi_q(P, e) = 0\) for \(q > 0\) and \(P\) projective in \(\text{Ab} < 0\). As in the proof of Theorem 4.1, \(\pi_q(P, e)\) is a summand of \(\pi_q(\Sigma U, e) = \pi_q(\Sigma \Sigma Z^n, e)\). But \(\pi_q(\Sigma \Sigma Z^n, e) \simeq \Sigma \Sigma \pi_q(Z^n, e)\) (this follows from the definition of \(\pi_q\) and \(Z^n\) is contractible for each \(n \geq 0\), \(\pi_q(\Sigma U, e) = 0\) for \(q > 0\), so is \(\pi_q(P, e)\).

The original proof of the theorem is a consequence of the equivalence between simplicial modules and chain modules; we shall establish the equivalence (without proof) and some of its consequences for use in later sections.

For \(A\) in \(\mathfrak{M} < 0\), let \(CA\) be the normalized chain module of \(A\).
CA is the quotient of ∂A by the chain submodule generated by the degenerate simplexes of A [8, p. 236]; then the correspondence A → CA defines a functor C : \( \mathcal{M}^{<0} \rightarrow \mathcal{M} \), where \( \mathcal{M} \) is the category of left (positive) chain modules.

**Lemma 5.2.** The functor C is a homotopy preserving equivalence of categories (C carries simplicial homotopy to chain homotopy).

For details see, e.g., [3].

Let \( H^C \) denote the chain homology functor; then by the normalization theorem of Eilenberg-Mac Lane we have \( H^C(\partial A) \cong H^C(CA) \).

Thus Lemma 5.2 shows that \( \mathcal{M}^{<0} \) is identified with \( \mathcal{M} \) when homology is concerned. As a direct consequence of Theorem 3.1 and Lemma 5.2 we have:

**Lemma 5.3.** \( \mathcal{M} \) has enough projectives and injectives; the chain homology functor \( H^C \) is the left derived functor of \( H_0^C \).

This is true for all \( \partial A \), when \( A \) has enough projectives and injectives. In such a case, one constructs projectives (resp. injectives) as a coproduct (resp. product) from some «basic» projectives (resp. injectives).

### 6. Hyperhomology and derived spectral functors.

Let \( \partial \mathcal{M}^K \) be the category of complexes of precostacks of modules over \( K \) of the form

\[
\ldots \rightarrow X_n \rightarrow X_{n-1} \rightarrow \ldots \rightarrow X_0 \rightarrow A \rightarrow 0, \text{ or } X_* \rightarrow A;
\]

then \( \partial \mathcal{M}^K \) has enough projectives and injectives. \( X_* \) is projective in \( \partial \mathcal{M}^K \) if and only if it is a coproduct of the «basic» projectives of the form

\[
\ldots \rightarrow 0 \rightarrow P \rightarrow P \rightarrow 0 \rightarrow \ldots
\]

with a projective precostack \( P \) in positions \( n \) and \( n-1 \) (see, e.g., [4]).

It is easy to see that a «basic» projective is projective and that \( X_* \) is projective if and only if

1. each \( X_n \) is projective in \( \mathcal{M}^K \), and
2. \( X_* \) is split exact, i.e. \( H_q^C(X_*) = 0 \) for \( q \neq 0 \) and \( \partial X_n \) is a summand of \( X_{n-1} \) (recall that \( H^C \) denotes the chain homology functor).

Thus for any additive functor \( F : \mathcal{M}^K \rightarrow \mathcal{A} \) we have \( H_q^C(FX_*) \cong FH_q^C(X_*) = 0 \).
for \( q \neq 0 \), if \( X_* \) is projective (here \( FX_* \) is the complex obtained by applying \( F \) to \( X_* \) termwise).

Remarks. a) For \( A_* \) in \( \mathcal{M}_L \), let \( X_{**} \) be a left complex over \( A_* \); then there results a bicomplex \( X \) of the form

\[
\begin{array}{cccccc}
& & & & & \\
& & & & & \\
& & & & & \\
& & & & & \\
& & & & & \\
& & & & & \\
\vdots & & \vdots & & \vdots & & \vdots \\
& & & & & \\
& & & & & \\
& & & & & \\
& & & & & \\
& & & & & \\
0 & & 0 & & 0 & & 0 \\
\end{array}
\]

\( X_{p,q} \in \mathcal{M}_K \), and homology precostacks \( H^I_p(X_{*,q}) \) and \( H^{II}_q(X_{p,**}) \). The total homology precostack of \( X \) is denoted by \( H^{tot}(X) \). \( X_{**} \) is called a strong projective resolution of \( A_* \) if it is a «projective resolution» in the sense of Cartan and Eilenberg, i.e. if, for each \( p \geq 0 \), \( H^I_p(X) \) is a projective resolution of \( H^C_p(A_*) \) in \( \mathcal{M}_L \), and if \( B^I_p(X) \), the \( p \)-boundaries of the first boundary operation, is a projective resolution of \( B^I_p(A_*) \), see [1, p. 363].

b) Let \( F : \mathcal{M}_K \rightarrow \mathcal{A} \) be an additive functor, let

\[
(6.1) \quad E^I(FX) = H^I_p H^{II}_q(FX), \quad E^{II}(FX) = H^{II}_p H^I_q(FX)
\]

be the spectral homology sequences of \( FX \). If \( X_{**} \) is a strong projective resolution of \( A_* \), then (6.1) becomes

\[
(6.2) \quad E^I(FA_*) = H^C_p((\mathcal{Q}_q F) A_*), \quad E^{II}(FA_*) = (\mathcal{Q}_p F) (H^C_q A_*),
\]

where \( \mathcal{Q}_n F \) is the \( n \)-th left derived functor of \( F \). Both spectral sequences of (6.2) converge to the hyperhomology \( (HF)(A_*) = H^{tot}(FX) \); \( HF \) is the hyperhomology functor of \( F \).

We shall show that the hyperhomology functor \( HF : \mathcal{M}_K \rightarrow \mathcal{A} \) is
a derived functor.

**Lemma 6.1.** Let $P^{**}$ be a projective resolution of $A_{\ast}$ and let $X^{**}$ be a strong projective resolution, then

\[(6.3) \quad H^\text{tot}(FP) \cong H^\text{tot}(FX) = (HF)(A_{\ast}).\]

Indeed, the identity chain map of $A_{\ast}$ lifts uniquely (up to homotopy) to a map $P \rightarrow X$ of bicomplexes. This map induces a morphism

\[E^I(FP) \rightarrow E^I(FX)\]

of spectral sequences, which is an isomorphism since the two spectral sequences are isomorphic to $H^C_p((\mathcal{L}_q F)A_{\ast})$.

**Theorem 6.2.** The hyperhomology functor $HF$ is the left derived functor of the composite functor $FHo_{\mathcal{O}}^C : \mathcal{A}^{K} \rightarrow \mathcal{A}$, i.e. $HF \cong \mathcal{L}(FH^C_{\mathcal{O}})$.

**Proof.** Let $P^{**}$ be a projective resolution of $A_{\ast}$; the first spectral sequence $E^I(FP)$ converges to $H^\text{tot}(FP) \cong (HF)(A_{\ast})$ by (6.3). Compute the second spectral sequence

\[E^{II}(FP) = H^{II}_p H^I_q(FP);\]

since each column of $P$ is a projective in $\mathcal{A}^{K}$, it is split exact and so $H^I_q(FP) \cong F(H^I_q P) = 0$ for $q \neq 0$. Thus the second spectral sequence collapses to

\[H^{II}_p(FH^C_{\mathcal{O}} P) = \mathcal{L}_p(FH^C_{\mathcal{O}})(A_{\ast});\]

and yields

\[\mathcal{L}(FH^C_{\mathcal{O}})(A_{\ast}) \cong H^\text{tot}(FP) = (HF)(A_{\ast}).\]

This completes the proof.

Thus for obtaining the spectral sequences (6.2) one may use either a projective resolution of $A_{\ast}$ or a strong projective resolution of $A_{\ast}$; in either case the hyperhomology is $\mathcal{L}(FH^C_{\mathcal{O}})(A_{\ast})$.

The argument in this section remains valid when $^{K}$ is replaced by an abelian category with enough projectives. For example, let $A_{\ast}$ be a complex over $A$ in $\mathcal{M}^K$; then $\partial A_{\ast}$ (\partial applies to $A_{\ast}$ termwise) is a complex over $\mathcal{M}$ in $\mathcal{M}$ and yields a bicomplex of $\Lambda$-modules. (6.2) becomes
both converge to \( \mathcal{Q} \left( \mathcal{O}_* \right) A \). If \( A_* \) is a projective resolution of \( A \), then \( \partial A_* \) is a projective resolution of \( \partial A \) in \( \mathcal{M} \). In this case the second spectral sequence collapses and yields
\[
H^C_p \left( \mathcal{Q}_* F \left( \partial A_* \right) \right) = \mathcal{Q} \left( \mathcal{O}_* \right) A.
\]
Moreover, if \( F \) is the identity functor of \( \mathcal{M} \), then both spectral sequences collapse and yield \( H(K; A) = H(A) \simeq \mathcal{Q} \left( \mathcal{O}_* \right) (A) \) of Theorem 4.2.

7. Spectral sequences of a simplicial map.

Let \( A_* \) be a positive complex of precostacks of modules over \( K \), \( A_* \in \mathcal{M} \), and let \( F = H_o : \mathcal{M}^K \to \mathcal{M} \) be the cosection functor over \( K \). Then, since \( H \simeq \mathcal{Q} \), the spectral sequences of the previous section become
\[
E^1(H_{o} A_*) = H^C_p (H_q A_*), \quad E^{II}(H_{o} A_*) = H_p (K; H^C_q A_*),
\]
where \( H_q A_* \) is obtained by applying \( H_q = \mathcal{Q}_q H_o \) to \( A_* \) termwise. There are two interesting cases corresponding to the degeneracy of one or the other of the spectral sequences.

CASE 1. Assume that \( H^C_p (H_q A_*) = 0 \) for \( q > 0 \), e.g. when each term of \( A_* \) is an acyclic precostack. Then the first spectral sequence collapses and yields
\[
E^1_{p, o} (H_{o} A_*) = H^C_p (H_{o} A_*).
\]
There is a spectral sequence (the second spectral sequence).
\[
E^2_{p, q} = H_p (K; H^C_q A_*) \Rightarrow H^C_{p+q} (H_{o} A_*).
\]

CASE 2. Assume that \( H^C_q (A_*) = 0 \) for \( q > 0 \), e.g. when \( A_* \) is a resolution of \( A \). Then the second spectral sequence collapses and yields
\[
E^1_{p, o} (H_{o} A_*) = H_p (K; A), \quad A = H^C_o A_*.
\]
There is a spectral sequence (the first spectral sequence)
\[
E^2_{p, q} = H^C_p (H_q A_*) \Rightarrow H^C_{p+q} (K; A).
\]

As an application of case 1 we shall construct a simplicial dual of the Leray spectral sequence of a map.
Let $f : K \to L$ be a simplicial map. For each $A$ in $\mathcal{M}^K$, let $f_0 A$ be an object in $\mathcal{M}^L$ defined as follows: For $y \in L$, let $A_y(x)$ be the constant precostack over $K$ with value the module $\Sigma_{f(x)} = y A(x)$. Let 

$$(f_0 A)(y) = H_o(K; A_y(x))$$

be the $0$-th (classical) homology module of $K$ with coefficients in $\Sigma_{f(x)} = y A(x)$. Then, since $f$, $H_o$, $A$ are functors, $f_0 A$ is made a functor from $L$ to $\mathcal{M}$ in the natural way; $u : y \to uy$ in $L$ determines a homomorphism 

$$(u : \Sigma_{f(x)} = y A(x) \to \Sigma_{f(x)} = uy A(x))$$

which extends to a morphism $u : A_y(x) \to A_{uy}(x)$, which induces the morphism $u : (f_0 A)(y) \to (f_0 A)(uy)$. It is easily seen that the correspondence $A \to f_0 A$ defines a functor $f_0 : \mathcal{M}^K \to \mathcal{M}^L$, called the direct image functor (with respect to the cosection functor $H_o$). Since $H_o$ is right exact, we have:

**Proposition 7.1.** $f_0$ is right exact.

We shall show that $f_0 A$ is acyclic whenever $A$ is projective.

**Proposition 7.2.** If $A$ is projective in $\mathcal{M}^K$, then $H_q(L; f_0 A) = 0$ for $q > 0$.

**Proof.** As usual let $\partial(f_0 A)$ be the chain module of $f_0 A$,

$$... \to \partial_{n+1}(f_0 A) \to \partial_n(f_0 A) \to \partial_{n-1}(f_0 A) \to ... ;$$

then $\partial_n(f_0 A) = \Sigma_y H_o(K; A_y(x))$, where $y$ ranges over all $n$-simplexes of $L$. Since $H_o$ preserves coproducts, a simple computation shows that 

$$\partial_n(f_0 A) \simeq H_o(K; \partial_n A) = H_o(\partial_n A / K),$$

where $\partial_n A / K$ is the constant precostack over $K$ with value the $n$-chain module $\partial_n A$ of $\partial A$. Thus $\partial A$ is a chain of coefficients that gives the chain of constant precostacks

$$(7.2) \quad ... \to \partial_{n+1} A / K \to \partial_n A / K \to \partial_{n-1} A / K \to ...$$

from which $\partial(f_0 A)$ is obtained by applying the cosection functor $H_o$ termwise.

If $A$ is projective, so is $\partial A$. Thus each $\partial_n A$ is a projective module and $\partial A$ is split exact, see the first paragraph of section 6. This
gives the split exactness of (7.2), therefore $\partial(f \circ A)$ is exact. This proves the proposition.

For $A$ in $\mathbb{K}^K$, the cosection of $f \circ A$ is computed in the following proposition.

**Proposition 7.3.** $H_0(f \circ A) \simeq H_0(K; H_oA)$, the (classical) 0-th homology module of $K$ with coefficients in the module $H_oA$.

**Proof.** Since $H_o$ is right exact and preserves coproducts, we have the commutative diagram with exact rows and columns

$$
\begin{array}{cccc}
\Sigma_1 \partial_j A & \longrightarrow & \Sigma_1 \partial_o A & \longrightarrow & \Sigma_1 H_o A & \longrightarrow & 0 \\
\longrightarrow & & \downarrow & & \downarrow & & \downarrow \\
\Sigma_o \partial_j A & \longrightarrow & \Sigma_o \partial_o A & \longrightarrow & \Sigma_o H_o A & \longrightarrow & 0 \\
\longrightarrow & & \downarrow & & \downarrow & & \downarrow \\
H_o(\partial_j A/K) & \longrightarrow & H_o(\partial_o A/K) & \longrightarrow & H_o(K; H_oA) & \longrightarrow & 0 \\
\end{array}
$$

where $\Sigma_i \partial_j A$ is the coproduct indexed by $K_i$ of copies of $\partial_j A$, $i, j = 0, 1$. The last row of the diagram shows that

$$
H_o(K; H_oA) \simeq H_o(f \circ A).
$$

The following theorem is a simplicial dual of the Leray spectral sequence of a map.

**Theorem 7.4.** Let $f: K \to L$ be a simplicial map, let $A$ be in $\mathbb{K}^K$, and let $P_*$ be a projective resolution of $A$. If $K$ is connected (any two 0-simplexes can be joined by a chain of 1-simplexes), then there is a spectral sequence

$$
E^2_{p, q} = H_p(L; H^C_q(f \circ P_*)) \Rightarrow H_{p+q}(K; A),
$$

where $f \circ P_*$ is obtained by applying $f$ to $P_*$ termwise.

**Proof.** Since each term of $P_*$ is projective, each term of $f \circ P_*$ is acyclic by proposition 7.2. Case 1 applies and yields a spectral sequence

$$
E^2_{p, q} = H_p(L; H^C_q(f \circ P_*)) \Rightarrow H^C_{p+q}(f \circ P_*).
$$
Since $K$ is connected, (7.3) yields $H_\alpha(f_\alpha P\star) \cong H_\alpha P\star$. The right hand side of (7.4) becomes $H_{p+q}(A) = H_{p+q}(K; A)$. This proves the theorem.

Again $f : K \rightarrow L$ is a simplicial map. This time we consider the inverse image functor $f^\#: \mathfrak{M}^L \rightarrow \mathfrak{M}^K$. Let $B$ be in $\mathfrak{M}^L$ and let $X\star$ be a projective resolution of $B$. Since $f^\#$ is exact, $f^\#X\star$ is a resolution of $f^\#B$. Case 2 applies.

**Theorem 7.5.** Let $f : K \rightarrow L$ be a simplicial map, let $B$ be in $\mathfrak{M}^L$, and let $X\star$ be a projective resolution of $B$. There is a spectral sequence

(7.5) \[ E^2_{p,q} = H^C_p(H_q(f^\#X\star)) \Rightarrow H_{p+q}(K; f^\#B). \]

In particular when $B = G$ is a constant precostack over $L$ with value the module $G$ (a local system of coefficient modules), then $f^\#G$ is constant (a local system) over $K$ and $H(K; f^\#G)$ is the classical homology of $K$ with (local) coefficients in $G$, i.e. $H(K; f^\#G) = H(K; G)$. In this case (7.5) becomes

(7.6) \[ E^2_{p,q} = H^C_p(H_q(f^\#X\star)) \Rightarrow H_{p+q}(K; G). \]
CHAPTER III
COSTACKS

8. Subcategory of costacks:

Recall that a precostack of modules over $K$, $A \in \mathcal{M}^K$, is characterized by:

1. For each $x \in K$, $A(x)$ is a module, and
2. For each simplicial operator $u$ ($u$ can be decomposed in a unique way as a composite of face operators $d^i$ and degeneracy operators $s^j$), $A(u_x) : A(x) \to A(ux)$ is a homomorphism of modules.

It is realized as a simplicial set over $K$. If for each degeneracy $s$ of $K$ the homomorphism $A(s_x) : A(x) \to A(sx)$ is an isomorphism, then $A$ is a costack (of modules over $K$). By the definition of $K$,

$$d^i s^j = 1 \text{ for } j = i, i + 1,$$

we have for $A(s_x) : A(x) \to A(sx)$ and $A(d_{sx}) : A(sx) \to A(x)$,

$$A(d_{sx}) A(s_x) = A(d_{sx} s_x) = 1 : A(x).$$

Thus we have:

**Proposition 8.1.** A precostack $A$ is a costack if and only if $A(d_{sx})$ is an isomorphism for all $x \in K$.

Examples 1.2 and 1.3 of Section 1 are two examples of costacks.

Costacks over $K$ form a subcategory $\mathcal{G}^K$ of $\mathcal{M}^K$. The following properties of $\mathcal{G}^K$ are easy to check:

1. $\mathcal{G}^K$ is a faithful, exact subcategory of $\mathcal{M}^K$;
2. it is closed under the formation of subobjects, quotient objects, and extensions, i.e. it is a Serre subcategory of $\mathcal{M}^K$;
3. it is closed under the formation of products and coproducts.

Therefore, by a theorem of Freyd \[5\],

4. $\mathcal{G}^K$ is reflective and coreflective in $\mathcal{M}^K$ (c.f. section 0).

Let $R : \mathcal{M}^K \to \mathcal{G}^K$ be the coreflector; $R$ is the left adjoint functor of the inclusion functor $J : \mathcal{G}^K \to \mathcal{M}^K$; the coreflection $RA$ of $A \in \mathcal{M}^K$ is called the associated costack of $A$. (Strictly speaking, a coreflection is
a pair \((RA, \vartheta_A)\), where \(\vartheta_A : A \to RA\) is the canonical map displayed in (0.2). For example, let \(A\) be a simplicial module, \(A \in \mathcal{M}^{<0}\); the associated costack of \(A\) is the constant simplicial module with \(H_0(A)\) in each dimension. Indeed,

\[
\vartheta_A = (\ldots, \beta, \alpha, \varepsilon) : A \to RA
\]

is the simplicial homomorphism

\[
\begin{array}{ccc}
A_2 & \xrightarrow{\beta} & A_1 \\
\downarrow \beta & & \downarrow \alpha \\
A_0 & & \downarrow \varepsilon \\
\ldots H_o(A) = H_o(A) = H_o(A)
\end{array}
\]

with \(\varepsilon = \text{cokernel of } (d^1 - d^0)\),

\[
\alpha = \varepsilon d^0 = \varepsilon d^1, \quad \text{and} \quad \beta = \alpha d^0 = \alpha d^1 = \alpha d^2 = \varepsilon d^0 d^0, \text{ etc...}
\]

(that \(\varepsilon, \alpha, \beta, \ldots\) are compatible with degeneracy operators is obvious).

By property \(C_1\), the inclusion functor \(J\) is exact, therefore \(RA\) is projective in \(\mathcal{F}_A\) if \(A\) is projective in \(\mathcal{M}^K\) (see lemma 0.3).

**Construction of Reflections.** For \(A\) in \(\mathcal{M}^K\), let \(R'A\) be the costack defined as follows: \((R'A)(x) = A(x)\) for a nondegenerate simplex \(x \in K\) and \((R'A)(sx) = A(x)\) for any degeneracy operator \(s\) and any \(x \in K\). Then \(R'A\) is the reflection of \(A\) in \(\mathcal{F}_A^K\). Indeed, the canonical map

\[
\rho_A = \{ \rho_x : x \in K \} : R'A \to A
\]

is the natural transformation such that \(\rho_x = 1 : A(x)\) if \(x\) is nondegenerate, and \(\rho_{sx} = A(s_x) : A(x) \to A(sx)\); \(\rho_{dx}\) is determined by the compatibility of a simplicial map with \(s\) and \(d\). That \(\rho_A\) is well-defined follows from an easy but rather tedious computation which we shall not present here. Also, we omit the tedious proof that the \(R'A\) so defined is actually the reflection of \(A\). By recognizing all these, it is fairly easy to see that \(\rho_A\) is a monomorphism and that the reflector \(R'\) is an exact functor.

Since \(R'\) is the right adjoint functor of \(J\) and since both \(R'\) and
$J$ are exact, $R'$ preserves injectives and $J$ preserves projectives by Lemma 0.3.

We conclude this section with a theorem that summarizes some results scattered over the previous sections.

**Theorem 8.2.** Let $f: K \to L$ be a simplicial map. There is a diagram $D$

\[
\begin{array}{ccc}
\mathbb{M}^K & \xrightarrow{f^*} & \mathbb{M}^L \\
\downarrow R \downarrow & & \uparrow R \uparrow \\
\mathcal{G}_\Lambda^K & \xrightarrow{f^*} & \mathcal{G}_\Lambda^L \\
\uparrow J \uparrow & & \downarrow J \downarrow \\
\mathcal{G}_\Lambda^K & \xrightarrow{f^*} & \mathcal{G}_\Lambda^L \\
\end{array}
\]

(diagram $D$)

in which

1. All functors but $R$ and $Rf^*$ are exact, $R$ and $Rf^*$ are right exact;
2. $f^*$, $R$ and $Rf^*$ are the left adjoint functors of $f^*$, $J$, and $f^*/\mathcal{G}_\Lambda^L$ respectively.
3. $f^*$, $R$, $Rf^*$, and $J$ preserve projectives.

**Note.** $f^*$ takes costacks to costacks but $f^*$ usually not.

**9. Costack homology.**

Let $A$ be a costack of $\Lambda$-modules over $K$, $A \in \mathcal{G}_\Lambda^K$. The homology of $K$ with coefficients in $A$ is defined to be

$$H(K; JA) = H(JA) = H^C(\mathcal{G} JA)$$

as in the beginning of the section 4 and is denoted simply by $H(K; A)$ or $H(A)$. By regarding $A$ as a precostack, we proved in theorem 4.2 that $H(K; A)$ is isomorphic to $(\mathcal{G} H_o)(A)$ on $\mathcal{M}^K$. We shall show that this can also be done in $\mathcal{G}_\Lambda^K$ and that in computing $H(K; A)$ by resolutions there is no difference in doing it in $\mathcal{M}^K$ or in $\mathcal{G}_\Lambda^K$.

**Theorem 9.1.** Let $NK$ be the set of non-degenerate simplexes of $K$, let
Then \( \overline{U} \) is a projective generator of \( \mathcal{G}_\Lambda^K \) and therefore \( \mathcal{G}_\Lambda^K \) has enough projectives and injectives (by Lemma 0.2).

**Proof.** \( U' \) being a coproduct of projectives is itself a projective, therefore, by part (3) of Theorem 8.2, \( \overline{U} = RU' \) is projective. Compute \( \mathcal{G}_\Lambda^K(\overline{U}, A), \) \( A \in \mathcal{G}_\Lambda^K \); since \( R \) is the left adjoint of \( J \), we have

\[
(9.2) \quad \mathcal{G}_\Lambda^K(\overline{U}, A) \cong \mathcal{M}_K(\sum x \cdot \Lambda^n, JA) \cong \prod \mathcal{M}_K(\sum x \cdot \Lambda^n, A).
\]

By (3.2) the last term of (9.2) is isomorphic to \( \prod_{x \in NK} A(x) \). Thus

\[
(9.3) \quad \mathcal{G}_\Lambda^K(\overline{U}, A) \cong \prod_{x \in NK} A(x),
\]

and is nontrivial if \( A \neq 0 \). This shows that \( \overline{U} \) is a generator and the theorem is proved.

Thus, let \( P_* \) be a projective resolution of \( A \) in \( \mathcal{G}_\Lambda^K \), then since \( J \) is exact and preserves projectives, \( P_* \) is a projective resolution of \( A \) in \( \mathcal{M}_K \) and \( H(K; A) \) is naturally isomorphic to \( (LH_o)(A) \) where \( H_o \) is either the cosection functor on \( \mathcal{G}_\Lambda^K \) or the cosection functor on \( \mathcal{M}_K \).

We shall now consider the relative homology of a pair of simplicial sets with coefficients in a costack. Let \( (K, K') \) be a simplicial pair; \( K' \) is a simplicial subset of \( K \). Then the inclusion map \( i : K' \to K \) induces the functors \( i_* \) and \( i^* \) on precostacks. For \( A' \) in \( \mathcal{M}_K' \), \( i_* A' \) is a functor (a precostack) on \( K \) to \( \mathcal{M} \) with supports in \( K' \), i.e.

\[
(i_* A')(x) = A'(x) \quad \text{for } x \in K',
\]

\[
(i_* A')(x) = 0 \quad \text{for } x \in K - K'.
\]

This shows that if \( A' \) is a costack, so is \( i_* A' \). Thus we have a pair of adjoint functors \( i_* \rightleftharpoons i^* : (\mathcal{G}_\Lambda^K, \mathcal{G}_\Lambda^K) \). It is clear that \( i_* \) is an exact full embedding of categories; we shall identify \( A' \) with \( i_* A' \) and regard \( \mathcal{G}_\Lambda^K \) as a subcategory of \( \mathcal{G}_\Lambda^K \).

Let \( (K, K') \) be a simplicial pair; for \( A \) in \( \mathcal{G}_\Lambda^K \), let \( qA \) be the quotient

\[
qA = A / (i_* i^* A) = A / (i^* A);
\]

then \( qA \) has supports in \( K - K' \). Any costack over \( K \) having supports in
$K - K'$ is of the form $qA$ for some $A \in \mathcal{A}_\Lambda^K$ and is called a relative co-stack over $K$ (relative to $K'$). All $qA$ over $K$ form a subcategory $q\mathcal{A}_\Lambda^K$ of $\mathcal{A}_\Lambda^K$. Relative precostacks and $q\mathfrak{M}^K$ are defined in the same way.

For a simplicial pair $(K, K')$ and a costack $A \in \mathcal{A}_\Lambda^K$, the relative homology of $(K, K')$ with coefficients in $A$ is defined to be the homology of $qA$ and is written $H(K, K'; A) = H(qA)$. Thus $H(K, K'; A)$ can be computed by resolutions either in $\mathfrak{M}^K$ or in $\mathcal{A}_\Lambda^K$ as $(\mathcal{L}H_\omega)(qA)$. In particular, if $K'$ is empty, one identifies $(K, K')$ with $K$ and obtains the (absolute) homology $H(K; A)$.

Let $f : (K, K') \rightarrow (L, L')$ be a simplicial map of simplicial pairs; let $B$ be a costack over $L$. Since $f^#$ maps costacks onto costacks, the homomorphism $f_*$ of (4.3) induces a homomorphism

$$f_* : H(K, K'; f^#B) \rightarrow H(L, L'; B).$$

If $g = (M, M') \rightarrow (K, K')$ is another simplicial map, one has

$$(fg)_* = f_* g_* : H(M, M'; g^* f^#B) \rightarrow H(L, L'; B).$$

It is proved in [2] that this $H$ is a unique homology theory on the category of simplicial pairs in the sense of Eilenberg and Steenrod (actually the uniqueness theorem in [2] is more general than this).

For computing $H(qA)$, we prefer to do it in $\mathfrak{M}^K$. Define $H(K, K'; A)$ for $A \in \mathfrak{M}^K$ and extend the homology theory of previous chapters to the relative case. The details will be left to the reader.

In particular, if $K$ is finite ($K$ has a finite number of non-degenerate simplices) and $\Lambda = \mathbb{Z}$ is the ring of integers, the homology of a costack can be computed by torsion groups of modules (see [2]).
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