CAHIERS DU SÉMINAIRE D'HISTOIRE DES MATHÉMATIQUES

THOMAS W. KÖRNER

Sets of uniqueness

Cahiers du séminaire d'histoire des mathématiques 2e série, tome 2 (1992), p. 51-63
<http://www.numdam.org/item?id=CSHM_1992_2_2__51_0>

Sets of Uniqueness

Thomas W. Körner

Cambridge University (U. K.)

When I first started research my supervisor, Nick Varopoulos, told me to buy two essential works—Zygmund's *Trigonometric Series* [27] and Kahane and Salem's *Ensembles parfaits et séries trigonométriques* [8]. It was easy to buy the Zygmund but Kahane and Salem had just sold out. A request to the publishers produced the last copy in their warehouse. That well thumbed copy lies before me as I write, but I hardly need to refer to it to recall an introduction which so struck me when I first read it.

The authors first quote Queneau.

> Ce n'est pas à l'architecture, à la maçonnerie, qu'il faut comparer a la géométrie ou l'analyse, mais à la botanique, à la géographie, aux sciences physiques même. Il s'agit de décrire un monde, de le découvrir et non de le construire ou de l'inventer, car il existe en dehors de l'esprit humain et indépendant de lui.

They go on to say.

> Il y a quelques dizaines d'années, ce livre put se passer de cette préface, qui est écrite en guise d'apologie. Aujourd'hui, venant à un moment où la plupart des mathématiciens—et les meilleurs—s'intéressent surtout aux questions de structure, il peut paraître suranné et ressembler en quelque sorte à un herbier. Les auteurs se doivent donc d'expliquer que leur propos n'est en acune façon réactionaire. Il savent la beauté des grandes théories modernes, et que leur puissance est irremplaçable, car sans elles on serait souvent condamné (comme l'a dit Lebesgue) à renoncer à la solution de bien des problèmes à énoncés simples posés depuis fort longtemps. Mais ils pensent que, sans ignorer l'architecture qui domine les êtres mathématiques, il est permis de s'intéresser à ces êtres eux-mêmes qui, pour isolés qu'ils puissent paraître, cachent souvent en eux des propriétés, qui, considérées avec attention, posent des problèmes passionants. Plusieurs de nos amis appellent cela: faire des mathématiques 'fines', et les auteurs se sont souvent demandé si dans leur bouche ce terme était d'appréciation ou de mépris...

The book of Kahane and Salem deals with the subject of 'thin sets' which arose out of the attempt to understand the structure of the 'sets of uniqueness' introduced by Cantor.

Since the time of Fourier mathematicians have been interested in the relation between a function $f : \mathbf{T} \to \mathbf{C}$ and its Fourier coefficients

$$\hat{f}(n) = \int_{\mathbf{T}} f(t) \exp(-int)dt.$$

(Here $\mathbf{T} = \mathbf{R}/2\pi\mathbf{Z}$, and, as elsewhere, we hope that the anachronisms forced on us by the nature of a one hour talk will not distort our view too much.) Fourier showed how useful relations of the form

$$\sum_{n=-N}^{n=N} \hat{f}(n) \exp(int) \to f(t) \tag{1}$$

could be and Dirichlet showed rigorously in 1829 [6] that equation 1 was true for all t for a wide range of continuous functions. For the next 45 years it was generally believed that the equation was true for all t for all continuous functions but no proof was found. (In 1873 Du Bois-Reymond [7] constructed a continuous function f such that

$$\sum_{n=-N}^{n=N} \hat{f}(n) \to \infty$$

and so equation 1 fails at $t = 0$.)

In his Habilitatsionschrift [22] of 1854, Riemann adopted the different approach of studying

$$\sum_{n=-\infty}^{n=\infty} a_n \exp(int) = \lim_{N\to\infty} \sum_{n=-N}^{n=N} a_n \exp(int)$$

when it exists. (Note that, if the sum is to converge anywhere, we must have $|a_n|$ bounded.) Apart from the question of convergence, the other natural question to ask is that of uniqueness. If $\sum_{n=-\infty}^{n=\infty} a_n \exp(int)$ and $\sum_{n=-\infty}^{n=\infty} b_n \exp(int)$ converge everywhere to the same sum, is it necessarily true that $a_n = b_n$ for all n? By subtraction we see that this reduces to the question whether, if

$$\sum_{n=-N}^{n=N} a_n \exp(int) \to 0,$$

for all t then, automatically, $a_n = 0$.

Even if we were to assume (which we do not) that the sum $\sum_{n=-\infty}^{n=\infty} a_n \exp(int)$ converges everywhere the resultant function could be very badly behaved. In order to produce a better function Riemann integrates twice term by term to obtain the *formally* twice integrated function

$$F(t) = \frac{a_0}{2}t^2 - \sum_{n=-\infty, n\neq 0}^{n=\infty} \frac{a_n}{n^2} \exp(int) \frac{a_0 t^2}{2} + A + Bt. \tag{2}$$

Since the sequence $|a_n|$ is bounded, the sum $\sum_{n=-\infty,n\neq 0}^{n=\infty} \frac{a_n}{n^2}\exp(int)$ is uniformly absolutely convergent and everything is well behaved. We can not, of course recover our original sum $\sum_{n=-\infty}^{n=\infty} a_n \exp(int)$ by twice differentiating, but Riemann shows that, if $\sum_{n=-\infty}^{n=\infty} a_n \exp(int) \to s$ as $n \to \infty$, then

$$\frac{F(t+h)-2F(t)+F(t-h)}{2h^2} \to s$$

as $h \to 0$. (Thus we have an early example of a summation method.)

Riemann also proves his justly famous localisation principle which states that, provided that $a_n \to 0$, the convergence of $\sum_{n=-\infty}^{n=\infty} a_n \exp(int)$ depends only on the behaviour of $F(x)$ for x close to t. (Combined with Riemann's Lemma which states that, if f is Riemann integrable, then $\hat{f}(n) \to 0$ as $|n| \to \infty$ this shows that the convergence of the Fourier sums $\sum_{n=-N}^{n=N} \hat{f}(n)\exp(int)$ as $N \to \infty$ depends only on the behaviour of $f(x)$ for x close to t.)

The question of uniqueness was tackled by Cantor. In response to his enquiry, Schwartz proved that if $F : [a,b] \to \mathbf{R}$ satisfies

$$\frac{F(t+h)-2F(t)+F(t-h)}{2h^2} \to 0,$$

as $h \to 0$ then F is linear. (The proof is readily extended to higher dimensions, to show that a harmonic function can not have strict local maxima.) We thus obtain the theorem.

Theorem 1 *If $a_n \in \mathbf{C}$ for all $n \in \mathbf{Z}$ and*

$$\sum_{n=-N}^{n=N} n \exp(int) \to 0$$

as $N \to \infty$ for all $t \in \mathbf{T}$, then $a_n = 0$ for all $n \in \mathbf{Z}$.

What can we say if we only know that the sum converges to zero on some subset of the circle? Let us say that E is a *set of uniqueness* if whenever

$$\sum_{n=-N}^{n=N} a_n \exp(int) \to 0$$

as $N \to \infty$ for all $t \in \mathbf{T}\setminus E$, then $a_n = 0$ for all $n \in \mathbf{Z}$. (Thus Theorem 1 says that $\emptyset$ is a set of uniqueness.) Cantor's key observation here is that if $\sum_{n=-N}^{n=N} n\exp(int) \to 0$ as $N \to \infty$ on an interval, then $a_n \to 0$ as $|n| \to \infty$ and Riemann's localising argument gives the following version of Theorem 1.

Theorem 2 *Let $0 \leq b < a \leq 2\pi$. If*

$$\sum_{n=-N}^{n=N} a_n \exp(int) \to 0$$

as $N \to \infty$ for all $t \in [a,b]$, and $F(t)$ is defined as in Equation 2 then F is linear on $[a,b]$.

In a series of papers (written at the begining of the 1870's and reprinted in his Collected Works [4]) Cantor used this to show first that every finite set is a set of uniqueness, then that every set with a finite set of limit points is of uniqueness, then that every set whose set of limit points has only a finite set of limit points is of uniqueness.... We see here the begining of Cantor's theory of ordinals and, indeed, the genesis of set theory and point set topology.

Cantor's methods show that any closed countable set is of uniqueness[1] but it was left to Young [26] to show in 1909 that every countable set is of uniqueness. By this time Lebesgue had introduced his theory of measure and applied it with notable success to the theory of Fourier series (see e.g. [13]). In this context we note that two results mentioned earlier find natural generalisations as the Riemann-Lebesgue Lemma (if $f \in L^1$ then $\hat{f}(n) \to 0$ as $|n| \to \infty$) and the Cantor-Lebesgue Lemma (if $\sum_{n=-N}^{n=N} a_n \exp(int) \to 0$ as $N \to \infty$ on a set of non zero Lebesgue meaure then $a_n \to 0$ as $|n| \to \infty$).

Since much of the thrust of Lebesgue's theory was to suggest that sets of measure zero were irrelevant to the processes of analysis, it was natural to suppose that all sets of Lebesgue measure zero would (like countable sets) turn out to be of uniqueness. It was thus rather a shock when, in 1916, Mensŏv [18] produced a closed set of zero Lebesgue measure which was not of uniqueness. To see how he could have arrived at his discovery we look once more at the function

$$F(t) = \frac{a_0}{2t^2} - \sum_{n=-\infty,n\neq 0}^{n=\infty} \frac{a_n}{n^2} \exp(int) + A + Bt$$

introduced by Riemann. Suppose that $\sum_{n=-\infty,n\neq 0}^{n=\infty} a_n \exp(int)$ converges to zero outside a closed set of Lebesgue measure zero. If $|a_n| \to 0$ then localisation shows that F is linear on each interval (a,b) lying wholly outside E. Thus if we try to differentiate F we obtain a function which is constant on each interval (a,b) lying wholly outside E (but which may not even be defined on E). This reminds us of the famous devil's staircase function H of Cantor which is constant on each interval outside the Cantor middle third set and suggests looking at the Lebesgue-Stieltjes integral of series like

$$c_n = \int_{\mathbb{T}} \exp(-int)dH(t).$$

Mensŏv observed that, if G is a function of bounded variation and

$$b_n = \int_0^{2\pi} \exp(-int)dG(t)$$

then, if G is constant on an interval (a,b), and if $b_n \to 0$ as $|n| \to \infty$, the standard localisation argument shows that

$$\sum_{n=-N}^{n=N} b_n \exp(int) \to 0$$

[1]Cantor never published the full transfinite induction argument but [5] (pp 43–45) shows that he was already on his way to it when his interests shifted from the specific problem of uniqueness to the more general problems of set theory.

as $N \to \infty$. If G is increasing on $[0, 2\pi]$ and non-constant

$$b_0 = \int_0^{2\pi} dG(t) = G(2\pi) - G(0) \neq 0,$$

and it can be shown, in general, that $b_n = 0$ for all n only if G is constant. Thus to show that a closed set E is not of uniqueness we need only produce a non constant function G of bounded variation which is constant on intervals which do not intersect E and which satisfies the condition

$$\int_0^{2\pi} \exp(-int) dG(t) \to 0 \text{ as } n \to 0.$$

This Mensŏv did for a clever modification of the Cantor set (removing only a decreasing part of the intervals remaining at each step, but enough to make the set E of Lebesgue measure zero) and the associated staircase function.

Nowadays we would write Mensŏv's argument as follows.

Theorem 3 *If μ is a non zero measure on $\mathbb{T}$, such that $\hat{\mu}(n) \to 0$ as $|n| \to \infty$, then*

$$\sum_{n=-N}^{n=N} \hat{\mu}(n) \exp(int) \to 0$$

as $N \to \infty$ for all $t \notin$ suppμ. *In particular, no set containing* suppμ *can be of uniqueness.*

(Here measure means Borel measure, suppμ is the (closed) support of μ and $\hat{\mu}(n) = \int_{\mathbb{T}} \exp(-int) d\mu(t)$.) Since the old and the modern versions are mathematically identical ($\mu(a, b] = G(b) - G(a)$) the choice between them must depend on other criteria. The classic summaries of Zygmund [27] published in 1959 and Bari [3] published in 1961 use functions of bounded variation whilst the book of Kahane and Salem [8] published in 1963 uses measures. Since the study of Riemann's F leads more naturally to functions rather than measures we should not be surprised that change took so long in coming. I would speculate that the influence of distribution theory was decisive but that the use of general measures in probability theory and Haar measure in Fourier analysis on groups also played a role. (The corresponding changes in the university first courses in measure theory would also predispose research students in favour of Borel measures and against functions of bounded variation.) As a child of my time, I will, anhistorically, use measures from now on.

It could now be conjectured that no uncountable set is of uniqueness but in 1922 and 1923 Rajchman [21] and Bari [1] and independently constructed uncountable closed sets which were of uniqueness. It now became an open problem to classify closed sets of Lebesgue measure zero into sets of uniqueness and sets of non-uniqueness (which we call *sets of multiplicity*). To see why we need only consider sets of Lebesgue measure zero, recall that if A is a set of positive Lebesgue measure then A has a closed subset B, say, of positive Lebesgue measure. If I_B is the indicator function of B (i.e. $I_B(x) = 1$ when $x \in B$, $I_B(x) = 0$ otherwise) then, writing $d\mu(x) = I_B(x)dx$, the Riemann-Lebesgue Lemma tells us that $\hat{\mu}(n) \to 0$ as $|n| \to \infty$, and Theorem 3 tells us that A is of multiplicity.

The restriction to closed sets is intended mainly to give this essay a reasonably sharp focus and the reader who requires some indications of the work done on non-closed sets will find them in Bari's book [3]. However the reader should observe that localisation results of the type given in Theorem 3 cease to give useful information so that one of our main tools fails for the more general problem. Further, we note that if $f_0, f_1, f_2, \ldots$ are continuous functions on $\mathbf{T}$ then

$$\{t:\ f_N(t)\to 0 \text{ as } N\to\infty\} = \bigcap_{m=1}^{\infty}\bigcup_{n=1}^{\infty}\bigcap_{r=n}^{\infty}\{t:\ |f_r(t)|<m^{-1}\},$$

and so $\{t:\ f_N(t)\to 0 \text{ as } N\to\infty\}$ must be of a rather simple topological type (a $G_{\delta\sigma}$ set). This means that the study of general sets of uniqueness must involve disentangling topological and analytic considerations. (Bari's example [1] of two sets of uniqueness whose union is the whole circle is an example of the kind of thing that can arise for sufficiently wild, in this case, necessarally non-measurable, sets.) Finally, if we can not even classify closed sets, we can hardly expect greater success in the general case.

An early success was recorded by Bari in her 1923 paper [1].

Theorem 4 *The countable union of closed sets of uniqueness is a set of uniqueness.*

Further success was achieved in the study of the symmetric sets of constant ratio ξ given by

$$E_\xi = \{2\pi(1-\xi)\sum_{r=1}^{\infty}\epsilon_r\xi^r:\ \epsilon_r\in\{0,1\}\ [1\le r]\}.$$

(Observe that if $\xi = 1/3$ we recover the classical Cantor middle third set; E_ξ is a middle $1-\xi$ set.) Rajchman's proof in [21] of the existence of uncountable closed sets of uniqueness is based on the concept of an H-set.

Definition 5 *A closed set E is called an H-set if we can find real numbers α and δ with $0\le\delta<2$, and a sequence n_k of integers with $n_k\to\infty$ as $k\to\infty$ such that*

$$|\exp(in_kx)-\exp(i\alpha)|<\delta$$

for all $x\in E$ and all $k\ge 1$.

Thus, for example, Cantor's middle third set $E_{1/3}$ is an H-set (Since $3^kE_{1/3}=E_{1/3}$ and

$$\sup\{|\exp(ix)-1|:\ x\in E\}<2$$

so we can take $\alpha=0$, $n_k=3^k$.) Rajchman proved the following theorem.

Theorem 6 *Every closed H-set is a set of uniqueness.*

In particular, therefore, Cantor's middle third set is of uniqueness and, if F is the associated staircase, Theorem 3 tells us that

$$\int_0^{2\pi}\exp(-int)df(t)\not\to 0 \text{ as } n\to 0).$$

In 1937 Bari [2] carried this much further .

Theorem 7 *If ξ is rational, then E_ξ is of uniqueness if and only if ξ^{-1} is an integer.*

In 1943 Salem [23] extended this as follows.

Definition 8 *Suppose $\theta_1, \theta_2, \ldots, \theta_N$ are the roots of the equation*

$$z^N + \sum_{j=0}^{N-1} a_j z^j = 0,$$

where the coefficients $a_1, a_2, \ldots, a_{N-1}$ are integers. If $|\theta_j| < 1$ for each $1 \le j \le N-1$ but $|\theta_N| > 1$ then θ_N is called a Pisot number.

Theorem 9 *If ξ^{-1} is not a Pisot number, then E_ξ is of multiplicity.*

There are two points of interest here. The first is that the Pisot numbers were first investigated in number theory in connection with problems of uniform distribution. The second is that Salem shows that the most obvious candidate for a measure μ with $\mathrm{supp}\mu \subseteq E_\xi$ and $\hat{\mu}(n) \to 0$ as $|n| \to \infty$ actually works. If F_ξ is the standard staircase function then

$$\int_0^{2\pi} \exp(-int)dF_\xi(t) \to 0 \text{ as } n \to 0.$$

Finally in 1955, by using a generalisation of Rajchman's Theorem 6 due to Piatetski-Shapiro, Salem and Zygmund [24] completed the classification of the symmetric sets of constant ratio by proving the converse of Theorem 9.

Theorem 10 *If ξ^{-1} is a Pisot number, then E_ξ is of uniqueness.*

It can be shown that as ξ^{-1} increases from 0 to 1, the Hausdorff dimension of E_ξ increases continuously from 0 to 1. It can also be shown that the set of Pisot numbers and its complement are dense in $(1, \infty)$. The results of the previous paragraph thus showed that it was unlikely that any metric characterisation of sets of uniqueness could be found. This was confirmed by a theorem obtained by Ivašev-Musatov in 1952 [19]. (I state it without assuming prior knowledge of Hausdorff measure on the part of the reader.)

Theorem 11 *Let $h : [0,1] \to \mathbf{R}$ be any strictly increasing continuous function with $h(0) = 0$. Then we can find a closed set E and a x positive measure μ with the following properties.*

(i) $E \supseteq \mathrm{supp}\mu$, $||\mu|| = 1$ and $\hat{\mu}(n) \to 0$ as $|n| \to \infty$, and so E is of multiplicity.

(ii) Given any $\epsilon > 0$ we can find closed intervals $I_1, I_2, \ldots$ of lengths $|I_1|, |I_2|, \ldots$ such that

$$\bigcup_{r=1}^{\infty} I_r \supseteq E, \quad but \quad \sum_{r=1}^{\infty} h(|I_r|) < \epsilon.$$

Thus, however sharply cusped we make h at zero, in an attempt to make E 'thin' it can remain sufficiently 'thick' to support a measure whose Fourier coefficients die away towards infinity. (There are very interesting relations between the metric thinness of the support of a measure and the *rate* of decay of its Fourier coefficients; the book of Kahane and Salem [8] gives a clear account of the discoveries of Salem and others in this direction. However the *possibility* of decay seems to be a different matter.)

The nature of the results obtained so far gave rise to a strong suspicion which the reader will find explicitly stated in the treatises of both Zygmund ([27], Preface) and Bari ([3], Chapter XIV §23) that the arithmetic structure of a closed set would determine whether it was of uniqueness or not. This hope was dealt a severe blow by Rudin [25] when, in 1960, he constructed a set which was 'arithmetically very thin' but still sufficiently 'thick' to support a measure whose Fourier coefficients died away towards infinity.

Theorem 12 *We can find a closed set $E \subseteq \mathbb{T}$ and a positive measure μ with the following properties.*

(i) $E \supseteq \operatorname{supp}\mu$, $\|\mu\| = 1$ and $\hat{\mu}(n) \to 0$ as $|n| \to \infty$, and so E is of multiplicity.

(ii) If $e_1, e_2, \ldots, e_M$, are distinct points of E then the only solution of

$$\sum_{m=1}^{M} n_m e_m = 0$$

with $n_1, n_2, \ldots, n_M$ integers, is the trival one with $n_1 = n_2 = \ldots = n_M = 0$

There is another interesting point which appears when we inspect the examples of sets of multiplicity given above. They are all obtained using measures. Let us call a closed set E a set of *strong multiplicity* if we can find a non-zero measure μ with support contained in E such that $\hat{\mu}(n) \to 0$ as $|n| \to \infty$. Are all sets of multiplicity of strong multiplicity? This question can be posed rather naturally in terms of distributions. Recall that every distribution T on $\mathbb{T}$ (more formally every $T \in \mathcal{D}'(\mathbb{T})$) has a Fourier series $\hat{T}(n) = (T, \chi_{-n})$ (where $\chi_n(t) = \exp(int)$) which increases no faster than polynomially as $|n| \to \infty$. Further

$$T = \sum_{-\infty}^{\infty} \hat{T}(n)\chi_n,$$

where the convergence is in the distributional sense. Conversely if a_n increases no faster than polynomially as $|n| \to \infty$ then

$$T = \sum_{-\infty}^{\infty} a_n \chi_n,$$

defines a distribution T with $\hat{T}(n) = a_n$. The Riemann localisation principle now gives the following distributional theorem.

Theorem 13 *A closed set E is of multiplicity if and only if there exists a non-zero distribution T with support contained in E and $\hat{T}(n) \to 0$ as $|n| \to \infty$.*

We call a distribution T with $\sup|\hat{T}(n)| < \infty$ a *pseudomeasure* and a distribution T with $\hat{T}(n) \to 0$ as $|n| \to \infty$ a *pseudofunction.*

The problem of the last paragraph can thus be stated as the question 'Does there exist a closed set which supports a non-zero pseudofunction but does not support a non-zero measure which is also a pseudofunction'. Stated in this way it appears to have links with the problem of synthesis.

Definition 14 *A closed set E is called a set of synthesis if the distributional closure of the set of measures with support contained in E is the set of all pseudomeasures with support contained in E.*

It was unknown for a long time whether there were any closed sets which were not of synthesis. However the problem of the existence of the two kinds of sets mentioned were solved seperately and by different methods. In 1954 Piatetski-Shapiro [20] constructed a set of multiplicity which was not of strong multiplicity and in 1959 Malliavin [14] constructed a set which was not of synthesis.

None the less the two problems are linked (for example, in 1962 Malliavin [15] showed that if every closed subset of a closed set E is of synthesis then E is of uniqueness). A very close link is provided by the study of Helson sets.

Definition 15 *A closed set E is called a Helson-K set if, given any $\epsilon > 0$ and any continuous function $f : E \to \mathbb{C}$ with $|f(t)| \leq 1$ for all $t \in E$, we can find $a_r \in \mathbb{C}$ with*

$$\sum_{r=-\infty}^{\infty} a_r \exp(irt) = f(t) \text{ for all } t \in E, \text{ and } \sum_{r=-\infty}^{\infty} |a_r| \leq K + \epsilon.$$

It is easy to check that, if E is a Helson set, then the distributional closure of the set of measures with support contained in E can not contain a non-zero pseudofunction. Thus any Helson set which supports a non-zero pseudofunction is both a set of non-synthesis and a set of multiplicity but not of strong multiplicity. In 1973 [12] I constructed such a set. Almost immediately Kaufman [9] produced a much simpler proof of an improved result.

Theorem 16 *Suppose that S is a non-zero pseudofunction. Then we can find a non-zero pseudofunction T with* $\operatorname{supp}T \subseteq \operatorname{supp}S$ *and* $\operatorname{supp}T$ *a Helson-1 set.*

(In other words any closed set of multiplicity contains a Helson-1 set of mutiplicity. This set is automatically not of strong multiplicity and not of synthesis.)

Remark 1 Narative history and mathematical logic may impose patterns where none exist. When I worked on the construction of a pseudofunction on a Helson set it was as a problem on synthesis and I was, at most, dimly aware of the connection with problems on multiplicity (which graduate students know the full background of the problems they work on?). In retrospect (see the introductory paragraph of [9] and the last two paragraphs of page 239 of [12]) it is clear that Piatetski-Shapiro's 1954 example is very closely related to the later Helson set results. In particular a fairly simple distributional perturbation transforms Piatetski-Shapiro's example into a Helson-K set (with very large K) supporting a non-zero pseudofunction whilst Kaufman explicitly presents his proof as an adaptation of that of Piatetski-Shapiro.

Remark 2 The cynical reader may have remarked, correctly, that Theorem 13 is just a 19th century theorem in modern garb. But if the distributional approach has produced little in the way of general theorems it has, in my view, been invaluable in the construction of the intricate examples above. In his proof of Theorem 16 Kaufman starts with the pseudofunction $S_0 = S$ and constructs inductively a sequence $S_{n+1} = h_n S_n$ obtained by multiplying S_n by a C^∞ function h_n. The distributional limit of the S_n gives the required new pseudofunction T. Of course, the procedure can be followed using only formal trigonometric sums, but it seems unlikely to me that such a construction could be discovered by someone who viewed the objects involved as formal constructs with no real existence. The failure to follow up Piatetski-Shapiro's work sooner gives some evidence for my view

All these negative results suggest that the classification problem for sets of uniqueness is probably impossible and we might expect matters to stop here. However, our story now takes a typically 20th century twist with the discovery that what we vaguely feel to be impossible is, in fact, provably impossible! To get an idea of what is involved, consider first the space $C([0,1])$ of continuous functions on $[0,1]$ with metric the uniform norm

$$d(f,g) = \sup_{t\in[0,1]} |f(t) - g(t)|.$$

It is easy to see that the set C^1 of continuously differentiable functions is neither open nor closed in $C([0,1])$ with this metric and Mazurkiewicz [17] has shown that it is not Borel. (The set of Borel sets is the smallest collection of sets which contains every open set and which is closed under the set operations of countable union, countable intersection and complementation. Borel sets can have much a much more complex structure than a naive student might think, but there are sets whose structure is so complex that they can not be Borel.) It follows that we can not characterise C^1 as a subset of $C([0,1])$ in any reasonable way using only the uniform norm. Around the end of 1983 Solovay (unpublished, for details see [11]) and Kaufman [10] independently showed, using Theorem 16, that, if we put a certain natural metric on the space of closed subsets of $\mathbf{T}$, the collection of closed sets of uniqueness is not Borel. Thus it is impossible to characterise closed sets of uniqueness using the type of arithmetical and metric conditions investigated up to now. (Of course, although the set C^1 is not Borel in $C([0,1])$, most mathematicians feel that they understand it from other points of view. It is thus conceivable that there is some other way of looking at sets of uniqueness which would show them as understandable objects.)

The Solovay Kaufman theorem belongs to the subject of 'Descriptive Set Theory'. The relevant parts of harmonic analysis and descriptive set theory together with a proof of the theorem and many interesting further results that have been discovered in the subsequent spate of activity are collected in an excellent book [11] by Kechris and Louveau.

History studies men and women and their relation to society. It thus includes the study of the development of ideas and of technologies. The history of mathematics qualifies as important under both heads. (Whitehead says that the history of ideas without mathematics would be, not like Hamlet without the prince, but like Hamlet without Ophelia. Like Ophelia, mathematics is charming, essential to the plot, and a little mad.) The historian thus requires no further justification for the study of the history of mathematics—but why should mathematicians support it?

There are at least three possible reasons. The first is to supply anecdotes to lighten and add human interest to our courses. This reason is not to be lightly dismissed. Whatever its failings as history, E.T.Bell's *Men of Mathematics* has been an inspiration to many young mathematicians. However the repetition of amusing and instructive stories hardly qualifies as an academic discipline deserving our strong support.

The second reason is, for me, the strongest. If we forget our predecessors then our successors will forget us. The love of fame is a powerful driving force for many, perhaps for almost all, mathematicians. Without historians to record and celebrate our achievements the prize of fame will be an illusion. For this reason alone we should cherish our historians.

The third reason is the desire to learn from history. We know that some lines of mathematical research prove immensely fruitful whilst others just peter out. Might it not be possible, by studying programs of research which have been successful in the past, to predict those which are likely to be successful in the future? The history of sets of uniqueness shows that, even if history does provide such hints, they are unlikely to be easy to decypher. The problem of sets of uniqueness fails to satisfy several of the standard criteria for a good problem.

1. At no stage during the history of the problem would even the most favourable outcome have produced a theorem useful in other parts of mathematics.

2. The problem fails to generalise naturally to higher dimensions. In two dimensions we must decide whether to consider

$$\lim_{N\to\infty} \sum_{n^2+m^2\leq N} a_{n,m} \exp(inx + imy), \quad \lim_{N\to\infty} \sum_{|n|,|m|\leq N} a_{n,m} \exp(inx + imy),$$

   or some other mode of convergence. Whichever we choose, the key idea of Riemann is essentially one dimensional in character.

3. The problem appears to be devoid of any computational character.

4. There exist clean and simple results on L^2 convergence and uniform density (Féjer's Theorem) which cover our practical needs in most of pure and applied mathematics. Pointwise convergence is just a sideshow.

In spite of these interlinked, but powerful, objections we have seen that the problem formed the seed from which Cantorian set theory grew. The problem (together with other related 'thin set' problems) provided a testing ground for three more major theories—measure theory, distribution theory and descriptive set theory—giving rise to new techniques and speeding general acceptance of the utility of the theories themselves. It promoted interesting (though not earth shattering) work in number theory and provided many of the earliest examples of probabilistic constructions in analysis (both developments being particularly associated with the name of Salem). Finally, to end on a fashionable note, the book of Kahane and Salem [8] from which I took my opening quotation is a veritable bestiary of what would now be called 1 dimensional fractals. (Mandelbrot acknowledges his debt to our subject in several places in his book *The Fractal Geometry of Nature* [16]).

The history of sets of uniqueness shows how hard it is to

> ...look into the seeds of time,
> and say which grain will grow and which will not.

References

[1] N.K. Bari. Sur l'unicité du dévelloppement trigonométrique. *CRASP*, 177:62–118, 1923.

[2] N.K Bari. Sur le rôle des lois diophantiques dans le problème d'unicité du dévelloppement trigonométrique. *Mat Sbornik*, 2 (44):699–722, 1937.

[3] N.K. Bari. *A Treatise on Trigonometric Series.* Fizmatgiz, 1961. In Russian. An English translation was published by Pergamon in 1964.

[4] G. Cantor. *Gesammelte Abhandlungen mathematischen und philosophischen Inhalts.* Springer, 1932.

[5] J.W. Dauben. *Georg Cantor.* Harvard UP, 1979.

[6] P.G.L. Dirichlet. Sur la convergence des séries trigonométriques. *Jour. für Math.*, 4, 1829. Reprinted in his Collected Works.

[7] P. Du Bois-Reymond. *Nachrichten König. Ges. der Wiss. zu Gött.*, pages 571–82, 1873.

[8] J.P. Kahane and R. Salem. *Ensembles parfaits et séries trigonométriques.* Hermann, 1963.

[9] R. Kaufman. M-sets and distributions. *Astérisque*, 5:225–230, 1973.

[10] R. Kaufman. Fourier transforms and discriptive set theory. *Mathematika*, 31:336–339, 1984.

[11] A.S. Kechris and A.Louveau. *Descriptive Set Theory and the Structure of Sets of Uniqueness.* CUP, 1987.

[12] T.W. Körner. A pseudofunction on a helson set i and ii. *Astérisque*, 5:3–224 and 231–239, 1973.

[13] H. Lebesgue. *Leçons sur les séries trigonométriques.* Gauthier-Villars, 1906.

[14] P. Malliavin. Impossibilité de la synthése spectrale sur les groupes abéliens non compacts. *Publ. Math. IHES*, pages 61–68, 1959.

[15] P. Malliavin. Ensembles de résolution spectrale. *Proc. Internat. Congr. Math.*, pages 368–378, 1962.

[16] B. Mandelbrot. *The Fractal Geometry of Nature.* W.H.Freeman, 1977. Extended version of *Les Objects Fractals* (Flammarion 1975).

[17] S. Mazurkiewicz. Über die menge der differenzierbaren functionen. *Fund. Math.*, 27:244–249, 1936.

[18] D.E. Mensǒv. Sur l'unicité du dévelloppement trigonométrique. *CRASP*, 163:433–436, 1916.

[19] Ivašev-Musatov O.S. On the fourier-stieljes coefficients of singular functions. *Dokl. Akad. Nauk. SSSR (N.S.)*, 82:9–11, 1952. In Russian. There is an translation in the Amer. Math. Soc. Translation Series.

[20] Piatetski-Shapiro. Supplement to the work 'on the problems of uniqueness of expansion of a function in trigonometric series'. *Moscov. Gos. Univ. Uč. Zap.*, 165, Mat 7:79–97, 1954. In Russian. There is an translation in the Amer. Math. Soc. Translation Series.

[21] A. Rajchman. Sur l'unicité du dévelloppement trigonométrique. *Fund Math*, 3:286–302, 1922.

[22] G.F.B. Riemann. Habilitatsionschrift. *Abh. der Ges. der Wiss. zu Gött.*, 13:87–132, 1868. Reprinted in his Collected Works.

[23] R. Salem. Sets of uniqueness and sets of multiplicity. *Trans AMS*, 54:218–228, 1943. Only part of this paper is correct. Corrections appear on pages 595–598 of Trans AMS, vol 63 (1948). Or see Salem's Collected Works.

[24] R. Salem and A. Zygmund. Sur un théorème de piatetski-shapiro. *CRASP*, 240:2040–2042, 1954.

[25] Rudin W. Fourier-stieljes transforms of measures on independent sets. *Bull. Amer. Math. Soc.*, 66:199–202, 1960.

[26] W.H. Young. A note on trigonometric series. *Mess. for Maths.*, 38:44–48, 1909.

[27] A. Zygmund. *Trigonometric Series.* CUP, 2nd edition, 1959.