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UPPER BOUNDS FOR A CLASS OF ENERGIES CONTAINING A NON-LOCAL TERM

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Abstract. In this paper we construct upper bounds for families of functionals of the form

$$E_\varepsilon(\phi) := \int_\Omega \Big(\varepsilon|\nabla\phi|^2 + \frac{1}{\varepsilon}W(\phi)\Big)\mathrm{d}x + \frac{1}{\varepsilon}\int_{\mathbb{R}^N}|\nabla \bar{H}_{F(\phi)}|^2\mathrm{d}x$$

where $\Delta\bar{H}_u = \mathrm{div}\,\{\chi_\Omega u\}$. Particular cases of such functionals arise in Micromagnetics. We also use our technique to construct upper bounds for functionals that appear in a variational formulation of the method of vanishing viscosity for conservation laws.

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1. Introduction

Consider the energy functional defined for every $\varepsilon > 0$ by

$$E_\varepsilon(\phi) := \int_\Omega \Big(\varepsilon|\nabla\phi|^2 + \frac{1}{\varepsilon}W(\phi)\Big)\mathrm{d}x + \frac{1}{\varepsilon}\int_{\mathbb{R}^N}|\nabla \bar{H}_{F(\phi)}|^2\mathrm{d}x. \tag{1.1}$$

Here $W : \mathbb{R}^k \to \mathbb{R}$ satisfying $W \geq 0$ and $F : \mathbb{R}^k \to \mathbb{R}^{l\times N}$ are given functions, $\phi : \Omega \subset \mathbb{R}^N \to \mathcal{M} \subset \mathbb{R}^k$ and, given $u : \Omega \to \mathbb{R}^{l\times N}$, $\bar{H}_u : \mathbb{R}^N \to \mathbb{R}^l$ is defined by

$$\begin{cases} \Delta\bar{H}_u = \mathrm{div}\,\{\chi_\Omega u\} & \text{in the sense of distributions in } \mathbb{R}^N,\\ \nabla\bar{H}_u \in L^2(\mathbb{R}^N, \mathbb{R}^{l\times N}), \end{cases} \tag{1.2}$$

where χ_Ω is the characteristic function of Ω. One of the fields where functionals of type (1.1) are relevant is Micromagnetics (see [1,3,10,11] and other). The full 3-dimensional model of ferromagnetic materials deals

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with an energy functional, which, up to a rescaling, has the form

$$E_\varepsilon(m) := \varepsilon \int_\Omega |\nabla m|^2 \mathrm{d}x + \frac{1}{\delta_\varepsilon} \int_\Omega W(m)\mathrm{d}x + \frac{1}{\varepsilon} \int_{\mathbb{R}^3} |\nabla \bar{H}_m|^2 \mathrm{d}x, \tag{1.3}$$

where $\Omega \subset \mathbb{R}^3$ is a bounded domain, $m : \Omega \to S^2$ stands for the magnetization, $\delta_\varepsilon > 0$ is a material parameter and $\bar{H}_m : \mathbb{R}^3 \to \mathbb{R}$ is defined, as before, by

$$\begin{cases} \Delta \bar{H}_m = \operatorname{div}\{\chi_\Omega m\} & \text{in } \mathbb{R}^3, \\ \nabla \bar{H}_m \in L^2(\mathbb{R}^3, \mathbb{R}^3). \end{cases} \tag{1.4}$$

The first term in (1.3) is usually called exchange energy while the second is called the anisotropy energy and the third is called demagnetization energy. One can consider the infinite cylindrical domain $\Omega = G \times \mathbb{R}$ and configurations which don't depend on the last coordinate. These reduce the original model to a 2-dimensional one, where the energy, up to rescaling, has the form

$$E_\varepsilon(m) := \varepsilon \int_G |\nabla m|^2 \mathrm{d}x + \frac{1}{\delta_\varepsilon} \int_G W(m)\mathrm{d}x + \frac{1}{\varepsilon} \int_{\mathbb{R}^2} |\nabla \bar{H}_{m'}|^2 \mathrm{d}x, \tag{1.5}$$

where $G \subset \mathbb{R}^2$ is a bounded domain, $m = (m_1, m_2, m_3) : G \to S^2$ stands for the magnetization, $m' := (m_1, m_2) \in \mathbb{R}^2$ denotes the first two components of m, $\delta_\varepsilon > 0$ and $\bar{H}_{m'} : \mathbb{R}^2 \to \mathbb{R}$ is defined, as before, by

$$\begin{cases} \Delta \bar{H}_{m'} = \operatorname{div}\{\chi_G m'\} & \text{in } \mathbb{R}^2, \\ \nabla \bar{H}_{m'} \in L^2(\mathbb{R}^2, \mathbb{R}^2). \end{cases} \tag{1.6}$$

Note that in the case $\delta_\varepsilon = \varepsilon$ (*i.e.* the anisotropy and the demagnetization energies have the same order as $\varepsilon \to 0$) the energy-functionals in (1.3) and (1.5) are special cases of the energy in (1.1).

The asymptotic behavior of the energies E_ε for $\varepsilon \to 0$ can be understood by studying the the Γ-limit of E_ε. It is well known that if $\psi_0 = \lim_{\varepsilon\to 0} \psi_\varepsilon$, where ψ_ε are minimizers of E_ε then ψ_0 will be a minimizer of the Γ-limit functional. Usually, in order to prove that E_ε Γ-converges to E one has to prove two bounds.

* A lower bound, namely a functional $\underline{E}(\phi)$ such that for every family $\{\phi_\varepsilon\}_{\varepsilon>0}$, satisfying $\phi_\varepsilon \to \phi$ as $\varepsilon \to 0^+$, we have $\underline{\lim}_{\varepsilon\to 0^+} E_\varepsilon(\phi_\varepsilon) \geq \underline{E}(\phi)$.

** An upper bound, namely a functional $\overline{E}(\phi)$ such that for every ϕ there exists a family $\{\psi_\varepsilon\}_{\varepsilon>0}$, satisfying $\psi_\varepsilon \to \phi$ as $\varepsilon \to 0^+$, and $\overline{\lim}_{\varepsilon\to 0^+} E_\varepsilon(\psi_\varepsilon) \leq \overline{E}(\phi)$.

*** If this can be done with $\underline{E}(\phi) = \overline{E}(\phi) := E(\phi)$, then $E(\phi)$ will be the Γ-limit of $E_\varepsilon(\phi)$.

It is clear that if $\underline{E}_1(\phi)$ and $\underline{E}_2(\phi)$ are two lower bounds then $\max\{\underline{E}_1, \underline{E}_2\}(\phi)$ is also a lower bound. Therefore, there exists the sharp (maximal) lower bound which we call Γ-$\underline{\lim}\, E_\varepsilon$. The same holds for upper bounds *i.e.* there exists the sharp (minimal) upper bound which we call Γ-$\overline{\lim}\, E_\varepsilon$. Clearly only the sharp lower and upper bounds can be equal to the Γ-limit.

The treatment of lower and upper bounds is often based on completely different techniques. We focus here on upper bounds and generalize our results in [7,8]. In [8] we constructed the upper bound for the general singular perturbation functional without the non-local term having the form

$$E_\varepsilon(\phi) := \int_\Omega \Big(\varepsilon|\nabla\phi|^2 + \frac{1}{\varepsilon}W(\phi)\Big)\mathrm{d}x,$$

in the case where $\phi : \Omega \to \mathbb{R}^k$ is free (in this case our bound was sharp) and in the case of additional restriction $\phi = \nabla v$ where $v : \Omega \to \mathbb{R}^l$. In [7] we obtained the sharp upper bound for the simplest energy with a non-local term which is a particular case of (1.1). This energy is the so-called Rivière-Serfaty functional (see [10,11] for the motivation and the proof of the lower bound) and has the form

$$E_\varepsilon(m) := \varepsilon \int_G |\nabla m|^2 \mathrm{d}x + \frac{1}{\varepsilon} \int_{\mathbb{R}^2} |\nabla \bar{H}_m|^2 \mathrm{d}x,$$

where $G \subset \mathbb{R}^2$, $m : G \to S^1$ and $\bar{H}_m : \mathbb{R}^2 \to \mathbb{R}$ is defined, as before, by

$$\begin{cases} \Delta \bar{H}_m = \operatorname{div}\{\chi_G m\} & \text{in } \mathbb{R}^2, \\ \nabla \bar{H}_m \in L^2(\mathbb{R}^2, \mathbb{R}^2). \end{cases}$$

In this work, using the technique developed in [6–8], we construct the upper bound as $\varepsilon \downarrow 0$ for the general energy of the form (1.1) under certain conditions on $\mathcal{M}$ for functions $\phi \in BV \cap L^\infty$. This is included in Theorems 3.1 and 3.2. As a corollary of Theorem 3.2 in Section 4 we derive an upper bound for the functionals in (1.3) and (1.5) with $\delta_\varepsilon = \varepsilon$ (see Thm. 4.1, which treats a slightly more general situation). In Proposition 3.3 we show that in the case of scalar-valued functions (*i.e.* in the case $\mathcal{M} = \mathbb{R}$) the functional we get as upper bound is also a lower bound (and consequently the Γ-limit). One can ask whether we get the sharp upper bound in the general case. As it was mentioned in [8], at least in some cases our method does not give the sharp bound. This happens because our method is based on convolutions. It is clear that for function which depends only on one variable, convolution with standard smoothing kernels gives an approximating sequence which also depends on one variable. Although in our method the mollifying kernels are slightly different, for a function which depends only on one variable the convolution still gives asymptotically one-dimensional profiles. As it is known in Micromagnetics there are examples where the optimal profiles are not one-dimensional (see for example [1,4]). In [1] the authors found a functional which is always a lower bound and in particular cases an upper bound as $\varepsilon \to 0$ for the energy in (1.5) with $W(m) = m_3^2$ in the regime $\delta_\varepsilon \ll \varepsilon$. The optimal configurations, they obtain, are in some cases two-dimensional, the so called "cross-tie walls". In this paper we treat the different situation $\delta_\varepsilon \simeq \varepsilon$ and for this situation it is unknown whether one-dimensional interfaces are optimal, in other words, optimality of the upper bound obtained here is not known.

In Section 5, using the technique developed in the previous sections we construct the upper bound for the functional related to the variational study of symmetric Conservation Laws defined for every $\varepsilon > 0$ by

$$I_\varepsilon(u) := \int_0^T \int_{\mathbb{R}^N} \left\{ \varepsilon |\nabla_x u|^2 + \frac{1}{\varepsilon} |\nabla_x V_u|^2 \right\} \mathrm{d}x\mathrm{d}t + \int_{\mathbb{R}^N} |u(x,T)|^2 \mathrm{d}x, \tag{1.7}$$

where V_u is defined by

$$\Delta_x V_u = \partial_t u + \operatorname{div}_x F(u).$$

For the motivation of the study of this functional see [9]. The main result of Section 5 is Theorem 5.1.

2. Preliminaries

Throughout this paper we call domain an open set in $\mathbb{R}^N$. In this section we assume that Ω is a domain in $\mathbb{R}^N$ with Lipschitz boundary. We begin by introducing some notation. For a matrix valued function $F(x) := \{F_{ij}(x)\} : \mathbb{R}^N \to \mathbb{R}^{d\times N}$ we denote by $\operatorname{div} F$ the $\mathbb{R}^d$-valued vector field defined by $\operatorname{div} F := (l_1, \ldots, l_d)$

where $l_i = \sum_{j=1}^{N} \frac{\partial F_{ij}}{\partial x_j}$. For every $\boldsymbol{\nu} \in S^{N-1}$ (the unit sphere in $\mathbb{R}^N$) and $R > 0$ we denote

$$B_R^+(x,\boldsymbol{\nu}) = \{y \in \mathbb{R}^N \,:\, |y-x| < R,\, (y-x)\cdot\boldsymbol{\nu} > 0\}, \tag{2.1}$$

$$B_R^-(x,\boldsymbol{\nu}) = \{y \in \mathbb{R}^N : |y-x| < R,\, (y-x)\cdot\boldsymbol{\nu} < 0\}, \tag{2.2}$$

$$H_+^N(x,\boldsymbol{\nu}) = \{y \in \mathbb{R}^N : (y-x)\cdot\boldsymbol{\nu} > 0\}, \tag{2.3}$$

$$H_-^N(x,\boldsymbol{\nu}) = \{y \in \mathbb{R}^N : (y-x)\cdot\boldsymbol{\nu} < 0\}, \tag{2.4}$$

and

$$H_{\boldsymbol{\nu}} = \{y \in \mathbb{R}^N \,:\, y\cdot\boldsymbol{\nu} = 0\}\cdot \tag{2.5}$$

Definition 2.1. Consider a function $f \in BV(\Omega,\mathbb{R}^m)$ and a point $x \in \Omega$.
(i) We say that x is a point of *approximate continuity* of f if there exists $z \in \mathbb{R}^m$ such that

$$\lim_{\rho\to 0^+} \frac{\int_{B_\rho(x)} |f(y)-z|\,\mathrm{d}y}{\mathcal{L}^N\big(B_\rho(x)\big)} = 0.$$

In this case z is called an *approximate limit* of f at x and we denote z by $\tilde{f}(x)$. The set of points of approximate continuity of f is denoted by G_f.
(ii) We say that x is an *approximate jump point* of f if there exist $a, b \in \mathbb{R}^m$ and $\boldsymbol{\nu} \in S^{N-1}$ such that $a \neq b$ and

$$\lim_{\rho\to 0^+} \frac{\int_{B_\rho^+(x,\boldsymbol{\nu})} |f(y)-a|\,\mathrm{d}y}{\mathcal{L}^N\big(B_\rho(x)\big)} = 0, \quad \lim_{\rho\to 0^+} \frac{\int_{B_\rho^-(x,\boldsymbol{\nu})} |f(y)-b|\,\mathrm{d}y}{\mathcal{L}^N\big(B_\rho(x)\big)} = 0. \tag{2.6}$$

The triple $(a, b, \boldsymbol{\nu})$, uniquely determined by (2.6) up to a permutation of (a, b) and a change of sign of $\boldsymbol{\nu}$, is denoted by $(f^+(x), f^-(x), \boldsymbol{\nu}_f(x))$. We shall call $\boldsymbol{\nu}_f(x)$ the *approximate jump vector* and we shall sometimes write simply $\boldsymbol{\nu}(x)$ if the reference to the function f is clear. The set of approximate jump points is denoted by J_f. A choice of $\boldsymbol{\nu}(x)$ for every $x \in J_f$ (which is unique up to sign) determines an orientation of J_f. At a point of approximate continuity x, we shall use the convention $f^+(x) = f^-(x) = \tilde{f}(x)$.

We refer to [2] for the results on BV-functions that we shall use in the sequel.

Consider a function $\Phi = (\varphi_1, \varphi_2, \ldots, \varphi_d) \in BV(\Omega,\mathbb{R}^d)$. By [2], Proposition 3.21, we may extend Φ to a function $\bar{\Phi} \in BV(\mathbb{R}^N,\mathbb{R}^d)$ such that $\bar{\Phi} = \Phi$ a.e. in Ω and $\|D\bar{\Phi}\|(\partial\Omega) = 0$ (the proof also works in the case of an unbounded domain). From the proof of Proposition 3.21 in [2] it follows that if $\Phi \in BV(\Omega,\mathbb{R}^d) \cap L^\infty(\Omega,\mathbb{R}^d)$ then its extension $\bar{\Phi}$ is also in $BV(\mathbb{R}^N,\mathbb{R}^d)\cap L^\infty(\mathbb{R}^N,\mathbb{R}^d)$. Consider also a matrix valued function $\Xi \in (C^2 \cap Lip \cap L^\infty)(\mathbb{R}^N \times \mathbb{R}^N, \mathbb{R}^{l\times d})$, such that there exists a compact set $K \subset\subset \mathbb{R}^N$ with the property that $\operatorname{supp}\Xi \subset K \times \mathbb{R}^N$. For every $\varepsilon > 0$ define a function $\Psi_\varepsilon(x) : \mathbb{R}^N \to \mathbb{R}^l$ by

$$\Psi_\varepsilon(x) := \frac{1}{\varepsilon^N}\int_{\mathbb{R}^N} \Xi\Big(\frac{y-x}{\varepsilon}, x\Big)\cdot\bar{\Phi}(y)\mathrm{d}y = \int_{\mathbb{R}^N} \Xi(z,x)\cdot\bar{\Phi}(x+\varepsilon z)\,\mathrm{d}z, \quad \forall x \in \mathbb{R}^N. \tag{2.7}$$

We recall the following statement from [8] (Prop. 3.2).

Proposition 2.1. *Let $W \in C^1(\mathbb{R}^l \times \mathbb{R}^q, \mathbb{R})$ be such that*

$$\nabla_a W(a,b) = 0 \quad \textit{whenever } W(a,b) = 0. \tag{2.8}$$

Consider $\Phi \in BV \cap L^\infty(\Omega, \mathbb{R}^d)$ *and* $u \in BV \cap L^\infty(\Omega, \mathbb{R}^q)$ *satisfying*

$$W\left(\left\{\int_{\mathbb{R}^N} \Xi(z,x)\,\mathrm{d}z\right\} \cdot \Phi(x),\; u(x)\right) = 0 \quad \textit{for a.e. } x \in \Omega,$$

where $\Xi \in C^2(\mathbb{R}^N \times \mathbb{R}^N, \mathbb{R}^{l\times d}) \cap Lip \cap L^\infty$, *and such that there exists a compact set* $K \subset\subset \mathbb{R}^N$ *with* $\operatorname{supp}\Xi \subset K \times \mathbb{R}^N$, *as above. Let* $\Psi_\varepsilon(x)$ *be as in* (2.7). *Then,*

$$\lim_{\varepsilon\to 0}\int_\Omega \frac{1}{\varepsilon} W\big(\Psi_\varepsilon(x), u(x)\big)\mathrm{d}x = \int_{J_\Phi}\left\{\int_{-\infty}^0 W\big(\Gamma(t,x), u^+(x)\big)\mathrm{d}t + \int_0^{+\infty} W\big(\Gamma(t,x), u^-(x)\big)\mathrm{d}t\right\}\mathrm{d}\mathcal{H}^{N-1}(x), \quad (2.9)$$

where

$$\Gamma(t,x) = \left(\int_{-\infty}^t P(s,x)\,\mathrm{d}s\right)\cdot \Phi^-(x) + \left(\int_t^{+\infty} P(s,x)\,\mathrm{d}s\right)\cdot \Phi^+(x), \quad (2.10)$$

with

$$P(t,x) = \int_{H_{\boldsymbol{\nu}(x)}} \Xi(t\boldsymbol{\nu}(x) + y, x)\,\mathrm{d}\mathcal{H}^{N-1}(y), \quad (2.11)$$

$\boldsymbol{\nu}(x)$ *is the jump vector of* Φ *and it is assumed that the orientation of* J_u *coincides with the orientation of* J_Φ $\mathcal{H}^{N-1}$ *a.e. on* $J_u \cap J_\Phi$.

Remark 2.1. In Proposition 3.2 in [8] we considered a bounded domain, but the same proof works when we drop this assumption.

Definition 2.2. Given $f \in L^\infty(\mathbb{R}^N, \mathbb{R}^k)$ with compact support we define its Newtonian potential

$$(\Delta^{-1} f)(x) := \begin{cases} \int_{\mathbb{R}^N} \dfrac{1}{2\pi} \ln|x-y|\, f(y)\,\mathrm{d}y & \text{if } N = 2,\\ \int_{\mathbb{R}^N} \dfrac{c_N}{|x-y|^{N-2}}\, f(y)\,\mathrm{d}y & \text{if } N > 2. \end{cases}$$

Here $c_N := \frac{1}{(2-N)\mathcal{H}^{N-1}(S^{N-1})}$. Then it is well known that

$$\int_{\mathbb{R}^N} \left|\nabla^2\big(\Delta^{-1} f\big)(x)\right|^2 \mathrm{d}x = \int_{\mathbb{R}^N} |f(x)|^2\,\mathrm{d}x. \quad (2.12)$$

So, by continuity, we can consider the linear operator $\nabla^2\big(\Delta^{-1}\big) : L^2(\mathbb{R}^N, \mathbb{R}^k) \to L^2(\mathbb{R}^N, \mathbb{R}^{k\times N\times N})$, satisfying (2.12) and $\Delta\big(\Delta^{-1} f\big) = f$.

Definition 2.3. Given a domain $\Omega \subset \mathbb{R}^N$ let $\mathcal{V}^{(d)}(\Omega)$ be the class of all functions $\eta(z,x) \in C^\infty(\mathbb{R}^N \times \mathbb{R}^N, \mathbb{R}^{d\times d}) \cap Lip \cap L^\infty$, such that there exists a compact set $K \subset\subset \mathbb{R}^N$, with the property that $\operatorname{supp}\eta \subset K \times \mathbb{R}^N$ and $\operatorname{supp}\nabla_x \eta(z,x) \subset K \times K$ and such that

$$\int_{\mathbb{R}^N} \eta(z,x)\,\mathrm{d}z = I \quad \forall x \in \Omega. \quad (2.13)$$

Here I is the identity matrix. We also denote $\mathcal{V} := \mathcal{V}^{(1)}$.

Let $\mathcal{U}^{(l,d)}(\Omega)$ be the class of all functions $l(z,x) \in C_c^\infty(\mathbb{R}^N \times \Omega, \mathbb{R}^{l\times d})$ such that

$$\int_{\mathbb{R}^N} l(z,x)\,\mathrm{d}z = O \quad \forall x \in \mathbb{R}^N. \quad (2.14)$$

Here O is the null matrix. We also denote $\mathcal{U} := \mathcal{U}^{(1,1)}$.

We will write $\mathcal{V}^{(d)}$ or $\mathcal{U}^{(l,d)}$ if the reference to the domain is clear.

In [8] (Lem. 5.1) we proved the following statement. This approximation result generalize Claim 3 of Lemma 3.4 from [6] and was an essential tool in the optimizing the upper bound in [8].

Lemma 2.1. *Let μ be a positive finite Borel measure on Ω and let $\boldsymbol{\nu}_0(x):\Omega\to\mathbb{R}^N$ be a Borel measurable function with $|\boldsymbol{\nu}_0|=1$. Let $\mathcal{W}_1^{(d)}$ denote the set of the functions $p(t,x):\mathbb{R}\times\Omega\to\mathbb{R}^{d\times d}$ satisfying the following conditions:*

(i) p is Borel measurable and bounded;
(ii) there exists $M>0$ such that $p(t,x)=0$ for every $|t|>M$ and any $x\in\Omega$;
(iii) $\int_{\mathbb{R}}p(t,x)\,\mathrm{d}t=I,\ \forall x\in\Omega$.

Then for every $p(t,x)\in\mathcal{W}_1^{(d)}$ there exists a sequence of functions $\{\eta_n\}\subset\mathcal{V}^{(d)}$ (see Def. 2.3), such that the sequence of functions $\{p_n(t,x)\}$ defined on $\mathbb{R}\times\Omega$ by

$$p_n(t,x)=\int_{H_{\boldsymbol{\nu}_0(x)}}\eta_n(t\boldsymbol{\nu}_0(x)+y,x)\mathrm{d}\mathcal{H}^{N-1}(y),$$

has the following properties:

(i) there exists C_0 such that $|p_n(t,x)|\le C_0$ for every n, every $x\in\Omega$ and every $t\in\mathbb{R}$;
(ii) there exists $M>0$ such that $p_n(t,x)=0$ for every $|t|>M$, every $x\in\Omega$ and all n;
(iii) $\lim_{n\to\infty}\int_\Omega\int_{\mathbb{R}}|p_n(t,x)-p(t,x)|\,\mathrm{d}t\,\mathrm{d}\mu(x)=0$.

By the same method we can prove the following approximation result.

Lemma 2.2. *Let μ be a positive finite Borel measure on Ω and let $\boldsymbol{\nu}_0(x):\Omega\to\mathbb{R}^N$ be a Borel measurable function with $|\boldsymbol{\nu}_0|=1$. Let $\mathcal{W}_0^{(l,d)}$ denote the set of the functions $q(t,x):\mathbb{R}\times\Omega\to\mathbb{R}^{l\times d}$ satisfying the following conditions:*

(i) q is Borel measurable and bounded;
(ii) there exists $M>0$ such that $q(t,x)=0$ for every $|t|>M$ and every $x\in\Omega$;
(iii) $\int_{\mathbb{R}}q(t,x)\,\mathrm{d}t=0,\ \forall x\in\Omega$.

Then for every $q(t,x)\in\mathcal{W}_0^{(l,d)}$ there exists a sequence of functions $\{l_n\}\subset\mathcal{U}^{(l,d)}$ (see Def. 2.3), such that the sequence of functions $\{q_n(t,x)\}$ defined on $\mathbb{R}\times\Omega$ by

$$q_n(t,x)=\int_{H_{\boldsymbol{\nu}_0(x)}}l_n(t\boldsymbol{\nu}_0(x)+y,x)\mathrm{d}\mathcal{H}^{N-1}(y),$$

has the following properties:

(i) there exists C_0 such that $|q_n(t,x)|\le C_0$ for every n, every $x\in\Omega$ and every $t\in\mathbb{R}$;
(ii) there exists $M>0$ such that $q_n(t,x)=0$ for every $|t|>M$, every $x\in\Omega$ and all n;
(iii) $\lim_{n\to\infty}\int_\Omega\int_{\mathbb{R}}|q_n(t,x)-q(t,x)|\,\mathrm{d}t\,\mathrm{d}\mu(x)=0$.

Let $\varphi\in BV(\Omega,\mathbb{R}^d)$ and $\eta\in\mathcal{V}^{(d)}(\Omega)$. As before, by [2], Proposition 3.21, we extend φ to a function $\bar\varphi\in BV(\mathbb{R}^N,\mathbb{R}^d)$ such that $\bar\varphi=\varphi$ a.e. in Ω and $\|D\bar\varphi\|(\partial\Omega)=0$ (again, if φ is bounded, then the extension may be chosen bounded). For every $\varepsilon>0$ define a function $\psi_\varepsilon\in C^1(\mathbb{R}^N,\mathbb{R}^d)$ by

$$\psi_\varepsilon(x):=\frac{1}{\varepsilon^N}\int_{\mathbb{R}^N}\eta\Big(\frac{y-x}{\varepsilon},x\Big)\cdot\bar\varphi(y)\,\mathrm{d}y=\int_{\mathbb{R}^N}\eta(z,x)\cdot\bar\varphi(x+\varepsilon z)\,\mathrm{d}z. \tag{2.15}$$

Then, by Lemma 3.1 in [8], we have

$$\int_\Omega\big|\psi_\varepsilon(x)-\varphi(x)\big|\,\mathrm{d}x=O(\varepsilon)\qquad\text{as }\varepsilon\to0. \tag{2.16}$$

Due to [8] (Thm. 4.1), we have the following statement.

Theorem 2.1. *Let $\Omega \subset \mathbb{R}^N$ be a Lipschitz domain and let $H \in C^1(\mathbb{R}^{d\times N}\times\mathbb{R}^d\times\mathbb{R}^q,\mathbb{R})$ be such that $H(a,b,c) \geq 0$. Consider $u \in BV(\Omega,\mathbb{R}^q) \cap L^\infty$ and $\varphi \in BV(\Omega,\mathbb{R}^d) \cap L^\infty$ satisfying*

$$H\big(O,\,\varphi(x),\,u(x)\big) = 0 \quad \text{for a.e. } x \in \Omega.$$

For any $\eta \in \mathcal{V}^{(d)}$, let ψ_ε be defined by (2.15). Then,

$$\lim_{\varepsilon\to 0}\int_\Omega \frac{1}{\varepsilon}H\big(\varepsilon\nabla\psi_\varepsilon,\,\psi_\varepsilon,\,u\big)\,\mathrm{d}x = \int_{J_\varphi}\Bigg\{\int_{-\infty}^0 H\Big(p(t,x)\cdot\big(\varphi^+(x)-\varphi^-(x)\big)\otimes\boldsymbol{\nu}(x),\gamma(t,x)\;,u^+(x)\Big)\mathrm{d}t \\ + \int_0^\infty H\Big(p(t,x)\cdot\big(\varphi^+(x)-\varphi^-(x)\big)\otimes\boldsymbol{\nu}(x),\gamma(t,x)\;,u^-(x)\Big)\mathrm{d}t\Bigg\}\mathrm{d}\mathcal{H}^{N-1}(x), \quad (2.17)$$

where

$$\gamma(t,x) = \left(\int_{-\infty}^t p(s,x)\,\mathrm{d}s\right)\cdot\varphi^-(x) + \left(\int_t^\infty p(s,x)\,\mathrm{d}s\right)\cdot\varphi^+(x), \quad (2.18)$$

with

$$p(t,x) = \int_{H_{\boldsymbol{\nu}(x)}} \eta(t\boldsymbol{\nu}(x)+y,x)\,\mathrm{d}\mathcal{H}^{N-1}(y) \quad (2.19)$$

and it is assumed that the orientation of J_u coincides with the orientation of J_φ $\mathcal{H}^{N-1}$ a.e. on $J_u \cap J_\varphi$.

Remark 2.2. Again, in Theorem 4.1 in [8] we considered a bounded domain, but the same proof works for the general case.

3. First estimates

Throughout this section we assume that Ω is a domain in $\mathbb{R}^N$ with Lipschitz boundary.

Let $l \in \mathcal{U}(\Omega)$ (see Def. 2.3). Consider $r(z,x) := \Delta_z^{-1}l(z,x)$. Then $r \in C^2(\mathbb{R}^N\times\mathbb{R}^N,\mathbb{R})$ with $\operatorname{supp} r \subset \mathbb{R}^N \times K$, where $K \subset\subset \Omega$. Moreover, since $\int_{\mathbb{R}^N} l(z,x)\mathrm{d}z = 0$, for every $k = 0,1,2\ldots$ we have the estimates

$$\begin{cases} |\nabla_x^k r(z,x)| \leq \dfrac{C_k}{|z|^{N-1}+1}, \\ \big|\nabla_x^k\big(\nabla_z r(z,x)\big)\big| \leq \dfrac{C_k}{|z|^N+1}, \\ \big|\nabla_x^k\big(\nabla_z^2 r(z,x)\big)\big| \leq \dfrac{C_k}{|z|^{N+1}+1}, \end{cases} \quad (3.1)$$

where $C_k > 0$ does not depend on z and x.

The following lemma can be proved almost by the same method as Lemmas 3.1 and 3.2 in [7].

Lemma 3.1. *Let $\varphi = (\varphi_1,\varphi_2) \in BV(\Omega,\mathbb{R}^2) \cap L^\infty$ and $l_1, l_2 \in \mathcal{U}(\Omega)$. For every $k = 1,2$ and every $\varepsilon > 0$ consider the function $\varphi_{k,\varepsilon} \in C^1(\mathbb{R}^N,\mathbb{R})$ by*

$$\varphi_{k,\varepsilon}(x) := \frac{1}{\varepsilon^N}\int_{\mathbb{R}^N} l_k\Big(\frac{y-x}{\varepsilon},x\Big)\bar{\varphi}_k(y)\,\mathrm{d}y = \int_{\mathbb{R}^N} l_k(z,x)\bar{\varphi}_k(x+\varepsilon z)\,\mathrm{d}z, \quad (3.2)$$

where $\bar{\varphi}_k$ *is the extension of* φ_k *to* $\mathbb{R}^N$ *by* 0*. Then for every* $1 \le i, j \le N$ *we have*

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^N}\frac{1}{\varepsilon}\nabla\Big(\frac{\partial}{\partial x_i}(\Delta^{-1}\varphi_{1,\varepsilon})\Big)\cdot\nabla\Big(\frac{\partial}{\partial x_j}(\Delta^{-1}\varphi_{2,\varepsilon})\Big)\mathrm{d}x$$
$$= \int_{J_\varphi}(\varphi_1^+-\varphi_1^-)(\varphi_2^+-\varphi_2^-)\Big\{\int_{-\infty}^{+\infty}\Big(\nu_i\int_t^{+\infty}q_1(s,\cdot)\mathrm{d}s\Big)\Big(\nu_j\int_t^{+\infty}q_2(s,\cdot)\mathrm{d}s\Big)\mathrm{d}t\Big\}\mathrm{d}\mathcal{H}^{N-1}, \quad (3.3)$$

where

$$q_k(t,x) = \int_{H_{\boldsymbol{\nu}(x)}} l_k(t\boldsymbol{\nu}(x)+y,x)\,\mathrm{d}\mathcal{H}^{N-1}(y), \quad (3.4)$$

and $\boldsymbol{\nu}(x) = \big(\nu_1(x),\ldots,\nu_N(x)\big)$ *is the jump vector of* φ*.*

As a corollary of Lemma 3.1 we have the following lemma.

Lemma 3.2. *Let* $\varphi \in BV(\Omega,\mathbb{R}^d)\cap L^\infty$ *and* $l \in \mathcal{U}^{(N,d)}$ *(see Def. 2.3). For every* $\varepsilon > 0$ *and every* $x \in \mathbb{R}^N$ *consider the function* $\psi_\varepsilon \in C^1(\mathbb{R}^N,\mathbb{R}^N)$ *defined by*

$$\psi_\varepsilon(x) := \frac{1}{\varepsilon^N}\int_{\mathbb{R}^N} l\Big(\frac{y-x}{\varepsilon},x\Big)\cdot\bar{\varphi}(y)\,\mathrm{d}y = \int_{\mathbb{R}^N} l(z,x)\cdot\bar{\varphi}(x+\varepsilon z)\,\mathrm{d}z,$$

where $\bar{\varphi}$ *is some bounded* BV *extension of* φ *to* $\mathbb{R}^N$*. Then we have*

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^N}\frac{1}{\varepsilon}\Big|\nabla\big(\mathrm{div}\,(\Delta^{-1}\psi_\varepsilon)\big)\Big|^2\mathrm{d}x$$
$$= \int_{J_\varphi}\left(\int_{-\infty}^{+\infty}\left|\int_t^{+\infty}\big\{q(s,x)\cdot\big(\varphi^+(x)-\varphi^-(x)\big)\big\}\cdot\boldsymbol{\nu}(x)\,\mathrm{d}s\right|^2\mathrm{d}t\right)\mathrm{d}\mathcal{H}^{N-1}(x), \quad (3.5)$$

where

$$q(t,x) = \int_{H_{\boldsymbol{\nu}(x)}} l(t\boldsymbol{\nu}(x)+y,x)\,\mathrm{d}\mathcal{H}^{N-1}(y), \quad (3.6)$$

and $\boldsymbol{\nu}(x)$ *is the jump vector of* φ*.*

Next let $\varphi \in BV(\Omega,\mathbb{R}^d)\cap L^\infty$. As before, we may extend φ to a function $\bar{\varphi}\in BV(\mathbb{R}^N,\mathbb{R}^d)\cap L^\infty$ satisfying $\bar{\varphi}=\varphi$ a.e. in Ω and $\|D\bar{\varphi}\|(\partial\Omega)=0$. If φ is bounded then we consider its extension bounded too. Consider $\eta\in\mathcal{V}^{(d)}$. For any $\varepsilon>0$ define a function $\psi_\varepsilon(x):\mathbb{R}^N\to\mathbb{R}$ by

$$\psi_\varepsilon(x) := \frac{1}{\varepsilon^N}\int_{\mathbb{R}^N}\eta\Big(\frac{y-x}{\varepsilon},x\Big)\cdot\bar{\varphi}(y)\mathrm{d}y = \int_{\mathbb{R}^N}\eta(z,x)\cdot\bar{\varphi}(x+\varepsilon z)\,\mathrm{d}z, \quad \forall x\in\mathbb{R}^N. \quad (3.7)$$

Proposition 3.1. *Let* $F\in C^1(\mathbb{R}^d,\mathbb{R}^N)$ *and let* $\varphi\in BV(\Omega,\mathbb{R}^d)\cap L^\infty$ *and* $u := F(\varphi)$*, satisfying* $\mathrm{div}\, u = 0$ *in* Ω *as a distribution and* $u\cdot\boldsymbol{n}=0$ *on* $\partial\Omega$ *(*$\boldsymbol{n}$ *is the unit normal to* $\partial\Omega$*). Consider* $\psi_\varepsilon(x)$ *defined by* (3.7)*. Then,*

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^N}\frac{1}{\varepsilon}\Big|\nabla\Big(\mathrm{div}\,\Big\{\Delta^{-1}\big(\chi_\Omega F(\psi_\varepsilon)\big)\Big\}\Big)\Big|^2\mathrm{d}x = \mathcal{I}_\eta(\varphi)$$
$$:= \int_{J_\varphi}\left(\int_{-\infty}^{+\infty}\Big|\Big(F\big(\gamma(t,x)\big)-F\big(\varphi^-(x)\big)\Big)\cdot\boldsymbol{\nu}(x)\Big|^2\mathrm{d}t\right)\mathrm{d}\mathcal{H}^{N-1}(x) \quad (3.8)$$

where

$$\gamma(t,x) = \Big(\int_{-\infty}^{t} p(s,x)\mathrm{d}s\Big)\cdot\varphi^-(x) + \Big(\int_t^{+\infty} p(s,x)\,\mathrm{d}s\Big)\cdot\varphi^+(x), \quad (3.9)$$

with

$$p(t,x)=\int_{H_{\boldsymbol{\nu}(x)}}\eta(t\boldsymbol{\nu}(x)+y,x)\,\mathrm{d}\mathcal{H}^{N-1}(y),\tag{3.10}$$

and χ_Ω *is the characteristic function of* Ω.

Proof. Without loss of generality we may assume that $F(0)=0$. Since $(u^+-u^-)\cdot\boldsymbol{\nu}=0$, the r.h.s. in (3.8) does not depend on the orientation of J_φ and therefore we may assume that $\boldsymbol{\nu}(x)$ is Borel measurable.

Together with $\eta\in\mathcal{V}^{(d)}$ we consider a second kernel $\bar\eta\in\mathcal{V}^{(N)}$ and set $\bar u:=F(\bar\varphi)$. Then $\bar u\in BV(\mathbb{R}^N,\mathbb{R}^N)\cap L^\infty$, satisfying $\bar u=u$ a.e. in Ω and, by Volpert's chain rule, $\|D\bar u\|(\partial\Omega)=0$. Then consider

$$\bar p(t,x)=\int_{H_{\boldsymbol{\nu}(x)}}\bar\eta(t\boldsymbol{\nu}(x)+y,x)\,\mathrm{d}\mathcal{H}^{N-1}(y).\tag{3.11}$$

For any $\varepsilon>0$ define a function $u_\varepsilon(x):\mathbb{R}^N\to\mathbb{R}^N$ by

$$u_\varepsilon(x):=\frac{1}{\varepsilon^N}\int_{\mathbb{R}^N}\bar\eta\Big(\frac{y-x}{\varepsilon},x\Big)\cdot\bar u(y)\,\mathrm{d}y=\int_{\mathbb{R}^N}\bar\eta(z,x)\cdot F\big(\bar\varphi(x+\varepsilon z)\big)\,\mathrm{d}z,\quad\forall x\in\mathbb{R}^N.\tag{3.12}$$

Denote $\big(\varphi_1(x),\ldots,\varphi_d(x)\big):=\varphi(x)$. Define

$$D_1:=\{\,x\in J_\varphi:\ |\varphi_1^+(x)-\varphi_1^-(x)|\geq|\varphi_j^+(x)-\varphi_j^-(x)|\ \ \forall j\,\}.$$

Then by induction define

$$D_k:=\{\,x\in J_\varphi\setminus\cup_{j=1}^{k-1}D_j:\ |\varphi_k^+(x)-\varphi_k^-(x)|\geq|\varphi_j^+(x)-\varphi_j^-(x)|\ \ \forall j\,\}\quad\text{for every }1<k\leq d.$$

Then $J_\varphi=\bigcup_{j=1}^d D_j$, D_j are Borel measurable and disjoint. Moreover, $|\varphi_j^+(x)-\varphi_j^-(x)|\geq\frac{1}{\sqrt d}|\varphi^+(x)-\varphi^-(x)|$ for every $x\in D_j$. Next define $Q:=\big(Q_1,\ldots,Q_N\big):\mathbb{R}\times J_\varphi\to\mathbb{R}^N$ by

$$Q(t,x):=F\big(\gamma(t,x)\big)-\Gamma(t,x),\tag{3.13}$$

where $\gamma(t,x)$ is defined by (3.9) and $\Gamma(t,x)$ is given by

$$\Gamma(t,x):=\Big(\int_{-\infty}^t\bar p(s,x)\mathrm{d}s\Big)\cdot F\big(\varphi^-(x)\big)+\Big(\int_t^{+\infty}\bar p(s,x)\,\mathrm{d}s\Big)\cdot F\big(\varphi^+(x)\big).\tag{3.14}$$

Then define $S:=\{S_{ij}\}(1\leq i\leq N,\,1\leq j\leq d):\mathbb{R}\times J_\varphi\to\mathbb{R}^{N\times d}$ by

$$S_{ij}(t,x)=\begin{cases}\dfrac{Q_i(t,x)}{\varphi_k^+(x)-\varphi_k^-(x)} & \text{if }\ j=k\\ 0 & \text{if }\ j\neq k\end{cases}\qquad\text{for every }x\in D_k\ (1\leq k\leq d).\tag{3.15}$$

Next define $q:\mathbb{R}\times\Omega\to\mathbb{R}^{N\times d}$ by

$$q(t,x)=\begin{cases}-\dfrac{\mathrm{d}S(t,x)}{\mathrm{d}t} & x\in J_\varphi,\\ 0 & x\in\Omega\setminus J_\varphi.\end{cases}\tag{3.16}$$

Then $q(t,x)$ is Borel measurable, q is bounded on $\mathbb{R}\times\Omega$, there exists $M>0$ such that $\operatorname{supp}q\subset[-M,M]\times\Omega$ and $\int_{\mathbb{R}}q(t,x)\,\mathrm{d}t=0$, $\forall x\in\Omega$. Moreover

$$\Big(\int_t^{+\infty}q(s,x)\,\mathrm{d}s\Big)\cdot\big(\varphi^+(x)-\varphi^-(x)\big)=Q(t,x)=F\big(\gamma(t,x)\big)-\Gamma(t,x).\tag{3.17}$$

Next fix any Borel measurable vector field $\boldsymbol{\nu}_0(x) : \Omega \to S^{N-1}$ such that $\boldsymbol{\nu}_0(x) = \boldsymbol{\nu}(x)$ for any $x \in J_\varphi$. Then by Lemma 2.2, there exists a sequence of functions $l_n \in \mathcal{U}^{(N,d)}$ (see Def. 2.3), such that the sequence of functions $\{q_n\}$ defined on $\mathbb{R} \times \Omega$ by

$$q_n(t,x) = \int_{H_{\boldsymbol{\nu}_0(x)}} l_n(t\boldsymbol{\nu}_0(x) + y, x) \mathrm{d}\mathcal{H}^{N-1}(y),$$

has the following properties:

$$\text{there exists } C_0 \text{ such that } \|q_n\|_{L^\infty} \le C_0 \text{ for all } n, \tag{3.18}$$

$$\text{there exists } M > 0 \text{ such that } q_n(t,x) = 0 \text{ for } |t| > M, \text{ and every } x \in \Omega, \tag{3.19}$$

$$\lim_{n\to\infty} \int_\Omega \int_\mathbb{R} |q_n(t,x) - q(t,x)| \,\mathrm{d}t \,\mathrm{d}\|D\varphi\|(x) = 0. \tag{3.20}$$

In particular,

$$\lim_{n\to\infty} \int_{J_\varphi} \int_\mathbb{R} \left|\varphi^+(x) - \varphi^-(x)\right| \cdot |q_n(t,x) - q(t,x)| \,\mathrm{d}t \,\mathrm{d}\mathcal{H}^{N-1}(x) = 0. \tag{3.21}$$

For every positive integer n and for every $\varepsilon > 0$ consider the function $\varphi_{n,\varepsilon} \in C^1(\mathbb{R}^N, \mathbb{R}^N)$ given by

$$\varphi_{n,\varepsilon}(x) := \frac{1}{\varepsilon^N} \int_{\mathbb{R}^N} l_n\Big(\frac{y-x}{\varepsilon}, x\Big) \cdot \bar{\varphi}(y) \,\mathrm{d}y = \int_{\mathbb{R}^N} l_n(z,x) \cdot \bar{\varphi}(x + \varepsilon z) \,\mathrm{d}z. \tag{3.22}$$

We will use the following inequality, valid for any $f(x), g(x), \lambda(x) \in L^2(\mathbb{R}^N, \mathbb{R}^p)$,

$$\left| \int_{\mathbb{R}^N} |f(x)|^2 \mathrm{d}x - \int_{\mathbb{R}^N} |g(x)|^2 \mathrm{d}x \right| \le \big(\|f - g - \lambda\|_{L^2} + \|\lambda\|_{L^2}\big) \cdot \left(2\int_{\mathbb{R}^N} |f(x)|^2 \mathrm{d}x + 2\int_{\mathbb{R}^N} |g(x)|^2 \mathrm{d}x\right)^{1/2}. \tag{3.23}$$

Therefore, since $\varphi_{n,\varepsilon}(x) = 0$ for $x \notin \Omega$ and since $\operatorname{div}(\chi_\Omega \bar{u}) = 0$ as a distribution, we obtain,

$$\begin{aligned}
&\left| \int_{\mathbb{R}^N} \left|\nabla \operatorname{div} \Delta^{-1}\big(\chi_\Omega(x) F(\psi_\varepsilon(x))\big)\right|^2 \mathrm{d}x - \int_{\mathbb{R}^N} \left|\nabla \operatorname{div} \Delta^{-1}\big(\varphi_{n,\varepsilon}(x)\big)\right|^2 \mathrm{d}x \right| \\
&\qquad \le 2\bigg(\Big\|\nabla \operatorname{div} \Delta^{-1}\Big(\chi_\Omega\big(F(\psi_\varepsilon) - \varphi_{n,\varepsilon} - u_\varepsilon\big)\Big)\Big\|_{L^2} + \big\|\nabla \operatorname{div} \Delta^{-1}(\chi_\Omega u_\varepsilon)\big\|_{L^2} \bigg) \\
&\qquad\quad \times \left(\int_{\mathbb{R}^N} \left|\nabla \operatorname{div} \Delta^{-1}\big(\chi_\Omega F(\psi_\varepsilon)\big)\right|^2 \mathrm{d}x + \int_{\mathbb{R}^N} \left|\nabla \operatorname{div} \Delta^{-1}\big(\varphi_{n,\varepsilon}\big)\right|^2 \mathrm{d}x \right)^{1/2} \\
&\qquad = 2\bigg(\Big\|\nabla \operatorname{div} \Delta^{-1}\Big(\chi_\Omega\big(F(\psi_\varepsilon) - \varphi_{n,\varepsilon} - u_\varepsilon\big)\Big)\Big\|_{L^2} + \big\|\nabla \operatorname{div} \Delta^{-1}\big(\chi_\Omega(u_\varepsilon - \bar{u})\big)\big\|_{L^2} \bigg) \\
&\qquad\quad \times \left(\int_{\mathbb{R}^N} \left|\nabla \operatorname{div} \Delta^{-1}\Big(\chi_\Omega\big(F(\psi_\varepsilon) - F(\bar{\varphi})\big)\Big)\right|^2 \mathrm{d}x + \int_{\mathbb{R}^N} \left|\nabla \operatorname{div} \Delta^{-1}\big(\varphi_{n,\varepsilon}\big)\right|^2 \mathrm{d}x \right)^{1/2}.
\end{aligned} \tag{3.24}$$

But since for every $f \in L^2(\mathbb{R}^N, \mathbb{R}^N)$ we have

$$\int_{\mathbb{R}^N} \left|\nabla\left(\operatorname{div} \Delta^{-1} f\right)\right|^2 \mathrm{d}x \le K_0^2 \int_{\mathbb{R}^N} \left|\nabla^2 \Delta^{-1} f\right|^2 \mathrm{d}x = K_0^2 \int_{\mathbb{R}^N} |f|^2 \mathrm{d}x,$$

where $K_0>0$ is some constant, by (3.24), we obtain

$$\left|\frac{1}{\varepsilon}\int_{\mathbb{R}^N}\left|\nabla\mathrm{div}\,\Delta^{-1}\big(\chi_\Omega F(\psi_\varepsilon)\big)\right|^2\mathrm{d}x-\frac{1}{\varepsilon}\int_{\mathbb{R}^N}\left|\nabla\mathrm{div}\,\Delta^{-1}\big(\varphi_{n,\varepsilon}\big)\right|^2\mathrm{d}x\right| \\ \leq 2K_0\left\{\left(\frac{1}{\varepsilon}\int_\Omega\left|F(\psi_\varepsilon)-\varphi_{n,\varepsilon}-u_\varepsilon\right|^2\mathrm{d}x\right)^{1/2}+\left(\frac{1}{\varepsilon}\int_\Omega|u_\varepsilon-u|^2\mathrm{d}x\right)^{1/2}\right\} \\ \times\left(\frac{1}{\varepsilon}\int_\Omega\left|F(\psi_\varepsilon)-F(\varphi)\right|^2\mathrm{d}x+\frac{1}{\varepsilon}\int_{\mathbb{R}^N}\left|\nabla\left(\mathrm{div}\,\Delta^{-1}\big(\varphi_{n,\varepsilon}\big)\right)\right|^2\mathrm{d}x\right)^{1/2}. \quad (3.25)$$

Therefore,

$$\left|\frac{1}{\varepsilon}\int_{\mathbb{R}^N}\left|\nabla\mathrm{div}\,\Delta^{-1}(\chi_\Omega F(\psi_\varepsilon))\right|^2\mathrm{d}x-\mathcal{I}_\eta(\varphi)\right|\leq\left|\mathcal{I}_\eta(\varphi)-\frac{1}{\varepsilon}\int_{\mathbb{R}^N}\left|\nabla\mathrm{div}\,\Delta^{-1}(\varphi_{n,\varepsilon})\right|^2\mathrm{d}x\right| \\ +2K_0\left\{\left(\frac{1}{\varepsilon}\int_\Omega\left|F(\psi_\varepsilon)-\varphi_{n,\varepsilon}-u_\varepsilon\right|^2\mathrm{d}x\right)^{1/2}+\left(\frac{1}{\varepsilon}\int_\Omega|u_\varepsilon-u|^2\mathrm{d}x\right)^{1/2}\right\} \\ \times\left(\frac{1}{\varepsilon}\int_\Omega\left|F(\psi_\varepsilon)-F(\varphi)\right|^2\mathrm{d}x+\frac{1}{\varepsilon}\int_{\mathbb{R}^N}\left|\nabla\left(\mathrm{div}\,\Delta^{-1}\big(\varphi_{n,\varepsilon}\big)\right)\right|^2\mathrm{d}x\right)^{1/2}, \quad (3.26)$$

where $\mathcal{I}_\eta$ is defined by (3.8). By Proposition 2.1, we obtain,

$$\lim_{\varepsilon\to0}\frac{1}{\varepsilon}\int_\Omega\left|F(\psi_\varepsilon)-\varphi_{n,\varepsilon}-u_\varepsilon\right|^2\mathrm{d}x=D_n \\ :=\int_{J_\varphi}\left\{\int_{-\infty}^{+\infty}\left|F\big(\gamma(t,x)\big)-\Big(\int_t^{+\infty}q_n(s,x)\mathrm{d}s\Big)\cdot\big(\varphi^+(x)-\varphi^-(x)\big)-\Gamma(t,x)\right|^2\mathrm{d}t\right\}\mathrm{d}\mathcal{H}^{N-1}(x), \quad (3.27)$$

where $\gamma(t,x)$ and $\Gamma(t,x)$ are defined by (3.9), and (3.14) respectively. By Proposition 2.1, we also infer,

$$\lim_{\varepsilon\to0}\frac{1}{\varepsilon}\int_\Omega|u_\varepsilon-u|^2\mathrm{d}x=T(\bar{\eta}):=\int_{J_\varphi}\left\{\int_{-\infty}^{0}\left|\Gamma(t,x)-u^+(x)\right|^2\mathrm{d}t+\int_0^{+\infty}\left|\Gamma(t,x)-u^-(x)\right|^2\mathrm{d}t\right\}\mathrm{d}\mathcal{H}^{N-1}(x) \\ =\int_{J_\varphi}\left\{\int_{-\infty}^{0}\left|\Big(\int_{-\infty}^{t}\bar{p}(s,\cdot)\mathrm{d}s\Big)\cdot(u^+-u^-)\right|^2\mathrm{d}t+\int_0^{+\infty}\left|\Big(\int_t^{+\infty}\bar{p}(s,\cdot)\,\mathrm{d}s\Big)\cdot(u^+-u^-)\right|^2\mathrm{d}t\right\}\mathrm{d}\mathcal{H}^{N-1}, \quad (3.28)$$

and

$$\lim_{\varepsilon\to0}\frac{1}{\varepsilon}\int_\Omega\left|F(\psi_\varepsilon)-F(\varphi)\right|^2\mathrm{d}x=M_0 \\ :=\int_{J_\varphi}\left\{\int_{-\infty}^{0}\left|F\big(\gamma(t,x)\big)-F\big(\varphi^+(x)\big)\right|^2\mathrm{d}t+\int_0^{+\infty}\left|F\big(\gamma(t,x)\big)-F\big(\varphi^-(x)\big)\right|^2\mathrm{d}t\right\}\mathrm{d}\mathcal{H}^{N-1}(x). \quad (3.29)$$

By Lemma 3.2 we obtain

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^N}\frac{1}{\varepsilon}\Big|\nabla\big(\mathrm{div}\,(\Delta^{-1}\varphi_{n,\varepsilon})\big)(x)\Big|^2\mathrm{d}x = L_n$$
$$:= \int_{J_\varphi}\left(\int_{-\infty}^{+\infty}\Big|\int_t^{+\infty}\{q_n(s,x)\cdot\big(\varphi^+(x)-\varphi^-(x)\big)\}\cdot\boldsymbol{\nu}(x)\,\mathrm{d}s\Big|^2\mathrm{d}t\right)\mathrm{d}\mathcal{H}^{N-1}(x). \quad (3.30)$$

Therefore, letting ε tend to 0 in (3.26), we obtain,

$$\overline{\lim_{\varepsilon\to 0^+}}\left|\frac{1}{\varepsilon}\int_{\mathbb{R}^N}\Big|\nabla\mathrm{div}\,\Delta^{-1}\big(\chi_\Omega F(\psi_\varepsilon)\big)\Big|^2\mathrm{d}x-\mathcal{I}_\eta(\varphi)\right| \le \big|\mathcal{I}_\eta(\varphi)-L_n\big| + 2K_0\Big(\sqrt{D_n}+\sqrt{T(\bar\eta)}\Big)\sqrt{M_0+L_n}. \quad (3.31)$$

Using (3.17), (3.21), (3.18) and (3.19) we obtain

$$\lim_{n\to\infty} D_n = 0, \quad (3.32)$$

and since $(u^+-u^-)\cdot\boldsymbol{\nu}=0$ (by $\mathrm{div}\,u=0$), we also infer

$$\lim_{n\to\infty} L_n = \mathcal{I}_\eta(\varphi) = \int_{J_\varphi}\left(\int_{-\infty}^{+\infty}\Big|\Big(F\big(\gamma(t,x)\big)-F\big(\varphi^-(x)\big)\Big)\cdot\boldsymbol{\nu}(x)\Big|^2\mathrm{d}t\right)\mathrm{d}\mathcal{H}^{N-1}(x). \quad (3.33)$$

Therefore, letting n tend to $+\infty$ in (3.31), we obtain,

$$\overline{\lim_{\varepsilon\to 0^+}}\left|\frac{1}{\varepsilon}\int_{\mathbb{R}^N}\big|\nabla\mathrm{div}\,\Delta^{-1}\big(\chi_\Omega F(\psi_\varepsilon)\big)\big|^2\mathrm{d}x-\mathcal{I}_\eta(\varphi)\right| \le 2K_0\sqrt{T(\bar\eta)}\sqrt{M_0+\mathcal{I}_\eta(\varphi)}. \quad (3.34)$$

This inequality is valid for any $\bar\eta\in\mathcal{V}^{(N)}$, where M_0 and $\mathcal{I}_\eta(\varphi)$ do not depend on $\bar\eta$. For every $\delta>0$ we can always choose $\bar\eta_\delta := \bar\eta_\delta(z,x)=\eta_\delta^{(0)}(z)\,I$, such that $\eta_\delta^{(0)}(z)\in C^2(\mathbb{R}^N,\mathbb{R})$, satisfying $\eta_\delta^{(0)}(z)\ge 0$, $\mathrm{supp}\,\eta_\delta^{(0)}\subset B_\delta(0)$ and $\int_{\mathbb{R}^N}\eta_\delta^{(0)}(z)\mathrm{d}z=1$. Then, as before, define $p_\delta^{(0)}(t):\mathbb{R}\times J_\varphi\to\mathbb{R}$ by

$$p_\delta^{(0)}(t)=\int_{H_{\boldsymbol{\nu}(x)}}\eta_\delta^{(0)}(t\boldsymbol{\nu}(x)+y)\,\mathrm{d}\mathcal{H}^{N-1}(y).$$

Since $p_\delta^{(0)}\ge 0$, $\mathrm{supp}\,p_\delta^{(0)}\in[-\delta,\delta]$ and $\int_{-\infty}^{\infty}p_\delta^{(0)}(t)\mathrm{d}t=1$, by (3.28) we infer

$$T(\bar\eta_\delta)\le\int_{J_\varphi}\left\{\int_{-\delta}^{0}\Big|(u^+-u^-)\int_{-\infty}^{t}p_\delta^{(0)}(s)\mathrm{d}s\Big|^2\mathrm{d}t+\int_0^\delta\Big|(u^+-u^-)\int_t^{+\infty}p_\delta^{(0)}(s)\,\mathrm{d}s\Big|^2\mathrm{d}t\right\}\mathrm{d}\mathcal{H}^{N-1}$$
$$\le 2\delta\int_{J_\varphi}|u^+-u^-|^2\mathrm{d}\mathcal{H}^{N-1}\le C_0\delta\int_{J_\varphi}|u^+-u^-|\mathrm{d}\mathcal{H}^{N-1}\le C_0\delta\|Du\|(\Omega).$$

Therefore, by (3.34) we obtain

$$\overline{\lim_{\varepsilon\to 0^+}}\left|\frac{1}{\varepsilon}\int_{\mathbb{R}^N}\Big|\nabla\mathrm{div}\,\Delta^{-1}\big(\chi_\Omega F(\psi_\varepsilon)\big)\Big|^2\mathrm{d}x-\mathcal{I}_\eta(\varphi)\right|\le 2K_0\sqrt{\delta}\sqrt{C_0\|Du\|(\Omega)}\sqrt{M_0+\mathcal{I}_\eta(\varphi)}. \quad (3.35)$$

For $\delta\to 0$ in (3.35), gives (3.8). □

Combining Proposition 3.1 with Theorem 2.1, we infer the following proposition.

Proposition 3.2. *Let $\Omega\subset\mathbb{R}^N$ be a Lipschitz domain and let $H(a,b,c):\mathbb{R}^{d\times N}\times\mathbb{R}^d\times\mathbb{R}^m\to\mathbb{R}$ be a C^1 function satisfying $H(a,b,c)\geq 0$ for every $a\in\mathbb{R}^{d\times N}$, $b\in\mathbb{R}^d$ and $c\in\mathbb{R}^m$. In addition let $F\in C^1(\mathbb{R}^d,\mathbb{R}^{l\times N})$. Consider $u\in BV(\Omega,\mathbb{R}^m)\cap L^\infty$ and $\varphi\in BV(\Omega,\mathbb{R}^d)\cap L^\infty$ satisfying $\operatorname{div} F(\varphi)=0$ in Ω as a distribution and $F(\varphi)\cdot\boldsymbol{n}=0$ on $\partial\Omega$ ($\boldsymbol{n}$ is the unit normal to $\partial\Omega$). Moreover assume that*

$$H\big(O,\,\varphi(x),\,u(x)\big)=0\quad \text{for a.e. } x\in\Omega.$$

For any $\eta\in\mathcal{V}^{(d)}$ let ψ_ε be defined by

$$\psi_\varepsilon(x):=\frac{1}{\varepsilon^N}\int_{\mathbb{R}^N}\eta\Big(\frac{y-x}{\varepsilon},x\Big)\cdot\bar\varphi(y)\mathrm{d}y=\int_{\mathbb{R}^N}\eta(z,x)\cdot\bar\varphi(x+\varepsilon z)\,\mathrm{d}z,\quad\forall x\in\mathbb{R}^N,\tag{3.36}$$

where $\bar\varphi$ is some bounded BV extension of φ to $\mathbb{R}^N$, having no jump on $\partial\Omega$. Then,

$$\begin{aligned}\lim_{\varepsilon\to0}\int_\Omega\frac{1}{\varepsilon}H\big(\varepsilon\nabla\psi_\varepsilon,\,\psi_\varepsilon,\,u\big)\,\mathrm{d}x+\int_{\mathbb{R}^N}\frac{1}{\varepsilon}\Big|\nabla\Big(\operatorname{div}\Big\{\Delta^{-1}\big(\chi_\Omega F(\psi_\varepsilon)\big)\Big\}\Big)\Big|^2\mathrm{d}x\\ =Y_\varphi(\eta):=\int_{J_\varphi}\Big\{\int_{-\infty}^0 H\Big(p(t,x)\cdot\big(\varphi^+(x)-\varphi^-(x)\big)\otimes\boldsymbol{\nu}(x),\gamma(t,x),u^+(x)\Big)\mathrm{d}t\\ +\int_0^\infty H\Big(p(t,x)\cdot\big(\varphi^+(x)-\varphi^-(x)\big)\otimes\boldsymbol{\nu}(x),\gamma(t,x),u^-(x)\Big)\mathrm{d}t\Big\}\mathrm{d}\mathcal{H}^{N-1}(x)\\ +\int_{J_\varphi}\bigg(\int_{-\infty}^{+\infty}\Big|\Big(F\big(\gamma(t,x)\big)-F\big(\varphi^-(x)\big)\Big)\cdot\boldsymbol{\nu}(x)\Big|^2\mathrm{d}t\bigg)\mathrm{d}\mathcal{H}^{N-1}(x),\end{aligned}\tag{3.37}$$

where

$$\gamma(t,x)=\bigg(\int_{-\infty}^t p(s,x)\,\mathrm{d}s\bigg)\cdot\varphi^-(x)+\bigg(\int_t^\infty p(s,x)\,\mathrm{d}s\bigg)\cdot\varphi^+(x),\tag{3.38}$$

with

$$p(t,x)=\int_{H_{\boldsymbol{\nu}(x)}}\eta(t\boldsymbol{\nu}(x)+y,x)\,\mathrm{d}\mathcal{H}^{N-1}(y),\tag{3.39}$$

and it is assumed that the orientation of J_u coincides with the orientation of J_φ $\mathcal{H}^{N-1}$ a.e. on $J_u\cap J_\varphi$.

Next we turn to the minimization problem of the term on the r.h.s. of (3.37), over all kernels $\eta\in\mathcal{V}^{(d)}$ analogously to what was done in [6,8]. By the same method as there, we can obtain the following result.

Lemma 3.3. *Let H, F, u and φ be as in Proposition 3.2. Let $Y_\varphi(\eta):\mathcal{V}\to\mathbb{R}$ be defined as the r.h.s. of (3.37). Then,*

$$\begin{aligned}\inf_{\eta\in\mathcal{V}^{(d)}}Y_\varphi(\eta)=\mathcal{J}_0(\varphi):=\int_{J_\varphi}\bigg(\inf_{r\in\mathcal{R}^{(0)}_{\varphi^+(x),\varphi^-(x)}}\bigg\{\int_{-\infty}^0 H\Big(-r'(t)\otimes\boldsymbol{\nu}(x),r(t),u^+(x)\Big)\mathrm{d}t\\ +\int_0^\infty H\Big(-r'(t)\otimes\boldsymbol{\nu}(x),r(t),u^-(x)\Big)\mathrm{d}t\\ +\int_{-\infty}^\infty\Big|\big(F(r(t))-F(\varphi^-(x))\big)\cdot\boldsymbol{\nu}(x)\Big|^2\mathrm{d}t\bigg\}\bigg)\mathrm{d}\mathcal{H}^{N-1}(x),\end{aligned}\tag{3.40}$$

where $\mathcal{R}^{(0)}_{\boldsymbol{a},\boldsymbol{b}}$ is defined by

$$\mathcal{R}^{(0)}_{\boldsymbol{a},\boldsymbol{b}}:=\Big\{r(t)\in C^1\big(\mathbb{R},\mathbb{R}^d\big):\exists L>0 \text{ s.t. } r(t)=\boldsymbol{a}\ \ \forall t\leq-L,\ r(t)=\boldsymbol{b}\ \ \forall t\geq L\Big\}.$$

Then, using Proposition 3.2 and Lemma 3.3, exactly the same argument as in the proofs of [8], Theorems 1.1 and 1.2, gives the following result.

Theorem 3.1. *Let $\Omega\subset\mathbb{R}^N$ be a Lipschitz domain and let H, F, u and φ be as in Proposition 3.2. Then there exists a family of functions $\{\psi_\varepsilon\}\subset C^2(\mathbb{R}^N,\mathbb{R}^d)$, $0<\varepsilon<1$ such that*

$$\lim_{\varepsilon\to0^+}\psi_\varepsilon(x)=\varphi(x)\ \text{ in } L^p(\Omega,\mathbb{R}^d)\quad\forall p\in[1,\infty),$$

and

$$\lim_{\varepsilon\to0^+}\left(\int_\Omega\frac{1}{\varepsilon}H\big(\varepsilon\nabla\psi_\varepsilon,\,\psi_\varepsilon,\,u\big)\,\mathrm{d}x+\int_{\mathbb{R}^N}\frac{1}{\varepsilon}\Big|\nabla\Big(\operatorname{div}\Big\{\Delta^{-1}\big(\chi_\Omega F(\psi_\varepsilon)\big)\Big\}\Big)\Big|^2\mathrm{d}x\right)=\mathcal{J}_0(\varphi),$$

where $\mathcal{J}_0(\varphi)$ is defined by (3.40) and we assume that the orientation of J_u coincides with the orientation of J_φ $\mathcal{H}^{N-1}$ a.e. on $J_u\cap J_\varphi$.

We have also the following variant of Theorem 3.1 for elementary manifolds.

Theorem 3.2. *Let $\Omega\subset\mathbb{R}^N$ be a bounded Lipschitz domain and let $\mathcal{M}$ be a bounded d-dimensional elementary C^2-submanifold of $\mathbb{R}^l$, diffeomorphic to $\mathbb{R}^d$ $(l\geq d)$. Consider $H(a,b):\mathbb{R}^{l\times N}\times\mathcal{M}\to\mathbb{R}$ be a C^1-function satisfying $H(a,b)\geq0$ for every $a\in\mathbb{R}^{l\times N}$ and $b\in\mathcal{M}$. In addition let $G\in C^1(\mathcal{M},\mathbb{R}^{r\times N})$. Consider $\varphi\in BV(\Omega,\mathbb{R}^l)$ satisfying $\varphi(x)\in K$ for a.e. $x\in\Omega$, where $K\subset\subset\mathcal{M}$ is a compact. Moreover, assume that $\operatorname{div}G(\varphi)=0$ in Ω as a distribution, $G(\varphi)\cdot\boldsymbol{n}=0$ on $\partial\Omega$ ($\boldsymbol{n}$ is the unit normal to $\partial\Omega$) and*

$$H\big(O,\,\varphi(x)\big)=0\quad\text{for a.e. } x\in\Omega.$$

Then there exists a family of functions $\{\psi_\varepsilon\}\subset C^2(\mathbb{R}^N,\mathcal{M})$, $0<\varepsilon<1$ such that

$$\lim_{\varepsilon\to0^+}\psi_\varepsilon(x)=\varphi(x)\ \text{ in } L^p(\Omega,\mathbb{R}^l)\quad\forall p\in[1,\infty),$$

and

$$\begin{aligned}\lim_{\varepsilon\to0^+}&\left(\int_\Omega\frac{1}{\varepsilon}H\big(\varepsilon\nabla\psi_\varepsilon,\,\psi_\varepsilon\big)\,\mathrm{d}x+\int_{\mathbb{R}^N}\frac{1}{\varepsilon}\Big|\nabla\Big(\operatorname{div}\Big\{\Delta^{-1}\big(\chi_\Omega G(\psi_\varepsilon)\big)\Big\}\Big)\Big|^2\mathrm{d}x\right)=\mathcal{I}_0(\varphi)\\&:=\int_{J_\varphi}\left(\inf_{r\in\mathcal{R}^{(0,\mathcal{M})}_{\varphi^+(x),\varphi^-(x)}}\left\{\int_{-\infty}^{+\infty}H\Big(-r'(t)\otimes\boldsymbol{\nu}(x),r(t)\Big)\mathrm{d}t\right.\right.\\&\qquad\left.\left.+\int_{-\infty}^{\infty}\Big|\big\{G(r(t))-G(\varphi^-(x))\big\}\cdot\boldsymbol{\nu}(x)\Big|^2\mathrm{d}t\right\}\right)\mathrm{d}\mathcal{H}^{N-1}(x),\end{aligned}\tag{3.41}$$

where

$$\mathcal{R}^{(0,\mathcal{M})}_{\boldsymbol{a},\boldsymbol{b}}:=\Big\{r(t)\in C^1\big(\mathbb{R},\mathcal{M}\big):\exists L>0\ \text{s.t.}\ r(t)=\boldsymbol{a}\ \ \forall t\leq-L,\quad r(t)=\boldsymbol{b}\ \ \forall t\geq L\Big\}.$$

Moreover, if, in addition, $H(a,b)=|a|^2+W(b)$ for every $a\in\mathbb{R}^{l\times N}$ and $b\in\mathcal{M}$, where $W\in C^1(\mathcal{M},\mathbb{R})$, then

$$\begin{aligned}\lim_{\varepsilon\to0^+}&\left\{\int_\Omega\Big(\varepsilon|\nabla\psi_\varepsilon|^2+\frac{1}{\varepsilon}W\big(\psi_\varepsilon\big)\Big)\,\mathrm{d}x+\int_{\mathbb{R}^N}\frac{1}{\varepsilon}\Big|\nabla\big(\operatorname{div}\big\{\Delta^{-1}\big(\chi_\Omega G(\psi_\varepsilon)\big)\big\}\big)\Big|^2\mathrm{d}x\right\}=\mathcal{I}_0(\varphi)\\&=\int_{J_\varphi}\left(\inf_{r\in\mathcal{R}^{\mathcal{M}}_{\varphi^+,\varphi^-}}\left\{\int_{-1}^{1}2|r'(t)|\sqrt{W\big(r(t)\big)+\Big|\big\{G(r(t))-G(\varphi^-)\big\}\cdot\boldsymbol{\nu}\Big|^2}\mathrm{d}t\right\}\right)\mathrm{d}\mathcal{H}^{N-1},\end{aligned}\tag{3.42}$$

where

$$\mathcal{R}^{\mathcal{M}}_{\boldsymbol{a},\boldsymbol{b}} := \Big\{ r(t) \in C^1\big([-1,1],\mathcal{M}\big) : \, r(-1) = \boldsymbol{a}, \, r(1) = \boldsymbol{b} \Big\}.$$

Proof. Let $g : \mathcal{M} \to \mathbb{R}^d$ be a C^2 diffeomorphism, *i.e.*, one-to-one map such that $F := g^{-1} : \mathbb{R}^d \to \mathcal{M}$ belongs to $C^2(\mathbb{R}^d, \mathbb{R}^l)$ and satisfies $\operatorname{rank}(\nabla F) = d$ everywhere. Define $\Phi(x) := g\big(\varphi(x)\big)$ so that $\varphi(x) = F\big(\Phi(x)\big)$. Then $\Phi \in BV(\Omega, \mathbb{R}^d)$ and since $\varphi(x) \in K$ for a.e. $x \in \Omega$ we obtain in addition $\Phi \in L^\infty(\Omega, \mathbb{R}^d)$. Define $\overline{H}(\alpha,\beta) : \mathbb{R}^{d\times N} \times \mathbb{R}^d \to \mathbb{R}$ by the formula

$$\overline{H}(\alpha,\beta) := H\big(\nabla F(\beta)\cdot\alpha, F(\beta)\big). \tag{3.43}$$

Then $\overline{H} \in C^1$, $\overline{H} \geq 0$ and $H\big(O, \Phi(x)\big) = 0$ a.e. in Ω. Applying Theorem 3.1 with $\overline{H}$, $(G \circ F)$ and Φ instead of H F and φ we obtain existence of the family of functions $\{\Psi_\varepsilon\} \subset C^2(\mathbb{R}^N, \mathbb{R}^d)$, $0 < \varepsilon < 1$ such that

$$\lim_{\varepsilon\to0^+} \Psi_\varepsilon(x) = \Phi(x) \text{ in } L^p(\Omega,\mathbb{R}^d) \quad \forall p \in [1,\infty),$$

and

$$\begin{aligned}\lim_{\varepsilon\to0^+} \Bigg(\int_\Omega \frac{1}{\varepsilon}\overline{H}\big(\varepsilon\nabla\Psi_\varepsilon, \, \Psi_\varepsilon\big)\,\mathrm{d}x + \int_{\mathbb{R}^N} \frac{1}{\varepsilon}\Big|\nabla\Big(\operatorname{div}\Big\{\Delta^{-1}\big(\chi_\Omega G(F(\Psi_\varepsilon))\big)\Big\}\Big)\Big|^2 \mathrm{d}x \Bigg) \\ = \int_{J_\Phi} \Bigg(\inf_{r\in\mathcal{R}^{(0)}_{\Phi^+(x),\Phi^-(x)}} \Bigg\{ \int_{-\infty}^{+\infty} \overline{H}\Big(-r'(t)\otimes\boldsymbol{\nu}(x)\,, r(t)\Big)\mathrm{d}t \\ + \int_{-\infty}^{\infty} \Big|\Big\{G\big(F(r(t))\big) - G\big(F(\Phi^-(x))\big)\Big\}\cdot\boldsymbol{\nu}(x)\Big|^2 \mathrm{d}t\Bigg\}\Bigg)\mathrm{d}\mathcal{H}^{N-1}(x).\end{aligned} \tag{3.44}$$

Define $\psi_\varepsilon(x) := F\big(\Psi_\varepsilon(x)\big)$. Then $\psi_\varepsilon \in C^2(\mathbb{R}^N, \mathcal{M})$. Since $\mathcal{M}$ is a bounded subset of $\mathbb{R}^l$ we obtain

$$\lim_{\varepsilon\to0^+} \psi_\varepsilon(x) = \varphi(x) \text{ in } L^p(\Omega,\mathbb{R}^l) \quad \forall p \in [1,\infty),$$

and by (3.44) and (3.43) we infer

$$\begin{aligned}\lim_{\varepsilon\to0^+} \Bigg(\int_\Omega \frac{1}{\varepsilon}H\big(\varepsilon\nabla\psi_\varepsilon, \, \psi_\varepsilon\big)\,\mathrm{d}x + \int_{\mathbb{R}^N} \frac{1}{\varepsilon}\Big|\nabla\Big(\operatorname{div}\Big\{\Delta^{-1}\big(\chi_\Omega G(\psi_\varepsilon)\big)\Big\}\Big)\Big|^2 \mathrm{d}x \Bigg) \\ = \int_{J_\Phi} \Bigg(\inf_{r\in\mathcal{R}^{(0)}_{\Phi^+(x),\Phi^-(x)}} \Bigg\{ \int_{-\infty}^{+\infty} H\Big(-\frac{\mathrm{d}}{\mathrm{d}t}F\big(r(t)\big)\otimes\boldsymbol{\nu}(x)\,, F\big(r(t)\big)\Big)\mathrm{d}t \\ + \int_{-\infty}^{\infty} \Big|\Big\{G\big(F(r(t))\big) - G\big(\varphi^-(x)\big)\Big\}\cdot\boldsymbol{\nu}(x)\Big|^2 \mathrm{d}t\Bigg\}\Bigg)\mathrm{d}\mathcal{H}^{N-1}(x).\end{aligned} \tag{3.45}$$

Since $F(\Phi) = \varphi$, we obtain equality (3.41).

Next assume that, in addition, $H(a,b) = |a|^2 + W(b)$ for every $a \in \mathbb{R}^{l\times N}$ and $b \in \mathcal{M}$, where $W \in C^1(\mathcal{M},\mathbb{R})$. Then, using Lemma A.1 from Appendix, we infer from (3.45) that

$$\int_{J_\Phi}\left(\inf_{r\in\mathcal{R}^{(0)}_{\Phi^+,\Phi^-}}\left\{\int_{-\infty}^{+\infty} H\Big(-\frac{\mathrm{d}}{\mathrm{d}t}F\big(r(t)\big)\otimes\boldsymbol{\nu}, F\big(r(t)\big)\Big)\mathrm{d}t + \int_{-\infty}^{\infty}\Big|\Big\{G\big(F(r(t))\big) - G\big(\varphi^-\big)\Big\}\cdot\boldsymbol{\nu}\Big|^2\mathrm{d}t\right\}\right)\mathrm{d}\mathcal{H}^{N-1}$$
$$= \int_{J_\varphi}\left(\inf_{r\in\mathcal{R}_{\Phi^+,\Phi^-}}\left\{\int_{-1}^{1} 2\Big|\frac{\mathrm{d}F(r(t))}{\mathrm{d}t}\Big|\sqrt{W\big(F(r(t))\big) + \Big|\Big\{G\big(F(r(t))\big) - G\big(\varphi^-\big)\Big\}\cdot\boldsymbol{\nu}\Big|^2}\mathrm{d}t\right\}\right)\mathrm{d}\mathcal{H}^{N-1}.$$

Since $F(\Phi) = \varphi$, the result follows. □

The following proposition provide the lower bound for functionals depending on scalar valued functions.

Proposition 3.3. *Let $\Omega \subset \mathbb{R}^N$ be an open set and $F : \mathbb{R} \to \mathbb{R}^{l\times N}$ be a Lipschitz function. Assume that $\{u_\varepsilon\}_{\varepsilon>0} \subset H^1(\Omega,\mathbb{R})$ be such that, for a subsequence $\varepsilon_n \downarrow 0$,*

$$\varliminf_{n\to\infty}\int_\Omega \varepsilon_n|\nabla u_{\varepsilon_n}|^2 + \frac{1}{\varepsilon_n}\Big|\nabla\Big(\operatorname{div}\big\{\Delta^{-1}(\chi_\Omega F(u_{\varepsilon_n}))\big\}\Big)\Big|^2\,\mathrm{d}x < \infty$$

*and $u_{\varepsilon_n} \to u$ in $L^1_{\mathrm{loc}}(\Omega,\mathbb{R})$. Then the distribution $\mu = \operatorname{div}_x F(u(x)\wedge s) \in \mathcal{D}(\Omega_x\times\mathbb{R}_s,\mathbb{R}^l)$ belongs to $\mathcal{M}(\Omega\times\mathbb{R},\mathbb{R}^l)$ (*i.e.* it is a finite $\mathbb{R}^l$-valued Radon measure), where $a\wedge s := \min\{a,s\}$. Moreover,*

$$\varliminf_{n\to\infty}\int_\Omega \varepsilon_n|\nabla u_{\varepsilon_n}|^2 + \frac{1}{\varepsilon_n}\Big|\nabla\Big(\operatorname{div}\big\{\Delta^{-1}(\chi_\Omega F(u_{\varepsilon_n}))\big\}\Big)\Big|^2\,\mathrm{d}x \geq 2\|\mu\|_{\Omega\times\mathbb{R}}. \tag{3.46}$$

Proof. We follow the strategy for proving lower bounds used in the paper of Rivière and Serfaty [11]. Set

$$H_{u_\varepsilon} := \operatorname{div}\big\{\Delta^{-1}(\chi_\Omega F(u_\varepsilon))\big\}.$$

Consider $\delta \in C_c^\infty(\Omega\times\mathbb{R},\mathbb{R}^l)$ satisfying $|\delta| \leq 1$ everywhere. Then we have

$$\int_\Omega \frac{\varepsilon}{2}|\nabla u_\varepsilon|^2 + \frac{1}{2\varepsilon}\Big|\nabla\Big(\operatorname{div}\big\{\Delta^{-1}(\chi_\Omega F(u_\varepsilon))\big\}\Big)\Big|^2\,\mathrm{d}x \geq \int_\Omega \delta(x,u_\varepsilon)\cdot(\nabla_x H_{u_\varepsilon}\cdot\nabla_x u_\varepsilon)\,\mathrm{d}x. \tag{3.47}$$

We then use the Co-area Formula to deduce that we have,

$$\begin{aligned}\int\limits_\Omega \delta(x,u_\varepsilon)\cdot(\nabla_x H_{u_\varepsilon}\cdot\nabla_x u_\varepsilon)\,\mathrm{d}x &= \int\limits_{\mathbb{R}}\Bigg(\int\limits_{\partial^*\{u_\varepsilon(x)\leq s\}} \delta(x,u_\varepsilon)\cdot(\nabla_x H_{u_\varepsilon}\cdot\boldsymbol{\nu})\,\mathrm{d}\mathcal{H}^{N-1}(x)\Bigg)\mathrm{d}s\\ &= \int_{\mathbb{R}}\Bigg(\int_{\partial^*\{u_\varepsilon(x)\leq s\}} \delta(x,s)\cdot(\nabla_x H_{u_\varepsilon}\cdot\boldsymbol{\nu})\,\mathrm{d}\mathcal{H}^{N-1}(x)\Bigg)\mathrm{d}s\\ &= \int_{\mathbb{R}}\Bigg(\int_{\{u_\varepsilon(x)\leq s\}} \operatorname{div}_x\Big((\nabla_x H_{u_\varepsilon})^T\cdot\delta(x,s)\Big)\,\mathrm{d}x\Bigg)\mathrm{d}s\\ &= \int_{\mathbb{R}}\int_\Omega 1_{\{u_\varepsilon(x)\leq s\}}\Big(\delta(x,s)\cdot\Delta_x H_{u_\varepsilon} + \nabla_x\delta(x,s) : \nabla_x H_{u_\varepsilon}\Big)\,\mathrm{d}x\mathrm{d}s.\end{aligned} \tag{3.48}$$

But by the definition, we have

$$\Delta_x H_{u_\varepsilon}(x) = \operatorname{div}_x F(u_\varepsilon(x)).$$

Therefore, by (3.48), we obtain

$$\int_\Omega \delta(x,u_\varepsilon)\cdot(\nabla_x H_{u_\varepsilon}\cdot\nabla_x u_\varepsilon)\,\mathrm{d}x = \int_{\mathbb{R}}\int_\Omega 1_{\{u_\varepsilon(x)\le s\}}\,\mathrm{div}_x F(u_\varepsilon(x))\cdot\delta(x,s)\,\mathrm{d}x\mathrm{d}s \\ + \int_{\mathbb{R}}\left(\int_\Omega 1_{\{u_\varepsilon(x)\le s\}}\nabla_x\delta(x,s):\nabla_x H_{u_\varepsilon}(x)\,\mathrm{d}x\right)\mathrm{d}s. \quad (3.49)$$

Next we observe that

$$1_{\{u_\varepsilon(x)\le s\}}\,\mathrm{div}_x F(u_\varepsilon(x)) = \mathrm{div}_x F\big(u_\varepsilon(x)\wedge s\big).$$

Then, by (3.49), we infer

$$\int_\Omega \delta(x,u_\varepsilon)\cdot(\nabla_x H_{u_\varepsilon}\cdot\nabla_x u_\varepsilon)\,\mathrm{d}x = \int_{\mathbb{R}}\int_\Omega \mathrm{div}_x F\big(u_\varepsilon(x)\wedge s\big)\cdot\delta(x,s)\,\mathrm{d}x\mathrm{d}s \\ + \int_{\mathbb{R}}\left(\int_\Omega 1_{\{u_\varepsilon(x)\le s\}}\nabla_x\delta(x,s):\nabla_x H_{u_\varepsilon}(x)\,\mathrm{d}x\right)\mathrm{d}s. \quad (3.50)$$

Thus, by (3.47) and (3.50) we obtain

$$\underline{\lim}_{n\to\infty}\int_\Omega \frac{\varepsilon_n}{2}|\nabla u_{\varepsilon_n}|^2 + \frac{1}{2\varepsilon_n}\Big|\nabla\Big(\mathrm{div}\,\{\Delta^{-1}(\chi_\Omega F(u_{\varepsilon_n}))\}\Big)\Big|^2\,\mathrm{d}x \ge \underline{\lim}_{n\to\infty}\left\{\int_{\mathbb{R}}\int_\Omega \mathrm{div}_x F\big(u_{\varepsilon_n}(x)\wedge s\big)\cdot\delta(x,s)\,\mathrm{d}x\mathrm{d}s \right. \\ \left. + \int_{\mathbb{R}}\left(\int_\Omega 1_{\{u_{\varepsilon_n}(x)\le s\}}\nabla_x\delta(x,s):\nabla_x H_{u_{\varepsilon_n}}(x)\,\mathrm{d}x\right)\mathrm{d}s\right\}\cdot \quad (3.51)$$

Since, $\int_\Omega |\nabla_x H_{u_{\varepsilon_n}}|^2\,\mathrm{d}x \to 0$, the last term in (3.51) goes to 0. Then by (3.51) we obtain

$$\underline{\lim}_{n\to\infty}\int_\Omega \frac{\varepsilon_n}{2}|\nabla u_{\varepsilon_n}|^2 + \frac{1}{2\varepsilon_n}\Big|\nabla\Big(\mathrm{div}\,\{\Delta^{-1}(\chi_\Omega F(u_{\varepsilon_n}))\}\Big)\Big|^2\,\mathrm{d}x \ge \underline{\lim}_{n\to\infty}\int_{\mathbb{R}}\int_\Omega \mathrm{div}_x F\big(u_{\varepsilon_n}(x)\wedge s\big)\cdot\delta(x,s)\,\mathrm{d}x\mathrm{d}s \\ = -\lim_{n\to\infty}\int_{\mathbb{R}}\int_\Omega F\big(u_{\varepsilon_n}(x)\wedge s\big):\nabla_x\delta(x,s)\,\mathrm{d}x\mathrm{d}s = -\int_{\mathbb{R}}\int_\Omega F\big(u(x)\wedge s\big):\nabla_x\delta(x,s)\,\mathrm{d}x\mathrm{d}s. \quad (3.52)$$

Since $\delta(x,s)\in C_c^\infty(\Omega\times\mathbb{R},\mathbb{R}^l)$ satisfying $|\delta|\le 1$ was arbitrary, by (3.52) we deduce that the distribution $\mu\in\mathcal{D}(\Omega\times\mathbb{R},\mathbb{R}^l)$ is a finite vector-valued Radon measure on $\Omega\times\mathbb{R}$. Moreover we obtain (3.46). □

Remark 3.1. Under the conditions of Proposition 3.3, assume that in addition Ω is Lipschitz and $u\in BV\cap L^\infty$. Since the conditions of Proposition 3.3 imply that $\mathrm{div}\,(\chi_\Omega F(u_\varepsilon))=0$, then, by Theorem 3.1, there exists a family of functions $\{v_\varepsilon\}\subset C^2(\mathbb{R}^N,\mathbb{R})$, $0<\varepsilon<1$ such that

$$\lim_{\varepsilon\to0^+} v_\varepsilon(x) = u(x) \text{ in } L^p(\Omega,\mathbb{R}^d)\quad \forall p\in[1,\infty),$$

and

$$\lim_{\varepsilon\to0^+}\left(\int_\Omega \varepsilon|\nabla v_\varepsilon|^2\mathrm{d}x + \int_{\mathbb{R}^N}\frac{1}{\varepsilon}\Big|\nabla\Big(\mathrm{div}\,\Big\{\Delta^{-1}\big(\chi_\Omega F(v_\varepsilon)\big)\Big\}\Big)\Big|^2\mathrm{d}x\right) \\ = 2\int_{J_u}\left|\int_{u^-}^{u^+}\big|\{F(s)-F(u^-)\}\cdot\boldsymbol{\nu}\big|\mathrm{d}s\right|\mathrm{d}\mathcal{H}^{N-1} = 2\|\mu\|_{\Omega\times\mathbb{R}},$$

where μ defined in Proposition 3.3. So in this case the upper and the lower bounds coincide.

4. An application

Theorem 4.1. *Let $\Omega \subset \mathbb{R}^N$ be a bounded Lipschitz domain. Consider a function $W \in C^1(S^{k-1}, \mathbb{R})$, which is not identically zero and satisfies $W \geq 0$. Let in addition $F \in C^1(S^{k-1}, \mathbb{R}^N)$. Consider $\varphi \in BV(\Omega, \mathbb{R}^k)$ satisfying $\varphi(x) \in S^{k-1}$ for a.e. $x \in \Omega$. Moreover, assume that* $\operatorname{div} F(\varphi) = 0$ *in Ω as a distribution, $F(\varphi) \cdot \boldsymbol{n} = 0$ on $\partial\Omega$ ($\boldsymbol{n}$ is the unit normal to $\partial\Omega$) and*

$$W\big(\varphi(x)\big) = 0 \quad \text{for a.e. } x \in \Omega.$$

Then there exists a family of functions $\{\psi_\varepsilon\} \subset C^2(\mathbb{R}^N, S^{k-1})$, $0 < \varepsilon < 1$ such that

$$\lim_{\varepsilon \to 0^+} \psi_\varepsilon(x) = \varphi(x) \text{ in } L^p(\Omega, \mathbb{R}^k) \quad \forall p \in [1, \infty),$$

and

$$\lim_{\varepsilon \to 0^+} \left\{ \int_\Omega \Big(\varepsilon |\nabla \psi_\varepsilon|^2 + \frac{1}{\varepsilon} W\big(\psi_\varepsilon\big) \Big) \,\mathrm{d}x + \int_{\mathbb{R}^N} \frac{1}{\varepsilon} \Big| \nabla \Big(\operatorname{div} \Big\{ \Delta^{-1} \big(\chi_\Omega F(\psi_\varepsilon) \big) \Big\} \Big) \Big|^2 \mathrm{d}x \right\}$$
$$= \int_{J_\varphi} \left(\inf_{r \in \mathcal{R}^{S^{k-1}}_{\varphi^+, \varphi^-}} \left\{ \int_{-1}^{1} 2|r'(t)| \sqrt{W\big(r(t)\big) + \big| \big(F(r(t)) - F(\varphi^-) \big) \cdot \boldsymbol{\nu} \big|^2} \mathrm{d}t \right\} \right) \mathrm{d}\mathcal{H}^{N-1}, \quad (4.1)$$

where

$$\mathcal{R}^{S^{k-1}}_{\boldsymbol{a}, \boldsymbol{b}} := \Big\{ r(t) \in C^1\big([-1,1], S^{k-1}\big) : r(-1) = \boldsymbol{a},\, r(1) = \boldsymbol{b} \Big\}.$$

Proof. Set

$$K := \big\{ a \in S^{k-1} : W(a) = 0 \big\}.$$

It is clear that K is a compact proper subset of S^{k-1}. In particular there exists $a_0 \in S^{k-1}$ such that $a_0 \notin K$. Define $\mathcal{M} := S^{k-1} \setminus \{a_0\}$. Then $\mathcal{M}$ is a bounded $(k-1)$-dimensional elementary C^2-submanifold of $\mathbb{R}^k$, diffeomorphic to $\mathbb{R}^{k-1}$. We have $\varphi(x) \in K \subset\subset \mathcal{M}$ for a.e. x in Ω. Therefore, by Theorem 3.2 we obtain the existence of the family of functions $\{\psi_\varepsilon\} \subset C^2(\mathbb{R}^N, \mathcal{M})$, $0 < \varepsilon < 1$ such that

$$\lim_{\varepsilon \to 0^+} \psi_\varepsilon(x) = \varphi(x) \text{ in } L^p(\Omega, \mathbb{R}^k) \quad \forall p \in [1, \infty),$$

and

$$\lim_{\varepsilon \to 0^+} \left\{ \int_\Omega \Big(\varepsilon |\nabla \psi_\varepsilon|^2 + \frac{1}{\varepsilon} W\big(\psi_\varepsilon\big) \Big) \,\mathrm{d}x + \int_{\mathbb{R}^N} \frac{1}{\varepsilon} \Big| \nabla \Big(\operatorname{div} \Big\{ \Delta^{-1} \big(\chi_\Omega F(\psi_\varepsilon) \big) \Big\} \Big) \Big|^2 \mathrm{d}x \right\}$$
$$= \int_{J_\varphi} \left(\inf_{r \in \mathcal{R}^{\mathcal{M}}_{\varphi^+, \varphi^-}} \left\{ \int_{-1}^{1} 2|r'(t)| \sqrt{W\big(r(t)\big) + \big| \big(F(r(t)) - F(\varphi^-) \big) \cdot \boldsymbol{\nu} \big|^2} \mathrm{d}t \right\} \right) \mathrm{d}\mathcal{H}^{N-1}. \quad (4.2)$$

But, by density arguments, we have

$$\inf_{r \in \mathcal{R}^{\mathcal{M}}_{\varphi^+(x), \varphi^-(x)}} \left\{ \int_{-1}^{1} |r'(t)| \sqrt{W\big(r(t)\big) + \big| \big(F(r(t)) - F(\varphi^-(x)) \big) \cdot \boldsymbol{\nu}(x) \big|^2} \mathrm{d}t \right\}$$
$$= \inf_{r \in \mathcal{R}^{S^{k-1}}_{\varphi^+(x), \varphi^-(x)}} \left\{ \int_{-1}^{1} |r'(t)| \sqrt{W\big(r(t)\big) + \big| \big(F(r(t)) - F(\varphi^-(x)) \big) \cdot \boldsymbol{\nu}(x) \big|^2} \mathrm{d}t \right\},$$

and the result follows. □

5. Application to the variational functional related to conservation laws

5.1. Some definitions and preliminaries

Definition 5.1. We denote by $\tilde{H}_0^1(\mathbb{R}^N,\mathbb{R}^k)$ the closure of $C_c^\infty(\mathbb{R}^N,\mathbb{R}^k)$ with respect to the norm $|||\varphi||| := \big(\int_{\mathbb{R}^N}|\nabla\varphi|^2\mathrm{d}x\big)^{1/2}$ and by $\tilde{H}^{-1}(\mathbb{R}^N,\mathbb{R}^k)$ the space dual to $\tilde{H}_0^1(\mathbb{R}^N,\mathbb{R}^k)$.

Remark 5.1. It is obvious that $u\in\mathcal{D}'(\mathbb{R}^N,\mathbb{R}^k)$ belongs to $\tilde{H}^{-1}(\mathbb{R}^N,\mathbb{R}^k)$ if and only if there exists $w\in\tilde{H}_0^1(\mathbb{R}^N,\mathbb{R}^k)$ such that

$$\int_{\mathbb{R}^N}\nabla w:\nabla\delta\,\mathrm{d}x=-\langle u,\delta\rangle\qquad\forall\delta\in\tilde{H}_0^1(\mathbb{R}^N,\mathbb{R}^k).$$

In particular $\Delta w=u$ as distributions and

$$|||w|||=\sup_{\delta\in\tilde{H}_0^1(\mathbb{R}^N,\mathbb{R}^k),\ |||\delta|||\le 1}\langle u,\delta\rangle=|||u|||_{-1}.$$

Definition 5.2. Let $u\in\tilde{H}^{-1}(\mathbb{R}^N,\mathbb{R}^k)$. Then we define

$$\nabla(\tilde{\Delta}^{-1}u):=\nabla w,\tag{5.1}$$

where w is as in Remark 5.1. It is clear that for every $f\in L^2(\mathbb{R}^N,\mathbb{R}^{k\times N})$ we have

$$\nabla\big(\tilde{\Delta}^{-1}(\operatorname{div}f)\big)=\nabla\big(\operatorname{div}(\Delta^{-1}f)\big)$$

where $\nabla\big(\operatorname{div}(\Delta^{-1}f)\big)$ was defined by Definition 2.2.

Remark 5.2. It is clear that given a distribution $u\in\mathcal{D}(\mathbb{R}^N,\mathbb{R}^l)$, there exists a distribution $H\in\mathcal{D}(\mathbb{R}^N,\mathbb{R}^l)$ such that $\Delta H=u$ and $\nabla H\in L^2(\mathbb{R}^N,\mathbb{R}^{l\times N})$ if and only if $u\in\tilde{H}^{-1}(\mathbb{R}^N,\mathbb{R}^l)$. Moreover in the latter case we have $\nabla H=\nabla(\tilde{\Delta}^{-1}u)$.

Definition 5.3. Consider $T>0$. Let $\mathcal{V}_T^{(d)}$ be the class of all functions $\eta(z,x,t)\in C^2(\mathbb{R}^N\times\mathbb{R}^N\times[0,T],\mathbb{R}^{d\times d})\cap Lip\cap L^\infty$, such that there exists a compact set $K\subset\subset\mathbb{R}^N$, with the property that $\operatorname{supp}\eta\subset K\times\mathbb{R}^N\times[0,T]$, $\operatorname{supp}\nabla_x\eta(z,x,t)\subset K\times K\times[0,T]$ and $\operatorname{supp}\partial_t\eta(z,x,t)\subset K\times K\times[0,T]$ and such that

$$\int_{\mathbb{R}^N}\eta(z,x,t)\,\mathrm{d}z=I\quad\forall(x,t)\in\mathbb{R}^N\times[0,T].\tag{5.2}$$

Here I is the identity matrix. We note here that for every $t\in[0,T]$ we have $\eta(\cdot,\cdot,t)\in\mathcal{V}^{(d)}(\mathbb{R}^N)$ (see Def. 2.3).

Definition 5.4. For every $\eta_0\in C_c^2(\mathbb{R}^N,\mathbb{R})$ satisfying $\int_{\mathbb{R}^N}\eta_0(z)\mathrm{d}z=1$ let $V_{T,\eta_0}^{(d)}$ be the class of all $\eta(z,x,t)\in\mathcal{V}_T^{(d)}$, such that $\eta(z,x,0)=\eta(z,x,T)=\eta_0(z)\cdot I$.

Definition 5.5. Let $F\in C^1(\mathbb{R}^d,\mathbb{R}^{d\times N})\cap Lip$ satisfying $F(0)=0$. Denote by $\mathcal{E}_T^F$ the class of all $v(x,t)\in BV(\mathbb{R}^N\times(0,T),\mathbb{R}^d)\cap L^\infty(0,T;L^2(\mathbb{R}^N,\mathbb{R}^d))\cap L^\infty$ such that $v(x,t)$ is continuous in $[0,T]$ as a function of t with the values in $L^\infty(\mathbb{R}^N,\mathbb{R}^d)$ with respect to L^∞-weak* topology and satisfy

$$\partial_t v(x,t)+\operatorname{div}_x F\big(v(x,t)\big)=0\qquad\forall(x,t)\in\mathbb{R}^N\times(0,T).\tag{5.3}$$

Definition 5.6. For every set $Y\subset\mathbb{R}^N\times[0,T]$ and every $t\in[0,T]$ consider the set $Y_t:=\{x\in\mathbb{R}^N:\ (x,t)\in Y\}$. Moreover for $v\in BV(\mathbb{R}^N\times(0,T),\mathbb{R}^d)$ and every $t\in(0,T)$ consider $v_{(t)}(x):\mathbb{R}^N\to\mathbb{R}^d$ by $v_{(t)}(x)=v(x,t)$. By the results of Section 3.11 in [2] we obtain that for a.e. $t\in(0,T)$ we have

$$v_{(t)}(x)\in BV(\mathbb{R}^N,\mathbb{R}^d),\quad J_{v_{(t)}}=(J_v)_t,\tag{5.4}$$

and there exists an orientation of $J_{v_{(t)}}$, such that

$$v^+_{(t)}(x) = v^+(x,t), \quad v^-_{(t)}(x) = v^-(x,t), \quad \boldsymbol{\nu}_{v_{(t)}}(x) = \frac{(\boldsymbol{\nu}_v(x,t))_x}{|(\boldsymbol{\nu}_v(x,t))_x|}, \tag{5.5}$$

where for the vector $\boldsymbol{a} = (a_1, a_2, \ldots, a_N, a_{N+1}) \in \mathbb{R}^N \times \mathbb{R}$ we set $(\boldsymbol{a})_x = (a_1, a_2, \ldots, a_N) \in \mathbb{R}^N$. Moreover, we also have the equality of the measures

$$D_x v = D_x v_{(t)} \otimes \mathcal{L}^1 \llcorner [0,T], \qquad (v^+ - v^-) \otimes (\boldsymbol{\nu}_v)_x \mathcal{H}^N \llcorner J_v = \big((v^+_{(t)} - v^-_{(t)}) \otimes \boldsymbol{\nu}_{v_{(t)}} \mathcal{H}^{N-1} \llcorner J_{v_{(t)}}\big) \otimes \mathcal{L}^1 \llcorner [0,T]. \tag{5.6}$$

5.2. **The results and the proofs**

Given $v \in \mathcal{E}^F_T$ and $\eta \in \mathcal{V}^{(d)}_T$ define $\bar{u}_\varepsilon(x,t) : \mathbb{R}^N \times [0,T] \to \mathbb{R}^d$ and $u_\varepsilon(x,t) : \mathbb{R}^N \times [0,T] \to \mathbb{R}^d$ by

$$\bar{u}_\varepsilon(x,t) := \frac{1}{\varepsilon^N} \int_{\mathbb{R}^N} \eta\Big(\frac{y-x}{\varepsilon}, x, t\Big) \cdot v(y,t)\, \mathrm{d}y = \int_{\mathbb{R}^N} \eta(z,x,t) \cdot v(x+\varepsilon z, t)\, \mathrm{d}z,$$

$$u_\varepsilon(x,t) := \frac{1}{\varepsilon^N} \int_{\mathbb{R}^N} \eta\Big(\frac{y-x}{\varepsilon}, y, t\Big) \cdot v(y,t)\, \mathrm{d}y = \int_{\mathbb{R}^N} \eta(z, x+\varepsilon z, t) \cdot v(x+\varepsilon z, t)\, \mathrm{d}z. \tag{5.7}$$

Then we clearly have $\nabla_x u_\varepsilon(x,t) \in L^\infty$.

Lemma 5.1.

$$\lim_{\varepsilon\to 0} \int_0^T \int_{\mathbb{R}^N} \frac{1}{\varepsilon} \Big|\nabla_x \Big(\tilde{\Delta}_x^{-1} \big\{\partial_t u_\varepsilon + \mathrm{div}_x F(u_\varepsilon)\big\}\Big)\Big|^2 \mathrm{d}x\mathrm{d}t$$
$$= \int_0^T \int_{(J_v)_t} \Big(\int_{-\infty}^{+\infty} \Big| \Big(\big(F\big(\gamma(s,x,t)\big) - \Gamma(s,x,t)\big)\Big) \cdot \boldsymbol{\nu}_{v_{(t)}}(x) \Big|^2 \mathrm{d}s \Big) \mathrm{d}\mathcal{H}^{N-1}(x)\, \mathrm{d}t \tag{5.8}$$

where

$$\gamma(s,x,t) = \Big(\int_{-\infty}^{s} p(\tau,x,t)\mathrm{d}\tau\Big) \cdot v^-(x,t) + \Big(\int_{s}^{+\infty} p(\tau,x,t)\, \mathrm{d}\tau\Big) \cdot v^+(x,t), \tag{5.9}$$

and

$$\Gamma(s,x,t) = \Big(\int_{-\infty}^{s} p(\tau,x,t)\mathrm{d}\tau\Big) \cdot F\big(v^-(x,t)\big) + \Big(\int_{s}^{+\infty} p(\tau,x,t)\, \mathrm{d}\tau\Big) \cdot F\big(v^+(x,t)\big), \tag{5.10}$$

with

$$p(s,x,t) = \int_{H_{\boldsymbol{\nu}_{v_{(t)}}(x)}} \eta(s\boldsymbol{\nu}_{v_{(t)}}(x) + y, x, t)\, \mathrm{d}\mathcal{H}^{N-1}(y). \tag{5.11}$$

Proof. Let $\delta(x,t) \in C^\infty_c(\mathbb{R}^N \times (0,T), \mathbb{R}^d)$. Consider

$$\delta_\varepsilon(x,t) := \frac{1}{\varepsilon^N} \int_{\mathbb{R}^N} \eta^T\Big(\frac{x-y}{\varepsilon}, x, t\Big) \cdot \delta(y,t)\, \mathrm{d}y,$$

where η^T is a transpose of the matrix η. Then $\delta_\varepsilon(x,t) \in C^\infty_c(\mathbb{R}^N \times (0,T), \mathbb{R}^d)$ and by (5.3) we obtain

$$\int_0^T \int_{\mathbb{R}^N} v(x,t) \cdot \partial_t \delta_\varepsilon(x,t)\, \mathrm{d}x\mathrm{d}t + \int_0^T \int_{\mathbb{R}^N} F\big(v(x,t)\big) : \nabla_x \delta_\varepsilon(x,t)\, \mathrm{d}x\mathrm{d}t = 0. \tag{5.12}$$

But

$$\int_0^T\int_{\mathbb{R}^N} F\big(v(x,t)\big):\nabla_x\delta_\varepsilon(x,t)\,\mathrm{d}x\mathrm{d}t=\int_0^T\int_{\mathbb{R}^N} F\big(v(x,t)\big):\nabla_x\Big\{\frac{1}{\varepsilon^N}\int_{\mathbb{R}^N}\eta^T\Big(\frac{x-y}{\varepsilon},x,t\Big)\cdot\delta(y,t)\,\mathrm{d}y\Big\}\,\mathrm{d}x\mathrm{d}t$$
$$=\int_0^T\int_{\mathbb{R}^N}\Big\{\frac{1}{\varepsilon^N}\int_{\mathbb{R}^N}\eta\Big(\frac{y-x}{\varepsilon},y,t\Big)\cdot F\big(v(y,t)\big)\,\mathrm{d}y\Big\}:\nabla_x\delta(x,t)\,\mathrm{d}x\mathrm{d}t$$
$$+\int_0^T\int_{\mathbb{R}^N}\sum_{1\le i,k\le d}\sum_{1\le j\le N}\Big\{\frac{1}{\varepsilon^N}\int_{\mathbb{R}^N}\frac{\partial\eta_{ki}}{\partial l_j}\Big(\frac{y-x}{\varepsilon},y,t\Big)F_{ij}\big(v(y,t)\big)\,\mathrm{d}y\Big\}\delta_k(x,t)\,\mathrm{d}x\mathrm{d}t,\quad(5.13)$$

where $\nabla_l\eta$ is a partial gradient of the function $\eta(z,x,t)$ by the second argument x. On the other hand,

$$\int_0^T\int_{\mathbb{R}^N} v(x,t)\cdot\partial_t\delta_\varepsilon(x,t)\,\mathrm{d}x\mathrm{d}t=\int_0^T\int_{\mathbb{R}^N} v(x,t)\cdot\partial_t\Big\{\frac{1}{\varepsilon^N}\int_{\mathbb{R}^N}\eta^T\Big(\frac{x-y}{\varepsilon},x,t\Big)\cdot\delta(y,t)\,\mathrm{d}y\Big\}\,\mathrm{d}x\mathrm{d}t$$
$$=\int_0^T\int_{\mathbb{R}^N}\Big\{\frac{1}{\varepsilon^N}\int_{\mathbb{R}^N}\eta\Big(\frac{y-x}{\varepsilon},y,t\Big)\cdot v(y,t)\,\mathrm{d}y\Big\}\cdot\partial_t\delta(x,t)\,\mathrm{d}x\mathrm{d}t$$
$$+\int_0^T\int_{\mathbb{R}^N}\Big\{\frac{1}{\varepsilon^N}\int_{\mathbb{R}^N}\partial_t\eta\Big(\frac{y-x}{\varepsilon},y,t\Big)\cdot v(y,t)\,\mathrm{d}y\Big\}\cdot\delta(x,t)\,\mathrm{d}x\mathrm{d}t$$
$$=\int_0^T\int_{\mathbb{R}^N} u_\varepsilon(x,t)\cdot\partial_t\delta(x,t)\,\mathrm{d}x\mathrm{d}t+\int_0^T\int_{\mathbb{R}^N}\Big\{\frac{1}{\varepsilon^N}\int_{\mathbb{R}^N}\partial_t\eta\Big(\frac{y-x}{\varepsilon},y,t\Big)\cdot v(y,t)\,\mathrm{d}y\Big\}\cdot\delta(x,t)\,\mathrm{d}x\mathrm{d}t.\quad(5.14)$$

Therefore, by (5.12), (5.13) and (5.14) we obtain

$$\int_0^T\int_{\mathbb{R}^N} u_\varepsilon(x,t)\cdot\partial_t\delta(x,t)\,\mathrm{d}x\mathrm{d}t=-\int_0^T\int_{\mathbb{R}^N}\Big\{\frac{1}{\varepsilon^N}\int_{\mathbb{R}^N}\eta\Big(\frac{y-x}{\varepsilon},y,t\Big)\cdot F\big(v(y,t)\big)\,\mathrm{d}y\Big\}:\nabla_x\delta(x,t)\,\mathrm{d}x\mathrm{d}t$$
$$-\int_0^T\int_{\mathbb{R}^N}\sum_{1\le i,k\le d}\Big\{\frac{1}{\varepsilon^N}\int_{\mathbb{R}^N}\Big(\sum_{1\le j\le N}\Big(\frac{\partial\eta_{ki}}{\partial l_j}\Big(\frac{y-x}{\varepsilon},y,t\Big)F_{ij}\big(v(y,t)\big)\Big)$$
$$+\partial_t\eta_{ki}\Big(\frac{y-x}{\varepsilon},y,t\Big)v_i(y,t)\Big)\,\mathrm{d}y\Big\}\delta_k(x,t)\,\mathrm{d}x\mathrm{d}t.\quad(5.15)$$

Thus, since we have an equality in (5.15) for every $\delta\in C_c^\infty(\mathbb{R}^N\times(0,T),\mathbb{R}^d)$ we deduce

$$\partial_t u_\varepsilon(x,t)=-\mathrm{div}_x\Big\{\frac{1}{\varepsilon^N}\int_{\mathbb{R}^N}\eta\Big(\frac{y-x}{\varepsilon},y,t\Big)\cdot F\big(v(y,t)\big)\,\mathrm{d}y\Big\}$$
$$+\frac{1}{\varepsilon^N}\int_{\mathbb{R}^N}\Big\{\nabla_l\eta\Big(\frac{y-x}{\varepsilon},y,t\Big):F\big(v(y,t)\big)+\partial_t\eta\Big(\frac{y-x}{\varepsilon},y,t\Big)\cdot v(y,t)\Big\}\,\mathrm{d}y,\quad(5.16)$$

where we denote by $\nabla_l\eta:F$ the vector in R^d with the k-th component equal to $\sum_{1\le i\le d}\sum_{1\le j\le N}\partial_{l_j}\eta_{ki}F_{ij}$. In particular we obtain that $\partial_t u_\varepsilon\in L^\infty$. So all distributional derivatives of u_ε are in L^∞. Therefore, since $v(\cdot,t)$ is continuous in $[0,T]$ with respect to L^∞-weak* topology, we obtain that $u_\varepsilon\in Lip(\mathbb{R}^N\times[0,T],\mathbb{R}^d)$.

Next set

$$\bar{\Psi}_\varepsilon(x,t) := \frac{1}{\varepsilon^N}\int_{\mathbb{R}^N}\eta\Big(\frac{y-x}{\varepsilon},x,t\Big)\cdot F\big(v(y,t)\big)\,\mathrm{d}y,$$
$$\Psi_\varepsilon(x,t) := \frac{1}{\varepsilon^N}\int_{\mathbb{R}^N}\eta\Big(\frac{y-x}{\varepsilon},y,t\Big)\cdot F\big(v(y,t)\big)\,\mathrm{d}y. \tag{5.17}$$

By Proposition 3.1, for a.e. $t\in[0,T]$ we obtain

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^N}\frac{1}{\varepsilon}\Big|\nabla_x\Big(\tilde{\Delta}_x^{-1}\big\{\mathrm{div}_x\big(F(\bar{u}_\varepsilon)-\bar{\Psi}_\varepsilon\big)\big\}\Big)\Big|^2\,\mathrm{d}x = \lim_{\varepsilon\to 0}\int_{\mathbb{R}^N}\frac{1}{\varepsilon}\Big|\nabla_x\Big(\mathrm{div}_x\big(\Delta_x^{-1}\big\{F(\bar{u}_\varepsilon)-\bar{\Psi}_\varepsilon\big\}\big)\Big)\Big|^2\,\mathrm{d}x$$
$$= \int_{(J_v)_t}\left(\int_{-\infty}^{+\infty}\left|\Big(\Big(F\big(\gamma(s,x,t)\big)-\Gamma(s,x,t)\Big)\Big)\cdot\boldsymbol{\nu}_{v(t)}(x)\right|^2\mathrm{d}s\right)\mathrm{d}\mathcal{H}^{N-1}(x), \tag{5.18}$$

where γ and Γ are defined by (5.9) and (5.10) with p defined by (5.11). Next we have the following estimates

$$\int_{\mathbb{R}^N}\big|u_\varepsilon(x,t)-\bar{u}_\varepsilon(x,t)\big|^2\mathrm{d}x = \int_{\mathbb{R}^N}\left|\int_{\mathbb{R}^N}\big(\eta(z,x+\varepsilon z,t)-\eta(z,x,t)\big)\cdot v(x+\varepsilon z,t)\,\mathrm{d}z\right|^2\mathrm{d}x$$
$$= \varepsilon^2\int_{\mathbb{R}^N}\left|\int_0^1\int_{\mathbb{R}^N}\left(\sum_{k=1}^N z_k\,\partial_{x_k}\eta(z,x+\varepsilon sz,t)\right)\cdot v(x+\varepsilon z,t)\,\mathrm{d}z\mathrm{d}s\right|^2\mathrm{d}x \le C\varepsilon^2\int_{\mathbb{R}^N}|v(x,t)|^2\mathrm{d}x, \tag{5.19}$$

where C is a constant which does not depend on ε and t. By the same way,

$$\int_{\mathbb{R}^N}\big|\Psi_\varepsilon(x,t)-\bar{\Psi}_\varepsilon(x,t)\big|^2\mathrm{d}x \le C\varepsilon^2\int_{\mathbb{R}^N}|F(v(x))|^2\mathrm{d}x \le C_1\varepsilon^2\int_{\mathbb{R}^N}|v(x,t)|^2\mathrm{d}x. \tag{5.20}$$

In particular

$$\int_{\mathbb{R}^N}\frac{1}{\varepsilon}\Big|\nabla_x\Big(\tilde{\Delta}_x^{-1}\Big\{\mathrm{div}_x\Big(\big(F(u_\varepsilon)-\Psi_\varepsilon\big)-\big(F(\bar{u}_\varepsilon)-\bar{\Psi}_\varepsilon\big)\Big)\Big\}\Big)\Big|^2\,\mathrm{d}x$$
$$\le 2\int_{\mathbb{R}^N}\frac{1}{\varepsilon}\Big|F(u_\varepsilon)-F(\bar{u}_\varepsilon)\Big|^2\,\mathrm{d}x + 2\int_{\mathbb{R}^N}\frac{1}{\varepsilon}\Big|\Psi_\varepsilon-\bar{\Psi}_\varepsilon\Big|^2\,\mathrm{d}x \le C_2\varepsilon\int_{\mathbb{R}^N}|v(x,t)|^2\mathrm{d}x.$$

Therefore, by (5.18) we obtain

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^N}\frac{1}{\varepsilon}\Big|\nabla_x\Big(\tilde{\Delta}_x^{-1}\big\{\mathrm{div}_x\big(F(u_\varepsilon)-\Psi_\varepsilon\big)\big\}\Big)\Big|^2\,\mathrm{d}x$$
$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^N}\frac{1}{\varepsilon}\Big|\nabla_x\Big(\tilde{\Delta}_x^{-1}\big\{\mathrm{div}_x\big(F(\bar{u}_\varepsilon)-\bar{\Psi}_\varepsilon\big)\big\}\Big)\Big|^2\,\mathrm{d}x$$
$$= \int_{(J_v)_t}\left(\int_{-\infty}^{+\infty}\left|\Big(\Big(F\big(\gamma(s,x,t)\big)-\Gamma(s,x,t)\Big)\Big)\cdot\boldsymbol{\nu}_{v(t)}(x)\right|^2\mathrm{d}s\right)\mathrm{d}\mathcal{H}^{N-1}(x). \tag{5.21}$$

Since for every $t \in [0,T]$ we have $\partial_{x_j}\eta_{ki}(z,x,t) \in \mathcal{U}(\mathbb{R}^N)$ and $\partial_t\eta_{ki}(z,x,t) \in \mathcal{U}(\mathbb{R}^N)$, by Lemma A.2 from the Appendix and (5.16) from (5.21) we obtain

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^N}\frac{1}{\varepsilon}\Big|\nabla_x\Big(\tilde{\Delta}_x^{-1}\{\partial_t u_\varepsilon + \operatorname{div}_x F(u_\varepsilon)\}\Big)\Big|^2 \mathrm{d}x = \lim_{\varepsilon\to 0}\int_{\mathbb{R}^N}\frac{1}{\varepsilon}\Big|\nabla_x\Big(\operatorname{div}_x\big(\Delta_x^{-1}\{F(u_\varepsilon)-\Psi_\varepsilon\}\big)\Big)\Big|^2 \mathrm{d}x$$
$$= \int_{(J_v)_t}\left(\int_{-\infty}^{+\infty}\Big|\Big(\big(F\big(\gamma(s,x,t)\big)-\Gamma(s,x,t)\big)\Big)\cdot\boldsymbol{\nu}_{v(t)}(x)\Big|^2\mathrm{d}s\right)\mathrm{d}\mathcal{H}^{N-1}(x) \qquad \text{for a.e. } t\in[0,T]. \quad (5.22)$$

But

$$\int_{\mathbb{R}^N}\frac{1}{\varepsilon}\Big|\nabla_x\Big(\operatorname{div}_x\big(\Delta_x^{-1}\{F(u_\varepsilon)-\Psi_\varepsilon\}\big)\Big)\Big|^2\mathrm{d}x \le C\int_{\mathbb{R}^N}\frac{1}{\varepsilon}\Big|F(u_\varepsilon)-\Psi_\varepsilon\Big|^2\mathrm{d}x \le C_1\|D_x v_{(t)}\|(\mathbb{R}^N).$$

Therefore, by (5.22), (5.6), (5.16) and the Dominated Convergence Theorem (see also Lem. A.2) we obtain (5.8). □

Next, by Theorem 2.1, we obtain

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^N}\varepsilon|\nabla_x\bar{u}_\varepsilon|^2\,\mathrm{d}x = \int_{(J_v)_t}\left(\int_{-\infty}^{+\infty}\big|p(s,x,t)\cdot\big(v^+(x,t)-v^-(x,t)\big)\big|^2\mathrm{d}s\right)\mathrm{d}\mathcal{H}^{N-1}(x)$$
$$\text{for a.e. } t\in[0,T]. \quad (5.23)$$

But for any $1\le j\le N$ we have

$$\frac{\partial\bar{u}_\varepsilon(x,t)}{\partial x_j} = \int_{\mathbb{R}^N}\Big\{-\frac{1}{\varepsilon}\frac{\partial}{\partial z_j}\eta(z,x,t)+\frac{\partial}{\partial x_j}\eta(z,x,t)\Big\}\cdot v(x+\varepsilon z,t)\,\mathrm{d}z,$$
$$\frac{\partial u_\varepsilon(x,t)}{\partial x_j} = \int_{\mathbb{R}^N}\Big\{-\frac{1}{\varepsilon}\frac{\partial}{\partial z_j}\eta(z,x+\varepsilon z,t)\Big\}\cdot v(x+\varepsilon z,t)\,\mathrm{d}z.$$

Therefore by the similar way as we did in (5.19)–(5.21), using (5.23) we obtain

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^N}\varepsilon|\nabla_x u_\varepsilon|^2\,\mathrm{d}x = \int_{(J_v)_t}\left(\int_{-\infty}^{+\infty}\big|p(s,x,t)\cdot\big(v^+(x,t)-v^-(x,t)\big)\big|^2\mathrm{d}s\right)\mathrm{d}\mathcal{H}^{N-1}(x)$$
$$\text{for a.e. } t\in[0,T]. \quad (5.24)$$

Moreover,

$$\int_{\mathbb{R}^N}\varepsilon|\nabla_x u_\varepsilon|^2\,\mathrm{d}x \le C\|D_x v_{(t)}\|(\mathbb{R}^N).$$

So, as before, using (5.24), we deduce

$$\lim_{\varepsilon\to 0}\int_0^T\int_{\mathbb{R}^N}\varepsilon|\nabla_x u_\varepsilon|^2\,\mathrm{d}x\mathrm{d}t = \int_0^T\int_{(J_v)_t}\left(\int_{-\infty}^{+\infty}\big|p(s,x,t)\cdot\big(v^+(x,t)-v^-(x,t)\big)\big|^2\mathrm{d}s\right)\mathrm{d}\mathcal{H}^{N-1}(x)\,\mathrm{d}t. \quad (5.25)$$

Then, by linking (5.8) with (5.25) we can prove the following proposition.

Proposition 5.1. *Given $v \in \mathcal{E}_T^F$ (see Def. 5.5) and $\eta \in \mathcal{V}_T^{(d)}$ (see Def. 5.3) consider $u_\varepsilon(x,t) : \mathbb{R}^N \times [0,T] \to \mathbb{R}^d$ as in (5.7). Then*

$$\lim_{\varepsilon\to 0} \int_0^T \int_{\mathbb{R}^N} \left\{\varepsilon|\nabla_x u_\varepsilon|^2 + \frac{1}{\varepsilon}\Big|\nabla_x\Big(\tilde{\Delta}_x^{-1}\{\partial_t u_\varepsilon + \operatorname{div}_x F(u_\varepsilon)\}\Big)\Big|^2\right\} \mathrm{d}x\mathrm{d}t = H_v(\eta)$$
$$:= \int_{J_v} \left(\int_{-\infty}^{+\infty} \left\{|(\boldsymbol{\nu}_v)_x| \cdot \Big|\frac{\partial\gamma}{\partial s}(s,x,t)\Big|^2 + \frac{1}{|(\boldsymbol{\nu}_v)_x|} \cdot \Big|L\big(\gamma(s,x,t),v,x,t\big)\Big|^2\right\}\mathrm{d}s\right)\mathrm{d}\mathcal{H}^N(x,t), \quad (5.26)$$

where

$$L(\gamma,v,x,t) := (\boldsymbol{\nu}_v)_t\big(\gamma - v^-(x,t)\big) + \Big(F(\gamma) - F\big(v^-(x,t)\big)\Big)\cdot(\boldsymbol{\nu}_v)_x \quad (5.27)$$

and where γ, and p are defined by (5.9), and (5.11) respectively.

Proof. By (5.8) and (5.25) we obtain

$$\lim_{\varepsilon\to 0} \int_0^T \int_{\mathbb{R}^N} \left\{\varepsilon|\nabla_x u_\varepsilon|^2 + \frac{1}{\varepsilon}\Big|\nabla_x\Big(\tilde{\Delta}_x^{-1}\{\partial_t u_\varepsilon + \operatorname{div}_x F(u_\varepsilon)\}\Big)\Big|^2\right\} \mathrm{d}x\mathrm{d}t$$
$$= \int_0^T \int_{(J_v)_t} \left(\int_{-\infty}^{+\infty} \left\{\Big|\frac{\partial\gamma}{\partial s}(s,x,t)\Big|^2 + \Big|\Big(F\big(\gamma(s,x,t)\big) - \Gamma(s,x,t)\Big)\Big)\cdot\boldsymbol{\nu}_{v_{(t)}}(x)\Big|^2\right\}\mathrm{d}s\right)\mathrm{d}\mathcal{H}^{N-1}(x)\,\mathrm{d}t, \quad (5.28)$$

where γ, Γ and p are defined by (5.9), (5.10) and (5.11) respectively. On the other hand, by (5.3) we deduce

$$(\boldsymbol{\nu}_v)_t\big(v^+(x,t) - v^-(x,t)\big) + \Big(F\big(v^+(x,t)\big) - F\big(v^-(x,t)\big)\Big)\cdot(\boldsymbol{\nu}_v)_x = 0 \quad \forall (x,t) \in J_v. \quad (5.29)$$

In particular there exists $c > 0$ independent on (x,t) such that $c \le |(\boldsymbol{\nu}_v)_x(x,t)| \le 1$. Therefore, by (5.5), (5.6) and (5.29) we can rewrite (5.28) in the form

$$\lim_{\varepsilon\to 0} \int_0^T \int_{\mathbb{R}^N} \left\{\varepsilon|\nabla_x u_\varepsilon|^2 + \frac{1}{\varepsilon}\Big|\nabla_x\Big(\tilde{\Delta}_x^{-1}\{\partial_t u_\varepsilon + \operatorname{div}_x F(u_\varepsilon)\}\Big)\Big|^2\right\} \mathrm{d}x\mathrm{d}t$$
$$= \int_0^T \int_{(J_v)_t} \frac{1}{|(\boldsymbol{\nu}_v)_x|}\left(\int_{-\infty}^{+\infty} \left\{|(\boldsymbol{\nu}_v)_x| \cdot \Big|\frac{\partial\gamma}{\partial s}(s,x,t)\Big|^2 + \frac{1}{|(\boldsymbol{\nu}_v)_x|} \cdot \Big|L\big(\gamma(s,x,t),v,x,t\big)\Big|^2\right\}\mathrm{d}s\right)\mathrm{d}\mathcal{H}^{N-1}(x)\,\mathrm{d}t$$
$$= \int_{J_v} \left(\int_{-\infty}^{+\infty} \left\{|(\boldsymbol{\nu}_v)_x| \cdot \Big|\frac{\partial\gamma}{\partial s}(s,x,t)\Big|^2 + \frac{1}{|(\boldsymbol{\nu}_v)_x|} \cdot \Big|L\big(\gamma(s,x,t),v,x,t\big)\Big|^2\right\}\mathrm{d}s\right)\mathrm{d}\mathcal{H}^N(x,t), \quad (5.30)$$

where $L(\gamma,v,x,t)$ is defined by (5.27). □

Remark 5.3. For the proof of Proposition 5.1, we used the results of Section 3. We proved these results for the case $N \ge 2$, but we can prove these results, with an even simpler proof, also for $N = 1$.

Next, as in [8] and as above, we can prove the following proposition.

Proposition 5.2.

$$\inf_{\eta\in V_{T,\eta_0}^{(d)}} H_v(\eta) = \int_{J_v}\left(\inf_{r\in\mathcal{R}_{v^+,v^-}}\left\{\int_{-1}^{1} 2\big|r'(s)\big|\cdot\Big|(\boldsymbol{\nu}_v)_t\big(r(s)-v^-\big) + \Big(F\big(r(s)\big) - F\big(v^-\big)\Big)\cdot(\boldsymbol{\nu}_v)_x\Big|\mathrm{d}s\right\}\right)\mathrm{d}\mathcal{H}^N(x,t), \quad (5.31)$$

where $V^{(d)}_{T,\eta_0}$ *is defined in Definition* 5.4, $H_v(\eta)$ *is defined by* (5.26) *and*

$$\mathcal{R}_{\boldsymbol{a},\boldsymbol{b}} := \Big\{ r(t) \in C^1\big([-1,1],\mathbb{R}^d\big) : r(-1) = \boldsymbol{a},\, r(1) = \boldsymbol{b} \Big\}\cdot \tag{5.32}$$

Then, as before, by Proposition 5.1 and Proposition 5.2 we can deduce the following proposition.

Proposition 5.3. *Given* $v \in \mathcal{E}^F_T$ *(see Def.* 5.5*) and* $\eta_0 \in C^2_c(\mathbb{R}^N,\mathbb{R})$ *satisfying* $\int_{\mathbb{R}^N}\eta_0(z)\mathrm{d}z = 1$ *set* $w_{\{\eta_0,\varepsilon\}}(x)$ *by* $w_{\{\eta_0,\varepsilon\}}(x) := \frac{1}{\varepsilon^N}\int_{\mathbb{R}^N}\eta_0((y-x)/\varepsilon)v(y,0)\,\mathrm{d}y$*. Then there exist a sequence of functions* $\{u_{\{\eta_0,\varepsilon\}}(x,t)\}_{\varepsilon>0} \subset L^\infty(0,T;L^2(\mathbb{R}^N,\mathbb{R}^d)) \cap Lip(\mathbb{R}^N\times[0,T],\mathbb{R}^d)\cap L^\infty$ *such that* $u_{\{\eta_0,\varepsilon\}} \to v$ *in* $\bigcap_{q\geq 1} L^q(\mathbb{R}^N\times(0,T),\mathbb{R}^d)$ *as* $\varepsilon\to 0$*,* $u_{\{\eta_0,\varepsilon\}}(x,0) = w_{\{\eta_0,\varepsilon\}}(x)$*,*

$$\lim_{\varepsilon\to0}\left(\int_0^T\int_{\mathbb{R}^N}\left\{\varepsilon|\nabla_x u_{\{\eta_0,\varepsilon\}}|^2+\frac{1}{\varepsilon}\Big|\nabla_x\Big(\tilde{\Delta}_x^{-1}\big\{\partial_t u_{\{\eta_0,\varepsilon\}}+\mathrm{div}_x F(u_{\{\eta_0,\varepsilon\}})\big\}\Big)\Big|^2\right\}\mathrm{d}x\mathrm{d}t+\int_{\mathbb{R}^N}|u_{\{\eta_0,\varepsilon\}}(x,T)|^2\mathrm{d}x\right)$$
$$=\int_{J_v}\left(\inf_{r\in\mathcal{R}_{v^+,v^-}}\left\{\int_{-1}^1 2\big|r'(s)\big|\cdot\Big|(\boldsymbol{\nu}_v)_t\big(r(s)-v^-\big)+\Big(F\big(r(s)\big)-F\big(v^-\big)\Big)\cdot(\boldsymbol{\nu}_v)_x\Big|\mathrm{d}s\right\}\right)\mathrm{d}\mathcal{H}^N(x,t)+\int_{\mathbb{R}^N}|v(x,T)|^2\mathrm{d}x, \tag{5.33}$$

where $\mathcal{R}_{\boldsymbol{a},\boldsymbol{b}}$ *is defined by* (5.32) *and* $u_\varepsilon(x,T)\to v(x,T)$ *in* $L^2(\mathbb{R}^N,\mathbb{R}^d)$*.*

Now we can prove the main theorem of this section.

Theorem 5.1. *Given* $v \in \mathcal{E}^F_T$ *(see Def.* 5.5*) there exist a sequence of functions* $\{\bar{u}_\varepsilon(x,t)\}_{\varepsilon>0} \subset L^2(0,T;H^1_0(\mathbb{R}^N,\mathbb{R}^d))\cap L^\infty(0,T;L^2(\mathbb{R}^N,\mathbb{R}^d))\cap L^\infty$ *such that* $\bar{u}_\varepsilon\to v$ *in* $\bigcap_{q\geq1}L^q(\mathbb{R}^N\times(0,T),\mathbb{R}^d)$*,* $\bar{u}_\varepsilon$ *is* L^2*-strongly continuous in* $[0,T]$ *as a function of* t*,* $\bar{u}_\varepsilon(x,0) = v(x,0)$*,*

$$\lim_{\varepsilon\to0}\left(\int_0^T\int_{\mathbb{R}^N}\left\{\varepsilon|\nabla_x\bar{u}_\varepsilon|^2+\frac{1}{\varepsilon}\Big|\nabla_x\Big(\tilde{\Delta}_x^{-1}\big\{\partial_t\bar{u}_\varepsilon+\mathrm{div}_x F(\bar{u}_\varepsilon)\big\}\Big)\Big|^2\right\}\mathrm{d}x\mathrm{d}t+\int_{\mathbb{R}^N}|\bar{u}_\varepsilon(x,T)|^2\mathrm{d}x\right)$$
$$=\int_{J_v}\left(\inf_{r\in\mathcal{R}_{v^+,v^-}}\left\{\int_{-1}^1 2\big|r'(s)\big|\cdot\Big|(\boldsymbol{\nu}_v)_t\big(r(s)-v^-\big)+\Big(F\big(r(s)\big)-F\big(v^-\big)\Big)\cdot(\boldsymbol{\nu}_v)_x\Big|\mathrm{d}s\right\}\right)\mathrm{d}\mathcal{H}^N(x,t)+\int_{\mathbb{R}^N}|v(x,T)|^2\mathrm{d}x, \tag{5.34}$$

where $\mathcal{R}_{\boldsymbol{a},\boldsymbol{b}}$ *is defined by* (5.32) *and* $\bar{u}_\varepsilon(x,T)\to v(x,T)$ *in* $L^2(\mathbb{R}^N,\mathbb{R}^d)$*.*

Proof. Let $\eta_1\in C^2_c(\mathbb{R}^N,\mathbb{R})$ satisfying $\int_{\mathbb{R}^N}\eta_1(z)\mathrm{d}z = 1$ and $\eta_1\geq 0$. For every $h>0$ set $\eta_h(z) := \eta_1(z/h)/h^N$. For every $\varepsilon>0$ and $h>0$ let $u_{h,\varepsilon} := u_{\{\eta_h,\varepsilon\}}$ and $w_{h,\varepsilon} := w_{\{\eta_h,\varepsilon\}}(x)$ be as in Proposition 5.3 corresponding to η_h. We want to modify $u_{h,\varepsilon}$ in order to satisfy the initial conditions. Let $\chi_\varepsilon(x,t)\in L^\infty(0,T;L^2(\mathbb{R}^N,\mathbb{R}^d))\cap L^2(0,T;H^1_0(\mathbb{R}^N,\mathbb{R}^d))\cap L^\infty$ be the solution of the heat equation

$$\begin{cases}\varepsilon\Delta_x\chi_\varepsilon = \partial_t\chi_\varepsilon,\\ \chi_\varepsilon(x,0) = v(x,0).\end{cases} \tag{5.35}$$

It is clear that we may assume that χ_ε is L^2-strongly continuous in $[0,T]$ as a function of t and $\chi_\varepsilon(x,0)=v(x,0)$. Moreover for every $0\leq\bar t\leq T$ we have

$$2\int_0^{\bar t}\int_{\mathbb{R}^N}\varepsilon|\nabla\chi_\varepsilon|^2=\int_{\mathbb{R}^N}v^2(x,0)\mathrm{d}x-\int_{\mathbb{R}^N}\chi_\varepsilon^2(x,\bar t)\mathrm{d}x. \tag{5.36}$$

Next let $\theta(t)\in C^\infty(\mathbb{R},[0,1])$ be a cut-off function satisfying $\theta(t)=0$ for every $t\geq 1$ and $\theta(t)=1$ for every $t\leq 1/2$. For every small $\varepsilon>0$ define $\bar u_{h,\varepsilon}(x,t)\in L^\infty(0,T;L^2(\mathbb{R}^N,\mathbb{R}^d))\cap L^2(0,T;H_0^1(\mathbb{R}^N,\mathbb{R}^d))\cap L^\infty$ by

$$\bar u_{h,\varepsilon}(x,t):=u_{h,\varepsilon}(x,t)+\theta(t(h\varepsilon)^{-1})\big(\chi_\varepsilon(x,t)-u_{h,\varepsilon}(x,0)\big)=u_{h,\varepsilon}(x,t)+\theta(t(h\varepsilon)^{-1})\big(\chi_\varepsilon(x,t)-w_{h,\varepsilon}(x)\big). \tag{5.37}$$

Then $\bar u_{h,\varepsilon}$ is L^2-strongly continuous in $[0,T]$ as a function of t and $\bar u_{h,\varepsilon}(x,0)=v(x,0)$. Moreover, $\bar u_{h,\varepsilon}(x,t)=u_{h,\varepsilon}(x,t)$ whenever $t\geq h\varepsilon$. Now we will want to prove that

$$\varlimsup_{\varepsilon\to0^+}\int_0^{h\varepsilon}\int_{\mathbb{R}^N}\left\{\varepsilon\big|\nabla_x(\bar u_{h,\varepsilon}-u_{h,\varepsilon})\big|^2+\frac{1}{\varepsilon}\Big|\nabla_x\Big(\tilde\Delta_x^{-1}\big\{\partial_t(\bar u_{h,\varepsilon}-u_{h,\varepsilon})+\mathrm{div}_x\big(F(\bar u_{h,\varepsilon})-F(u_{h,\varepsilon})\big)\big\}\Big)\Big|^2\right\}\mathrm{d}x\mathrm{d}t\leq Ch, \tag{5.38}$$

where $C>0$ is a constant which does not depend on h and ε. First of all by (5.37) we observe that

$$\begin{aligned}\varlimsup_{\varepsilon\to0^+}\int_0^{h\varepsilon}\int_{\mathbb{R}^N}\varepsilon\big|\nabla_x(\bar u_{h,\varepsilon}-u_{h,\varepsilon})\big|^2\mathrm{d}x\mathrm{d}t&\leq\varlimsup_{\varepsilon\to0^+}2\varepsilon^2h\int_{\mathbb{R}^N}\big|\nabla_x w_{h,\varepsilon}(x)\big|^2\mathrm{d}x\\&+\varlimsup_{\varepsilon\to0^+}\int_0^{h\varepsilon}\int_{\mathbb{R}^N}2\varepsilon\big|\nabla_x\chi_\varepsilon\big|^2\mathrm{d}x\mathrm{d}t=\varlimsup_{\varepsilon\to0^+}2h\int_{\mathbb{R}^N}\left|\int_{\mathbb{R}^N}\nabla\eta_h(z)\otimes v(x+\varepsilon z)\mathrm{d}z\right|^2\mathrm{d}x\\&+\varlimsup_{\varepsilon\to0^+}\left(\int_{\mathbb{R}^N}\chi_\varepsilon^2(x,0)\mathrm{d}x-\int_{\mathbb{R}^N}\chi_\varepsilon^2(x,h\varepsilon)\mathrm{d}x\right)=0.\end{aligned} \tag{5.39}$$

On the other hand

$$\begin{aligned}\varlimsup_{\varepsilon\to0^+}\int_0^{h\varepsilon}\int_{\mathbb{R}^N}\frac{1}{\varepsilon}\Big|\nabla_x\Big(\tilde\Delta_x^{-1}\big\{\mathrm{div}_x\big(F(\bar u_{h,\varepsilon})-F(u_{h,\varepsilon})\big)\big\}\Big)\Big|^2\mathrm{d}x\mathrm{d}t&\leq C_0\varlimsup_{\varepsilon\to0^+}\int_0^{h\varepsilon}\int_{\mathbb{R}^N}\frac{1}{\varepsilon}|\bar u_{h,\varepsilon}-u_{h,\varepsilon}|^2\mathrm{d}x\mathrm{d}t\\&\leq C_0\varlimsup_{\varepsilon\to0^+}\int_0^{h\varepsilon}\int_{\mathbb{R}^N}\frac{2}{\varepsilon}\big(|\chi_\varepsilon(x,t)|^2+|w_{h,\varepsilon}(x)|^2\big)\mathrm{d}x\mathrm{d}t\leq 4C_0h\int_{\mathbb{R}^N}v^2(x,0)\mathrm{d}x=O(h).\end{aligned} \tag{5.40}$$

Next we have

$$\begin{aligned}\partial_t(\bar u_{h,\varepsilon}-u_{h,\varepsilon})&=\partial_t\Big\{\theta(t(h\varepsilon)^{-1})\big(\chi_\varepsilon(x,t)-w_{h,\varepsilon}(x)\big)\Big\}\\&=\theta(t(h\varepsilon)^{-1})\partial_t\chi_\varepsilon(x,t)+(h\varepsilon)^{-1}\theta'(t(h\varepsilon)^{-1})\big(\chi_\varepsilon(x,t)-w_{h,\varepsilon}(x)\big)\\&=\theta(t(h\varepsilon)^{-1})\varepsilon\Delta_x\chi_\varepsilon(x,t)+(h\varepsilon)^{-1}\theta'(t(h\varepsilon)^{-1})\big(\chi_\varepsilon(x,t)-w_{h,\varepsilon}(x)\big).\end{aligned} \tag{5.41}$$

Therefore, by (5.41) we obtain

$$\varlimsup_{\varepsilon\to 0^+}\int_0^{h\varepsilon}\int_{\mathbb{R}^N}\frac{1}{\varepsilon}\Big|\nabla_x\Big(\tilde{\Delta}_x^{-1}\{\partial_t(\bar{u}_{h,\varepsilon}-u_{h,\varepsilon})\}\Big)\Big|^2\mathrm{d}x\mathrm{d}t\le C\Bigg(\varlimsup_{\varepsilon\to 0^+}\int_0^{h\varepsilon}\int_{\mathbb{R}^N}\frac{1}{h^2\varepsilon^3}$$
$$\times\Big|\nabla_x\Big(\tilde{\Delta}_x^{-1}\big(\chi_\varepsilon(x,t)-w_{h,\varepsilon}(x)\big)\Big)\Big|^2\mathrm{d}x\mathrm{d}t+\varlimsup_{\varepsilon\to 0^+}\int_0^{h\varepsilon}\int_{\mathbb{R}^N}\varepsilon\big|\nabla_x\chi_\varepsilon\big|^2\mathrm{d}x\mathrm{d}t\Bigg)$$
$$\le C\Bigg(\varlimsup_{\varepsilon\to 0^+}\frac{1}{2}\bigg(\int_{\mathbb{R}^N}\chi_\varepsilon^2(x,0)\mathrm{d}x-\int_{\mathbb{R}^N}\chi_\varepsilon^2(x,h\varepsilon)\mathrm{d}x\bigg)$$
$$+\varlimsup_{\varepsilon\to 0^+}\int_0^{h\varepsilon}\int_{\mathbb{R}^N}\frac{1}{h^2\varepsilon^3}\Big|\nabla_x\Big(\tilde{\Delta}_x^{-1}\big\{\big((\chi_\varepsilon(x,t)-v(x,0))-(w_{h,\varepsilon}(x)-v(x,0))\big)\big\}\Big)\Big|^2\mathrm{d}x\mathrm{d}t\Bigg)$$
$$\le 0+2C\varlimsup_{\varepsilon\to 0^+}\int_0^{h\varepsilon}\int_{\mathbb{R}^N}\frac{1}{h^2\varepsilon^3}\Big|\nabla_x\Big(\tilde{\Delta}_x^{-1}\big\{\big((\chi_\varepsilon(x,t)-\chi_\varepsilon(x,0))\big)\big\}\Big)\Big|^2\mathrm{d}x\mathrm{d}t$$
$$+2C\varlimsup_{\varepsilon\to 0^+}\int_{\mathbb{R}^N}\frac{1}{h\varepsilon^2}\Big|\nabla_x\Big(\tilde{\Delta}_x^{-1}\big\{\big((w_{h,\varepsilon}(x)-v(x,0))\big)\big\}\Big)\Big|^2\mathrm{d}x. \quad (5.42)$$

But

$$\chi_\varepsilon(x,t)-\chi_\varepsilon(x,0)=\varepsilon\Delta_x\bigg(\int_0^t\chi_\varepsilon(x,s)\mathrm{d}s\bigg). \quad (5.43)$$

Therefore,

$$\varlimsup_{\varepsilon\to 0^+}\int_0^{h\varepsilon}\int_{\mathbb{R}^N}\frac{1}{h^2\varepsilon^3}\Big|\nabla_x\Big(\tilde{\Delta}_x^{-1}\big\{\big((\chi_\varepsilon(x,t)-\chi_\varepsilon(x,0))\big)\big\}\Big)\Big|^2\mathrm{d}x\mathrm{d}t=\varlimsup_{\varepsilon\to 0^+}\int_0^{h\varepsilon}\int_{\mathbb{R}^N}\frac{1}{h^2\varepsilon}\bigg|\int_0^t\nabla_x\chi_\varepsilon(x,s)\,\mathrm{d}s\bigg|^2\mathrm{d}x\mathrm{d}t$$
$$\le\varlimsup_{\varepsilon\to 0^+}\int_0^{h\varepsilon}\int_{\mathbb{R}^N}\frac{t}{h^2\varepsilon}\bigg(\int_0^t\big|\nabla_x\chi_\varepsilon(x,s)\big|^2\mathrm{d}s\bigg)\mathrm{d}x\mathrm{d}t\le\varlimsup_{\varepsilon\to 0^+}\int_0^{h\varepsilon}\int_{\mathbb{R}^N}\varepsilon\big|\nabla_x\chi_\varepsilon\big|^2\mathrm{d}x\mathrm{d}t$$
$$=\varlimsup_{\varepsilon\to 0^+}\frac{1}{2}\bigg(\int_{\mathbb{R}^N}\chi_\varepsilon^2(x,0)\mathrm{d}x-\int_{\mathbb{R}^N}\chi_\varepsilon^2(x,h\varepsilon)\mathrm{d}x\bigg)=0. \quad (5.44)$$

Then, by (5.42) and (5.44) we obtain

$$\varlimsup_{\varepsilon\to 0^+}\int_0^{h\varepsilon}\int_{\mathbb{R}^N}\frac{1}{\varepsilon}\Big|\nabla_x\Big(\tilde{\Delta}_x^{-1}\{\partial_t(\bar{u}_{h,\varepsilon}-u_{h,\varepsilon})\}\Big)\Big|^2\mathrm{d}x\mathrm{d}t\le 2C\varlimsup_{\varepsilon\to 0^+}\int_{\mathbb{R}^N}\frac{1}{h\varepsilon^2}\Big|\nabla_x\Big(\tilde{\Delta}_x^{-1}\big\{\big((w_{h,\varepsilon}(x)-v(x,0))\big)\big\}\Big)\Big|^2\mathrm{d}x$$
$$\le\bar{C}h\int_{\mathbb{R}^N}v^2(x,0)\mathrm{d}x=O(h). \quad (5.45)$$

Here we deduce the last inequality in the similar way as (A.11) in Lemma A.2. So combining (5.39), (5.40) and (5.45) we obtain (5.38). Then we obtain

$$\lim_{\varepsilon\to 0}\Bigg(\int_0^T\int_{\mathbb{R}^N}\bigg\{\varepsilon|\nabla_x\bar{u}_{h,\varepsilon}|^2+\frac{1}{\varepsilon}\Big|\nabla_x\Big(\tilde{\Delta}_x^{-1}\{\partial_t\bar{u}_{h,\varepsilon}+\mathrm{div}_xF(\bar{u}_{h,\varepsilon})\}\Big)\Big|^2\bigg\}\mathrm{d}x\mathrm{d}t+\int_{\mathbb{R}^N}|\bar{u}_{h,\varepsilon}(x,T)|^2\mathrm{d}x\Bigg)$$
$$=\int_{J_v}\Bigg(\inf_{r\in\mathcal{R}_{v^+,v^-}}\bigg\{\int_{-1}^1 2\big|r'(s)\big|\cdot\Big|(\boldsymbol{\nu}_v)_t\big(r(s)-v^-\big)+\Big(F\big(r(s)\big)-F\big(v^-\big)\Big)\cdot(\boldsymbol{\nu}_v)_x\Big|\mathrm{d}s\bigg\}\Bigg)\mathrm{d}\mathcal{H}^N(x,t)$$
$$+\int_{\mathbb{R}^N}|v(x,T)|^2\mathrm{d}x+o_h(1). \quad (5.46)$$

Finally we complete the prove by taking a diagonal subsequence from $\{\bar{u}_{h,\varepsilon}\}$, as we did before. □

Appendix A

Lemma A.1. *Let $G(x) \in C^1(\mathbb{R}^d,\mathbb{R})$ satisfy $G \geq 0$ and $G(\boldsymbol{a}) = G(\boldsymbol{b}) = 0$, for some $\boldsymbol{a} \neq \boldsymbol{b}$ in $\mathbb{R}^d$ and let $\Theta(x) : \mathbb{R}^d \to \mathbb{R}^{l\times d}$ satisfy $\Theta \in C^1$ and $\operatorname{rank}\big(\Theta(x)\big) = d$ for every $x \in \mathbb{R}^d$. Then,*

$$\inf_{\zeta\in\mathcal{R}^{(0)}_{\boldsymbol{a},\boldsymbol{b}}} \left(\int_{-\infty}^{+\infty} \big|\Theta\big(\zeta(t)\big)\cdot\zeta'(t)\big|^2 + G\big(\zeta(t)\big)\,\mathrm{d}t\right) = \inf_{r\in\mathcal{R}_{\boldsymbol{a},\boldsymbol{b}}} \left(\int_{-1}^{1} 2\big|\Theta\big(r(t)\big)\cdot r'(t)\big|\sqrt{G\big(r(t)\big)}\mathrm{d}t\right), \tag{A.1}$$

where

$$\mathcal{R}^{(0)}_{\boldsymbol{a},\boldsymbol{b}} := \Big\{\zeta(t) \in C^1\big(\mathbb{R},\mathbb{R}^d\big) : \exists L > 0 \text{ s.t. } \zeta(t) = \boldsymbol{a}\ \ \forall t \leq -L,\ \zeta(t) = \boldsymbol{b}\ \ \forall t \geq L\Big\} \tag{A.2}$$

and

$$\mathcal{R}_{\boldsymbol{a},\boldsymbol{b}} := \Big\{r(t) \in C^1\big([-1,1],\mathbb{R}^d\big) : r(-1) = \boldsymbol{a},\, r(1) = \boldsymbol{b}\Big\}. \tag{A.3}$$

Proof. For any $\zeta \in \mathcal{R}^{(0)}_{\boldsymbol{a},\boldsymbol{b}}$ define $r : [-1,1] \to \mathbb{R}$ by

$$r(t) := \zeta(L\,t),$$

where $L > 0$ is as in (A.2). Then $r \in \mathcal{R}_{\boldsymbol{a},\boldsymbol{b}}$ and, by the inequality $a^2 + b^2 \geq 2ab$, we have

$$\begin{aligned}\int_{-L}^{L} \Big\{\big|\Theta\big(\zeta(t)\big)\cdot\zeta'(t)\big|^{\,2} + G\big(\zeta(t)\big)\Big\}\,\mathrm{d}t &\geq \int_{-L}^{L} 2\big|\Theta\big(\zeta(t)\big)\cdot\zeta'(t)\big|\sqrt{G\big(\zeta(t)\big)}\,\mathrm{d}t\\ &= \int_{-1}^{1} 2\big|\Theta\big(r(t)\big)\cdot r'(t)\big|\sqrt{G\big(r(t)\big)}\,\mathrm{d}t.\end{aligned}$$

Therefore, we infer

$$\inf_{\zeta\in\mathcal{R}^{(0)}_{\boldsymbol{a},\boldsymbol{b}}} \left(\int_{-\infty}^{+\infty} \big|\Theta\big(\zeta(t)\big)\cdot\zeta'(t)\big|^2 + G\big(\zeta(t)\big)\,\mathrm{d}t\right) \geq \inf_{r\in\mathcal{R}_{\boldsymbol{a},\boldsymbol{b}}} \left(\int_{-1}^{1} 2\big|\Theta\big(r(t)\big)\cdot r'(t)\big|\sqrt{G\big(r(t)\big)}\mathrm{d}t\right). \tag{A.4}$$

Next let $r(t) \in \mathcal{R}_{\boldsymbol{a},\boldsymbol{b}}$, such that $r'(t) \neq 0$ for any $t \in [-1,1]$. Define $\tau_n(t) \in C^1(\mathbb{R},[-1,1])$ as the solution of

$$\begin{cases}\tau_n'(t) = \dfrac{\sqrt{G\left(r(\tau_n(t))\right)}}{|\Theta(r(\tau_n(t)))\cdot r'(\tau_n(t))|} + \frac{1}{n}\big(1 - |\tau_n(t)|\big) & \forall t \in \mathbb{R},\\ \tau_n(0) = 0.\end{cases} \tag{A.5}$$

Then τ_n is nondecreasing on $\mathbb{R}$ and $\lim\limits_{t\to-\infty}\tau_n(t) = -1$, $\lim\limits_{t\to+\infty}\tau_n(t) = 1$. Moreover,

$$0 \leq 1 - |\tau_n(t)| \leq \mathrm{e}^{-|t|/n} \quad \forall t \in \mathbb{R}. \tag{A.6}$$

Next for any $n \in \mathbb{N}$ define $\zeta_n(t) : \mathbb{R} \to \mathbb{R}^d$ by

$$\zeta_n(t) := r\big(\tau_n(t)\big).$$

Consider

$$I_0(\zeta) := \int_{-\infty}^{+\infty} \big|\Theta\big(\zeta(t)\big)\cdot\zeta'(t)\big|^2 + G\big(\zeta(t)\big)\,\mathrm{d}t,$$

and

$$I(r) := \int_{-1}^{1} 2\big|\Theta\big(r(t)\big)\cdot r'(t)\big|\,\sqrt{G\big(r(t)\big)}\mathrm{d}t.$$

Then

$$\begin{aligned} I_0\big(\zeta_n(t)\big) &= \int_{-\infty}^{+\infty} \big|\Theta\big(\zeta_n(t)\big)\cdot\zeta_n'(t)\big|^2 + G\big(\zeta_n(t)\big)\,\mathrm{d}t \\ &= \int\limits_{-\infty}^{+\infty} \big(\tau_n'(t)\big)^2\Big|\Theta\big(r(\tau_n(t))\big)\cdot r'(\tau_n(t))\Big|^2 + \int\limits_{-\infty}^{+\infty} G\big(r(\tau_n(t))\big)\mathrm{d}t. \end{aligned} \tag{A.7}$$

Therefore, by (A.5),

$$\begin{aligned} I_0\big(\zeta_n(t)\big) =& \int\limits_{-\infty}^{+\infty} \tau_n'(t)\Big|\Theta\big(r(\tau_n(t))\big)\cdot r'(\tau_n(t))\Big|\left(\sqrt{G\big(r(\tau_n(t))\big)} + \frac{1}{n}\big(1-|\tau_n(t)|\big)\Big|\Theta\big(r(\tau_n(t))\big)\cdot r'(\tau_n(t))\Big|\right)\mathrm{d}t \\ &+ \int\limits_{-\infty}^{+\infty}\left(\tau_n'(t)\Big|\Theta\big(r(\tau_n(t))\big)\cdot r'(\tau_n(t))\Big| - \frac{1}{n}\big(1-|\tau_n(t)|\big)\Big|\Theta\big(r(\tau_n(t))\big)\cdot r'(\tau_n(t))\Big|\right)\sqrt{G\big(r(\tau_n(t))\big)}\mathrm{d}t \\ =& I(r) + \frac{1}{n}\int\limits_{-\infty}^{+\infty}\big(1-|\tau_n(t)|\big)\Big|\Theta\big(r(\tau_n(t))\big)\cdot r'(\tau_n(t))\Big|\left(\tau_n'(t)\Big|\Theta\big(r(\tau_n(t))\big)\cdot r'(\tau_n(t))\Big| - \sqrt{G\big(r(\tau_n(t))\big)}\right)\mathrm{d}t \\ =& I(r) + \frac{1}{n^2}\int\limits_{-\infty}^{+\infty}\big(1-|\tau_n(t)|\big)^2\Big|\Theta\big(r(\tau_n(t))\big)\cdot r'(\tau_n(t))\Big|^2\mathrm{d}t = I(r) + O\Big(\frac{1}{n}\Big), \end{aligned} \tag{A.8}$$

where we infer the last equality from (A.6). Next for $m > 0$ define the function $\zeta^{\{n,m\}}(t) : \mathbb{R} \to \mathbb{R}^d$ by,

$$\zeta^{\{n,m\}}(t) = \begin{cases} \zeta_n(t) + \frac{1}{2}\Big(\big(b-\zeta_n(m)\big) - \big(a-\zeta_n(-m)\big)\Big)\frac{t}{m} + \frac{1}{2}\Big(\big(b-\zeta_n(m)\big) + \big(a-\zeta_n(-m)\big)\Big) \\ \quad \forall t \in [-m, m], \\ a \qquad\qquad \forall t \in (-\infty, -m), \\ b \qquad\qquad \forall t \in (m, +\infty). \end{cases}$$

Using (A.6) we see easily that for every n,

$$\lim_{m\to+\infty} I_0(\zeta^{\{n,m\}}) = I_0(\zeta_n).$$

From (A.8), a density argument and taking also diagonal subsequence it follows that there exists a sequence $\hat{\zeta}_n \in \mathcal{R}^{(0)}_{\boldsymbol{a},\boldsymbol{b}}$ such that

$$\lim_{n\to+\infty} I_0(\hat{\zeta}_n) = I(r).$$

Then,

$$\inf_{\zeta\in\mathcal{R}^{(0)}_{\boldsymbol{a},\boldsymbol{b}}} I_0(\zeta) \le \inf\big\{I(r):\ r\in\mathcal{R}_{\boldsymbol{a},\boldsymbol{b}},\ r'(t)\neq 0\ \ \forall t\in[-1,1]\big\}.$$

But by density arguments

$$\inf\big\{I(r):\ r\in\mathcal{R}_{\boldsymbol{a},\boldsymbol{b}},\ r'(t)\neq 0\ \ \forall t\in[-1,1]\big\} = \inf_{r\in\mathcal{R}_{\boldsymbol{a},\boldsymbol{b}}} I(r).$$

Therefore,

$$\inf_{\zeta\in\mathcal{R}^{(0)}_{\boldsymbol{a},\boldsymbol{b}}} I_0(\zeta) \le \inf_{r\in\mathcal{R}_{\boldsymbol{a},\boldsymbol{b}}} I(r). \tag{A.9}$$

This inequality, together with (A.4), gives

$$\inf_{\zeta\in\mathcal{R}^{(0)}_{\boldsymbol{a},\boldsymbol{b}}} I_0(\zeta) = \inf_{r\in\mathcal{R}_{\boldsymbol{a},\boldsymbol{b}}} I(r).$$

□

The following lemma is used in Section 5.

Lemma A.2. *Let $\varphi\in L^2(\mathbb{R}^N,\mathbb{R})$ and let $l(z,x)\in\mathcal{U}(\mathbb{R}^N)$ (see Def. 2.3). For every $\varepsilon>0$ consider the function $\varphi_\varepsilon\in C^1(\mathbb{R}^N,\mathbb{R})$ by*

$$\varphi_\varepsilon(x) := \frac{1}{\varepsilon^N}\int_{\mathbb{R}^N} l\Big(\frac{y-x}{\varepsilon},y\Big)\varphi(y)\,\mathrm{d}y = \int_{\mathbb{R}^N} l(z,x+\varepsilon z)\,\varphi(x+\varepsilon z)\,\mathrm{d}z. \tag{A.10}$$

Then there exists $C_0>0$ that depend only on $\|l\|_{L^\infty}$, $\|\nabla_x l\|_{L^\infty}$ and $\operatorname{supp} l$ such that

$$\int_{\mathbb{R}^N} |\nabla(\tilde\Delta^{-1}\varphi_\varepsilon)|^2\mathrm{d}x \le C_0\varepsilon^2\|\varphi\|^2_{L^2}. \tag{A.11}$$

Proof. It is enough to prove (A.11) for $\varphi\in C^1\cap L^2$, otherwise we approximate φ by smooth functions. So without loss of generality we may assume that $\varphi\in C^1$.

It is clear that there exists a compact set $K\subset\subset\mathbb{R}^N$, depending only on $\operatorname{supp} l$, such that $l(z,x)=0$ whenever $z\in\mathbb{R}^N\setminus K$ or $x\in\mathbb{R}^N\setminus K$. Since we assumed $\varphi\in C^1$, by definition of φ_ε we obtain

$$\begin{aligned}\varphi_\varepsilon(x) &= \int_{\mathbb{R}^N} l(z,x+\varepsilon z)\,\varphi(x+\varepsilon z)\,\mathrm{d}z = \int_{\mathbb{R}^N}\Big(l(z,x+\varepsilon z)\,\varphi(x+\varepsilon z) - l(z,x)\,\varphi(x)\Big)\,\mathrm{d}z\\ &= \varepsilon\int_0^1\int_{\mathbb{R}^N} z\cdot\nabla_x\Big(l(z,x+\varepsilon tz)\,\varphi(x+\varepsilon tz)\Big)\,\mathrm{d}z\mathrm{d}t\\ &= \varepsilon\,\mathrm{div}_x\bigg\{\int_0^1\int_K l(z,x+\varepsilon tz)\,\varphi(x+\varepsilon tz)\,z\,\mathrm{d}z\mathrm{d}t\bigg\}.\end{aligned} \tag{A.12}$$

Therefore,

$$\begin{aligned}\int_{\mathbb{R}^N}|\nabla(\tilde\Delta^{-1}\varphi_\varepsilon)|^2\mathrm{d}x &\le 2\varepsilon^2\int_{\mathbb{R}^N}\bigg|\nabla\bigg(\tilde\Delta^{-1}\bigg\{\mathrm{div}\int_0^1\int_K l(z,x+\varepsilon tz)\,\varphi(x+\varepsilon tz)\,z\,\mathrm{d}z\mathrm{d}t\bigg\}\bigg)\bigg|^2\mathrm{d}x\\ &\le 2\varepsilon^2\int_{\mathbb{R}^N}\bigg|\int_0^1\int_K l(z,x+\varepsilon tz)\,\varphi(x+\varepsilon tz)\,z\,\mathrm{d}z\mathrm{d}t\bigg|^2\mathrm{d}x\\ &\le C_1\varepsilon^2\int_0^1\int_K\int_{\mathbb{R}^N}|\varphi(x+\varepsilon tz)|^2\mathrm{d}x\mathrm{d}z\mathrm{d}t \le C_0\varepsilon^2\int_{\mathbb{R}^N}|\varphi(x)|^2\mathrm{d}x,\end{aligned} \tag{A.13}$$

where C_0, C_1 depend only on K and $\|l\|_{L^\infty}$. This completes the proof. □

References

[1] F. Alouges, T. Rivière and S. Serfaty, Néel and cross-tie wall energies for planar micromagnetic configurations. *ESAIM: COCV* **8** (2002) 31–68.

[2] L. Ambrosio, N. Fusco and D. Pallara, *Functions of Bounded Variation and Free Discontinuity Problems, Oxford Mathematical Monographs.* Oxford University Press, New York (2000).

[3] A. DeSimone, S. Müller, R.V. Kohn and F. Otto, Recent analytical developments in micromagnetics, in *The Science of Hysteresis* **2**, G. Bertotti and I. Mayergoyz Eds., Elsevier Academic Press (2005) 269–381.

[4] A. Hubert and R. Schäfer, *Magnetic domains.* Springer (1998).

[5] W. Jin and R.V. Kohn, Singular perturbation and the energy of folds. *J. Nonlinear Sci.* **10** (2000) 355–390.

[6] A. Poliakovsky, Upper bounds for singular perturbation problems involving gradient fields. *J. Eur. Math. Soc.* **9** (2007) 1–43.

[7] A. Poliakovsky, Sharp upper bounds for a singular perturbation problem related to micromagnetics. *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* **6** (2007) 673–701.

[8] A. Poliakovsky, A general technique to prove upper bounds for singular perturbation problems. *J. Anal. Math.* **104** (2008) 247–290.

[9] A. Poliakovsky, On a variational approach to the Method of Vanishing Viscosity for Conservation Laws. *Adv. Math. Sci. Appl.* **18** (2008) 429–451.

[10] T. Rivière and S. Serfaty, Limiting domain wall energy for a problem related to micromagnetics. *Comm. Pure Appl. Math.* **54** (2001) 294–338.

[11] T. Rivière and S. Serfaty, Compactness, kinetic formulation and entropies for a problem related to mocromagnetics. *Comm. Partial Differential Equations* **28** (2003) 249–269.