CORRIGENDUM TO: “ON THE CIRCLE CRITERION FOR BOUNDARY CONTROL SYSTEMS IN FACTOR FORM: LYAPUNOV STABILITY AND LUR’E EQUATIONS”

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1. INTRODUCTION

The authors are deeply indebted to Hartmut Logemann, Department of Mathematics, University of Bath, UK for pointing out a counterexample, repeated below, showing that the statement of [2], Theorem 4.1, p. 186, is wrong.

With the notation of [2] all assumptions of that theorem are met for

\[
H = \mathbb{R}, \quad A = -1 = A^{-1}, \quad h = -1 (\iff e^# x = x), \quad d = 1, \quad \delta = 1, \quad e = \frac{8}{3}, \quad q = \frac{16}{3},
\]

however the system (3.1) has exactly two solutions \((H, G) = (-\frac{8}{3}, 0), (H, G) = (-\frac{2}{3}, 2)\) and none of them is such that \(H \geq 0\). This counterexample demonstrates that the assumptions of [2], Theorem 4.1, p. 186, are not enough to ensure non-negativity of \(H\).

The aim of this note is to correct the result by adding reasonable and non-restrictive assumptions which can be verified without solving (3.1) explicitly.

2. CORRIGENDUM OF [2], THEOREM 4.1 (i), P. 186

Theorem 2.1. Let assumptions (H1)–(H5) hold. Moreover assume that:

(H6) The operator \(A: (D(A) \subset H) \longrightarrow H\) is such that the semigroup generated by \(A^{-1}\) is AS.

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Then:

(i) The system (3.1) has a solution \((\mathcal{H}, \mathcal{G}), \mathcal{H} \in \mathcal{L}(H), \mathcal{H} = \mathcal{H}^* \geq 0\), provided that if \(q > 0\) then, in addition, the assumption \((\text{A3})\) holds and

\[
\frac{1}{1 + \mu_0 g} \in H^\infty(\mathbb{C}^+) \quad \text{for} \quad \mu_0 := \frac{k_1 + k_2}{2},
\]

\((2.1)\)

\(\mathcal{G} \in \mathcal{H}\), where in particular: \(\mathcal{G}\) is the solution of the realization equation (4.4), where \(\phi\) is the spectral factor of the Popov function \(\pi\) (given by (4.2)) such that \(\phi(0) = \sqrt{\delta}\), and both \(\phi\) and \(1/\phi\) are in \(H^\infty(\mathbb{C}^+)\).

**Remark 2.1.** It should be emphasized that if \(q \leq 0\) the statement of [2], Theorem 4.1(i), p. 186, is fully correct, i.e., the assertion holds without \((\text{A3})\) and (2.1). The claim [2], Theorem 4.1(ii), p. 186, does not require any correction.

**Proof.** The whole reasoning of the existing proof remains correct after removing: the sentence starting from the words: “The symbol of the Toeplitz operator . . .”, the footnote on p. 186 and after dropping the inequality \(\mathcal{H} \geq 0\) in the sentence just following (4.17). Having this done, we may correct the proof as follows. Since \(X\) is a solution of (4.15) given by (4.10) it is clear that

\[
\mathcal{H} = -X = \psi^* \left[ (q^2 - eI)R^{-1}(q^2 - eI)^* - qI \right] \psi \geq 0 \tag{2.2}
\]

if \(q \leq 0\), whence the claim of the remark above is met.

Now, consider the case \(q > 0\) \(\implies \mu_0 \neq 0\) where, in addition \((\text{A3})\) (i.e., \(d\) is an admissible factor control vector) and (2.1) hold. Observe that

\[
1 - \mu_0 \frac{c^\#d}{c^\#} \neq 0,
\]

for if not, by (4.2), we would have \(\pi(0) = \delta = \left(1 - \frac{\mu_0}{\mu_0}\right) \left(1 - \frac{\frac{k_1}{k_1 + k_2}}{\frac{c^\#}{c^\#}}\right)^2 < 0\), which contradicts (4.3). Since the LHS of (2.2) satisfies the Riccati equation

\[
(A^{-1})^*\mathcal{H} + \mathcal{H}A^{-1} + \left[ \frac{1}{\sqrt{\delta}}(-\mathcal{H}d + eh) \right] \left[ \frac{1}{\sqrt{\delta}}(-\mathcal{H}d + eh) \right]^* - qhh^* = 0 \tag{2.3}
\]

then, adding \(
\frac{\mu_0}{1 - \mu_0 c^\#d}hh^*\mathcal{H} + \frac{\mu_0}{1 - \mu_0 c^\#d}hh^*\mathcal{H}d\mathcal{H}^*
\)

to both sides of (2.3), we conclude that \(\mathcal{H}\) satisfies the Lyapunov operator equation

\[
\left[ A^{-1} + \frac{\mu_0}{1 - \mu_0 c^\#d}dh^* \right]^* \mathcal{H} + \mathcal{H} \left[ A^{-1} + \frac{\mu_0}{1 - \mu_0 c^\#d}dh^* \right] = -(\mathcal{G} - q_1 h)(\mathcal{G} - q_1 h)^* - q_0 hh^*
\]

with

\[
q_1 := \frac{\mu_0 \sqrt{\delta}}{1 - \mu_0 c^\#d}, \quad q_0 = \frac{(k_2 - k_1)^2}{4(1 - \mu_0 c^\#d)^2} > 0,
\]

or equivalently,

\[
\langle A_0 x, \mathcal{H} x \rangle_H + \langle \mathcal{H} x, A_0 x \rangle_H = -[\langle (\mathcal{G} - q_1 h)^* A_0 x \rangle^2 - q_0 \langle h^* A_0 x \rangle^2] \quad \forall x \in D(A_0), \tag{2.4}
\]

where

\[
A_0 x := A(x - \mu_0 d c^\# x), \quad D(A_0) = \{ x \in D(d^\#): x - \mu_0 d c^\# x \in D(A) \}.
\]
This is because $A_0^{-1} = A^{-1} + \frac{\mu_0}{1 - \mu_0 c^\# d} \hat{h}^* \in \mathbb{L}(H)$. The operator $A_0$ arises by applying negative linear feedback $u = -\mu_0 y$ to

$$\begin{align*}
\dot{x} &= A(x + ud) \\
y &= c^* x
\end{align*}$$

(2.5)

and it corresponds to the Lur'e control system of [2], Figure 1.1, p. 170, with $f(y) = \mu_0 y$. Since $c^\#$ is admissible and $\hat{g} \in H^\infty(\mathbb{C}^+)$, for $L^2(0, \infty)$-controls the output is given by

$$y = \overline{P} x_0 + \overline{F} u$$

where $\overline{P}$ and $\overline{F}$ stand for the extended observability map and the extended input-output operator, both associated with (2.5). Thus, for the closed-loop system, by the Paley-Wiener theory, one has

$$(I + \mu_0 \overline{P}) y = \overline{P} x_0 \iff (1 + \mu_0 \hat{g}) y = \overline{F} x_0,$$

and, due to (2.1), the last equation has a unique solution $\hat{y} \in H^2(\mathbb{C}^+)$. Via the feedback law equation $u = -\mu_0 y$ this implies that for any $x_0 \in L^2(0, \infty)$, Now [2], Lemma 2.11, p. 177, implies that for every initial condition $x_0$ the first equation of (2.5) has a unique weak solution, whence, by Ball’s theorem [1], p. 371 (see also [4], p. 259), the operator $A_0$ generates a $C_0$-semigroup $\{S_0(t)\}_{t \geq 0}$ on $H$ which is $\mathcal{A}$.

Now, for every $x_0 \in D(A_0)$ and each $t \geq 0$, (2.4) yields

$$\frac{d}{dt} \langle S_0(t)x_0, \mathcal{H} S_0(t)x_0 \rangle_H = -\langle (\mathcal{G} - q_1 h)^* A_0 S_0(t)x_0 \rangle^2 - q_0 \langle h^* A_0 S_0(t)x_0 \rangle^2.$$.

Integrating both sides from 0 to $t$ and employing $\mathcal{A}$ we obtain

$$\langle x_0, \mathcal{H} x_0 \rangle_H = \int_0^t \left\{ \langle (\mathcal{G} - q_1 h)^* A_0 S_0(t)x_0 \rangle^2 + q_0 \langle h^* A_0 S_0(t)x_0 \rangle^2 \right\} dt \geq 0 \quad \forall x_0 \in D(A_0).$$

Since $D(A_0)$ is dense in $H$ as a $C_0$-semigroup generator and $\mathcal{H} = \mathcal{H}^* \in \mathbb{L}(H)$ we get $\mathcal{H} \geq 0$. $\square$

**Remark 2.2.** The above proof may be slightly, but not essentially, modified by concluding $\mathcal{A}$ of the semigroup $\{e^{t A_0^{-1}}\}_{t \geq 0}$ from the reciprocal system

$$\begin{align*}
\dot{x} &= A^{-1} x + ud \\
y &= -h^* x
\end{align*}$$

with the feedback law $u = -\frac{\mu_0}{1 - \mu_0 c^\# d} y$, with an aid of [3], Lemma 12, p. 959. This is possible if $d$ is admissible with respect to $\{e^{t A^{-1}}\}_{t \geq 0}$ and $u \in L^2(0, \infty)$ for any initial condition $x_0 \in H$. It is not difficult to see, using duality between observation and control (see [2], p. 173) and the arguments which led to [2], Lemma 2.6, p. 174, that the first condition holds iff $d$ is admissible. Since in the frequency-domain the closed-loop output equation reads as

$$\hat{y}(s) = -h^* \left( sI - A^{-1} \right)^{-1} x_0 - h^* \left( sI - A^{-1} \right)^{-1} d \left[ \frac{\mu_0}{1 - \mu_0 c^\# d} \hat{g}(s) \right]$$

$$= \left( U \overline{P} x_0 \right)(s) + G(s) \left[ -\frac{\mu_0}{1 - \mu_0 c^\# d} \right] \hat{g}(s),$$

where $U$ is the unitary operator introduced in [2], p. 174, and $G$ is given by [2], (4.12), p. 187, then the second condition holds if $\frac{1}{1 + \frac{\mu_0}{1 - \mu_0 c^\# d}} G \in H^\infty(\mathbb{C}^+)$. By [2], (4.13), p. 187, the last condition is equivalent to (2.1).
Next, our Lyapunov operator equation
\[
(A_0^{-1})^* \mathcal{H} + \mathcal{H} A_0^{-1} = -(\mathcal{G} - q_1 h)(\mathcal{G} - q_1 h)^* - q_0 hh^*
\]
allows to get directly
\[
\langle x_0, \mathcal{H} x_0 \rangle_{\mathcal{H}} = \int_0^\infty \left\{ \left( (\mathcal{G} - q_1 h)e^{tA_0^{-1}} x_0 \right)^2 + q_0 \left[ h^* e^{tA_0^{-1}} x_0 \right]^2 \right\} dt \geq 0 \quad \forall x_0 \in \mathcal{H}.
\]

3. Correction of [2], Example

Just before the sentence starting from the words ([2], Sect. 5.2, p. 1927): “Thus all assumptions of Theorem 4.1 are met . . . ” the following text should be inserted:

Recall that \( d \) is an admissible factor control vector and for \( b \in (0, 1) \) the assumption (2.1) holds. Indeed, here
\[
\frac{1}{1 + \mu_0 \hat{g}(s)} = \frac{1 + \frac{4b}{a(1 + b)} e^{-s} \frac{a}{1 + b} e^{-2s}}{1 + be^{-2s} + \frac{4b}{(1 + b)} e^{-s} + 1}.
\]
The numerator is bounded by \( 1 + b \) on \( \mathbb{C}^+ \), while for the denominator one has
\[
be^{-2s} + \frac{4b}{(1 + b)} e^{-s} + 1 = b \left( z_0 - e^{-s} \right) \left( \bar{z}_0 - e^{-s} \right), \quad \text{Re} \ z_0 = \frac{-2b}{1 + b}, \quad |z_0|^2 = \frac{1}{b},
\]
whence
\[
\left| be^{-2s} + \frac{4b}{(1 + b)} e^{-s} + 1 \right| = b |z_0 - e^{-s}| |\bar{z}_0 - e^{-s}| \geq b (|z_0| - 1)^2 = (1 - \sqrt{b})^2,
\]
and consequently:
\[
\left\| \frac{1}{1 + \mu_0 \hat{g}} \right\|_{L^\infty(\mathbb{C}^+)} \leq \frac{1 + b}{(1 - \sqrt{b})^2} < \infty.
\]

References


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\(^3\)Since \( q = k_1 k_2 < 0 \) for \( b \in (0, 3 - 2\sqrt{2}) \) and sufficiently small \( \nu \) then, in fact, corrections are needed only for \( b \in [3 - 2\sqrt{2}, 1) \).