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ON THE HIGHER ORDER CONNECTIONS

by Bohumil CENKL

The non-holonomic connection of order r on a principal fibre bundle H which is geometrically defined in this paper is considered in relation to that one studied in [5]. A given connection gives rise to $r - 1$ tensor forms on the considered principal bundle H . Some characterisation of these forms is given.

After this paper has been finished, there appeared quite a few papers on differential geometry of higher order and particularly on the higher order connections. For example, the papers of C. Ehresmann, E.A. Feldman, P. Libermann, Ngo-Van-Que and others. Then some of the results, or similar theorems, have been proved by other mathematicians in the mean time.

1. Non-holonomic jets.

Let $\mathfrak{M}$ be the set of the C^s -mappings ($s \geq 1$) f of R into the C^∞ -manifold V , such that $f(0) = x$ for some fixed $x \in V$. Two such mappings $f, g \in \mathfrak{M}$ are said to be equivalent if $f(0) = g(0)$ and if the partial derivatives of the first order of the functions which give the mappings f, g in some local coordinates in the neighborhood of x on V are equal at the point 0. We shall denote by $T_x(V)$ the set of all so defined equivalence classes from $\mathfrak{M}$. This vector space is called the tangent (vector) space to V at x and $T(V) = \bigcup_{x \in V} T_x(V)$ the tangent bundle of the manifold V .

Let $p : W \rightarrow V$ be a projection of a manifold W onto a manifold V . (Throughout this paper will be considered only C^s -manifolds for s sufficiently large). If $\text{Hom}(T_x(V), T_y(W))$ denotes the set of homomorphisms of vector spaces, we shall consider the sets

$$\text{hom}(T_x(V), T_y(W)) = \{ \lambda \in \text{Hom}(T_x(V), T_y(W)) \mid p_* \circ \lambda = id \}$$

and

Part of this work was done during the author's stay at the Tata Institute of Fundamental Research, Bombay.

$$\text{hom}(T(V), T(W)) = \bigcup_{p, y = x \in V} \text{hom}(T_x(V), T_y(W)).$$

If we consider the product $V \times W$ of any manifolds V and W with the canonical projections $\tau_1 : V \times W \rightarrow V$, $\tau_2 : V \times W \rightarrow W$, we shall take

$$\text{hom}(T(V), T(V \times W)) = J^1.$$

We have well defined projections

$$\alpha : J^1 \rightarrow V, \quad \beta_1 : J^1 \rightarrow V \times W, \quad \beta = \tau_2 \circ \beta_1 : J^1 \rightarrow W.$$

Analogously can be defined by induction $\tilde{J}^r = \text{hom}(T(V), T(\tilde{J}^{r-1}))$, $r > 0$, $J^1 \equiv \tilde{J}^1$. Let us denote $V \times W \equiv J^0 \equiv \tilde{J}^0$. For each $r \geq 0$ there are the projections $\alpha : \tilde{J}^r \rightarrow V$, $\beta_r : \tilde{J}^r \rightarrow \tilde{J}^{r-1}$, $\beta = \tau_2 \circ \beta_1 \circ \beta_2 \circ \dots \circ \beta_r$.

DEFINITION 1.1. The elements of the manifold $\tilde{J}^r = \text{hom}(T(V), T(\tilde{J}^{r-1}))$ are called the non-holonomic jets of order r of the manifold V into W .

We shall denote by $\tilde{J}^r(V, W)$ or briefly $\tilde{J}^r$ the manifold of the non-holonomic r -jets of V into W . For $X \in \tilde{J}^r(V, W)$, $x = \alpha(X)$ is called the source and $y = \beta(X)$ the target of the jet X . The set of non-holonomic r -jets of $\tilde{J}^r$ with source x (resp. target y , resp. with source x and target y) is denoted by $\tilde{J}_x^r$ (resp. $\tilde{J}_{,y}^r$, resp. $\tilde{J}_{x,y}^r$).

DEFINITION 1.2. The elements of the manifold

$$\bar{J}^r(V, W) = \bar{J}^r = \{ X \in \text{hom}(T(V), T(\bar{J}^{r-1})) \mid \beta_r X = (\beta_{r-1})_* \circ X \},$$

$$\text{for } r \geq 2; \bar{J}^r = J^r, \quad r = 0, 1.$$

are called the semi-holonomic jets of order r of the manifold V into W .

It is not difficult to prove that there exists a local mapping $\mathcal{Q}$ of the manifold $\bar{J}^r$ into the vector bundle $\sum_{s=1}^r T(W) \otimes (\otimes^s T^*(V))$ which is injective. A map of an element from $\bar{J}^r$ is called its tensorial representation. Let us denote by $\odot^r(T_x^*(V))$ the r -tuple symmetric product of $T_x^*(V)$. It is clear that

$$\sum_{s=1}^r T_y(W) \otimes \odot^s(T_x^*(V)) \subset \sum_{s=1}^r T_y(W) \otimes (\otimes^s T_x^*(V)).$$

The injection $\mathcal{Q}$ depends on the coordinates chosen in the respective neighborhoods. But if for some $X \in \bar{J}^r$, $\mathcal{Q}(X) \in \sum_{s=1}^r T_y(W) \otimes \odot^s(T_x^*(V))$, then the

fact that $\mathcal{Q}(X)$ belongs to the manifold

$$\sum_{s=1}^r T_y(W) \otimes \odot^s (T_x^*(V))$$

is independent of the coordinate systems. Or briefly, the symmetry of the tensorial representation of an element $X \in \tilde{J}^r$ does not depend on the coordinate system.

DEFINITION. *The elements of*

$$J^r = \{ X \in \tilde{J}^r \mid \mathcal{Q}(X) \in \sum_{s=1}^r T(W) \otimes \odot^s (T^*(V)) \}$$

are called the holonomic jets of order r of the manifold V into W .

It is not difficult to see that each element from J^r can be given by some local mapping f of V into W . The element from $J_{x,y}^r$, if f is defined in the neighborhood of a point $x \in V$ and $f(x) = y \in W$, which is given by f , is denoted by $j_x^r f$. An element $X \in \tilde{J}^r$ is said to be *regular* if the corresponding linear mapping

$$\tau_2(\beta_2(\beta_3(\dots(\beta_r(X)\dots) \in \text{Hom}(T_x(V), T_y(W)), x = \alpha(X), y = \beta(X)$$

is of the maximal rank. It is not difficult to show that there exists a tensorial representation (a local mapping) of the manifold $\tilde{J}^r$ into the space

$$\sum_{s=1}^r T(W) \otimes \left(\bigotimes_{\substack{\alpha=1 \\ 1 \leq i_1 < \dots < i_\alpha \leq r}}^s T^*(V_{i_\alpha}) \right),$$

V_{i_α} being different copies of the manifold V . The composition of non-holonomic jets is defined analogously to that one defined in [3]. Let V, N, W be three manifolds. Let $X \in \tilde{J}_{x,z}^r(V, N)$, $Y \in \tilde{J}_{z,y}^r(N, W)$. An element $YX \in \tilde{J}_{x,y}^r(V, W)$ is called *composition of X and Y* and is defined as follows: let us denote $X' = \beta_r(X)$, $Y' = \beta_r(Y)$. There exists a neighborhood of the point x in V and a mapping f of this neighborhood into some neighborhood of X' in $\tilde{J}^{r-1}(V, N)$ such that $\alpha \circ f$ is the identity on V and $f_* = X$. Analogously there exists a mapping g of a neighborhood of a point z in N into some neighborhood of a point Y' in $\tilde{J}^{r-1}(N, W)$ such that $\alpha \circ g$ is the identity on N and $g_* = Y$. It is easy to see that the composition YX is trivially defined for $r = 1$ as the composition of the linear

mappings. Let us assume that the composition is well defined for jets of order $r-1$. Then there exists the composition $g \circ f$ which is a mapping defined on the neighborhood of the point x on V into some neighborhood of a point $Y'X'$ such that $\alpha \circ g \circ f$ is the identity on V . Let us define $YX = (g \circ f)_*$. It is clear that $YX \in \tilde{J}^r(V, W)$. The composition of semi-holonomic jets is a semi-holonomic jet and the composition of holonomic jets is a holonomic jet. Let n be the dimension of the manifold V . The regular non-holonomic r -jets $\in J_{o,x}^r(R^n, V)$ are called *non-holonomic r -frames at a point $x \in V$* and we shall denote their set by $\tilde{H}_x^r(V)$ and further $\tilde{H}^r(V) = \bigcup_{x \in V} \tilde{H}_x^r(V)$. $\tilde{L}_{n,m}^r$ denotes the set of non-holonomic n^r -velocities of the manifold R^m at the point O , i.e. the elements of $\tilde{J}_{o,o}^r(R^n, R^m)$. The regular jets of order r of $\tilde{J}_{o,o}^r(R^n, R^n)$ form the group $\tilde{L}_n^r$, so called non-holonomic prolongation of order r of the linear group $L_n = GL(n, R)$. Let us denote further

$$\tilde{T}_n^r(V) = \tilde{J}_{o,o}^r(R^n, V) \text{ and } \tilde{T}_n^{r*}(V) = \tilde{J}_{o,o}^r(V, R^n).$$

We shall take similar notations for semi-holonomic and holonomic jets. It follows easily from the definition, that the manifold $\tilde{J}^r(V, W)$, where $\dim V = n$, $\dim W = m$, has three natural structures of fibre bundle [2], namely

$$\begin{aligned} & \tilde{J}^r\{V \times W, \tilde{L}_{n,m}^r, L_n^r \times L_m^r, H^r(V) \times H^r(W)\}, \\ & \tilde{J}^r[V, \tilde{T}_n^r(W), L_n^r, H^r(V)], \tilde{J}^r[W, \tilde{T}_m^{r*}(W), L_m^r, H^r(W)]. \end{aligned}$$

The groupoid $\tilde{\Pi}^r(V)$ contained in $\tilde{J}^r(V, V)$ is a groupoid acting on $\tilde{J}^r(V, W)$. The *class of intransitivity* of the element $z \in \tilde{J}^r(V, W)$ with respect to $\tilde{\Pi}^r(V)$ is the set of all the elements $z \oplus \in \tilde{J}^r(V, W)$, $\oplus \in \tilde{\Pi}^r(V)$. To the class of intransitivity of $z \in \tilde{J}^r(V, W)$ with respect to $\tilde{\Pi}^r(V)$ there corresponds in $\tilde{T}_n^r(V)$ the class YL_n^r , $Y = zb$, $b \in H^r(V)$. The class YL_n^r is called the *non-holonomic element of contact* associated to Y or z . We speak also about a non-holonomic n^r -element of contact of W at the point $\beta(z) = y$. A non-holonomic element of contact X of W at y is said to be regular if all the non-holonomic n^r -velocities in the class X are regular, i.e. the corresponding n^1 -velocities are regular 1-jets of dimension n .

REMARK. Let V and W be two differentiable manifolds and let X be a non-holonomic r -jet of V into W , $\alpha(X) = x$, $\beta(X) = y$. The element X gives rise to a unique linear mapping X_* of the vector space $\tilde{T}_x^r(V)$ into $\tilde{T}_y^r(W)$ and a unique linear mapping X^* of $\tilde{T}_y^{r*}(W)$ into $\tilde{T}_x^{r*}(V)$, where

$$\tilde{T}^{r*}(V) = \tilde{J}_{\circ}^r(V, R), \quad \tilde{T}^r(V) = (\tilde{T}^{r*}(V))^*$$

PROPOSITION 1.1. Let $H(B, G)$ be a principal fibre bundle. The set

$$D^r = \{ X \in \tilde{J}^r(B, H) \mid p_* X = j_{\alpha(X)}^r \}$$

has a structure of a fibre bundle with the base B and the fibre

$$G \times G_n^r, \quad G_n^r \equiv \tilde{J}_{\circ, e}^r(R^n, G).$$

On the fibre $G \times G_n^r$ acts the r -th prolongation of the operation $L_n \times G$ on $R^n \times G$.

PROOF. We shall identify first $D^r(R^n, R^n \times G)$ with $R^n \times G \times G_n^r$. Let $X \in D^r(R^n, R^n \times G)$, $\alpha(X) = x$, $\beta(X) = (x, a)$. Let us denote by

$$\tau_1: R^n \times G \rightarrow R^n, \quad \tau_2: R^n \times G \rightarrow G$$

the canonical projections. The isomorphism $D^r(R^n, R^n \times G) \cong R^n \times G \times G_n^r$ is given by the identification $X \longleftrightarrow (x, a, \tau_2(j_{x, a}^r t_{x, a}^{-1})X(j_{\circ}^r t_x^{-1}))$, where

$$t_x: R^n \rightarrow R^n, \quad t_x(y) = x - y.$$

The operation of the pseudogroup ψ_n of operations on $R^n \times G$ is given by the formula $\bar{\psi}: (x, a) \rightarrow (\psi(x), ga)$, $\bar{\psi} \equiv (\psi, g) \in \psi_n$, $(x, a) \in R^n \times G$. Let us consider the prolongation ψ_n^r of this pseudogroup on $R^n \times G \times G_n^r$.

The prolongation of the atlas $\mathcal{Q} \times \mathcal{H}$ of $R^n \times R^n \times G$ onto $B \times H$ on the atlas $\bar{\mathcal{Q}} \times \bar{\mathcal{H}}$ of $D^r(R^n, R^n \times G)$ onto $D^r(B, H)$ is given for the above chosen $X \in D^r(R^n, R^n \times G)$ and $(g, b) \in \mathcal{Q} \times \mathcal{H}$ by the formula $(g, b): X \rightarrow (j_{x, a}^r b)(j_{\circ, e}^r t_{x, a}^{-1})(j_{x, a}^r t_{x, a})(j_{\circ}^r t_x^{-1})(j_x^r t_x)(j_{g(x)}^r g^{-1})$. This prolonged atlas is compatible with the operation of the pseudogroup ψ_n^r on $D^r(R^n, R^n \times G)$.

$$X \in D^r(R^n, R^n \times G), \quad \bar{\psi} \equiv (\psi, g) \in \psi_n, \quad X \rightarrow (j_{x, a}^r \bar{\psi})(j_{\psi(x)}^r \psi^{-1}).$$

The operation of the group $G \times L_n^r$ on the fibre $G \times G_n^r$ is given in a natural way: $(g, s)(a, w) = (ga, ws^{-1})$, $(g, s) \in G \times L_n^r$, $(a, w) \in G \times G_n^r$.

The operation of $\psi_n \times G \times L_n^r$ on $R^n \times G \times G_n^r$ is in fact the operation of the pseudogroup ψ_n^r . The prolongation of the operation R_g , $g \in G$ on H is the right translation on $\tilde{J}^r(B, H)$ given by the elements of G . Then D^r/G as a quotient has a structure of fibre bundle with the base B , fibre G_n^r and structural group L_n^r ($s \in L_n^r$, $w \in G_n^r$, $s : w \rightarrow ws^{-1}$). Let us now consider some global section σ^r of the fibre bundle D^r/G over B . We shall consider the restriction $\tilde{\mathcal{D}}^r$ of the tangent bundle $T(\tilde{J}^r(B, H))$ on $D^r(B, H)$. The section σ^r can be looked at as a section of $D^r(B, H)$ over H which is invariant under the transformations of G . The restriction of $\tilde{\mathcal{D}}^r$ to that section σ^r is a manifold $\mathcal{D}^r$ and because $\mathcal{D}^r \times G \rightarrow \mathcal{D}^r$ is a natural mapping we have the vector bundle $Q^r = \mathcal{D}^r/G$ with the base B . Analogously let us consider the sub-bundle $F(\tilde{J}^r(B, H))$ of $T(\tilde{J}^r(B, H))$ of vertical tangent vectors on $\tilde{J}^r(B, H)$. If we consider the restriction $\tilde{\mathcal{F}}^r$ of F onto $D^r(B, H)$ and the restriction $\mathcal{F}^r$ of $\tilde{\mathcal{F}}^r$ on the section $\sigma^r : B \rightarrow D^r$, we have a well defined vector-bundle $R^r = \mathcal{F}^r/G$ over B . Then holds the following

THEOREM 1.1. *Let H be a principal fibre bundle with the base B , structural group G . Let σ^r be a global section of the fibre bundle D^r/G over B . Then there exists a canonical exact sequence $\mathcal{Q}(H, \sigma^r)$*

$$0 \rightarrow R^r \rightarrow Q^r \rightarrow T(B) \rightarrow 0$$

of vector bundles over B .

It is immediately clear, that the splittings of the exact sequence $\mathcal{Q}(H, \sigma^r)$ are in 1-1 correspondence with the global sections σ^{r+1} of D^{r+1}/G over B . Let us take B as a section of the respective fibre bundle over B for $r=0$. $\mathcal{Q}(H, \sigma^0)$ is the well known Atiyah's sequence [1]. We shall use later the following.

PROPOSITION 1.2. *Let V, W, B be three differentiable manifolds. Let $\bar{\alpha}$ be a mapping of V into B , $\bar{\beta}$ a mapping of $V \times W$ into B so that $\bar{\alpha}(x) = \bar{\beta}(x, y), \forall (x, y) \in V \times W$. If $X \in \tilde{J}^r(B, V)$, $Y \in \tilde{J}^r(B, W)$, $\alpha(X) = \alpha(Y) = u$, $\beta(X) = x$, $\beta(Y) = y$, then $(j_x^r \bar{\alpha})X = (j_{(x,y)}^r \bar{\beta})(X, Y)$.*

PROOF. Let τ_1 be the canonical projection of the product $V \times W$ onto V . Then

$$j_{(x,y)}^r \tau_1(X, Y) = X$$

and

$$j_{(x,y)}^r \bar{\alpha} j_{(x,y)}^r \tau_1 = j_{(x,y)}^r \bar{\alpha} \tau_1 = j_{(x,y)}^r \bar{\beta},$$

so that

$$j_x^r \bar{\alpha} X = j_x^r \bar{\alpha} j_{(x,y)}^r \tau_1(X, Y) = j_{(x,y)}^r \bar{\beta}(X, Y),$$

as was to be shown.

2. The non-holonomic connections of order r .

Let $E(B, F, G, H)$ be a fibre bundle with the structural group G , basis B , $\dim B = n$, standard fibre F and projection p . Let $H(B, G)$ be the associated principal bundle to E . Basis B is a differentiable manifold of the dimension n . A vector τ_z of a non-holonomic tangent vector space of order r , $\tilde{T}_z^r(H)$ is said to be vertical if $p_* \tau_z$ is a zero vector of $\tilde{T}_{p(z)}^r(B)$.

DEFINITION 2.1. We say that a non-holonomic connection of order r on a principal bundle H is given if:

1) to each $z \in H$ there is a vector space $\tilde{H}_z^r$ (subspace of $\tilde{T}_z^r(H)$), assigned so that this assignment is C^∞ .

2) The field of spaces $\tilde{H}_z^r$ is invariant under the right translation on H , i.e. $\tilde{H}_{zg} = R_g * \tilde{H}_z$, $g \in G$, $R_g z = zg$.

3) There exists just one element $Z_z = Z \in \tilde{J}^r(B, H)$, $\alpha(Z) = p(z)$, $\beta(Z) = z$, $pZ = j_{p(z)}^r$, where $j_{p(z)}^r$ is the r -jet of the identical mapping of B onto itself with source $p(z)$, such that $\tilde{H}_z^r = Z_* \tilde{T}_{p(z)}^r(B)$. The space $\tilde{H}_z^r$ is called the horizontal space at the point $z \in H$.

THEOREM 2.1. Let H be a principal bundle. The set of non-holonomic connections of order r on H is in 1-1 correspondence with the fields of regular non-holonomic elements of contact on H such that:

1°) X_z being the regular n^r -element of contact at $z \in H$, $(j_z^r p)X_z$ (i.e. the set $(j_z^r p)Y_z L_n^r, Y_z L_n^r = X_z$) is a regular n^r -element of contact of B at $p(z)$.

$$2^\circ) X_{zg} = (j_z^r R_g)X_z.$$

An element of such a field is called a horizontal n^r -element of contact at a point z of the manifold H .

PROOF. We shall prove presently that the field of horizontal non-holonomic n^r -elements of contact gives rise to the non-holonomic connection of order r on H . Let Y_z be a representative of the class X_z . An r -jet Y_z defines just one linear mapping Y_z^* of $\tilde{T}_0^r(R^n)$ into $\tilde{T}_z^r(H)$. It is clear that the space $\tilde{H}_z^r = Y_z^* \tilde{T}_0^r(R^n)$ is independent of the choice of Y_z in the class X_z . X_z is namely the set of the elements $Y_z s$, $s \in L_n^r$ and L_n^r is the group of transformations on $\tilde{T}_0^r(R^n)$. The mapping Y_z^* is clearly C^∞ . On the basis of 2° follows that $\tilde{H}_{zg}^r = R_g^* \tilde{H}_z^r$, $g \in G$. We have $(j_z^r p) Y_z \in \tilde{J}^r(R^n, B)$, $\alpha\{(j_z^r p) Y_z\} = 0$. Let b be an element of $\tilde{H}^r(B)$, then $g = b^{-1}(j_z^r p) Y_z \in L_n^r$. We show that $Z_z = Y_z g^{-1} b^{-1}$ has the property 3 from the definition of connection. If we consider an element $Y_z s$, $s \in L_n^r$ instead of Y_z , then $Y_z s s^{-1} g^{-1} b^{-1} = Y_z g^{-1} b^{-1}$ and it is easy to see that Z_z depends only on the non-holonomic n^r -element of contact X_z . If we take namely instead of b any other isomorphism $k = ba$, $a \in \tilde{L}_n^r$, of $\tilde{T}_0^r(R^n)$ onto $\tilde{T}_{p(z)}^r(B)$ we have $Y_z \{a^{-1} b^{-1} (j_z^r p) Y_z\}^{-1} a^{-1} b^{-1} = Y_z g^{-1} b^{-1}$. And further $(j_z^r p) Z_z = b b^{-1} (j_z^r p) Z_z = b b^{-1} (j_z^r p) Y_z g^{-1} b^{-1} = j_{p(z)}^r$.

And now let us suppose on the contrary that there is given a non-holonomic connection of order r on H . Let $b \in H^r(B)$, $\gamma(b) = p(z)$, $z \in H$, γ being a projection of the fibre bundle H^r onto B , then $Z_z b \in \tilde{J}^r(R^n, H)$. Let us define an equivalence relation $Z_z b \sim Z_z bs$, $s \in L_n^r$. A class defined by Z_z (the connection is given) is a non-holonomic regular n^r -element of contact of H at the point z . The property 1 is satisfied on the basis of $(j_z^r p) Z_z b = j_{p(z)}^r b = b$. By the definition there is $\tilde{H}_z^r = Z_z^* \tilde{T}_{p(z)}^r(B)$, $\tilde{H}_{zg}^r = Z_{zg}^* \tilde{T}_{p(zg)}^r(B) = Z_{zg}^* \tilde{T}_{p(z)}^r(B) = R_g^* \tilde{H}_z^r = R_g^* Z_z^* \tilde{T}_{p(z)}^r(B)$, i.e. $Z_{zg} = (j_z^r R_g) Z_z$. And this finishes the proof of the theorem.

Let Φ be the groupoid associated to the principal bundle H , i.e. $\Phi = HH^{-1}$. If we denote by $\hat{B}$ the units of Φ , there are two projections a, b of Φ onto $\hat{B}$. $a\Theta$ is a right unit of $\Theta \in \Phi$ and $b\Theta$ is a left unit of $\Theta \in \Phi$. We have now a mapping $\varphi: H \times H \rightarrow HH^{-1}$ so that $\varphi(b, b') = b'b^{-1}$. The targets of the elements of the diagonal Δ of $H \times H$ by the mapping φ ,

$\varphi(b, b) = bb^{-1}$, belong to $\hat{B}$. bb^{-1} is a target of all elements $(bg, bg) \in \Delta$, $g \in G$. We can in a natural way identify Δ and H . We can also to each $\tilde{x} = bb^{-1} \in \hat{B}$ associate a point $x = p(b) \in B$ and on the contrary. Let $\hat{a}$ be the projection of Φ onto B defined as follows :

$$\hat{a} : (b'b^{-1}) \rightarrow a(b'b^{-1}) = bb^{-1} \rightarrow p(b).$$

In the same way can be defined a projection $\hat{b} : \Phi \rightarrow B$. Let $X \in \tilde{J}^r(B, \Phi)$, $\alpha(X) = x$, $\beta(X) = \tilde{x}$, $\hat{a}(X) = j_x^r$, (j_x^r is an r -jet of the retraction of B to the point $x \in B$; this notation is used throughout this paper), $\hat{b}(X) = j_x^r$. Let $\tilde{Q}^r$ be a set of all these r -jets. $\tilde{Q}^r$ is a fibre bundle with basis B [5]. There exists a cross-section of $\tilde{Q}^r$ over B .

THEOREM 2.2.[5]. *The cross-sections in $\tilde{Q}^r$ are in 1-1 correspondence with the non-holonomic connections of order r on the principal bundle H . An element X of this cross-section over a point $x \in B$ is said to be an element of the non-holonomic connection of order r (or an element of the connection at x).*

PROOF. Let be given a connection on H and let Z_x be the element of $\tilde{J}^r(B, H)$ mentioned in 3. Let k be a mapping of the neighborhood of $x = p(z)$ in B into the point $z \in H$. Then $(j_x^r k, Z_x) \in \tilde{J}^r(B, H \times H)$. Let $W = (j_x^r k) \bullet Z_x$, where $\bullet$ is the non-holonomic prolongation of order r of the composition rule $\varphi : H \times H \rightarrow HH^{-1}$. By the definition [3] we have $(j_x^r k) \bullet Z_x = (j_{(z, z)}^r \varphi)(j_x^r k, Z_x)$. Obviously

$$W \in \tilde{J}^r(B, \Phi), \alpha(W) = \alpha(j_x^r k \bullet Z_x) = x, \beta(W) = \beta(j_{(z, z)}^r \varphi) = \tilde{x}.$$

$\hat{a}(W)$ is an abbreviated notation for $(j_x^r \hat{a})W = (j_{(z, z)}^r \hat{a} \circ \varphi)(j_x^r k, Z_x)$. Because of $(\hat{a} \circ \varphi)(b, b') = p(b)$, $(b, b') \in H \times H$, we have on the basis of proposition 1.2 the following :

$$\hat{a}(W) = (j_{(z, z)}^r \hat{a} \circ \varphi)(j_x^r k, Z_x) = (j_z^r p) \quad (j_x^r k) = j_x^r.$$

Analogously we get the relation

$$\hat{b}(W) = (j_{(z, z)}^r \hat{b} \circ \varphi)(j_x^r k, Z_x) = (j_z^r p)Z_x = j_x^r.$$

We have to show now that W is independent on the choice of a point z on the fibre H_x over x . Because by the definition $Z_{zg} = (j_z^r R_g) Z_z$, we have

$$\begin{aligned} (j_x^r (R_g \circ k)) \bullet Z_{zg} &= \{(j_z^r R_g)(j_x^r k)\} \bullet \{(j_z^r R_g) Z_z\} = \\ &= (j_{(zg, zg)}^r \varphi) \{(j_z^r R_g)(j_x^r k), (j_z^r R_g) Z_z\} = \\ &= (j_{(zg, zg)}^r \varphi)(j_z^r R_g) \{j_x^r k, Z_z\} = (j_{(z, z)}^r \varphi) \{j_x^r k, Z_z\} = W, \end{aligned}$$

realizing $\varphi \circ R_g = \varphi$.

Let now on the contrary $X \in \tilde{J}^r(B, \Phi)$ be an element of a cross-section in $\tilde{Q}^r$ over B ; then $\alpha(X) = x$, $\beta(X) = \tilde{x}$. Consider the mapping $\psi : \Phi \times H \rightarrow H$, $\psi(\theta, z) = \theta z$. The groupoid acts thus on H . Let k be a mapping of the neighborhood of a point x on B into the point $z \in H$. The prolongation of the composition rule ψ gives rise to the composition $X \bullet z = X \bullet (j_x^r k) = (j_{(\tilde{x}, z)}^r \psi)(X, j_x^r k) \in \tilde{J}^r(B, H)$. On the basis of the relation $p(\psi(\theta, z)) = b(\theta)$ and by the proposition 1.2 we get

$$\begin{aligned} (j_z^r p)(X \bullet z) &= (j_z^r p)(j_{(\tilde{x}, z)}^r \psi)(X, j_x^r k) = (j_{(\tilde{x}, z)}^r p \circ \psi)(X, j_x^r k) = \\ &= (j_{\tilde{x}}^r \hat{b})X = j_x^r. \end{aligned}$$

A non-holonomic n^r -element of contact belonging to $X \bullet z$ is then regular. We have further

$$\begin{aligned} X \bullet zg &= (j_{(\tilde{x}, zg)}^r \psi)(X, (j_z^r R_g)(j_x^r k)) = (j_z^r R_g)(j_{(\tilde{x}, z)}^r \psi)(X, j_x^r k) = \\ &= (j_z^r R_g)(X \bullet z), \end{aligned}$$

because $\psi(\theta, zg) = R_g \{ \psi(\theta, z) \}$. The theorem is then proved.

Let W be any m -dimensional manifold and let $Z \in \tilde{J}^r(W, H)$.

DEFINITION 2.2. A horizontal projection of Z with respect to an element X of the non-holonomic connection of order r on H at a point $p(z)$ is a non-holonomic r -jet $X^{-1}Z = (X^{-1}pZ) \bullet Z$, where $\bullet$ is a non-holonomic prolongation of order r of the composition rule $\psi : \Phi \times H \rightarrow H$; $X \rightarrow X^{-1}$ is a prolongation of mapping $\theta \rightarrow \theta^{-1}$ defined on Φ .

Obviously $X^{-1}Z \in \tilde{J}^r(W, H)$. A horizontal projection is called by C. Ehresmann [5] an absolute differential. It can be shown that the following proposition holds.

PROPOSITION 2.1. [5]. An element $X^{-1}Z$ belongs to $\tilde{J}^r(W, H_x)$, H_x being the fibre of H over $x \in B$.

PROOF. We have by the definition

$$(j_x^r p)X^{-1}Z = (j_{(\tilde{x}, z)}^r p \circ \psi)(X^{-1}(j_x^r p)Z, Z) = (j_{\tilde{x}}^r \hat{b})X^{-1}(j_x^r p Z).$$

Let k be the mapping of the neighborhood of the point $\alpha(Z)$ on W into the point $x \in B$. On the basis of the relation $(j_x^r \hat{b})X^{-1} = \hat{j}_x^r$, $x = p(z)$, we have $\hat{d}X = \hat{j}_x^r$ and then $(j_x^r p)X^{-1}Z = \hat{j}_x^r(j_x^r p)Z = j_{\alpha(Z)}^r k$.

Let $W_z \in \tilde{H}^r(H)$, $\beta(W_z) = z$, then $(X^{-1}W_z)W_z^{-1} = X^{-1}j_x^r$, X being an element of connection at the point $x = p(z)$. W_z^{-1} is well defined because W_z is a regular element. Using the composition rule of non-holonomic r -jets, we have $W_z W_z^{-1} = j_x^r$. Now we have

$$\begin{aligned} (X^{-1}W_z)W_z^{-1} &= (j_{(\tilde{x}, z)}^r \psi)(X^{-1}(j_x^r p)W_z, W_z)W_z^{-1} = \\ &= (j_{(\tilde{x}, z)}^r \psi)(X^{-1}(j_x^r p)W_z W_z^{-1}, W_z W_z^{-1}) = \\ &= (j_{(\tilde{x}, z)}^r \psi)(X^{-1}(j_x^r p)j_x^r, j_x^r) = X^{-1}j_x^r. \end{aligned}$$

Let $j_x^r re$ be an r -jet of the injective mapping of the fibre H_x into H , with source $z \in H_x$. Let us consider now

$$\begin{aligned} X^{-1}(j_x^r re) &= (j_{(\tilde{x}, z)}^r \psi)(X^{-1}(j_x^r p)(j_x^r re), j_x^r re) = \\ &= (j_{(\tilde{x}, z)}^r \psi)(X^{-1}\hat{j}_x^r, j_x^r re). \end{aligned}$$

If x is a fixed point, then the mapping $\psi(\tilde{x}, z) = z$ is the projection $(\tilde{x}) \times H_x \rightarrow H_x$ and by the proposition 1.2 we have $X^{-1}(j_x^r re) = j_x^r re$.

We have now a linear mapping $P_{z,*}$ of $\tilde{T}_z^r(H)$ into $\tilde{T}_z^r(H_x)$ defined by the non-holonomic r -jet $X^{-1}j_x^r = P_z$. $\omega = \{(j_x^r z^{-1})P_z\}_*$ is then a linear mapping of $\tilde{T}_z^r(H)$ into $\tilde{T}_e^r(G)$.

Let φ_g be an inner endomorphism of G associated with $g \in G$, $Ad(g)$ the linear mapping $(j_e^r \varphi_g)_*$ of $\tilde{T}_e^r(G)$ onto itself and by

$$\tilde{T}^r(V) = \bigcup_{x \in V} \tilde{T}_x^r(V)$$

the non-holonomic tangent bundle of order r over V . A cross-section in $\tilde{T}^r(V)$ over V is called a non-holonomic vector field of order r . It is easy to see that the set of non-holonomic r -jets of V into $\tilde{T}^s(V)$ with

source x , denoted by $\tilde{S}_x^{r,s}(V)$, is a vector space.

REMARK. Let L_g be a left translation defined on G by the element $g \in G$. The structural group G is a right transformation group on H which is simply transitive on each fibre $H_x = p^{-1}(x)$. We shall now define, analogously as it is for an infinitesimal connection of the 1st-order, a non-holonomic fundamental vector field of order r on H associated to a vector $Y = {}_e Y \in \tilde{T}_e^r(G)$ (e is a unit of G). Let ${}_g Y = (j_e^r L_g)_* {}_e Y$. Let b_x be a homomorphism of G onto H_x so that $b_x e = z$. Consider the element ${}_e Y_z = (j_e^r b_x)_* {}_e Y$ at the point $z \in H_x$ and the element ${}_g Y_{zg} = (j_g^r b_x)_* {}_g Y$ at the point $zg \in H_x$. It is easy to see that ${}_g Y_{zg} = (j_e^r b_{xg})_* {}_e Y = {}_e Y_{zg}$. We have now on H a vector field which corresponds to the left invariant vector field on G , and this correspondence does not depend on the choice of $b_x \in H_x$. We shall speak about the non-holonomic *fundamental vector field* Y of order r associated to Y (or briefly about a fundamental vector field only). It is clear that $R_{g*} Y_z$ is a vector of the vector field associated to $Ad(g^{-1})Y$.

DEFINITION. Let V be a differentiable manifold, and φ_x a linear mapping of $\tilde{M} \tilde{S}_x^{r,s}(V)$ into a vector space M . A differentiable field $x \rightarrow \varphi_x$ is called an M -valued differentiable form φ on V of degree $m > 0$, order (r, s) .

THEOREM 2.3. A non-holonomic connection of order r on H can be defined by $\tilde{aT}_e^r(G)$ -valued differential form ω on H of the 1st-degree, order $(0, r)$ so that the following conditions are satisfied:

- 1) $\omega(Y_z) = Y$, $Y \in \tilde{T}_e^r(G)$, $z \in H$,
- 2) $\omega(R_{g*} X) = Ad(g^{-1})\omega(X)$, $X \in \tilde{T}_z^r(H)$,
- 3) There exists a non-holonomic r -jet $Z \in \tilde{J}^r(H, H_x)$, $\alpha(Z) = \beta(Z) = z$ so that $(j_e^r z)_* \omega = Z_*$.

PROOF. First let us suppose that a non-holonomic connection on H is given. We have a linear mapping $\omega = \{(j_z^r z^{-1})(X^{-1} j_z^r)\}_*$ of $\tilde{T}_z^r(H)$ into $\tilde{T}_e^r(G)$. We shall prove presently that ω is a form considered in the theorem. We know that H_x is a submanifold of H . Let $V_z \in \tilde{H}^r(H_x)$ and $W_z \in \tilde{H}^r(H)$, be the r -frames at the point z . Let Y_z be any vertical vector at $z \in H$, i.e. a vector of $\tilde{T}_z^r(H_x)$. Let m be the dimension of H and n the

dimension of H_x . Then $Y' = (V_z^{-1})_* Y_z \in \tilde{T}_o^r(R^n)$, $Y'' = (W_z^{-1})_* Y_z \in \tilde{T}_o^r(R^m)$. It is clear that

$$\begin{aligned} (X^{-1}W_z)_* Y'' &= (j_{(\tilde{x}, z)}^r \psi)_* \{ (X^{-1})_* p_* W_{z*} Y'', W_{z*} Y'' \} = \\ &= (j_{(\tilde{x}, z)}^r \psi)_* \{ (X^{-1})_* p_* V_{z*} Y', V_{z*} Y' \} = (X^{-1}V_z)_* Y'. \end{aligned}$$

We have then $\{ (X^{-1}W_z)W_z^{-1} \}_* Y_z = \{ (X^{-1}V_z)V_z^{-1} \}_* Y_z$. But $(X^{-1}W_z)W_z^{-1} = j_z^r$ and analogously $(X^{-1}V_z)V_z^{-1} = j_z^r re$. On the basis of the relation

$$X^{-1}(j_z^r re) = j_z^r re$$

we see that the mappings

$$\omega = \{ (j_z^r z^{-1})(X^{-1}j_z^r) \}_* \text{ and } \omega_1 = \{ (j_z^r z^{-1})(j_z^r re) \}_*$$

are linear mappings of $\tilde{T}_z^r(H_x)$ onto $\tilde{T}_e^r(G)$ so that $\omega_1(Y_z) = Y$, Y_z being a vector of a fundamental vector field belonging to $Y \in T_e^r(G)$. We have proved then that 1) is satisfied for ω .

Let $A_z \in \tilde{T}_z^r(H)$ and let $A_{zg} = R_g * A_z$. We have then $\omega(A_{zg}) = \{ (j_{zg}^r(zg)^{-1})(X^{-1}j_{zg}^r)(j_z^r R_g) \}_* A_z$. Let us prove first that

$$(X^{-1}j_{zg}^r)(j_z^r R_g) = (j_z^r R_g)(X^{-1}j_z^r).$$

We have namely the relations

$$\begin{aligned} (j_z^r R_g)(X^{-1}j_z^r) &= (j_z^r R_g)(j_{(\tilde{x}, z)}^r \psi)(X^{-1}p j_z^r, j_z^r) = \\ &= (j_{(\tilde{x}, zg)}^r \psi)(j_z^r R_g)(X^{-1}p j_z^r, j_z^r) = (j_{(\tilde{x}, zg)}^r \psi)(X^{-1}p j_{zg}^r, j_{zg}^r)(j_z^r R_g) = \\ &= (X^{-1}j_{zg}^r)(j_z^r R_g). \end{aligned}$$

Then $\omega(A_{zg}) = \{ (j_{zg}^r(zg)^{-1})(j_z^r R_g)(X^{-1}j_z^r) \}_* A_z$. Denoting by L_g a left translation on G defined by $g \in G$ we have

$$\begin{aligned} \omega(A_{zg}) &= \{ (j_g^r L_g^{-1})(j_{zg}^r z^{-1})(j_z^r R_g)(j_e^r z)(j_z^r z^{-1})(X^{-1}j_z^r) \}_* A_z = \\ &= \{ (j_g^r L_{g^{-1}})(j_e^r R_g) \}_* \{ (j_z^r z^{-1})(X^{-1}j_z^r) \}_* A_z = \\ &= (j_e^r \varphi_{g^{-1}})_* \omega(A_z) = Ad(g^{-1})\omega(A_z). \end{aligned}$$

An element $X^{-1}j_z^r$ is just a non-holonomic r -jet Z mentioned in 3. The mapping $(j_e^r Z)_* \omega$ is the mapping associated with it.

Let now on the contrary ω be a $\tilde{T}_e^r(G)$ -valued differential form of degree 1, order $(0, r)$ on H so that the properties 1, 2, 3 are fulfilled.

Let σ be a cross-section in H over a neighborhood of the point $x \in B$, $\sigma(x) = z$. Let $W = j_x^r \sigma$. Let Z be a non-holonomic r -jet from $\tilde{J}^r(H, H_x)$ with the properties contained in the theorem. Let $X = ZW \bullet W$, $\bullet$ being a prolongation of the composition rule $\varphi: H \times H \rightarrow HH^{-1}$. It is clear that $X \in \tilde{J}^r(B, \Phi)$. We have further $\alpha(X) = x$, $\beta(X) = \tilde{x}$. On the basis of $\hat{a}(b'b^{-1}) = p(b)$, $\hat{b}(b'b^{-1}) = p(b')$, $b, b' \in H$, we obtain

$$(j_x^r \hat{a})(j_{(z, z)}^r \varphi)(ZW, W) = (j_z^r p)ZW = j_x^r \hat{a},$$

$$(j_x^r \hat{b})(j_{(z, z)}^r \varphi)(ZW, W) = (j_z^r p)W = j_x^r \hat{b}.$$

X is then an element of a cross-section in $\tilde{Q}^r$ over B . We prove now the independence of X on the choice of the cross-section σ over a neighborhood of the point $x \in B$. Let σ' be another lifting, $\sigma'(y) = \sigma(y)g(y)$, $g(y) \in G$ for each y from the considered neighborhood of the point $x \in B$, $g(x) = e$. Let us notice that we have defined a holonomic prolongation of a composition rule [6]. Using this operation we have $j_x^r \sigma' = (j_x^r \sigma)(j_x^r g)$. Let us prove that the identity $Z\{W(j_x^r g)\} = \{ZW\}(j_x^r g)$ holds. Let first $r = 1$. Let f be a mapping of the neighborhood $U_1(z) \subset H$ onto the neighborhood $U_2(z) \subset H_x$ so that $f(z) = z$ and $Z = j_z^1 f$. Let σ be a cross-section in H over the neighborhood $V(x) \subset B$, $\sigma(V(x)) \subset U_1(z)$ and be $W = j_x^1 \sigma$. Let σ' be another cross-section in H over $V(x)$ so that $\sigma'(y) = \sigma(y)g(y)$, $y \in V(x)$. Then $j_x^1(f \circ \sigma') = (j_x^1 f)\{(j_x^1 \sigma)(j_x^1 g)\}$ and further

$$\begin{aligned} j_x^1(f \circ \sigma') &= j_x^1\{(f \circ \sigma)g\} = (j_x^1(f \circ \sigma))(j_x^1 g) = \\ &= \{(j_z^1 f)(j_x^1 \sigma)\}(j_x^1 g), \end{aligned}$$

because $f \circ \sigma'$, $f \circ \sigma$ are the cross-sections in H over $V(x)$. Let s be a cross-section in $\tilde{J}^{r-1}(B, H)$ over $V(x)$ defined as follows: $s(y) = j_y^{r-1} \sigma$, $y \in V(x)$. Let s' be a cross-section in $\tilde{J}^{r-1}(H, H_x)$ over $U_1(x)$ so that $j_z^1 s' = Z$. We have then a mapping $s'' = s's$ of $V(x)$ into H_x so that $s''(x) = z$. Let λ be a cross-section in $\tilde{J}^{r-1}(B, G)$ over $V(x)$ so that $\lambda(y) = j_y^{r-1} g$, $y \in V(x)$. We have then $j_x^1 \lambda = j_x^r g$. By the assumption

$$s'(w)\{s(y)\lambda(y)\} = \{s'(w)s(y)\}\lambda(y), \quad y \in V(x), \quad \sigma(y) = w.$$

We have then

$$\begin{aligned} j_x^1 \{ s'(s\lambda) \} &= (j_x^1 s') \{ (j_x^1 s)(j_x^1 \lambda) \}, \\ j_x^1 \{ (s's)\lambda \} &= \{ (j_x^1 s')(j_x^1 s) \} (j_x^1 \lambda) \end{aligned}$$

and, on the basis of the equality $j_x^1 \{ (s's)\lambda \} = j_x^1 \{ s'(s\lambda) \}$, the result

$$Z \{ W(j_x^r g) \} = (ZW)(j_x^r g).$$

But by the definition $ZW \bullet W = (j_{(z,z)}^r \varphi)(ZW, W)$. Denote $W' = W(j_x^r g)$.

It is then

$$\begin{aligned} ZW' \bullet W' &= (j_{(zg, zg)}^r \varphi)(ZW(j_x^r g), W(j_x^r g)) = \\ &= (j_{(z,z)}^r \varphi)(ZW, W). \end{aligned}$$

The last equality may be proved by induction. We must show now that $\{ Z(j_z^r R_g)W \} \bullet \{ (j_z^r R_g)W \} = ZW \bullet W$ holds. We know that $Z_{z*} = (j_z^r z)_* \omega$ is a linear mapping of $\tilde{T}_z^r(H)$ onto $\tilde{T}_z^r(H_x)$. Let $A_{zg} \in \tilde{T}_{zg}^r(H)$, $A_{zg} = R_{g*} A_z$; then

$$\begin{aligned} Z_{zg*}(A_{zg}) &= (j_e^r zg)_* \omega(A_{zg}) = \{ (j_g^r z)(j_e^r L_g) \} * Ad(g^{-1}) \omega(A_z) = \\ &= \{ (j_z^r R_g)(j_e^r z) \} * \omega(A_z) = (j_z^r R_g)_* Z_{z*}(A_z) \end{aligned}$$

because of $zg g^{-1} ag = z(Lg(\varphi_g^{-1}(a))) = R_g(z(a))$. Then

$$\{ Z(j_z^r R_g)W \} \bullet \{ (j_z^r R_g)W \} = \{ (j_z^r R_g)ZW \} \bullet \{ (j_z^r R_g)W \}.$$

On the basis of the relation $\varphi \circ R_g = \varphi$ we have the above result and so is the theorem completely proved.

THEOREM 2.4. Let $\mathcal{Q}(H, \sigma^k)$, $0 \leq k \leq r-1$, be the exact sequence associated to a global section σ^k of the bundle D^k/G over B , such that σ^k is given by a splitting ρ^k of the exact sequence $\mathcal{Q}(H, \sigma^{k-1})$, for $k \geq 1$ and σ^0 is B itself.

The non-holonomic connections of order r on the principal fibre bundle H are in 1-1 correspondence with the splittings ρ^s , $1 \leq s \leq r$, of the exact sequence $\mathcal{Q}(H, \sigma^{s-1})$ of vector bundles.

PROOF. First, let be given a non-holonomic connection of order r on H . Then to each $z \in H$ there is associated by definition a non-holonomic r -jet $Z \in \tilde{J}_{x,z}^r(B, H)$, s.t. $p_* Z = j_x^r$, $Z_{zg} = R_g Z_z$, $g \in G$. The projection $j^1 Z$ into $J^1(B, H)$ uniquely gives the section σ^1 of D^1/G over B . Assu-

ming that the section σ^{r-1} of D^{r-1}/G is given uniquely by the projection $j^{r-1}Z$ we shall prove that the section $\sigma^r: B \rightarrow D^r/G$ is given by the element Z . We know already that σ^{r-1} can be considered as a section of D^r over H which is invariant under the transformations of G . The jet Z is then defined as $j_x^1 \sigma$, where σ is some section of D^{r-1} over B such that $\sigma(x) = z$. Further let us take $Z_{zg} = j_{p(zg)}^1 (R_g \sigma)$, then $Z_{zg} = R_g j_x^1 \sigma = R_g Z_x$. We have thus a well defined section σ^r of D^r over H , which is invariant under the transformations of G , namely

$$\sigma^r: a \rightarrow j_{p(a)}^1 \sigma, \quad a = \beta(Z) \in H,$$

σ being the section mentioned above. And now on the contrary, let there be given r splittings ρ_s ($1 \leq s \leq r$) of the exact sequences $\mathcal{Q}(H, \sigma^{s-1})$ of vector bundles. We have to prove that there is given exactly one non-holonomic connection of order r on H by these splittings. This holds for $r = 1$. Let us assume that the statement be true for $s = r-1$. To the splitting ρ_{r-1} is uniquely associated the section σ^r of D^r/G over B or, what is the same, the G -invariant section σ^r of D^r over H and so we have the non-holonomic connection of order r on H (straight by the definition). From similar reasons as in [1] follows that a non-holonomic connection of order r on H in the real case always exists.

3. Induced connection and prolongation.

Let H be a principal bundle with the structural group G and let M be a vector space and R a representation of G in M . Let $\tilde{S}_z^{r,s}(H)$ be the vector space of all non-holonomic r -jets of H into $\tilde{T}^s(H)$ with source $z \in H$. A vector $X \in \tilde{S}_z^{r,s}(H)$ is said to be vertical if $p_* \beta(X)$ is a zero vector of $\tilde{T}_p^s(B)$. Denote by $R_{g*} X$ the element $(j_z^{r+s} R_g)_* X$ of $\tilde{S}_z^{r,s}(H)$. The operation $(j_z^{r+s} R_g)_* X$ is defined as follows: we know that $X \in \tilde{S}_z^{r,s}(H)$ is an element of $J^r(H, \tilde{T}^s(H))$. If $r = 1$, then $X = j_z^1 \sigma$, σ being a cross-section in $\tilde{T}^s(H)$ over a neighborhood of the point $z \in H$. We have now a cross-section $\tilde{\sigma}: z \rightarrow (j_z^s R_g)_* \sigma(z)$ in $\tilde{T}^s(H)$ over a neighborhood of the point $zg \in H$. We denote then $j_{zg}^1 \tilde{\sigma} = (j_z^{s+1} R_g)_* j_z^1 \sigma$. It is clear now how is the mapping $(j_z^{r+s} R_g)_*$ defined for $r > 1$. It is the prolongation of the composition rule defined by the transformations of $(j_z^s R_g)_*$ on $\tilde{T}^s(H)$.

DEFINITION 3.1. An M -valued differential form φ of degree $m > 0$, order (r, s) on a principal bundle H is said to be a tensorial form of degree $m > 0$, order (r, s) , type $\mathcal{R}(G)$, if the following conditions are satisfied:

- a) if at least one of the vectors $X_1, \dots, X_m$ is vertical, then $\varphi(X_1, \dots, X_m) = 0$.
 b) $\varphi(R_{g*}X_1, \dots, R_{g*}X_m) = \mathcal{R}(g^{-1})\varphi(X_1, \dots, X_m)$.

PROPOSITION 3.1. Let $Y \in \tilde{J}^r(V, N)$, $Z \in \tilde{J}^r(N, W)$ and let j^k be the projection of the non-holonomic r -jets into the non-holonomic k -jets. Then $(j^k Z)(j^k Y) = j^k(ZY)$.

PROOF. We know that

$$j^{k+l} = j^k \circ j^l.$$

Let σ_1 be a cross-section in $\tilde{J}^{r-1}(V, N)$ over a neighborhood of the point $x \in V$ and let σ_2 be a cross-section in $\tilde{J}^{r-1}(N, W)$ over a neighborhood of the point $\beta(\sigma_1(x)) = y \in N$ and let $Y = j_x^1 \sigma_1$, $Z = j_y^1 \sigma_2$. Let

$$\sigma : u \mapsto \sigma_2(v)\sigma_1(u), \quad v = \beta(\sigma_1(u))$$

be a cross-section in $\tilde{J}^{r-1}(V, W)$ over a neighborhood of $x \in V$. We have then $ZY = j_x^1 \sigma$ and then $j^{r-1}(j_x^1 \sigma) = \sigma(x)$, $j^{r-1}Z = \sigma_2(y)$, $j^{r-1}Y = \sigma_1(x)$. Then $\sigma(x) = \sigma_2(y)\sigma_1(x)$. We have proved then the theorem for the case $k = r-1$, but it is clear that by induction one can easily prove that the theorem is true for an arbitrary $k < r$.

THEOREM 3.1. [5]. Let C be a cross-section in $\tilde{Q}^r$ over B , i.e. a non-holonomic connection of order r . Denoting by X the element $C(x)$, $x \in B$, we have the cross-section $x \mapsto j^1(X) = C_1(x)$ in Q^1 over B . Let

$$X' = (j_{(X, \tilde{x})}^1 \psi)(j_x^1 C, j^1 X).$$

The mapping $x \mapsto X'$ is a cross-section in $\tilde{Q}^{r+1}$ over B , i.e. a non-holonomic connection of order $r+1$ which is called the prolongation of the non-holonomic connection C . ψ is the composition rule

$$\psi : \tilde{J}^r(B, \Phi) \times \Phi \rightarrow \tilde{J}^r(B, \Phi).$$

PROOF. We first show what ψ , the composition rule, is looking like. Let $\lambda : \Phi * \Phi \rightarrow \Phi$ be the composition of the groupoid Φ . Let $A \in \tilde{J}^r(B, \Phi)$,

$\alpha(A) = x \in B, \beta(A) = \theta' \in \Phi, D \in \tilde{J}^r(B, \Phi), \alpha(D) = x, \beta(D) = \theta$. Let $D = j_x^r k$, k being a mapping of a neighborhood of $x \in B$ into the point $\theta \in \Phi$. We have then $A \bullet D = (j_{(\theta', \theta)}^r \lambda)(A, D) \in \tilde{J}^r(B, \Phi)$.

We can identify D with the point θ and write then $\psi(A, \theta) = A \bullet D = A \bullet \theta$. On the basis of $(a \circ \lambda)(\theta', \theta) = a(\theta)$ we have

$$\{(j_{(\theta', \theta)}^r \hat{a}) \circ \psi\}(A, D) = (j_{\theta}^r \hat{a})D.$$

We have further $j_x^1 C \in \tilde{J}^{r+1}(B, \Phi), j^1 X \in J^1(B, \Phi)$ and then

$$(j_{\tilde{x}}^{r+1} \hat{a})(j_{(X, \tilde{x})}^1 \psi)(j_x^1 C, j^1 X) = (j_{\tilde{x}}^{r+1} \hat{a})j^1 X.$$

But we know that $(j_{\tilde{x}}^r \hat{a})X = \hat{j}_x^r$ and using the operation of projection j^1 we have $(j_{\tilde{x}}^1 \hat{a})j^1 X = \hat{j}_x^1$. Let l be a mapping of a neighborhood of the point x on B into the point $j^1 X \in J^1(B, \Phi)$ and let us identify $j_x^r l$ with the point $j^1 X$. We have then $(j_{\tilde{x}}^{r+1} \hat{a})j^1 X = \hat{j}_x^{r+1}$. Analogously on the basis of $(b \circ \lambda)(\theta', \theta) = b(\theta')$ the relation $\{(j_{(\theta', \theta)}^r \hat{b}) \circ \psi\}(A, D) = (j_{\theta}^r \hat{b})A$ holds and then

$$(j_{\tilde{x}}^{r+1} \hat{b})(j_{(X, \tilde{x})}^1 \psi)(j_x^1 C, j^1 X) = (j_{\tilde{x}}^{r+1} \hat{b})(j_x^1 C) = j_x^{r+1}.$$

THEOREM 3.2. *Let X' be a prolongation of order k of the element X of a non-holonomic connection of order r , C with respect to C . Then $j^r X' = X$.*

PROOF. Denote by τ the operation of prolongation of the 1st order of X with respect to C and further

$$\tau^k X = \underbrace{\tau(\tau(\dots(\tau(X))\dots))}_{k \text{ times}} = X';$$

$\tau^2 X$ is prolongation of the 1st order of the element τX with respect to the prolongation C' of the first order of C . It is sufficient to prove the theorem in the case $k = 1$. By the definition $X' = (j_{(X, \tilde{x})}^1 \psi)(j_x^1 C, j^1 X)$. Then

$$j^r X' = j^r(j_{(X, \tilde{x})}^1 \psi)j^r(j_x^1 C, j^1 X) = \psi(C(x), \tilde{x}) = X.$$

THEOREM 3.3. *Let be given a non-holonomic connection of order r on H by the form ω . This connection uniquely gives rise to the non-holonomic connection of order k , ($k \leq r$) on H with the form $\omega_{(k)} = j_k \omega$, j_k being the projection defined by the canonical projection j^k for non-holonomic r -jets into k -jets. The following diagram is commutative.*

$$\begin{array}{ccc}
 \tilde{T}_z^k(H) & \xrightarrow{\omega(k)} & \tilde{T}_e^k(G) \\
 (j^k)_* \downarrow & & \downarrow (j^k)_* \\
 \tilde{T}_z^r(H) & \xrightarrow{\omega} & \tilde{T}_e^r(G)
 \end{array}$$

$(j^k)_*$ is the linear mapping associated to the projection j^k .

PROOF. Let $X \in \tilde{Q}^r$ be an element of the non-holonomic connection of order r and let $X_{(k)} = j^k X$. It is clear that

$$\begin{aligned}
 \alpha(X_{(k)}) &= x, \quad \beta(X_{(k)}) = \tilde{x}, \\
 (j_x^k \hat{a})X_{(k)} &= \{j_x^k(j_x^r \hat{a})\} \{j^k X\} = j^k \{(j_x^r \hat{a})X\} = j^k(j_x^r) = j_x^k.
 \end{aligned}$$

Analogously we have

$$(j_x^k \hat{b})X_{(k)} = \{j_x^k(j_x^r \hat{b})\} \{j^k X\} = j^k \{(j_x^r \hat{b})X\} = j^k(j_x^r) = j_x^k.$$

Further

$$\begin{aligned}
 j^k \{X^{-1} j_z^r\} &= j^k [j_{(x,z)}^r \psi (X^{-1} p j_z^r, j_z^r)] = \\
 &= (j_{(x,z)}^k \psi) (X_{(k)}^{-1} p j_z^k, j_z^k) = X_{(k)}^{-1} j_z^k,
 \end{aligned}$$

ψ being the mapping $\psi: \Phi * H \rightarrow H$; $\psi(\theta, z) = \theta z$. We have $j^k(j_z^r z^{-1}) = j_z^k z^{-1}$. Then $\omega_{(k)} = \{(j_z^k z^{-1})(X_{(k)}^{-1} j_z^k)\}_* = j_k \omega$ the form of the non-holonomic connection of order k associated to ω and to the projection j^k . Let further $L \in \tilde{J}_e^r(G)$, then $j^k L \in \tilde{J}_e^k(G)$, $j^k L (j_z^k z^{-1})(X_{(k)}^{-1} j_z^k) \in \tilde{J}_z^k(H)$. But it is easy to see that $j^k L (j_z^k z^{-1})(X_{(k)}^{-1} j_z^k) = j^k L \{(j_z^r z^{-1})(X^{-1} j_z^r)\}$. If we take the dual vector spaces, we see then that the diagram is commutative.

Let $\omega^{(k)} = i^k \omega$ be the form of the non-holonomic connection of order $r+k$, which is the prolongation of order k of the non-holonomic connection of order r given by the form ω . The operator i^0 is the identity. On the basis of the theorem 3.3 we see that the diagram

$$\begin{array}{ccc}
 \tilde{T}_z^{r+k}(H) & \xrightarrow{\omega^{(k)}} & \tilde{T}_e^{r+k}(G) \\
 (j^r)_* \downarrow & & \downarrow (j^r)_* \\
 T_z^r(H) & \xrightarrow{\omega} & T_e^r(G)
 \end{array}$$

is commutative.

Let ω be the form of the non-holonomic connection C of order r on H . This connection gives rise to $(r-1)$ connections of order r on H . These connections are given by the forms $\pi_\alpha = i^{r-\alpha} j_\alpha \omega$; $\alpha = 1, 2, \dots, r-1$. The forms $\kappa_\alpha = \omega - \pi_\alpha$ are the $\tilde{T}_e^r(G)$ -valued tensorial forms of degree 1, order $(0, 1)$, type $Ad(G)$. In the notation introduced above $i^1 \omega$ is the form of a connection of order $s+1$ if ω is the form of a connection of order s . If κ_α is a zero form, then $\omega = i^{r-\alpha} j_\alpha \omega$ and on the basis of the theorem 3.3 we have $j_\beta \omega = i^{\beta-\alpha} j_\alpha \omega$, $\beta = \alpha, \alpha+1, \dots, r-1$. Then $\pi_\beta = i^{r-\beta} j_\beta \omega = i^{r-\alpha} j_\alpha \omega = \omega$ and then κ_β is a zero form for each $\beta = \alpha, \dots, r+1$. We have proved then the theorem.

THEOREM 3.4. *Let C be a non-holonomic connection of order r on H . This connection gives rise to $r-1$ non-holonomic connections of order s ($s = 1, 2, \dots, r-1$) on H and to $r-1$ $\tilde{T}_e^r(G)$ -valued tensorial forms κ_α of degree 1, order $(0, 1)$, type $Ad(G)$. The form κ_α is a zero form if and only if there exists a non-holonomic connection C_α so that the connection C is a prolongation of order $r-\alpha$ of the connection C_α .*

Let $H'(B', G')$, $H(B, G)$ be two principal bundles. Let $\varphi: B' \rightarrow B$ be an imbedding of B' into B , ρ a homomorphism of G' into G and f a mapping of H' into H compatible with ρ , i.e.

- (1) $f(z'g') = f(z')\rho(g')$,
- (2) $p(f(z')) = \varphi(p'(z'))$, $z' \in H'$, $g' \in G'$.

The mapping f is called an immersion of the principal bundle H' into H . A linear mapping π of $\tilde{T}_e^r(G)$ into $\tilde{T}_e^r(G')$ is called an invariant projection if

- a) $\pi(\rho_*(X')) = X'$, $X' \in \tilde{T}_{e'}^r(G')$,
- b) $\pi(Ad(\rho(g'))X') = Ad(g')\pi(\rho_*(X'))$, $g' \in G'$, $X' \in \tilde{T}_{e'}^r(G')$.

c) There exists a non-holonomic r -jet $Z \in \tilde{J}^r(G, G')$ so that $\alpha(Z) = e \in G$, $\beta(Z) = e' \in G'$, $Z_* = \pi$.

THEOREM 3.5. Let ω be the form of a connection of order r on H . Let $f : H' \rightarrow H$ be a homomorphism of the principal bundle H' into H and

$$\pi : \tilde{T}_e^r(G) \rightarrow \tilde{T}_{e'}^r(G')$$

an invariant projection. Then the form $\omega' = \pi \omega f_*$ is the form of a non-holonomic connection of order r on $H'(B', G')$. We shall speak about the induced non-holonomic connection of order r . The induction of the non-holonomic connection is invariant under prolongation and projection of the connection.

PROOF. Let Y_z be the vector of the fundamental vector field on H' at the point z' , which belongs to $Y' \in \tilde{T}_{e'}^r(G')$. Then $f_*(Y_z)$ is a vertical vector of H at the point $f(z')$. From the definition of an immersion f follows that $f_*(Y_z)$ is a vector of the fundamental vector field on H associated to $\rho_*(Y') \in \tilde{T}_e^r(G)$. Denoting by $h_z : G' \rightarrow H'_{p'(z')}$, $h_z : G \rightarrow H_{p(z)}$, $h_z(e') = z'$, $h_z(e) = z = f(z')$ the respective homomorphisms we see that the mapping f is identical with $h_z^{-1} \circ \rho \circ h_z$. Let $X_z \in \tilde{T}_z^r(H')$, $R_g * X_z = X_{z', g} \in \tilde{T}_{z', g}^r(H')$. We have $f_*(X_{z', g}) = R_{\rho(g')} * f_*(X_z)$. On the basis of the relation

$$\omega(R_g * X_z) = Ad(g^{-1})\omega(X_z), X_z = f_*(X_{z'})$$

we have

$$\begin{aligned} \pi \omega(R_g * X_z) &= \pi(Ad(g^{-1})\omega(X_z)) = Ad(g'^{-1})\pi \omega(X_z) = \\ &= Ad(g'^{-1})\omega'(X_{z'}). \end{aligned}$$

Let K be the r -jet associated to ω . Let Z be an r -jet with the property c from the definition of an invariant projection. Let

$$W = (j_e^r, z')Z(j_z^r z^{-1})K(j_z^r, f).$$

It is easy to see that $W \in \tilde{J}^r(H', H'_{p'(z')})$, $\alpha(W) = \beta(W) = z'$ and $W_* = (j_e^r, z')_* \omega'$. Because $\omega = \{(j_z^r z^{-1})(X^{-1} j_z^r)\}_*$, $\omega_{(k)} = \{(j_z^k z^{-1})(X_{(k)}^{-1} j_z^k)\}_*$ and $\omega' = \{Z(j_z^r z^{-1})(X^{-1} j_z^r)(j_z^r, f)\}_*$ we have

$$j_k \omega' = \{(j^k Z)(j_z^k z^{-1})(X_{(k)}^{-1} j_z^k)(j_z^k, f)\}_* = (j^k Z)_* j_k \omega(j_z^k, f)_* = \pi(j_k \omega) f_*.$$

On the basis of this condition it is immediately clear that the induction is invariant with respect to the prolongation.

REMARK. It is possible to show that the space $G/\rho(G')$ being of a certain special type (generalized weak reductivity) we have an invariant projection uniquely given.

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