REGULARITY AND VARIATIONALITY OF SOLUTIONS TO HAMILTON-JACOBI EQUATIONS. 
PART I: REGULARITY (ERRATA)

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Abstract. This errata corrects one error in the 2004 version of this paper [Mennucci, ESAIM: COCV 10 (2004) 426–451].

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After the publication of [7] in 2004, it became clear that the regularity of the form $\alpha$ in Lemma 4.4 had to be related to the regularity of $K$ and of $u_0$; this influences the minimal regularity of $K, u_0$, as needed in hypotheses in Lemma 4.4, in Theorem 4.1, and in many following relevant discussions. This errata corrects that error; to keep the matter short, all material that is unaffected by the error is omitted; whereas care was taken so that results and discussions that are here corrected retain the original numbering as in [7].

4.1. Regularity of conjugate points

We will prove in this section results regarding the set of focal points; each following result extends to the set $\Gamma$ of conjugate points that is a subset of the focal points.

Theorem 4.1. Assume (CC0,H1,H2). If $u_0, K, H$ are regular enough, then, by Lemma 4.4, there is a (at most) countable number of $n - 1$ dimensional submanifolds of $\mathbb{R} \times O$ that cover all the sets $G^i$; these submanifolds are graphs of functions $\lambda_{i,h} : A_{i,h} \to \mathbb{R}$ (for $h = 1, \ldots$) where $A_{i,h} \subset O$ are open sets. The least regular case is $i = n - 1$, and the regularity of the $\lambda$ functions is related to the regularity of $u_0, K, H$, and to the dimension $\dim(M) = n$ as in the following table:

\[
\begin{array}{|c|c|c|c|}
\hline
\dim(M) & u_0, K & H & \lambda \\
\hline
n = 2 & C^{(R+2,0)} & C^{(R+2,0)} & C^{(R,\theta)} \\
n \geq 3 & C^{(R+2,0)} & C^{(R+n-1,0)} \cap C^n & C^{(R,\theta)} \\
\hline
\end{array}
\]

(4.1)

where $R \in \mathbb{N}, \theta \in [0, 1]$.

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We now infer some explanatory results on the regularity of the focal points $X(\cup_i G^i)$ from the above theorem.

At the lowest regularity, when $u_0, K \in C^2, H \in C^n$, we know that $X \in C^1$ and that the sets $G^i$ are graphs; we conclude that the set of focal points has measure zero. When $u_0, K \in C^{(2,0)}, H \in C^n \cap C^{(2,0)}$, we know that the dimension of the sets $G^i$ does not exceed $n - \theta$; so again we conclude that the set of focal points has dimension at most $n - \theta$. In the case $\theta = 1$, we can obtain the set of all focal points is rectifiable; that is, if $u_0, K \in C^{(2,1)}, H \in C^n \cap C^{(2,1)}$, then the sets $G^i$ are covered by Lipschitz graphs, so (by known results in [2]) the set of focal points may be covered by $(n - 1)$-dimensional $C^1$ regular submanifolds of $M$, but for a set of Hausdorff $H^{n-1}$ measure zero.

When we further raise the regularity, we may suppose that $u_0, K \in C^{s+3}, H \in C^{s+n}$ (with $s \in \mathbb{N}$); then the sets $G^i$ are covered by graphs $(\lambda(y), y)$ inside $\mathbb{R} \times O$ of regularity $C^{1+s}$; while $X \in C^{2+s}$ (at least), and we restrict it to those graphs; we can then apply Theorem A.4 to state that the focal points are covered by $C^{1+s}$ regular submanifolds of $M$ but for a set of $H^\alpha$ measure zero, where $\alpha \doteq n - 2 + 1/(1 + s)$.

[... unchanged material deleted ...]

The main tool is this lemma; the complete proof of the lemma is in Section 6.

**Lemma 4.4.** We assume that the hypotheses (CC0,H1,H2) hold.

We set the regularity of the data $u_0, K, H$ by defining parameters $R, R' \in \mathbb{N}$, $\theta, \theta' \in [0, 1]$, and assuming that

$$u_0 \in C^{(R+s,2,\theta')}, \quad K \in C^{(R+2,\theta')}, \quad H \in C^{(R+2,\theta')}.$$ 

by Proposition 3.7, the flow $\Phi = (X, P)$ is $C^{(R+1,\theta)}$ regular; and $O$ is a $C^{(R+1,\theta')} \cup C^{(R+2,\theta)}$ manifold (that is, the least regular of the two).

Let $i \geq 1, i \leq n - 1$, and fix a point $(s', y') \in \mathbb{R} \times O$, such that $(s', y') \in G^i$.

Let $U$ be a neighbourhood of $0$ in $\mathbb{R}^{n-1}$ and let $\phi : U \to O$ be a local chart to the neighbourhood $\phi(U)$ of $y' = \phi(0)$. The map $\phi$ has regularity $C^{(R+1,\theta)} \cup C^{(R+2,\theta)}$. In the following, $y$ will be a point in $\phi(U)$.

To study $G^i$, we should study the rank of the Jacobian of the map $(t, x) \mapsto X(t, \phi(x))$; since the regularity of $X$ is related only to the regularity of $H$, it will be useful to decouple this Jacobian in two parts. To this end, we define a $n$-form $\alpha$ on $\mathbb{R} \times O$, with requirement that $\alpha(t, y) = \alpha(y)$ (that is, $\alpha$ does not depend on $t$).

Writing $X^{(t,y)}$ for $X(t, y)$, let

$$X^{(t,y)^*}\alpha$$

be the push-forward of $\alpha$ along $X$: $X^{(t,y)^*}\alpha$ is then a tangent form defined on $T_X^{(t,y)} M$; it will be precisely defined in equation (6.2). We remark that $X^{(t,y)^*}\alpha = 0$ iff $(t, y) \in \bigcup_j G^j$. Note that the pushforward $X^{(t,y)^*}$ is $C^{(R,\theta)}$ regular, while the form $\alpha$ is as regular as $TO$, that is, $\alpha \in C^{(R,\theta')} \cup C^{(R+1,\theta)}$.

Note that, since $X$ solves an O.D.E., then $X$ and $\frac{\partial}{\partial t}X$ have the same regularity; note moreover that

$$\frac{\partial}{\partial t}(X^{(s',y')^*}\alpha) = \left(\frac{\partial}{\partial t}X\right)^{(s',y')}^* \alpha$$

since $\alpha$ does not depend on $t$. So, by hypotheses and by the definition (6.2) of $X^{(t,y)^*}\alpha$, the forms $X^{(t,y)^*}\alpha$ and $\frac{\partial}{\partial t}(X^{(t,y)^*}\alpha)$ have regularity $C^{(R,\theta)} \cap C^{(R',\theta')}$ (see also Eq. (6.3)); the derivatives $\frac{\partial^j}{\partial t^j}X^{(s',y')^*}\alpha$ with $j \geq 1$ have regularity $C^{(R-j+1,\theta)} \cup C^{(R',\theta')}$.

Then, when $R + 1 \geq i$, we prove (in Sect. 6) that

$$X^{(s',y')^*}\alpha = 0, \quad \frac{\partial}{\partial t}X^{(s',y')^*}\alpha = 0, \quad \frac{\partial^{i-1}}{\partial t^{i-1}}X^{(s',y')^*}\alpha = 0$$

1A similar result may be obtained when $u_0, K \in C^{(s+3,\theta)}, H \in C^{(s+n,\theta)}$. 
whereas
\[ \frac{\partial}{\partial t^i} X(s',y')^\ast \alpha \neq 0. \]

We define eventually the map \( F : \mathbb{R} \times \mathbb{R}^{n-1} \rightarrow \mathbb{R} \) given by
\[ F(t,x) = \frac{\partial}{\partial t^{i-1}} X(t,\phi(x))^\ast \alpha; \]

since
\[ \frac{\partial}{\partial t} F(t,x) = \frac{\partial}{\partial t^i} X(t,\phi(x))^\ast \alpha \neq 0 \]

the above Dini lemma implies that the set \( G^{(i)} \) is locally covered by the graph of a function \( \lambda_i \) defined on an open subset of \( O \); \( \lambda \) has the same regularity of \( F \), so, if \( i = 1 \) then \( \lambda \) is in \( C^{R,\varrho} \) while for \( i \geq 2 \) it is \( C^{(R-\varrho+2,\varrho)} \).

The above directly implies Theorem 4.1.

[... all other results are unchanged ...]

5. APPLICATIONS

5.1. The Cauchy problem

We show now how the above theorems may be used for the Cauchy problem (1.2)
\[
\begin{cases}
\frac{\partial}{\partial t} w(t,x') + H'(t,x',\frac{\partial}{\partial x} w(t,x')) = 0 & \text{for } t > 0, x' \in M' \\
w(0,x') = w_0(x') & \forall x' \in M'.
\end{cases}
\]  

[... the preliminary discussion is unchanged ...]

This improves the results of 4.10, 4.12 and 4.17 in [1]; to provide for an easy comparison, we summarize these results

- if \( n' = \dim(M') \), \( n = n' + 1 \), if \( H' \in C^s \) with \( s = n \vee 3 \) and \( w_0 \in C^2 \), then the set \( \Gamma \) has measure zero, so the set \( \Sigma_n = \Sigma \cup \Gamma \) has measure zero;
- if \( H, w_0 \in C^{(2,1)} \), then the set \( \Gamma \) is rectifiable, so the set \( \Sigma_n = \Sigma \cup \Gamma \) is rectifiable;
- and when \( H' \in C^{R+1,\varrho} \), \( w_0 \in C^{R+1,\varrho} \), \( R \geq 2 \), \( w \) is continuous, we prove that the Hausdorff dimension of \( \Gamma \setminus \Sigma \) is at most \( \beta \), and moreover \( H^b(\Gamma \setminus \Sigma) = 0 \) if \( \theta = 0 \), where \( \beta = n' - 1 + 2/(R + \varrho) \).

In the counterexample in Section 4.4 in [1], \( w_0 \) is \( C^{1,1}(M') \) and not \( C^2(M') \); so our results close the gap between the counterexample, where \( w_0 \) is \( C^{1,1}(M') \), and the theorem, where \( w_0 \) is \( C^2(M') \); and actually, studying the counterexample, it is quite clear that, if \( w_0 \) is smoothed to become a \( C^2(M') \) function, then the counterexample would not work.

5.2. Eikonal equation and cutlocus

As in Section 3.5, consider a smooth Riemannian manifold \( M \), and a closed set \( K \subset M \) and let \( d_K(x) = d(x,K) \) be the distance to \( K \). We set \( u_0 = 0 \): then \( O \) is the bundle of unit covectors that are normal to \( TK \), and \( d_K(x) \) coincides with the \( \min \) solution \( u(x) \).

We define
\[ \Sigma_{d_K} = \{ x \mid 2\nabla d_K(x) \} \]

If \( K \) is \( C^1 \), then \( \Sigma_{d_K} \) coincides with \( \Sigma \) as defined in (4.1).

Since \( d_K \) is semiconcave in \( M \setminus K \), \( \Sigma_{d_K} \) is always rectifiable.

This primal problem is a good test bed to discuss the differences and synergies of the results in this paper and the results in Itoh and Tanaka [4] and Li and Nirenberg [5].
• In the example in Section 3 in [6], there is a curve \( K \subset \mathbb{R}^2 \), \( K \in C^{1,1} \) such that \( \Sigma_{dK} \) has positive Lebesgue measure. Note that in this example \( \Sigma_{dK} \neq \text{Cut}(K) = \Sigma_{dK} \), so the cutlocus \( \text{Cut}(K) \) is rectifiable (but not closed).

We do not know if there is a curve \( K \in C^{1,1} \) such that \( \text{Cut}(K) \) has measure zero. Note that in this example \( \Sigma_{dK} \neq \text{Cut}(K) = \Sigma_{dK} \), so the cutlocus \( \text{Cut}(K) \) is rectifiable (but not closed).

• Theorem 4.1 states that if \( K \) is \( C^2 \), then \( \Gamma \) has measure zero, so by (1.4) and 4.1.14, we obtain that \( \Sigma_{dK} \) has measure zero; so Theorem 4.1 closes the gap between the counterexample in Section 3 [6] and the previous available results.

5.2.1. Improvements

[...the discussion is unchanged...]

Corollary 5.1. Consider a 2-dimensional smooth Riemannian manifold \( M \); suppose that \( K \) is a compact \( C^{3+s} \) embedded submanifold.

Then, for any open bounded set \( A \subset M \), the set \( A \cap \Gamma \) is \( C^{s+1}M^{1/(s+1)} \)-rectifiable: that is, it can be covered by at most countably many \( C^{s+1} \) curves, but for a set \( E \) such that \( M^{1/(s+1)}(E) = 0 \).

6. PROOF OF 4.4

[...the two lemma are unchanged...]

Now we prove Lemma 4.4.

We want to define the \( n \) form \( \alpha \) so that \( \alpha \) does not depend on \( t \); and so that \( \alpha = e_1 \wedge \cdots \wedge e_n \) where the vectors fields \( e_n-1, \ldots, e_n \) span the kernel of \( \frac{\partial}{\partial s}X \) at the point \((s', y')\) (kernel that we will call \( V \)) while \( \frac{\partial}{\partial t}X \) is full rank on \( e_1, \ldots, e_{n-1} \) (that generate the space \( W \)).

One possible way to this is to fix the local chart \( \phi : U \subset \mathbb{R}^{n-1} \rightarrow O \), define

\[
\tilde{e}_1 \overset{\text{def}}{=} \phi \frac{\partial}{\partial x_1}, \ldots, \tilde{e}_{n-1} \overset{\text{def}}{=} \phi \frac{\partial}{\partial x_{n-1}}, \tilde{e}_n \overset{\text{def}}{=} \frac{\partial}{\partial t}
\]

and then choose a \( n \times n \) constant matrix \( A \), so that

\[
e_n \overset{\text{def}}{=} \sum_k A_{h,k} \tilde{e}_k
\]

satisfy the requirements.

[...the rest of the proof is unchanged...]

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REFERENCES


