HOMOGENIZATION OF VARIATIONAL PROBLEMS IN MANIFOLD VALUED SOBOLEV SPACES

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Abstract. Homogenization of integral functionals is studied under the constraint that admissible maps have to take their values into a given smooth manifold. The notion of tangential homogenization is defined by analogy with the tangential quasiconvexity introduced by Dacorogna et al. [Calc. Var. Part. Diff. Eq. 9 (1999) 185–206]. For energies with superlinear or linear growth, a Γ-convergence result is established in Sobolev spaces, the homogenization problem in the space of functions of bounded variation being the object of [Babadjian and Millot, Calc. Var. Part. Diff. Eq. 36 (2009) 7–47].

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1. INTRODUCTION

The homogenization theory aims to find an effective description of materials whose heterogeneities scale is much smaller than the size of the body. The simplest example is periodic homogenization for which the microstructure is assumed to be periodically distributed within the material. In the framework of the Calculus of Variations, periodic homogenization problems rest on the study of equilibrium states, or minimizers, of integral functionals of the form

\[ \int_{\Omega} f \left( \frac{x}{\varepsilon}, \nabla u \right) \, dx, \quad u : \Omega \to \mathbb{R}^d, \]

under suitable boundary conditions, where \( \Omega \subset \mathbb{R}^N \) is a bounded open set and \( f : \mathbb{R}^N \times \mathbb{R}^{d \times N} \to [0, +\infty) \) is some oscillating integrand with respect to the first variable. To understand the asymptotic behavior of (almost) minimizers of such energies, it is convenient to perform a Γ-convergence analysis (see [13] for a detailed description of this subject) which is an adequate theory to study such variational problems. It is usual to assume that the integrand \( f \) satisfies uniform \( p \)-growth and \( p \)-coercivity conditions (with \( 1 \leq p < +\infty \)) so that one should ask the admissible fields to belong to the Sobolev space \( W^{1,p} \). For energies with superlinear growth, i.e., \( p > 1 \), this problem has a quite long history, and we refer to [20] in the convex case. Then it has received the most general answer in the independent works of [7,21], showing that such materials asymptotically behave like homogeneous ones. These results have been subsequently generalized into a lot of different manners. Let us mention [9] where the authors add a surface energy term allowing for fractured media. In that case, Sobolev spaces are not adapted to take into account eventual discontinuities of the deformation field across the cracks.

Keywords and phrases. Homogenization, Γ-convergence, manifold valued maps.

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In many applications admissible fields have to satisfy additional constraints. This is for example the case in the study of equilibria for liquid crystals, in ferromagnetism or for magnetostrictive materials where the order parameters take their values into a given manifold. It then becomes necessary to understand the behaviour of integral functionals of the type \((1.1)\) under this additional constraint. For fixed \(\varepsilon > 0\), the possible lack of lower semicontinuity of the energy may prevent the existence of minimizers (with eventual boundary conditions). It leads to compute its relaxation under the manifold constraint. In the framework of Sobolev spaces, it has been studied in [1, 12], and the relaxed energy is obtained by replacing the integrand by its tangential quasiconvexification which is the analogue of the quasiconvex envelope in the non constrained case. We finally mention a slightly different problem originally introduced in \([6, 10]\), where the energy is assumed to be finite and all problems are formulated within the manifold constraint. In this case, \(\varepsilon\)-energy density is the tangentially homogenized energy density.

For energies with superlinear growth, we have the following result.

**Theorem 1.1.** Let \(\mathcal{M}\) be a connected smooth submanifold of \(\mathbb{R}^d\) without boundary, and \(f: \mathbb{R}^N \times \mathbb{R}^{d \times N} \to [0, +\infty)\) be a Carathéodory function satisfying \((H_1)\) and \((H_2)\) with 1 \(<\) \(p\) \(<\) \(+\infty\). Then the family \(\{\mathcal{F}_\varepsilon\}_{\varepsilon > 0}\) \(\Gamma\)-converges for the strong \(L^p\)-topology to the functional \(\mathcal{F}_{\text{hom}}: L^p(\Omega; \mathbb{R}^d) \to [0, +\infty]\) defined by

\[
\mathcal{F}_{\text{hom}}(u) := \begin{cases} 
\int_{\Omega} f \left( \frac{x}{\varepsilon}, \nabla u \right) \, dx & \text{if } u \in W^{1,p}(\Omega; \mathcal{M}), \\
+\infty & \text{otherwise.}
\end{cases}
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+\infty & \text{otherwise,}
\end{cases}
\]

where for every \(s \in \mathcal{M}\) and \(\xi \in [T_s(\mathcal{M})]^N\),

\[
Tf_{\text{hom}}(s, \xi) = \lim_{t \to +\infty} \inf_{\varphi} \left\{ \int_{(0,t)^N} f(y, \xi + \nabla \varphi(y)) \, dy : \varphi \in W^{1,\infty}_0((0,t)^N; T_s(\mathcal{M})) \right\}
\]

is the tangentially homogenized energy density.

If the integrand \(f\) has a linear growth in the \(\xi\)-variable, i.e., if \(f\) satisfies \((H_2)\) with \(p = 1\), we assume in addition that \(\mathcal{M}\) is compact, and that
(H1) there exists L > 0 such that
\[ |f(y, \xi) - f(y, \xi')| \leq L|\xi - \xi'| \quad \text{for a.e. } y \in \mathbb{R}^N \text{ and all } \xi, \xi' \in \mathbb{R}^{d \times N}. \]

Then the following representation result on \( W^{1,1}(\Omega; \mathcal{M}) \) holds:

**Theorem 1.2.** Let \( \mathcal{M} \) be a connected and compact smooth submanifold of \( \mathbb{R}^d \) without boundary, and \( f : \mathbb{R}^N \times \mathbb{R}^{d \times N} \to [0, +\infty) \) be a Carathéodory function satisfying (H1) to (H3) with \( p = 1 \). Then the family \( \{\mathcal{F}_\varepsilon\}_{\varepsilon > 0} \) \( \Gamma \)-converges for the strong \( L^1 \)-topology at every \( u \in W^{1,1}(\Omega; \mathcal{M}) \) to \( \mathcal{F}_{\text{hom}} : W^{1,1}(\Omega; \mathcal{M}) \to [0, +\infty) \), where
\[
\mathcal{F}_{\text{hom}}(u) := \int_{\Omega} T_{f_{\text{hom}}}(u, \nabla u) \, dx,
\]
and \( T_{f_{\text{hom}}} \) is given by (1.2).

We would like to emphasize that the use of hypothesis (H3) is not too restrictive. Indeed, the \( \Gamma \)-limit remains unchanged upon first relaxing the functional \( \mathcal{F}_\varepsilon \) (at fixed \( \varepsilon > 0 \)) in \( W^{1,1}(\Omega; \mathbb{R}^d) \). It would lead to replace the integrand \( f \) by its tangential quasiconvexification which, by virtue of the growth condition (H1), does satisfy such a Lipschitz continuity assumption (see [12]).

We finally underline that Theorem 1.2 is not completely satisfactory in its present form. Indeed, in the case of an integrand with linear growth, the domain of the \( \Gamma \)-limit is obviously larger than the Sobolev space \( W^{1,1}_0(\Omega; \mathcal{M}) \) and the analysis has to be performed in the space of functions of bounded variation. In fact Theorem 1.2 is a first step in this direction and the complete study in BV-spaces can be found in [3].

The paper is organized as follows. The study of the energy density \( T_{f_{\text{hom}}} \) and its main properties are presented in Section 2. A locality property of the \( \Gamma \)-limit is established in Section 3. The upper bound inequalities in Theorems 1.1 and 1.2 are the object of Section 4. The lower bounds are obtained in Section 5 where the proofs of both theorems are completed.

**Notations**

We start by introducing some notations. Let \( \Omega \) be a generic bounded open subset of \( \mathbb{R}^N \). We denote by \( \mathcal{A}(\Omega) \) the family of all open subsets of \( \Omega \). We write \( B_k(s, r) \) for the closed ball in \( \mathbb{R}^k \) of center \( s \in \mathbb{R}^k \) and radius \( r > 0 \), \( Q := (-1/2, 1/2)^N \) the open unit cube in \( \mathbb{R}^N \), and \( Q(x_0, \rho) := x_0 + \rho Q \).

The space of real valued Radon measures in \( \Omega \) with finite total variation is denoted by \( \mathcal{M}(\Omega) \). We denote by \( L^p \) the Lebesgue measure in \( \mathbb{R}^N \). If \( \mu \in \mathcal{M}(\Omega) \) and \( \lambda \in \mathcal{M}(\Omega) \) is a nonnegative Radon measure, we denote by \( \frac{d\mu}{d\lambda} \) the Radon-Nikodym derivative of \( \mu \) with respect to \( \lambda \). By a generalization of Besicovitch Differentiation Theorem (see [2], Prop. 2.2), there exists a Borel set \( E \) such that \( \lambda(E) = 0 \) and
\[
\frac{d\mu}{d\lambda}(x) = \lim_{\rho \to 0^+} \frac{\mu(Q(x, \rho))}{\lambda(Q(x, \rho))}
\]
for all \( x \in \text{Supp } \mu \setminus E \).

**2. Properties of the Homogenized Energy Density**

In this section we present the main properties of the energy density \( T_{f_{\text{hom}}} \) defined in (1.2). We consider the bulk energy density
\[
T_{f_{\text{hom}}}(s, \xi) := \liminf_{t \to +\infty} \inf_{\varphi} \left\{ \int_{(0,t)^N} f(y, \xi + \nabla \varphi(y)) \, dy : \varphi \in W^{1,\infty}_0((0,t)^N; T_s(\mathcal{M})) \right\}
\]
defined for \( s \in \mathcal{M} \) and \( \xi \in [T_s(\mathcal{M})]^N \). Our first concern is to show that the liminf above is actually a limit. To this purpose we shall introduce a new energy density \( \tilde{f} \) for which we can apply classical homogenization theories.
Proposition 2.1. Let \( f : \mathbb{R}^N \times \mathbb{R}^{d \times N} \to [0, +\infty) \) be a Carathéodory integrand satisfying assumptions (H1) and (H2) with \( 1 \leq p < +\infty \). Then the following properties hold:

(i) for every \( s \in \mathcal{M} \) and \( \xi \in [T_s(\mathcal{M})]^N \),

\[
T_{f_{\text{hom}}}(s, \xi) = \lim_{t \to +\infty} \inf_{\varphi} \left\{ \int_{(0,t)^N} f(y, \xi + \nabla \varphi(y)) \, dy : \varphi \in W_0^{1,\infty}((0,t)^N; T_s(\mathcal{M})) \right\},
\]

and

\[
T_{f_{\text{hom}}}(s, \xi) = f_{\text{hom}}(s, \xi),
\]

where

\[
f_{\text{hom}}(s, \xi) := \lim_{t \to +\infty} \inf_{\varphi} \left\{ \int_{(0,t)^N} f(y, s, \xi + \nabla \varphi(y)) \, dy : \varphi \in W_0^{1,\infty}((0,t)^N; \mathbb{R}^d) \right\}
\]
is the usual homogenized energy density of \( f \) (see, e.g., [8], Chap. 14);

(ii) the function \( T_{f_{\text{hom}}} \) is tangentially quasiconvex, i.e., for all \( s \in \mathcal{M} \) and all \( \xi \in [T_s(\mathcal{M})]^N \),

\[
T_{f_{\text{hom}}}(s, \xi) \leq \int_Q T_{f_{\text{hom}}}(s, \xi + \nabla \varphi(y)) \, dy
\]

for every \( \varphi \in W_0^{1,\infty}(Q; T_s(\mathcal{M})) \). In particular \( T_{f_{\text{hom}}}(s, \cdot) \) is rank one convex;

(iii) there exists \( C > 0 \) such that

\[
\alpha |\xi|^p \leq T_{f_{\text{hom}}}(s, \xi) \leq \beta (1 + |\xi|^p),
\]

and

\[
|T_{f_{\text{hom}}}(s, \xi) - T_{f_{\text{hom}}}(s, \xi')| \leq C(1 + |\xi|^{p-1} + |\xi'|^{p-1}) |\xi - \xi'|
\]

for every \( s \in \mathcal{M} \) and \( \xi, \xi' \in [T_s(\mathcal{M})]^N \).

Proof. Fix \( s \in \mathcal{M} \) and \( \xi \in [T_s(\mathcal{M})]^N \). For any \( t > 0 \), we introduce

\[
T_{f_t}(s, \xi) := \inf_{\varphi} \left\{ \int_{(0,t)^N} f(y, \xi + \nabla \varphi(y)) \, dy : \varphi \in W_0^{1,\infty}((0,t)^N; T_s(\mathcal{M})) \right\},
\]
and
\[ \tilde{f}_t(s, \xi) := \inf_{\varphi} \left\{ \int_{(0,t)^N} f(y, s, \xi + \nabla \varphi) \, dy : \varphi \in W^{1,\infty}_0((0,t)^N; \mathbb{R}^d) \right\}. \]

By classical results (see, e.g., [8], Prop. 14.4), there exists
\[ \lim_{t \to +\infty} \tilde{f}_t(s, \xi) \quad \text{for every } s \in \mathcal{M} \text{ and } \xi \in [T_s(\mathcal{M})]^N. \]

Hence to prove (i), it suffices to show that \( T \tilde{f}_t(s, \xi) = \tilde{f}_t(s, \xi) \) for every \( t > 0 \). For any \( \varphi \in W^{1,\infty}_0((0,t)^N; T_s(\mathcal{M})) \), we have
\[ \tilde{f}_t(s, \xi) \leq \int_{(0,t)^N} f(y, s, \xi + \nabla \varphi) \, dy \leq \int_{(0,t)^N} f(y, s, \xi + \nabla \varphi) \, dy, \]

since \( \xi + \nabla \varphi(y) \in [T_s(\mathcal{M})]^N \) for a.e. \( y \in (0,t)^N \). Taking the infimum over all such \( \varphi \)'s in the right hand side of the previous inequality yields \( \tilde{f}_t(s, \xi) \leq T \tilde{f}_t(s, \xi) \). To prove the converse inequality we pick up \( \psi \in W^{1,\infty}_0((0,t)^N; \mathbb{R}^d) \) and we set \( \psi = P_s(\hat{\psi}) \). One easily checks that \( \psi \in W^{1,\infty}_0((0,t)^N; T_s(\mathcal{M})) \) and \( \nabla \psi = P_s(\nabla \psi) \) a.e. in \( (0,t)^N \). Therefore
\[ T \tilde{f}_t(s, \xi) \leq \int_{(0,t)^N} f(y, s, \xi + \nabla \psi) \, dy \leq \int_{(0,t)^N} f(y, s, \xi + \nabla \psi) \, dy. \]

Then the converse inequality arises taking the infimum over all admissible \( \psi \)'s.

By standard results \( \tilde{f}_{\text{hom}}(s, \cdot) \) is a quasiconvex function for every \( s \in \mathcal{M} \) (see, e.g., [8], Thm. 14.5). As a consequence, for any \( s \in \mathcal{M}, \xi \in [T_s(\mathcal{M})]^N \) and \( \varphi \in W^{1,\infty}_0(Q; T_s(\mathcal{M})) \), we have
\[ T \tilde{f}_{\text{hom}}(s, \cdot) = \tilde{f}_{\text{hom}}(s, \cdot) \leq \int_Q T \tilde{f}_{\text{hom}}(s, \xi + \nabla \varphi) \, dy = \int_Q T \tilde{f}_{\text{hom}}(s, \xi + \nabla \varphi) \, dy, \]

which proves that \( T \tilde{f}_{\text{hom}} \) is tangentially quasiconvex. As a consequence of (2.3) and the fact that \( \tilde{f}_{\text{hom}}(s, \cdot) \) is rank one convex, it follows that \( T \tilde{f}_{\text{hom}}(s, \cdot) \) is rank one convex as well.

The proof of (2.4) is immediate in view of (H1) and the definition of \( T \tilde{f}_{\text{hom}} \). Moreover rank one convex functions satisfying uniform \( p \)-growth and \( p \)-coercivity conditions are \( p \)-Lipschitz (see, e.g., [11], Lem. 2.2, Chap. 4), and thus (2.5) holds.

\[ \square \]

**Remark 2.1.** It readily follows from the previous proof that Proposition 2.1 still holds for any Carathéodory integrand \( \tilde{f} : \mathbb{R}^N \times \mathcal{M} \times \mathbb{R}^{d \times N} \to [0, +\infty) \) instead of \( f \), provided that: \( \tilde{f}(x, s, \xi) = f(y, s, \xi) \) for every \( s \in \mathcal{M}, \) every \( \xi \in [T_s(\mathcal{M})]^N \) and a.e. \( y \in \mathbb{R}^N \); \( \tilde{f}(\cdot, s, \cdot) \) satisfies (H1) and (H2) for every \( s \in \mathcal{M} \) with uniform estimates with respect to \( s \).

**Remark 2.2.** If dim(\( \mathcal{M} \)) = 1 then \( T_s(\mathcal{M}) \) is a one dimensional linear subspace of \( \mathbb{R}^d \) for every \( s \in \mathcal{M} \). Hence, given \( s \in \mathcal{M} \), we can identify \( T_s(\mathcal{M}) \) with \( \mathbb{R} \) through some linear mapping \( i_s : \mathbb{R} \to T_s(\mathcal{M}) \). Using the application \( i_s \), we can also identify \( [T_s(\mathcal{M})]^N \) with \( \mathbb{R}^N \) setting for \( z = (z_1, \ldots, z_N) \in \mathbb{R}^N \), \( i_s(z) := (i_s(z_1), \ldots, i_s(z_N)) \). Define \( \tilde{f}(y, s, z) := f(y, i_s(z)) \) for \( (y, s, z) \in \Omega \times \mathcal{M} \times \mathbb{R}^N \). By (2.3) and [8], Remark 14.6, we can replace in formula (1.2) homogeneous boundary conditions by periodic boundary conditions, and the limit as \( t \to +\infty \) by the infimum over all \( t \in \mathbb{N} \). Moreover, in the scalar case the homogenization formula can be reduced to a single
Example 2.1. Assume that $\mathcal{M} = \mathbb{S}^1$ and for $x \in \mathbb{R}^N$, $\xi = (\xi_{ij}) \in \mathbb{R}^{2 \times N}$,
\[
f(x, \xi) = \sum_{j=1}^{N} (a(x_1)|\xi_{1j}|^2 + b(x_1)|\xi_{2j}|^2),
\]
where $a, b \in L^\infty(\mathbb{R})$ are 1-periodic and bounded from below by a positive constant. Arguing as in Remark 2.2 and [13], Example 25.6, one may compute for $s = (s_1, s_2) \in \mathbb{S}^1$ and $\xi \in [T_s(\mathbb{S}^1)]^N$,
\[
T_{f_{\text{hom}}}(s, \xi) = \sum_{j=1}^{N} \alpha_j(s) (|\xi_{1j}|^2 + |\xi_{2j}|^2),
\]
with
\[
\alpha_j(s) = \begin{cases} 
\left( \int_{-1/2}^{1/2} \frac{dt}{a(t)s_2^2 + b(t)s_1^2} \right)^{-1} & \text{if } j = 1, \\
\int_{-1/2}^{1/2} (a(t)s_2^2 + b(t)s_1^2) \, dt & \text{otherwise}.
\end{cases}
\]

Compare this result with [13], Example 25.6.

To treat the homogenization problem with $p = 1$, we will need to extend the function $\bar{f}$ to the whole space $\mathbb{R}^{N} \times \mathbb{R}^d \times \mathbb{R}^{d \times N}$. We state in the following lemma our extension procedure.

Lemma 2.1. Assume that $\mathcal{M}$ is compact. Let $f : \mathbb{R}^{N} \times \mathbb{R}^{d \times N} \to [0, +\infty)$ be a Carathéodory function satisfying $(H_1)$ to $(H_3)$ with $p = 1$. Then there exists a Carathéodory function $g : \mathbb{R}^{N} \times \mathbb{R}^d \times \mathbb{R}^{d \times N} \to [0, +\infty)$ such that
\[
g(y, s, \xi) = f(y, \xi) \quad \text{for } s \in \mathcal{M} \text{ and } \xi \in [T_s(\mathcal{M})]^N,
\]
and satisfying:
(i) $g$ is 1-periodic in the first variable;
(ii) there exist $0 < \alpha' \leq \beta'$ such that
\[
\alpha' |\xi| \leq g(y, s, \xi) \leq \beta'(1 + |\xi|) \tag{2.7}
\]
for every $(s, \xi) \in \mathbb{R}^d \times \mathbb{R}^{d \times N}$ and a.e. $y \in \mathbb{R}^N$;

(iii) there exist $C > 0$ and $C^1 > 0$ such that
\[
|g(y, s, \xi) - g(y, s', \xi)| \leq C|s - s'| |\xi|, \tag{2.8}
\]
and
\[
|g(y, s, \xi) - g(y, s', \xi')| \leq C'|\xi - \xi'| \tag{2.9}
\]
for every $s, s' \in \mathbb{R}^d$, every $\xi \in \mathbb{R}^{d \times N}$ and a.e. $y \in \mathbb{R}^N$.

Proof. For $\delta_0 > 0$ fixed, let $\mathcal{U} := \{s \in \mathbb{R}^d : \text{dist}(s, \mathcal{M}) < \delta_0\}$ be the $\delta_0$-neighborhood of $\mathcal{M}$. Choosing $\delta_0 > 0$ small enough, we may assume that the nearest point projection $\Pi : \mathcal{U} \to \mathcal{M}$ is a well defined Lipschitz mapping. Then the map $s \mapsto \Pi_{\Pi(s)}$ is Lipschitz. Now we introduce a cut-off function $\chi \in C_c^\infty(\mathbb{R}^d; [0, 1])$ such that $\chi(t) = 1$ if $\text{dist}(s, \mathcal{M}) \leq \delta_0/2$, and $\chi(s) = 0$ if $\text{dist}(s, \mathcal{M}) \geq 3\delta_0/4$. We define
\[
\mathbb{P}_s(\xi) := \chi(s)\mathbb{P}_{\Pi(s)}(\xi) \quad \text{for } (s, \xi) \in \mathbb{R}^d \times \mathbb{R}^{d \times N}.
\]
We consider the integrand $g : \mathbb{R}^N \times \mathbb{R}^d \times \mathbb{R}^{d \times N} \to [0, +\infty)$ given by
\[
g(y, s, \xi) = f(y, \mathbb{P}_s(\xi)) + |\xi - \mathbb{P}_s(\xi)|.
\]
One may check that $g$ is a Carathéodory function, that $g(\cdot, s, \xi)$ is 1-periodic for every $(s, \xi) \in \mathbb{R}^d \times \mathbb{R}^{d \times N}$, and that $(H_2)$ yields (2.7). Then (2.8) and (2.9) follow from $(H_3)$ and the Lipschitz continuity of $s \mapsto \mathbb{P}_s$. \qed

Remark 2.3. In view of (2.6), one may argue exactly as in the proof of (2.3) to show that
\[
Tf_{\text{hom}}(s, \xi) = g_{\text{hom}}(s, \xi) \quad \text{for every } s \in \mathcal{M} \text{ and } \xi \in [T_s(\mathcal{M})]^N, \tag{2.10}
\]
where
\[
g_{\text{hom}}(s, \xi) := \lim_{t \to +\infty} \inf_{\varphi} \left\{ \int_{(0, t)N} g(y, s, \xi + \nabla\varphi(y)) \, dy : \varphi \in W_0^{1, \infty}((0, t)N; \mathbb{R}^d) \right\}.
\]
Hence upon extending $Tf_{\text{hom}}$ by $g_{\text{hom}}$ outside the set $\{ (s, \xi) \in \mathbb{R}^d \times \mathbb{R}^{d \times N} : s \in \mathcal{M}, \xi \in [T_s(\mathcal{M})]^N \}$, we can tacitly assume $Tf_{\text{hom}}$ to be defined over the whole $\mathbb{R}^d \times \mathbb{R}^{d \times N}$.

3. Localization

In this section we show that a suitable functional larger than the $\Gamma$-limit is a measure. It will allow us to obtain the upper bound on the $\Gamma$-limit (see Lem. 4.1) through the blow-up method introduced in [15,16].

Let us consider an arbitrary sequence $\{ \varepsilon_n \} \searrow 0^+$. Along this sequence we define the $\Gamma(L^p)$-lower limit $\mathcal{F} : L^p(\Omega; \mathbb{R}^d) \to [0, +\infty]$ by
\[
\mathcal{F}(u) := \inf_{(u_n)} \left\{ \liminf_{n \to +\infty} \mathcal{F}_{\varepsilon_n}(u_n) : u_n \in W^{1, p}(\Omega; \mathcal{M}), u_n \rightharpoonup u \text{ in } L^p(\Omega; \mathbb{R}^d) \right\}.
\]
The idea is to localize the functionals $\{ \mathcal{F}_{\varepsilon_n} \}_{n \in \mathbb{N}}$ on the family $\mathcal{A}(\Omega)$ of all open subsets of $\Omega$. For every $u \in L^p(\Omega; \mathbb{R}^d)$ and every $A \in \mathcal{A}(\Omega)$, define
\[
\mathcal{F}_{\varepsilon_n}(u, A) := \begin{cases} \int_A f \left( \frac{\nabla u}{\varepsilon_n} \right) \, dx & \text{if } u \in W^{1, p}(\Omega; \mathcal{M}), \\ +\infty & \text{otherwise.} \end{cases}
\]
For every key point in the upcoming analysis is the following locality result.

Step 1. We start by proving that for any subsequence \( \{ \varepsilon_k \} \subset \{ \varepsilon_n \} \setminus 0^+ \), we introduce for \( u \in W^{1,p}(\Omega; M) \) and \( A \in \mathcal{A}(\Omega) \),

\[ F_{\varepsilon_k}^{(\varepsilon_k)}(u, A) := \inf_{\{ u_k \}} \left\{ \limsup_{k \to +\infty} F_{\varepsilon_k}(u_k, A) : u_k \rightharpoonup u \text{ weakly in } W^{1,p}(\Omega; \mathbb{R}^d), \right. \]

\[ u_k \to u \text{ uniformly and } u_k(x) = u(x) \text{ whenever } \text{dist}(u(x), K) > 1 \text{ for a.e. } x \in \Omega \} \]

A key point in the upcoming analysis is the following locality result.

**Lemma 3.1.** For every \( u \in W^{1,p}(\Omega; M) \), there exists a subsequence \( \{ \varepsilon_k \} \) such that the set function \( F_{\varepsilon_k}^{(\varepsilon_k)}(u, \cdot) \) is the restriction to \( \mathcal{A}(\Omega) \) of a Radon measure absolutely continuous with respect to the Lebesgue measure \( \mathcal{L}^N \).

**Proof.** From the \( p \)-growth condition \( (H_2) \) we infer that for any subsequence \( \{ \varepsilon_k \} \),

\[ F_{\varepsilon_k}^{(\varepsilon_k)}(u, A) \leq \beta \int_A (1 + |\nabla u|^p) \, dx, \quad (3.1) \]

so it remains to prove the existence of a suitable subsequence \( \{ \varepsilon_k \} \) for which \( F_{\varepsilon_k}^{(\varepsilon_k)}(u, \cdot) \) is (the trace of) a Radon measure.

**Step 1.** We start by proving that for any subsequence \( \{ \varepsilon_k \} \) the following subadditivity property holds:

\[ F_{\varepsilon_k}^{(\varepsilon_k)}(u, A) \leq F_{\varepsilon_k}^{(\varepsilon_k)}(u, B) + F_{\varepsilon_k}^{(\varepsilon_k)}(u, A \setminus B) \quad (3.2) \]

for every \( A, B \) and \( C \in \mathcal{A}(\Omega) \) such that \( C \subset B \subset A \). Given \( \eta > 0 \) arbitrary, there exist sequences \( \{ u_k \}, \{ v_k \} \subset W^{1,p}(\Omega; M) \) such that \( u_k \) and \( v_k \) converge weakly to \( u \) in \( W^{1,p}(\Omega; \mathbb{R}^d) \), \( u_k(x) = v_k(x) = u(x) \) if \( \text{dist}(u(x), K) > 1 \) for a.e. \( x \in \Omega \), \( u_k \) and \( v_k \) are uniformly converging to \( u \), and

\[ \limsup_{k \to +\infty} F_{\varepsilon_k}(u_k, B) \leq F_{\varepsilon_k}^{(\varepsilon_k)}(u, B) + \eta, \]

\[ \limsup_{k \to +\infty} F_{\varepsilon_k}(v_k, A \setminus B) \leq F_{\varepsilon_k}^{(\varepsilon_k)}(u, A \setminus B) + \eta. \quad (3.3) \]

Let \( K' := \{ s \in M : \text{dist}(s, K) \leq 1 \} \), then \( K' \) is a compact subset of \( M \) and \( u_k(x) = v_k(x) = u(x) \) if \( u(x) \notin K' \) for a.e. \( x \in \Omega \).

Consider \( L := \text{dist}(C, \partial B), M \in \mathbb{N} \), and for every \( i \in \{0, \ldots, M \} \) define

\[ B_i := \left\{ x \in B : \text{dist}(x, \partial B) > \frac{iL}{M} \right\}. \]

Given \( i \in \{0, \ldots, M - 1 \} \) let \( S_i := B_i \setminus \overline{B_{i+1}} \), and \( \zeta_i \in C^\infty(\Omega; [0, 1]) \) be a cut-off function satisfying

\[ \zeta_i(x) = \begin{cases} 1 & \text{in } B_{i+1}, \\ 0 & \text{in } \Omega \setminus B_i, \end{cases} \quad \text{and } |\nabla \zeta_i| \leq \frac{2M}{L}. \]
By Lemma 3.2 and Remark 3.3 in [12], there exist \( \delta > 0 \), \( c > 0 \), and a uniformly continuously differentiable mapping \( \Phi : D_\delta \times [0,1] \rightarrow \mathcal{M} \), where

\[
D_\delta := \{(s_0, s_1) \in \mathcal{M} \times \mathcal{M} : \text{dist}(s_0, \mathcal{K}) < \delta, \text{dist}(s_1, \mathcal{K}) < \delta, |s_0 - s_1| < \delta \},
\]

such that

\[
\Phi(s_0, s_1, 0) = s_0, \quad \Phi(s_0, s_1, 1) = s_1, \quad \frac{\partial \Phi}{\partial t}(s_0, s_1, t) \leq c|s_0 - s_1|,
\]

and

\[
|\Phi(s_0, s_1, t) - s_0| \leq c|s_0 - s_1|.
\]

Since \( \{u_k\} \) and \( \{v_k\} \) are uniformly converging to \( u \), one can choose \( k \) large enough to ensure that

\[
\|u_k - u\|_{L^\infty(\Omega, \mathbb{R}^d)} < \delta, \quad \|v_k - u\|_{L^\infty(\Omega, \mathbb{R}^d)} < \delta \quad \text{and} \quad \|u_k - v_k\|_{L^\infty(\Omega, \mathbb{R}^d)} < \delta.
\]

Therefore for a.e. \( x \in \Omega \), \( \text{dist}(u_k(x), \mathcal{K}') < \delta \) and \( \text{dist}(v_k(x), \mathcal{K}') < \delta \) whenever \( u(x) \in \mathcal{K}' \). Now we are allowed to define

\[
w_{k,i}(x) := \begin{cases} \Phi(v_k(x), u_k(x), \zeta_i(x)) & \text{if } u(x) \in \mathcal{K}', \\ u(x) & \text{if } u(x) \notin \mathcal{K}', \end{cases}
\]

and \( w_{k,i} \in W^{1,p}(\Omega, \mathcal{M}) \). Using the \( p \)-growth condition \((H_2)\) together with (3.4), we derive

\[
\int_A f \left( \frac{x}{\varepsilon_k}, \nabla w_{k,i} \right) dx \leq \int_B f \left( \frac{x}{\varepsilon_k}, \nabla u_k \right) dx + \int_{A \setminus \overline{\mathcal{M}}} f \left( \frac{x}{\varepsilon_k}, \nabla v_k \right) dx + C_0 \int_{S_1} \left( 1 + |\nabla u_k|^p + |\nabla v_k|^p + M^p |u_k - v_k|^p \right) dx,
\]

for some constant \( C_0 > 0 \) independent of \( k, i \) and \( M \). Summing up over \( i \in \{0, \ldots, M-1\} \) and dividing by \( M \) yields

\[
\frac{1}{M} \sum_{i=0}^{M-1} \int_A f \left( \frac{x}{\varepsilon_k}, \nabla w_{k,i} \right) dx \leq \int_B f \left( \frac{x}{\varepsilon_k}, \nabla u_k \right) dx + \int_{A \setminus \overline{\mathcal{M}}} f \left( \frac{x}{\varepsilon_k}, \nabla v_k \right) dx + \frac{C_0}{M} \int_{B \setminus \overline{\mathcal{M}}} \left( 1 + |\nabla u_k|^p + |\nabla v_k|^p + M^p |u_k - v_k|^p \right) dx.
\]

Hence one may find some \( i_k \in \{0, \ldots, M-1\} \) such that \( \tilde{w}_k := w_{k,i_k} \) satisfies

\[
\int_A f \left( \frac{x}{\varepsilon_k}, \nabla \tilde{w}_k \right) dx \leq \int_B f \left( \frac{x}{\varepsilon_k}, \nabla u_k \right) dx + \int_{A \setminus \overline{\mathcal{M}}} f \left( \frac{x}{\varepsilon_k}, \nabla v_k \right) dx + \frac{C_0}{M} \int_{B \setminus \overline{\mathcal{M}}} \left( 1 + |\nabla u_k|^p + |\nabla v_k|^p + M^p |u_k - v_k|^p \right) dx. \tag{3.6}
\]
From (3.4) and (3.5) we deduce that \( \bar{w}_k \to u \) uniformly, \( \bar{w}_k \rightharpoonup u \) in \( W^{1,p}(\Omega; \mathbb{R}^d) \), and \( \bar{w}_k(x) = u(x) \) if \( \text{dist}(u(x), K) > 1 \) for a.e. \( x \in \Omega \). Taking \( \{\bar{w}_k\} \) as competitor for \( F^{(\varepsilon_k)}_K(u, A) \), and using (3.6) together with (3.3) leads to

\[
F^{(\varepsilon_k)}_K(u, A) \leq \limsup_{k \to +\infty} F^{(\varepsilon_k)}_K(\bar{w}_k, A)
\leq \limsup_{k \to +\infty} \left\{ F^{(\varepsilon_k)}_K(u_k, B) + F^{(\varepsilon_k)}_K(v_k, A \setminus C) + \frac{C_0}{M} \int_{B \setminus C} (1 + |\nabla u_k|^p + |\nabla v_k|^p + M|u_k - v_k|^p) \, dx \right\}
\leq F^{(\varepsilon_k)}_K(u, B) + F^{(\varepsilon_k)}_K(u, A \setminus C) + 2\eta
\]

Then property (3.2) arises sending first \( M \to +\infty \), and then \( \eta \to 0 \).

**Step 2.** Now we complete the proof of Lemma 3.1. Using a standard diagonal argument, we construct a subsequence \( \{\varepsilon_k\} \subset \{\varepsilon\} \) and a sequence \( \{u_k\} \subset W^{1,p}(\Omega, \mathcal{M}) \) satisfying

\[
\lim_{k \to +\infty} F^{(\varepsilon_k)}_K(u_k, \Omega) = \inf_{\{v_k\}} \left\{ \liminf_{k \to +\infty} F^{(\varepsilon_k)}_K(v_k, \Omega) : v_k \rightharpoonup u \text{ weakly in } W^{1,p}(\Omega; \mathbb{R}^d), v_k \to u \text{ uniformly and } v_k(x) = u(x) \text{ whenever } \text{dist}(u(x), K) > 1 \text{ for a.e. } x \in \Omega \right\}.
\]

By construction of \( \{\varepsilon_k\} \) and \( \{u_k\} \), we have \( \lim_{k \to +\infty} F^{(\varepsilon_k)}_K(u_k, \Omega) = F^{(\varepsilon_k)}_K(u, \Omega) \). Up to the extraction of a further subsequence, we may assume that

\[
f \left( \frac{\cdot}{\varepsilon_k}, \nabla u_k \right) L^N \Omega \overset{\ast}{\rightharpoonup} \mu \text{ in } \mathcal{M}(\Omega),
\]

for some nonnegative Radon measure \( \mu \in \mathcal{M}(\Omega) \). By lower semicontinuity, we have

\[
\mu(\Omega) \leq \lim_{k \to +\infty} F^{(\varepsilon_k)}_K(u_k, \Omega) = F^{(\varepsilon_k)}_K(u, \Omega).
\]

We claim that

\[
F^{(\varepsilon_k)}_K(u, A) = \mu(A) \text{ for any } A \in \mathcal{A}(\Omega).
\]

We fix \( A \in \mathcal{A}(\Omega) \) and we start by proving the inequality “\( \leq \)”. Given \( \eta > 0 \) arbitrary we can select, in view of (3.1), \( C \subset \mathcal{A}(\Omega), C \subset \subset A \), such that \( F^{(\varepsilon_k)}_K(u, A \setminus C) \leq \eta \). Then inequality (3.2) implies that for any \( B \in \mathcal{A}(\Omega), C \subset \subset B \subset \subset A \),

\[
F^{(\varepsilon_k)}_K(u, A) \leq \eta + \limsup_{k \to +\infty} F^{(\varepsilon_k)}_K(u_k, B) \leq \eta + \mu(B) \leq \eta + \mu(A),
\]

and the conclusion follows from the arbitrariness of \( \eta \).

Conversely, for any \( B \in \mathcal{A}(\Omega), B \subset \subset A \), we have

\[
\mu(\Omega) \leq F^{(\varepsilon_k)}_K(u, A) \leq F^{(\varepsilon_k)}_K(u, A) + F^{(\varepsilon_k)}_K(u, \Omega \setminus B)
\leq F^{(\varepsilon_k)}_K(u, A) + \mu(\Omega \setminus B) \leq F^{(\varepsilon_k)}_K(u, A) + \mu(B) \leq F^{(\varepsilon_k)}_K(u, A) + \mu(\Omega) - \mu(B).
\]

Therefore \( \mu(B) \leq F^{(\varepsilon_k)}_K(u, A) \) and the conclusion follows by inner regularity of \( \mu \).

\[\square\]
4. The Upper Bound

We now make use of the previous locality result to show the upper bound. This will be done thanks to a blow-up analysis in the spirit of [12], Theorem 3.1.

Lemma 4.1. For every $p \in [1, +\infty)$ and $u \in W^{1,p}(\Omega; \mathcal{M})$, we have $\mathcal{F}(u) \leq \mathcal{F}_{\text{hom}}(u)$.

Proof. Step 1. Let $u \in W^{1,p}(\Omega; \mathcal{M})$. Given $R > 0$ arbitrary large, we set $\mathcal{K} := \mathcal{M} \cap B^d(0, R)$, and we consider the subsequence $\{\epsilon_k\}$ given by Lemma 3.1. Obviously $\mathcal{F}(u) \leq \mathcal{F}^{(\epsilon_k)}_{\mathcal{K}}(u, \Omega)$. We claim that

$$\mathcal{F}^{(\epsilon_k)}_{\mathcal{K}}(u, \Omega) \leq \int_{\Omega} \left\{ \chi_R(|u|) T_{\text{hom}}(u, \nabla u) + \beta \left( 1 - \chi_R(|u|) \right) \left( 1 + |\nabla u|^p \right) \right\} dx,$$

where $\chi_R(t) = 1$ for $t \leq R$ and $\chi_R(t) = 0$ otherwise. We postpone the proof of (4.1) to the next step, and we complete now the proof of Lemma 4.1.

Consider a sequence $R_j \to +\infty$ as $j \to +\infty$. Since $\chi_{R_j} \to 1$ pointwise, we deduce from Fatou’s lemma together with (2.4) that

$$\mathcal{F}(u) \leq \limsup_{j \to +\infty} \int_{\Omega} \left\{ \chi_{R_j}(|u|) T_{\text{hom}}(u, \nabla u) + \beta \left( 1 - \chi_{R_j}(|u|) \right) \left( 1 + |\nabla u|^p \right) \right\} dx \leq \int_{\Omega} T_{\text{hom}}(u, \nabla u) dx,$$

which is the announced estimate.

Step 2. Thanks to Lemma 3.1, to obtain (4.1) it suffices to prove that

$$\frac{d\mathcal{F}^{(\epsilon_k)}_{\mathcal{K}}(u, \cdot)}{d\mathcal{L}^N}(x_0) \leq \chi_R(|u(x_0)|) T_{\text{hom}}(u(x_0), \nabla u(x_0)) + \beta \left( 1 - \chi_R(|u(x_0)|) \right) \left( 1 + |\nabla u(x_0)|^p \right)$$

for $\mathcal{L}^N$-a.e. $x_0 \in \Omega$.

Let $x_0 \in \Omega$ be a Lebesgue point of $u$ and $\nabla u$ such that $u(x_0) \in \mathcal{M}$, $\nabla u(x_0) \in [T_{u(x_0)}(\mathcal{M})]^N$, and the Radon-Nikodym derivative of $\mathcal{F}^{(\epsilon_k)}_{\mathcal{K}}(u, \cdot)$ with respect to the Lebesgue measure $\mathcal{L}^N$ exists. Note that almost every points in $\Omega$ satisfy these properties. Now set $s_0 := u(x_0)$ and $\xi_0 := \nabla u(x_0)$.

Case 1. Assume that $s_0 \notin \mathcal{K}$. Then, using (H2), we derive that

$$\frac{d\mathcal{F}^{(\epsilon_k)}_{\mathcal{K}}(u, \cdot)}{d\mathcal{L}^N}(x_0) = \lim_{\rho \to 0^+} \frac{\mathcal{F}^{(\epsilon_k)}_{\mathcal{K}}(u, Q(x_0, \rho))}{\rho^N} \leq \limsup_{\rho \to 0^+} \frac{\mathcal{F}_{\mathcal{K}}(u, Q(x_0, \rho))}{\rho^N} \leq \lim_{\rho \to 0^+} \frac{\beta}{\rho^N} \int_{Q(x_0, \rho)} (1 + |\nabla u|^p) dx = \beta(1 + |\xi_0|^p),$$

which is the desired estimate.

Case 2. Now we assume that $s_0 \in \mathcal{K}$. Fix $0 < \eta < 1$ arbitrary. By Proposition 2.1, claim (i), there exist $j \in \mathbb{N}$ and $\varphi \in W^{1,\infty}_0((0, j)^N; T_{s_0}(\mathcal{M}))$ such that

$$\int_{(0,j)^N} f(y, \xi_0 + \nabla \varphi(y)) dy \leq T_{\text{hom}}(s_0, \xi_0) + \eta.$$

Extend $\varphi$ to $\mathbb{R}^N$ by $j$-periodicity, and define $\varphi_k(x) := \xi_0 x + \epsilon_k \varphi(x/\epsilon_k)$.
Let $\mathcal{U}$ be an open neighborhood of $\mathcal{M}$ such that the nearest point projection $\Pi : \mathcal{U} \to \mathcal{M}$ defines a $C^1$-mapping. Fix $\sigma, \delta_0 \in (0, 1)$ such that $B^d(s_0, 2\delta_0) \subset \mathcal{U}$, and consider $\delta = \delta(\sigma) \in (0, \delta_0)$ for which
\[
|\nabla \Pi(s) - \nabla \Pi(s')| < \sigma \text{ for all } s, s' \in B^d(s_0, \delta_0) \text{ satisfying } |s - s'| < \delta. \tag{4.3}
\]
Next we introduce a cut-off function $\zeta \in C^\infty_c(\mathbb{R}^d; [0, 1])$ satisfying
\[
\zeta(x) = \begin{cases} 
1 & \text{for } x \in B^d(0, \delta/4), \\
0 & \text{for } x \notin B^d(0, \delta/2), 
\end{cases} \quad \text{with } |\nabla \zeta| \leq \frac{C}{\delta},
\]
and we define
\[
w_k(x) := u(x) + \varepsilon_k \zeta(u(x) - s_0) \varphi(x/\varepsilon_k).
\]
Let $k_0 \in \mathbb{N}$ be such that
\[
\max \left\{ \varepsilon_k \| \varphi \|_{L^\infty((0, j)^N; \mathbb{R}^d)} \| \nabla \zeta \|_{L^\infty(\mathbb{R}^d; \mathbb{R}^d)}, \frac{2\varepsilon_k \| \varphi \|_{L^\infty((0, j)^N; \mathbb{R}^d)}}{\delta} \right\} < 1 \text{ for any } k \geq k_0. \tag{4.4}
\]
Define for every $k \geq k_0$,
\[
u_k(x) := \Pi(w_k(x)).
\]
Remark that by (4.4), for a.e. $x \in \Omega$ and all $k \geq k_0$, one has $w_k(x) \in B^d(s_0, \delta)$ whenever $|u(x) - s_0| < \delta/2$ while $w_k(x) = u(x)$ when $|u(x) - s_0| \geq \delta/2$. Hence $u_k$ is well defined, $\{u_k\} \subset W^{1,p}(\Omega; \mathcal{M})$, and for a.e. $x \in \Omega$, $u_k(x) = u(x)$ whenever $\text{dist}(u(x), K) > 1$. Moreover,
\[
\|w_k - u\|_{L^\infty(\Omega; \mathbb{R}^d)} = \|\Pi(w_k) - \Pi(u)\|_{L^\infty(\{|u(x) - s_0| < \delta/2\}; \mathbb{R}^d)} \leq \varepsilon_k \|\nabla \Pi\|_{L^\infty(B^d(s_0, \delta_0); \mathbb{R}^d)} \|\varphi\|_{L^\infty((0, j)^N; \mathbb{R}^d)} \to 0
\]
as $k \to +\infty$. Now the Chain Rule formula yields
\[
\nabla u_k(x) = \nabla \Pi(w_k(x))(\nabla u(x) + \varepsilon_k (\varphi(x/\varepsilon_k) \otimes \nabla \zeta(u(x) - s_0)) \nabla u(x) + \zeta(u(x) - s_0) \nabla \varphi(x/\varepsilon_k)),
\]
and consequently
\[
|\nabla u_k(x)| \leq \|\nabla \Pi\|_{L^\infty(B^d(s_0, \delta_0); \mathbb{R}^d)} \left(1 + \varepsilon_k \|\varphi\|_{L^\infty((0, j)^N; \mathbb{R}^d)} \|\nabla \zeta\|_{L^\infty(\mathbb{R}^d; \mathbb{R}^d)} \|\nabla u(x)\| + \|\nabla \varphi\|_{L^\infty((0, j)^N; \mathbb{R}^{d \times N})}\right).
\]
By (4.4) it follows that for any $k \geq k_0$,
\[
|\nabla u_k(x)| \leq C_0(|\nabla u(x) - \xi_0| + 1) \tag{4.5}
\]
for some constant $C_0 = C_0(s_0, \xi_0, \delta_0, \eta) > 0$ independent of $x$ and $k$. Hence the sequence $\{u_k\}$ is uniformly bounded in $W^{1,p}(\Omega; \mathbb{R}^d)$ so that $u_k \rightharpoonup u$ in $W^{1,p}(\Omega; \mathbb{R}^d)$.

Then we observe that $|\nabla u_k| \leq 2C_0$ a.e. in $\{|\nabla u - \xi_0| < \sigma\}$ while
\[
\|\nabla \varphi_k\|_{L^\infty(\Omega; \mathbb{R}^{d \times N})} \leq |\xi_0| + \|\nabla \varphi\|_{L^\infty((0, j)^N; \mathbb{R}^{d \times N})}.
\]
Set
\[
M := \max \{2C_0, |\xi_0| + \|\nabla \varphi\|_{L^\infty((0, j)^N; \mathbb{R}^{d \times N})}\}, \tag{4.6}
\]
(which only depends on $s_0, \xi_0, \delta_0$ and $\eta$) so that
\[
|\nabla u_k| \leq M \text{ and } |\nabla \varphi_k| \leq M \text{ a.e. in } \{|\nabla u - \xi_0| < \sigma\}. \tag{4.7}
\]
Next for a.e. \( x \in \{ |u - s_0| < \delta/4 \} \cap \{ |\nabla u - \xi_0| < \sigma \} \), we have \( \zeta(u(x) - s_0) = 1 \) and
\[
|\nabla u_k(x) - \nabla \varphi_k(x)| \leq |\nabla u_k(x) - \xi_0| + |\nabla \varphi_k(x/\varepsilon_k) - \nabla \varphi(x/\varepsilon_k)|
\leq |\nabla u_k(x) - \nabla \varphi_k(0)| + |\nabla \varphi(0)| |\nabla u(x) - \xi_0|
+ |\nabla \varphi_k(0) - \nabla \varphi_k(s_0)| ||\nabla \varphi||_{L^\infty((0, j), \mathbb{R}^{d \times N})},
\]
where, in the last inequality, we have used the fact that \( \nabla \varphi_k(s_0) \nabla \varphi(y) = \nabla \varphi(y) \) since \( \nabla \varphi(y) \in [T_{u_0}(\mathcal{M})]^N \) for a.e. \( y \in \mathbb{R}^N \). Using (4.3) and the fact that \( |u_k - s_0| < \delta \) a.e. in \( \{ |u - s_0| < \delta/4 \} \cap \{ |\nabla u - \xi_0| < \sigma \} \), we deduce
\[
|\nabla u_k(x) - \nabla \varphi_k(x)| \leq (|\nabla u(x)| + |\nabla \varphi(s_0)| + ||\nabla \varphi||_{L^\infty((0, j), \mathbb{R}^{d \times N})}) \sigma \leq C_1 \sigma
\]
for a.e. \( x \in \{ |u - s_0| < \delta/4 \} \cap \{ |\nabla u - \xi_0| < \sigma \} \), where \( C_1 = C_1(s_0, \xi_0, \delta_0, \eta) > 0 \) is a constant independent of \( \sigma, k \), and \( x \).

Now we estimate
\[
\frac{d\mathcal{F}^{(\varepsilon_k)}_{K} (u, \cdot)}{d\mathcal{L}^N}(x_0) = \lim_{\rho \to 0^+} \frac{\mathcal{F}^{(\varepsilon_k)}_{K}(u, Q(x_0, \rho))}{\rho^N}
\leq \limsup_{\rho \to 0^+} \limsup_{k \to +\infty} \frac{1}{\rho^N} \int_{Q(x_0, \rho)} f \left( \frac{x}{\varepsilon_k}, \nabla u_k \right) dx
\leq \limsup_{\rho \to 0^+} \limsup_{k \to +\infty} \frac{1}{\rho^N} \int_{Q(x_0, \rho) \cap \{ |u - s_0| \leq \delta / 4 \}} f \left( \frac{x}{\varepsilon_k}, \nabla u_k \right) dx
+ \limsup_{\rho \to 0^+} \limsup_{k \to +\infty} \frac{1}{\rho^N} \int_{Q(x_0, \rho) \cap \{ |u - s_0| < \delta / 4 \} \cap \{ |\nabla u - \xi_0| < \sigma \}} f \left( \frac{x}{\varepsilon_k}, \nabla u_k \right) dx
+ \limsup_{\rho \to 0^+} \limsup_{k \to +\infty} \frac{1}{\rho^N} \int_{Q(x_0, \rho) \cap \{ |u - s_0| < \delta / 4 \} \cap \{ |\nabla u - \xi_0| \geq \sigma \}} f \left( \frac{x}{\varepsilon_k}, \nabla u_k \right) dx
=: I_1 + I_2 + I_3.
\]
Thanks to (4.5), the \( p \)-growth condition \((H_2)\) and our choice of \( x_0 \), we have
\[
I_1 \leq C \limsup_{\rho \to 0^+} \frac{1}{\rho^N} \int_{Q(x_0, \rho) \cap \{ |u - s_0| \geq \delta / 4 \}} (1 + |\nabla u(x) - \xi_0|^p) dx
\leq C \limsup_{\rho \to 0^+} \int_{Q(x_0, \rho)} |\nabla u(x) - \xi_0|^p dx + \frac{4C}{\delta} \limsup_{\rho \to 0^+} \int_{Q(x_0, \rho)} |u(x) - s_0| dx = 0,
\]
while
\[
I_3 \leq C \limsup_{\rho \to 0^+} \frac{1}{\rho^N} \int_{Q(x_0, \rho) \cap \{ |u - s_0| < \delta / 4 \} \cap \{ |\nabla u - \xi_0| \geq \sigma \}} (1 + |\nabla u(x) - \xi_0|^p) dx
\leq C \limsup_{\rho \to 0^+} \int_{Q(x_0, \rho)} |\nabla u(x) - \xi_0|^p dx + \frac{C}{\sigma} \limsup_{\rho \to 0^+} \int_{Q(x_0, \rho)} |\nabla u(x) - \xi_0| dx = 0.
\]
Let us now treat the integral \( I_2 \). Since, for a.e. \( y \in \mathbb{R}^N \), the function \( f(y, \cdot) \) is continuous, it is uniformly continuous on \( B^{d \times N}(0, M) \) where \( M > 0 \) is given in (4.6). Define the modulus of continuity of \( f(y, \cdot) \) over \( B^{d \times N}(0, M) \) by
\[
\omega(y, t) := \sup\{|f(y, \xi) - f(y, \xi')| : \xi, \xi' \in B^{d \times N}(0, M) \text{ and } |\xi - \xi'| \leq t\}.
\]
It turns out that $\omega(y, \cdot)$ is increasing, continuous and $\omega(y, 0) = 0$, while $\omega(\cdot, t)$ is measurable (since the supremum can be restricted to all admissible $\xi$ and $\xi'$ having rational entries) and 1-periodic. Thanks to (4.7) and (4.8) we get that

$$
\left| f \left( \frac{x}{\varepsilon_k}, \nabla u_k(x) \right) - f \left( \frac{x}{\varepsilon_k}, \nabla \varphi_k(x) \right) \right| \leq \omega \left( \frac{x}{\varepsilon_k}, C_1 \sigma \right)
$$

for a.e. $x \in Q(x_0, \rho) \cap \{|u - s_0| < \delta/4 \} \cap \{|\nabla u - \xi_0| < \sigma\}$.

Integrating over the set $Q(x_0, \rho) \cap \{|u - s_0| < \delta/4 \} \cap \{|\nabla u - \xi_0| < \sigma\}$, and taking the limit as $k \to +\infty$, we obtain in view of the Riemann-Lebesgue lemma that

$$
\limsup_{k \to +\infty} \rho^{-N} \int_{Q(x_0, \rho) \cap \{|u - s_0| < \delta/4 \} \cap \{|\nabla u - \xi_0| < \sigma\}} \left| f \left( \frac{x}{\varepsilon_k}, \nabla u_k(x) \right) - f \left( \frac{x}{\varepsilon_k}, \nabla \varphi_k(x) \right) \right| \, dx \leq
$$

$$
\limsup_{k \to +\infty} \rho^{-N} \int_{Q(x_0, \rho)} \omega \left( \frac{x}{\varepsilon_k}, C_1 \sigma \right) \, dx = \int_Q \omega(y, C_1 \sigma) \, dy,
$$

where we have used the fact that $y \mapsto \omega(y, C_1 \sigma)$ is a measurable 1-periodic function. Observe that the Dominated Convergence Theorem together with $\omega(y, 0) = 0$ implies

$$
\lim_{\sigma \to 0^+} \int_Q \omega(y, C_1 \sigma) \, dy = 0. \tag{4.12}
$$

We have obtained

$$
I_2 \leq \limsup_{\rho \to 0^+} \limsup_{k \to +\infty} \frac{1}{\rho^N} \int_{Q(x_0, \rho)} f \left( \frac{x}{\varepsilon_k}, \nabla \varphi_k \right) \, dx + \int_Q \omega(y, C_1 \sigma) \, dy. \tag{4.13}
$$

Using the definition of $\varphi_k$ and the Riemann-Lebesgue lemma, we infer from (4.2) that

$$
\limsup_{\rho \to 0^+} \limsup_{k \to +\infty} \frac{1}{\rho^N} \int_{Q(x_0, \rho)} f \left( \frac{x}{\varepsilon_k}, \xi_0 + \nabla \varphi \left( \frac{x}{\varepsilon_k} \right) \right) \, dx = \int_{(0, j)^N} f(y, \xi_0 + \nabla \varphi(y)) \, dy \leq T_{f_{\text{hom}}}(s_0, \xi_0) + \eta. \tag{4.14}
$$

Hence gathering (4.9)–(4.11), (4.13) and (4.14) we deduce that

$$
\frac{d \mathcal{X}^{(\varepsilon_k)}(u_{\varepsilon_k})}{d\mathcal{L}^N}(x_0) \leq T_{f_{\text{hom}}}(s_0, \xi_0) + \int_Q \omega(y, C_1 \sigma) \, dy + \eta.
$$

Thanks to (4.12), the thesis follows sending first $\sigma \to 0$, and then $\eta \to 0$. \hfill $\Box$

5. The lower bound

We now investigate the $\Gamma$-lim inf inequality still through the blow-up method. In contrast with Lemma 4.1 we will distinguish energies with superlinear growth and energies with linear growth. We will conclude this section with the proofs of Theorems 1.1 and 1.2.

5.1. The case of superlinear growth

The case $p > 1$ is based on an equi-integrability result known as Decomposition Lemma [17], Lemma 1.2, which allows to consider sequences with $p$-equi-integrable gradients. It enables to use properties valid up to sets where the energy remains small.
Lemma 5.1. Assume $p \in (1, +\infty)$. Then $F(u) \geq F_{\text{hom}}(u)$ for every $u \in W^{1,p}(\Omega; \mathcal{M})$.

Proof. Let $u \in W^{1,p}(\Omega; \mathcal{M})$. By a standard diagonal argument, we first obtain a subsequence $\{\varepsilon_n\}$ (not relabeled) and $\{u_n\} \subset W^{1,p}(\Omega; \mathcal{M})$ such that $u_n \rightharpoonup u$ in $L^p(\Omega; \mathbb{R}^d)$ and

$$F(u) = \lim_{n \to +\infty} \int_{\Omega} f\left(\frac{x}{\varepsilon_n}, \nabla u_n\right) \, dx < +\infty.$$ 

Define the sequence of nonnegative Radon measures

$$\mu_n := f\left(\frac{x}{\varepsilon_n}, \nabla u_n\right) \mathcal{L}^N \mathbb{1}_\Omega.$$

Extracting a further subsequence if necessary, we may assume that there exists a nonnegative Radon measure $\mu \in \mathcal{M}(\Omega)$ such that $\mu_n \rightharpoonup^* \mu$ in $\mathcal{M}(\Omega)$. Using Lebesgue Differentiation Theorem one can split $\mu$ into the sum of two mutually disjoint nonnegative measures $\mu = \mu^a + \mu^s$ where $\mu^a \ll \mathcal{L}^N$ and $\mu^s$ is singular with respect to $\mathcal{L}^N$. Since $\mu^s(\Omega) \leq \mu(\Omega) \leq F(u)$, it is enough to check that

$$\frac{d\mu}{d\mathcal{L}^N}(x_0) \geq T_{\text{hom}}(u(x_0), \nabla u(x_0)) \text{ for } \mathcal{L}^N\text{-a.e. } x_0 \in \Omega.$$

Step 1. Select a point $x_0 \in \Omega$ which is a Lebesgue point of $u$ and $\nabla u$, a point of approximate differentiability of $u$ (so that $u(x_0) \in \mathcal{M}$, $\nabla u(x_0) \in [T_{u(x_0)}(\mathcal{M})]^N$), and such that the Radon-Nikodym derivative of $\mu$ with respect to the Lebesgue measure $\mathcal{L}^N$ exists and is finite. Note that almost every points $x_0 \in \Omega$ satisfy these properties. As in the proof of Lemma 4.1, set $s_0 := u(x_0)$ and $\xi_0 := \nabla u(x_0)$.

Let $\{\rho_k\} \setminus 0^+$ be such that $\mu(\partial Q(x_0, \rho_k)) = 0$ for every $k \in \mathbb{N}$. Using the integrand $f$ defined in (2.1) one obtains

$$\frac{d\mu}{d\mathcal{L}^N}(x_0) = \lim_{k \to +\infty} \frac{\mu(Q(x_0, \rho_k))}{\rho_k^N} \mathcal{L}^N \mathbb{1}_\Omega,$$

$$= \lim_{k \to +\infty} \frac{\mu_n(Q(x_0, \rho_k))}{\rho_k^N} \mathcal{L}^N \mathbb{1}_\Omega,$$

$$= \lim_{k \to +\infty} \lim_{n \to +\infty} \int_{Q} f\left(\frac{x_0 + \rho_k y}{\varepsilon_n}, \nabla u_n(x_0 + \rho_k y)\right) \, dy,$$

$$= \lim_{k \to +\infty} \lim_{n \to +\infty} \int_{Q} f\left(\frac{x_0 + \rho_k y}{\varepsilon_n}, u_n(x_0 + \rho_k y), \nabla u_n(x_0 + \rho_k y)\right) \, dy,$$

$$= \lim_{k \to +\infty} \lim_{n \to +\infty} \int_{Q} f\left(\frac{x_0 + \rho_k y}{\varepsilon_n}, s_0 + \rho_k v_n,k(y), \nabla v_n,k(y)\right) \, dy,$$

where we have set $v_{n,k}(y) := [u_n(x_0 + \rho_k y) - s_0]/\rho_k$. Note that since $x_0$ is a point of approximate differentiability of $u$ and $u_n \rightharpoonup u$ in $L^p(\Omega; \mathbb{R}^d)$, we have

$$\lim_{k \to +\infty} \lim_{n \to +\infty} \int_{Q} |v_{n,k}(y) - \xi_0 y|^p \, dy = \lim_{k \to +\infty} \int_{Q(x_0, \rho_k)} \left|u(y) - s_0 - \xi_0 (y - x_0)\right|^p \rho_k^{N+p} \, dy = 0.$$ 

Hence one can find a diagonal sequence $\varepsilon_k := \varepsilon_{nk} < \rho_k^2$ such that, setting $v_k(y) := v_{n,k}(y)$ and $v_0(y) := \xi_0 y$, $v_k \rightharpoonup v_0$ in $L^p(Q; \mathbb{R}^d)$ and

$$\frac{d\mu}{d\mathcal{L}^N}(x_0) = \lim_{k \to +\infty} \int_{Q} f\left(\frac{x_0 + \rho_k y}{\varepsilon_k}, s_0 + \rho_k v_k(y), \nabla v_k(y)\right) \, dy.$$ 

(5.1)
Next observe that $\{\nabla v_k\}$ is bounded in $L^p(Q; R^{d \times N})$ thanks to the coercivity condition (2.2). By the Decomposition Lemma [17], Lemma 1.2, we now find a sequence $\{v_k\} \subset W^{1,\infty}(Q; R^d)$ such that $\tilde{v}_k = v_0$ on a neighborhood of $\partial Q$, $v_k \to v_0$ in $L^p(Q; R^d)$, the sequence of gradients $\{|\nabla \tilde{v}_k|^p\}$ is equi-integrable, and

$$\lim_{k \to +\infty} \int_Q \tilde{f}\left(\frac{x_0 + \rho_k y}{\varepsilon_k}, s_0 + \rho_k v_k(y), \nabla v_k(y)\right) dy \geq \lim_{k \to +\infty} \int_Q \tilde{f}\left(\frac{x_0 + \rho_k y}{\varepsilon_k}, s_0 + \rho_k v_k(y), \nabla \tilde{v}_k(y)\right) dy.$$ (5.2)

**Step 2.** Write

$$\frac{x_0}{\varepsilon_k} = m_k + s_k \quad \text{with} \quad m_k \in \mathbb{Z}^N \quad \text{and} \quad s_k \in [0, 1)^N,$$

and define

$$x_k := \frac{\varepsilon_k}{\rho_k} s_k \to 0 \quad \text{and} \quad \delta_k := \varepsilon_k/\rho_k \to 0.$$

By the 1-periodicity of $\tilde{f}$ with respect to its first variable, (5.1) and (5.2), we infer

$$\frac{d\mu}{dL^N}(x_0) \geq \limsup_{k \to +\infty} \int_Q \tilde{f}\left(\frac{x_k + y}{\delta_k}, s_0 + \rho_k v_k(y), \nabla \tilde{v}_k(y)\right) dy$$

$$\geq \limsup_{k \to +\infty} \int_{x_k + Q} \tilde{f}\left(\frac{y}{\delta_k}, s_0 + \rho_k v_k(y - x_k), \nabla \tilde{v}_k(y - x_k)\right) dy.$$ (5.3)

Extend $v_k$ by 0, and $\tilde{v}_k$ by $v_0$ to the whole $\mathbb{R}^N$. As $x_k \to 0$ it follows that $L^N((Q - x_k)\Delta Q) \to 0$, and the equi-integrability of $\{|\nabla \tilde{v}_k|^p\}$ together with the $p$-growth condition (2.2) implies

$$\int_{Q \Delta (x_k + Q)} \tilde{f}\left(\frac{y}{\delta_k}, s_0 + \rho_k v_k(y - x_k), \nabla \tilde{v}_k(y - x_k)\right) dy \leq \beta' \int_{(Q - x_k)\Delta Q} (1 + |\nabla \tilde{v}_k|^p) dy \to 0.$$

Hence (5.3) yields

$$\frac{d\mu}{dL^N}(x_0) \geq \limsup_{k \to +\infty} \int_Q \tilde{f}\left(\frac{y}{\delta_k}, s_0 + \rho_k w_k, \nabla \tilde{w}_k\right) dy,$$ (5.4)

where $w_k(y) := v_k(y - x_k)$ and $\tilde{w}_k(y) := \tilde{v}_k(y - x_k)$ converge to $v_0$ in $L^p(Q; R^d)$, and $\{|\nabla \tilde{w}_k|^p\}$ is equi-integrable as well.

**Step 3.** For $M > 1$ and $k \in \mathbb{N}$, consider the set $E_{M,k} := \{x \in Q : |\nabla \tilde{w}_k| \leq M\}$. By Chebyshev inequality, (5.4) and (2.2), $L^N(Q \setminus E_{M,k}) \leq C/M^p$ for some constant $C > 0$ independent of $k$ and $M$.

By the Scorza-Drogni Theorem (see [14], p. 235), for any $\eta > 0$, we may find a compact set $K_\eta \subset Q$ such that $L^N(Q \setminus K_\eta) < \eta$ and $f : K_\eta \times R^{d \times N} \to [0, +\infty)$ is continuous. In particular the restriction of $f(\cdot, s, \cdot)$ to $K_\eta \times B^{d \times N}(0, M)$ is uniformly continuous for every $s \in \mathcal{M}$. Therefore the function $\Psi_{\eta,M} : [0, +\infty) \to [0, +\infty)$ defined by

$$\Psi_{\eta,M}(t) = \sup\left\{|f(y, \xi) - f(y, \xi')| : y \in K_\eta, \xi, \xi' \in B^{d \times N}(0, M), |\xi - \xi'| \leq t\right\},$$

is continuous, satisfies $\Psi_{\eta,M}(0) = 0$, and is bounded. In view of (2.1), we have

$$|f(y, s_1, \xi) - f(y, s_2, \xi)| \leq \Psi_{\eta,M}(M|P_{s_1} - P_{s_2}|) + C_M|P_{s_1} - P_{s_2}| =: \tilde{\Psi}_{\eta,M}(|P_{s_1} - P_{s_2}|)$$

for every $y \in K_\eta$, $s_1, s_2 \in \mathcal{M}$ and $\xi \in B^{d \times N}(0, M)$, where the constant $C_M > 0$ only depends on $M$ and $p$. Define

$$K^\per_{\eta} := \bigcup_{\xi \in \mathbb{Z}^N} (\ell + K_\eta).$$

Since $\tilde{f}$ is 1-periodic in the first variable,

$$|\tilde{f}(y, s_1, \xi) - \tilde{f}(y, s_2, \xi)| \leq \tilde{\Psi}_{\eta,M}(|P_{s_1} - P_{s_2}|)$$ (5.5)
for every $y \in K_{\delta_k}^{\text{per}}$, $s_1, s_2 \in M$ and $\xi \in B^{d \times N}(0, M)$. From (5.4) and (5.5) it follows that

$$\frac{d\mu}{d\mathcal{L}^N}(x_0) \geq \limsup_{k \to +\infty} \int_{E_{M,k} \cap (\delta_k K_{\delta_k}^{\text{per}})} f\left(\frac{y}{\delta_k}, s_0 + \rho_k w_k, \nabla \bar{w}_k\right) dy,$$

$$\geq \limsup_{k \to +\infty} \int_{E_{M,k} \cap (\delta_k K_{\delta_k}^{\text{per}})} f\left(\frac{y}{\delta_k}, s_0, \nabla \bar{w}_k\right) dy - \limsup_{k \to +\infty} \int_Q \tilde{\Psi}_{\eta,M}(|P_{s_0+\rho_k w_k(y)} - P_{s_0}|) dy.$$

Since $\tilde{\Psi}_{\eta,M}$ is continuous and bounded, $\tilde{\Psi}_{\eta,M}(0) = 0$, and (up to a subsequence) $P_{s_0+\rho_k w_k(y)} \to P_{s_0}$ for a.e. $y \in Q$, we obtain by Dominated Convergence that

$$\lim_{k \to +\infty} \int_Q \tilde{\Psi}_{\eta,M}(|P_{s_0+\rho_k w_k(y)} - P_{s_0}|) dy = 0,$$

and thus

$$\frac{d\mu}{d\mathcal{L}^N}(x_0) \geq \limsup_{k \to +\infty} \int_{E_{M,k} \cap (\delta_k K_{\delta_k}^{\text{per}})} f\left(\frac{y}{\delta_k}, s_0, \nabla \bar{w}_k\right) dy. \quad (5.6)$$

From the $p$-growth condition (2.2) and the Riemann-Lebesgue Lemma, we deduce that

$$\limsup_{k \to +\infty} \int_{E_{M,k} \cap (\delta_k K_{\delta_k}^{\text{per}})} f\left(\frac{y}{\delta_k}, s_0, \nabla \bar{w}_k\right) dy \leq \limsup_{k \to +\infty} \beta'(1 + M^p)\mathcal{L}^N(Q \setminus (\delta_k K_{\delta_k}^{\text{per}}))$$

$$= \beta'(1 + M^p)\mathcal{L}^N(Q \setminus K_{\delta_k}) \leq \beta'(1 + M^p)\eta.$$

Hence (5.6) yields

$$\frac{d\mu}{d\mathcal{L}^N}(x_0) \geq \limsup_{k \to +\infty} \int_{E_{M,k}} f\left(\frac{y}{\delta_k}, s_0, \nabla \bar{w}_k\right) dy - \beta'(1 + M^p)\eta,$$

and sending $\eta \to 0$, we derive

$$\frac{d\mu}{d\mathcal{L}^N}(x_0) \geq \limsup_{k \to +\infty} \int_{E_{M,k}} f\left(\frac{y}{\delta_k}, s_0, \nabla \bar{w}_k\right) dy. \quad (5.7)$$

Since $\mathcal{L}^N(Q \setminus E_{M,k}) \to 0$ as $M \to +\infty$ (uniformly with respect to $k$), the equi-integrability of $\{|\nabla \bar{w}_k|^p\}$ and the $p$-growth condition (2.2) imply

$$\sup_{k \in \mathbb{N}} \int_{Q \setminus E_{M,k}} f\left(\frac{y}{\delta_k}, s_0, \nabla \bar{w}_k\right) dy \leq \beta' \sup_{k \in \mathbb{N}} \int_{Q \setminus E_{M,k}} (1 + |\nabla \bar{w}_k|^p) dy \to 0 \quad \text{as } M \to +\infty.$$

Plugging this estimate in (5.7) leads to

$$\frac{d\mu}{d\mathcal{L}^N}(x_0) \geq \limsup_{k \to +\infty} \int_Q f\left(\frac{y}{\delta_k}, s_0, \nabla \bar{w}_k\right) dy.$$
Since \( \tilde{w}_k \to v_0 \) in \( L^p(Q; \mathbb{R}^d) \), we can invoke standard homogenization results (see, e.g., [8], Thm. 14.5) to infer that

\[
\limsup_{k \to +\infty} \int_Q f \left( \frac{y}{\delta_k}, s_0, \nabla \tilde{w}_k \right) dy \geq \int_Q f_{\text{hom}}(s_0, \nabla v_0) dy = \tilde{f}_{\text{hom}}(s_0, \xi_0).
\]

In view of Proposition 2.1 we finally conclude

\[
\frac{d\mu}{d\mathcal{L}^N}(x_0) \geq \tilde{f}_{\text{hom}}(s_0, \xi_0) = T_{\text{hom}}(s_0, \xi_0),
\]

and the proof is complete. \( \square \)

### 5.2. The case of linear growth

We now treat the case \( p = 1 \) assuming that the function \( u \) belongs to \( W^{1,1}(\Omega; \mathcal{M}) \). In contrast with the case \( p > 1 \), there is no equi-integrability result as the Decomposition Lemma. We follow here the approach of [15].

**Lemma 5.2.** Assume \( p = 1 \). Then \( F(u) \geq F_{\text{hom}}(u) \) for every \( u \in W^{1,1}(\Omega; \mathcal{M}) \).

**Proof.** Let \( u \in W^{1,1}(\Omega; \mathcal{M}) \). By a standard diagonal argument, we first obtain a subsequence \( \{\varepsilon_n\} \) (not relabeled) and \( \{u_n\} \subset W^{1,1}(\Omega; \mathcal{M}) \) such that \( u_n \to u \) in \( L^1(\Omega; \mathbb{R}^d) \) and

\[
F(u) = \lim_{n \to +\infty} \int_{\Omega} f \left( \frac{x}{\varepsilon_n}, \nabla u_n \right) dx < +\infty.
\]

Up to the extraction of a further subsequence, we may assume that there exists a nonnegative Radon measure \( \mu \in \mathcal{M}(\Omega) \) such that

\[
f \left( \frac{x}{\varepsilon_n}, \nabla u_n \right) \mathcal{L}^N \llcorner \Omega \rightharpoonup^* \mu \quad \text{in} \ \mathcal{M}(\Omega).
\] (5.8)

Hence it is enough to prove that \( \mu(\Omega) \geq F_{\text{hom}}(u) \). As in the proof of Lemma 5.1, it suffices to show that

\[
\frac{d\mu}{d\mathcal{L}^N}(x_0) \geq T_{\text{hom}}(u(x_0), \nabla u(x_0)) \quad \text{for} \ \mathcal{L}^N\text{-a.e.} \ x_0 \in \Omega.
\]

The proof will be divided into three steps. We first apply the blow-up method which reduces the study to affine limiting functions. Then we reproduce the argument of [16] which enables us to replace the original sequence by a uniformly converging one without increasing the energy. We will conclude using a classical homogenization result.

**Step 1.** Select a point \( x_0 \in \Omega \) which is a Lebesgue point of \( u \) and \( \nabla u \), a point of approximate differentiability of \( u \) (so that \( u(x_0) \in \mathcal{M}, \nabla u(x_0) \in [T_{u(x_0)}(\mathcal{M})]^N \) and such that the Radon-Nikodym derivative of \( \mu \) with respect to the Lebesgue measure \( \mathcal{L}^N \) exists and is finite. Note that \( \mathcal{L}^N \)-almost every points \( x_0 \in \Omega \) satisfy these properties. We write \( s_0 := u(x_0) \) and \( \xi_0 := \nabla u(x_0) \).

Up to a subsequence, we may assume that there exists a nonnegative Radon measure \( \lambda \in \mathcal{M}(\Omega) \) such that

\[
(1+|\nabla u_n|)\mathcal{L}^N \llcorner \Omega \rightharpoonup^* \lambda \text{ in } \mathcal{M}(\Omega).
\]

Consider a sequence \( \{\rho_k\} \setminus 0^+ \) such that \( Q(x_0, 2\rho_k) \subset \Omega \) and \( \mu(\partial Q(x_0, \rho_k)) = \lambda(\partial Q(x_0, \rho_k)) = 0 \) for each \( k \in \mathbb{N} \). Then (5.8) yields

\[
\mu(Q(x_0, \rho_k)) = \lim_{n \to +\infty} \int_{Q(x_0, \rho_k)} f \left( \frac{x}{\varepsilon_n}, \nabla u_n \right) dx.
\] (5.9)
Set $\tau_n := \varepsilon_n [x_0 / \varepsilon_n] \in \varepsilon_n \mathbb{Z}^N$. Since $\tau_n \to x_0$, given $r \in (1, 2)$ we have $Q(\tau_n, \rho_k) \subset Q(x_0, r \rho_k)$ whenever $n$ is large enough, and we may define for $x \in Q(0, \rho_k)$, $v_n(x) := u_n(x + \tau_n)$. By continuity of the translation in $L^1$, we get that

$$\int_{Q(0, \rho_k)} |v_n(x) - u(x + x_0)| \, dx = \int_{Q(\tau_n, \rho_k)} |u_n(x) - u(x + x_0 - \tau_n)| \, dx$$

$$\leq \int_{Q(x_0, r \rho_k)} |u_n(x) - u(x + x_0 - \tau_n)| \, dx \xrightarrow{n \to +\infty} 0. \quad (5.10)$$

Changing variable in (5.9) and using the periodicity condition $(H_1)$ of $f(\cdot, \xi)$ and the growth condition $(H_2)$, we are led to

$$\mu(Q(x_0, \rho_k)) = \lim_{n \to +\infty} \int_{Q(x_0 - \tau_n, \rho_k)} f \left( \frac{x + \tau_n}{\varepsilon_n}, \nabla u_n(x + \tau_n) \right) \, dx$$

$$= \lim_{n \to +\infty} \int_{Q(x_0 - \tau_n, \rho_k)} f \left( \frac{x}{\varepsilon_n}, \nabla v_n \right) \, dx$$

$$\geq \limsup_{n \to +\infty} \int_{Q(0, \rho_k)} f \left( \frac{x}{\varepsilon_n}, \nabla v_n \right) \, dx - \beta \limsup_{n \to +\infty} \int_{Q(\tau_n, \rho_k) \setminus Q(x_0, \rho_k)} (1 + |\nabla u_n|) \, dx. \quad (5.11)$$

On the other hand, by our choice of $\rho_k$,

$$\limsup_{n \to +\infty} \int_{Q(\tau_n, \rho_k) \setminus Q(x_0, \rho_k)} (1 + |\nabla u_n|) \, dx \leq \limsup_{r \to +\infty} \limsup_{n \to +\infty} \int_{Q(x_0, r \rho_k) \setminus Q(x_0, \rho_k)} (1 + |\nabla u_n|) \, dx$$

$$\leq \limsup_{r \to +\infty} \lambda \left( Q(x_0, r \rho_k) \setminus Q(x_0, \rho_k) \right)$$

$$\leq \lambda(\partial Q(x_0, \rho_k)) = 0,$$

so that the last term in (5.11) vanishes. Hence

$$\mu(Q(x_0, \rho_k)) \geq \limsup_{n \to +\infty} \int_{Q(0, \rho_k)} f \left( \frac{x}{\varepsilon_n}, \nabla v_n \right) \, dx,$$

where $\{v_n\} \subset W^{1,1}(Q(0, \rho_k); M)$ satisfies $v_n \to u(x_0 + \cdot)$ in $L^1(Q(0, \rho_k); \mathbb{R}^d)$ by (5.10).

Now we consider for every $n$, a sequence $\{v_{n,j}\} \subset C^\infty(Q(0, \rho_k); \mathbb{R}^d)$ such that $v_{n,j} \to v_n$ in $W^{1,1}(Q(0, \rho_k); \mathbb{R}^d)$,

$v_{n,j} \to v_n$ and $\nabla v_{n,j} \to \nabla v_n$ a.e. in $Q(0, \rho_k)$ as $j \to +\infty$ (we emphasize that in general, $v_{n,j}$ is not $M$-valued). Considering the integrand $g$ given by Lemma 2.1, one may check

$$\lim_{j \to +\infty} \int_{Q(0, \rho_k)} g \left( \frac{x}{\varepsilon_n}, v_{n,j}, \nabla v_{n,j} \right) \, dx = \int_{Q(0, \rho_k)} g \left( \frac{x}{\varepsilon_n}, v_n, \nabla v_n \right) \, dx = \int_{Q(0, \rho_k)} f \left( \frac{x}{\varepsilon_n}, \nabla v_n \right) \, dx,$$

so that we can find a diagonal sequence $\tilde{v}_n := v_{n,j,n}$ satisfying $\tilde{v}_n \to u(x_0 + \cdot)$ in $L^1(Q(0, \rho_k); \mathbb{R}^d)$ and

$$\mu(Q(x_0, \rho_k)) \geq \limsup_{n \to +\infty} \int_{Q(0, \rho_k)} g \left( \frac{x}{\varepsilon_n}, \tilde{v}_n, \nabla \tilde{v}_n \right) \, dx. \quad (5.12)$$
Changing variable in (5.12) yields

$$\frac{d\mu}{dL^N}(x_0) \geq \limsup_{k \to +\infty} \limsup_{n \to +\infty} \int_Q g \left( \frac{\rho_k x}{\varepsilon_n}, \varphi_n(\rho_k x), \nabla \varphi_n(\rho_k x) \right) dx$$

$$= \limsup_{k \to +\infty} \limsup_{n \to +\infty} \int_Q g \left( \frac{\rho_k x}{\varepsilon_n}, s_0 + \rho_k w_{n,k}, \nabla w_{n,k} \right) dx,$$

where we have set $w_{n,k}(x) := \left[ \varphi_n(\rho_k x) - s_0 \right]/\rho_k$. Since $x_0$ is a point of approximate differentiability of $u$ and $\varphi_n \to u(x_0 + \cdot)$ in $L^1(Q(0, \rho_k); \mathbb{R}^d)$, we have

$$\lim_{k \to +\infty} \lim_{n \to +\infty} \int_Q |w_{n,k}(x) - \xi_0 x| dx = \lim_{k \to +\infty} \int_{Q(x_0, \rho_k)} \frac{|u(y) - s_0 - \xi_0 (y - x_0)|}{\rho_k^{N+1}} dy = 0. \quad (5.14)$$

In view of (5.13) and (5.14), we can find a diagonal sequence $\varepsilon_{n_k} < \rho_k^2$ such that $w_k := w_{n_k,k} \to w_0$ in $L^1(Q; \mathbb{R}^d)$ with $w_0(x) := \xi_0 x$, and

$$\frac{d\mu}{dL^N}(x_0) \geq \limsup_{k \to +\infty} \int_Q g \left( \frac{x}{\delta_k}, s_0 + \rho_k w_k, \nabla w_k \right) dx,$$

where $\delta_k := \varepsilon_{n_k}/\rho_k \to 0$.

**Step 2.** We now argue as in Step 3 of the proof of [15], Theorem 2.1, to show that there exists a sequence $\{w_k\} \subset W^{1,\infty}(Q; \mathbb{R}^d)$ such that $w_k \to w_0$ in $L^\infty(Q; \mathbb{R}^d)$, $\{w_k\}$ is uniformly bounded in $W^{1,1}(Q; \mathbb{R}^d)$ and

$$\frac{d\mu}{dL^N}(x_0) \geq \limsup_{k \to +\infty} \int_Q g \left( \frac{x}{\delta_k}, s_0 + \rho_k \overline{w}_k, \nabla \overline{w}_k \right) dx. \quad (5.16)$$

Given $0 < s < t$, let $\zeta_{s,t} \in C_0^\infty(\mathbb{R}; [0, 1])$ be a cut-off function satisfying $\zeta_{s,t}(\tau) = 1$ if $|\tau| \leq s$, $\zeta_{s,t}(\tau) = 0$ if $|\tau| \geq t$ and $|\zeta'_{s,t}| \leq C/(t-s)$. Define

$$w^{k}_{s,t} := w_0 + \zeta_{s,t}(|w_k - w_0|)(w_k - w_0).$$

Obviously,

$$\|w^{k}_{s,t} - w_0\|_{L^\infty(Q; \mathbb{R}^d)} \leq t, \quad (5.17)$$

and the Chain Rule formula gives

$$\nabla w^{k}_{s,t} = \nabla w_0 + \zeta_{s,t}(|w_k - w_0|)(\nabla w_k - \nabla w_0) + \zeta'_{s,t}(|w_k - w_0|)(w_k - w_0) \otimes \nabla |w_k - w_0|. \quad (5.18)$$

In particular,

$$\int_Q g \left( \frac{x}{\delta_k}, s_0 + \rho_k w^{k}_{s,t}, \nabla w^{k}_{s,t} \right) dx = \int_{\{|w_k - w_0| \leq s\}} g \left( \frac{x}{\delta_k}, s_0 + \rho_k w_k, \nabla w_k \right) dx$$

$$+ \int_{\{s < |w_k - w_0| \leq t\}} g \left( \frac{x}{\delta_k}, s_0 + \rho_k w^{k}_{s,t}, \nabla w^{k}_{s,t} \right) dx$$

$$+ \int_{\{|w_k - w_0| > t\}} g \left( \frac{x}{\delta_k}, s_0 + \rho_k w_0, \xi_0 \right) dx. \quad (5.19)$$

From the growth condition (2.7), we infer that

$$\int_{\{|w_k - w_0| > t\}} g \left( \frac{x}{\delta_k}, s_0 + \rho_k w_0, \xi_0 \right) dx \leq \beta'(1 + |\xi_0|) L^N(\{ |w_k - w_0| > t \}), \quad (5.20)$$
and (5.18) yields
\[
\int_{\{s < |w_k - w_0| \leq t\}} g\left(\frac{x}{\delta_k}, s_0 + \rho_k w_{k,s,t}, \nabla w_{k,s,t}\right) dx \leq C \int_{\{s < |w_k - \xi_0| \leq t\}} (1 + |\nabla w_k - \xi_0|) dx \\
+ \frac{C}{t-s} \int_{\{s < |w_k - w_0| \leq t\}} |w_k - w_0| |\nabla w_k - w_0| dx. \tag{5.21}
\]

Observe that for $L^1$-a.e. $t > 0$,
\[
\lim_{s \to t^-} \int_{\{s < |w_k - w_0| \leq t\}} (1 + |\nabla w_k - \xi_0|) dy \leq C_k \lim_{s \to t^-} L^N(\{s < |w_k - w_0| \leq t\}) = 0, \tag{5.22}
\]
and by the Coarea formula,
\[
\lim_{s \to t^-} \frac{1}{t-s} \int_{\{s < |w_k - w_0| \leq t\}} |w_k - w_0| |\nabla w_k - w_0| dx \\
= \lim_{s \to t^-} \frac{1}{t-s} \int_s^t \tau H^{N-1}(\{|w_k - w_0| = \tau\}) d\tau = t H^{N-1}(\{|w_k - w_0| = t\}). \tag{5.23}
\]

Moreover, in view of (2.7) and (5.15) we infer that
\[
\int_Q |\nabla w_k - w_0| dx \leq C \int_Q (1 + |\nabla w_k|) dy \leq C_0.
\]

Applying [15], Lemma 2.6, there exists $t_k \in \left(\|w_k - w_0\|_{L^1(Q;R^d)}^{1/2}, \|w_k - w_0\|_{L^1(Q;R^d)}^{1/3}\right)$ such that (5.22) and (5.23) hold with $t = t_k$, and
\[
t_k H^{N-1}(\{|w_k - w_0| = t_k\}) \leq \frac{C_0}{\ln \left(\|w_k - w_0\|_{L^1(Q;R^d)}^{-1/6}\right)}. \tag{5.24}
\]

According to (5.22), (5.23) and (5.24), there exists $s_k \in \left(\|w_k - w_0\|_{L^1(Q;R^d)}^{1/2}, t_k\right)$ such that
\[
\int_{\{s_k < |w_k - w_0| \leq t_k\}} (1 + |\nabla w_k - \xi_0|) dx \leq \frac{1}{k}. \tag{5.25}
\]

and
\[
\frac{1}{t_k - s_k} \int_{\{s_k < |w_k - w_0| \leq t_k\}} |w_k - w_0| |\nabla w_k - w_0| dx \leq \frac{C_0}{\ln \left(\|w_k - w_0\|_{L^1(Q;R^d)}^{-1/6}\right)} + \frac{1}{k}. \tag{5.26}
\]

while (5.20) together with Chebyshev inequality yields
\[
\int_{\{|w_k - w_0| > t_k\}} g\left(\frac{x}{\delta_k}, s_0 + \rho_k w_k, \xi_0\right) dy \leq C \|w_k - w_0\|_{L^1(Q;R^d)}^{1/2}. \tag{5.27}
\]

Define now $\overline{w}_k := w_{k,s_k,t_k}$ so that $\overline{w}_k \to w_0$ in $L^\infty(Q;R^d)$ by (5.17). Moreover, gathering (5.19), (5.21), (5.25), (5.26) and (5.27), we deduce
\[
\limsup_{k \to +\infty} \int_Q g\left(\frac{x}{\delta_k}, s_0 + \rho_k \overline{w}_k, \nabla \overline{w}_k\right) dx \leq \limsup_{k \to +\infty} \int_Q g\left(\frac{x}{\delta_k}, s_0 + \rho_k w_k, \nabla w_k\right) dx,
\]
which proves (5.16). The fact that \( \{ \nabla w_k \} \) is uniformly bounded in \( L^1(Q; \mathbb{R}^{d \times N}) \) is a consequence of (5.16) and the coercivity condition (2.7).

**Step 3.** Since \( \{ \| w_k \|_{L^\infty(Q; \mathbb{R}^d)} \} \) and \( \{ \| \nabla w_k \|_{L^1(Q; \mathbb{R}^{d \times N})} \} \) are uniformly bounded, we derive from (2.8) that

\[
\lim_{k \to +\infty} \int_Q g \left( \frac{x}{\delta_k}, s_0 + \rho_k w_k, \nabla w_k \right) - g \left( \frac{x}{\delta_k}, s_0, \nabla w_k \right) \, dx = 0.
\]

In view of (5.16), it leads to

\[
\frac{d\mu}{d\mathcal{L}^N}(x_0) \geq \lim_{k \to +\infty} \int_Q g \left( \frac{x}{\delta_k}, s_0, \nabla w_k \right) \, dx.
\]

Using standard homogenization results (see e.g., [8], Thm. 14.5) together with (2.10), we finally conclude that

\[
\frac{d\mu}{d\mathcal{L}^N}(x_0) \geq g_{\text{hom}}(s_0, \xi_0) = T_{\text{f\text{hom}}}(s_0, \xi_0),
\]

which completes the proof of the lemma.

\[ \square \]

### 5.3. Proof of Theorems 1.1 and 1.2 completed

Since \( L^p(\Omega; \mathbb{R}^d) \) is separable \((1 \leq p < +\infty)\), there exists a subsequence \( \{ \varepsilon_{n_k} \} \) such that \( \mathcal{F} \) is the \( \Gamma \)-limit of \( \{ \mathcal{F}_{\varepsilon_{n_k}} \} \) for the strong \( L^p(\Omega; \mathbb{R}^d) \)-topology (see [13], Thm. 8.5).

**Case \( p > 1 \).** In view of (H2) and the closure of the pointwise constraint under strong \( L^p \)-convergence, we have \( \mathcal{F}(u) < +\infty \) if and only if \( u \in W^{1,p}(\Omega; \mathcal{M}) \). Hence, as a consequence of Lemmas 4.1 and 5.1, the functionals \( \{ \mathcal{F}_{\varepsilon_{n_k}} \} \) \( \Gamma \)-converge to \( \mathcal{F}_{\text{hom}} \) in \( L^p(\Omega; \mathbb{R}^d) \). Since the \( \Gamma \)-limit does not depend on the extracted subsequence, we get in light of [13], Proposition 8.3, that the whole sequence \( \{ \mathcal{F}_{\varepsilon_{n_k}} \} \) \( \Gamma \)-converges to \( \mathcal{F}_{\text{hom}} \).

**Case \( p = 1 \).** As a consequence of Lemmas 4.1 and 5.2, the functionals \( \{ \mathcal{F}_{\varepsilon_{n_k}} \} \) \( \Gamma \)-converge to \( \mathcal{F}_{\text{hom}} \) in \( W^{1,1}(\Omega; \mathcal{M}) \). Again, the \( \Gamma \)-limit does not depend on the extracted subsequence, so that the whole sequence \( \{ \mathcal{F}_{\varepsilon_{n_k}} \} \) \( \Gamma \)-converges to \( \mathcal{F}_{\text{hom}} \) in \( W^{1,1}(\Omega; \mathcal{M}) \).

\[ \square \]

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### References


