AN ELLIPTIC EQUATION WITH NO MONOTONICITY CONDITION ON THE NONLINEARITY

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Abstract. An elliptic PDE is studied which is a perturbation of an autonomous equation. The existence of a nontrivial solution is proven via variational methods. The domain of the equation is unbounded, which imposes a lack of compactness on the variational problem. In addition, a popular monotonicity condition on the nonlinearity is not assumed. In an earlier paper with this assumption, a solution was obtained using a simple application of topological (Brouwer) degree. Here, a more subtle degree theory argument must be used.

Mathematics Subject Classification. 35J20, 35J60.

Received July 6, 2005.

1. INTRODUCTION

In this paper we consider an elliptic equation of the form

$$-\Delta u + u = f(x,u), \quad x \in \mathbb{R}^N,$$

(1.1)

where $f$ is a “superlinear” function of $u$. For large $|x|$, the equation resembles an autonomous equation

$$-\Delta u + u = f_0(u), \quad x \in \mathbb{R}^N.$$

(1.2)

Under weak assumptions on $f$ and $f_0$, we prove the existence of a nontrivial solution $u$ of (1.1) with $|u(x)| \to 0$ as $|x| \to \infty$.

Let $f$ satisfy

$(f_1)$ $f \in C^2(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$.

$(f_2)$ $f(x,0) = 0 = f_q(x,0)$ for all $x \in \mathbb{R}^N$, where $f \equiv f(x,q)$.

$(f_3)$ If $N > 2$, there exist $a_1, a_2 > 0$, $s \in (1, (N + 2)/(N - 2))$ with $|f_q(x,q)| \leq a_1 + a_2|q|^{s-1}$ for all $q \in \mathbb{R}$, $x \in \mathbb{R}^N$. If $N = 2$, there exist $a_1 > 0$ and a function $\varphi : \mathbb{R}^+ \to \mathbb{R}$ with $|f_q(x,q)| \leq a_1 \exp(\varphi(|q|))$ for all $q \in \mathbb{R}$, $x \in \mathbb{R}^N$ and $\varphi(t)/t^2 \to 0$ as $t \to \infty$.

Keywords and phrases. Mountain-pass theorem, variational methods, Nehari manifold, Brouwer degree, concentration-compactness.

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There exists $\mu > 2$ such that
\[
0 < \mu F(x, q) \equiv \mu \int_0^q f(x, s) \, ds \leq f(x, q)q
\]
for all $q \in \mathbb{R}$, $x \in \mathbb{R}^N$.

Let $f_0 \in C^2(\mathbb{R}^N, \mathbb{R})$ with satisfy $(f_1)$-$,(f_4)$ (except there is no dependence on $x$). Let $f$ also satisfy
\[
(f_5) \quad (f(x, q) - f_0(q))/f_0(q) \to 0 \text{ as } |x| \to \infty, \text{ uniformly in } q \in \mathbb{R}^N \setminus \{0\}.
\]

In order to state the theorem, we need to outline the variational framework of the problem. Define functionals $I_0, I \in C^2(W^{1,2}(\mathbb{R}^N, \mathbb{R}), \mathbb{R})$ by
\[
I_0(u) = \frac{1}{2} \|u\|^2 - \int_{\mathbb{R}^N} F_0(u(x)) \, dx,
\]
\[
I(u) = \frac{1}{2} \|u\|^2 - \int_{\mathbb{R}^N} F(x, u(x)) \, dx,
\]
where $\|u\|$ is the standard norm on $W^{1,2}(\mathbb{R}^N, \mathbb{R})$ given by
\[
\|u\|^2 = \int_{\mathbb{R}^N} |\nabla u(x)|^2 + u(x)^2 \, dx.
\]

Critical points of $I_0$ correspond exactly to solutions $u$ of $(1.2)$ with $u(x) \to 0$ as $|x| \to \infty$, and critical points of $I$ correspond exactly to solutions $u$ of $(1.1)$ with $u(x) \to 0$ as $|x| \to \infty$.

By $(f_4)$, $F_0$ and $F$ are “superquadratic” functions of $q$, with, for example, $F(x, q)/q^2 \to 0$ as $q \to 0$ and $F(x, q)/q^2 \to \infty$ as $|q| \to \infty$ for all $x \in \mathbb{R}^N$, uniformly in $x$. Therefore $I(0) = I_0(0) = 0$, and there exists $r_0 > 0$ with $I(u) \geq \|u\|^2/3$ and $I_0(u) \geq \|u\|^2/3$ for all $u \in W^{1,2}(\mathbb{R}^N)$ with $\|u\| \leq r_0$, and there also exist $u, u_0 \in W^{1,2}(\mathbb{R}^N, \mathbb{R})$ with $I_0(u_0) < 0$ and $I(u) < 0$. So the sets of “mountain-pass curves” for $I_0$ and $I$,
\[
\Gamma_0 = \{ \gamma \in C([0,1], W^{1,2}(\mathbb{R}^N, \mathbb{R})) \mid \gamma(0) = 0, \ I_0(\gamma(1)) < 0 \},
\]
\[
\Gamma = \{ \gamma \in C([0,1], W^{1,2}(\mathbb{R}^N, \mathbb{R})) \mid \gamma(0) = 0, \ I(\gamma(1)) < 0 \},
\]
are nonempty, and the mountain-pass values
\[
c_0 = \inf_{\gamma \in \Gamma_0} \max_{\theta \in [0,1]} I_0(\gamma(\theta))
\]
\[
c = \inf_{\gamma \in \Gamma} \max_{\theta \in [0,1]} I(\gamma(\theta))
\]
are positive.

We are now ready to state the theorem.

**Theorem 1.1.** If $f_0$ and $f$ satisfy $(f_1)$-$(f_4)$ and $f$ satisfies $(f_5)$, and if there exists $\alpha > 0$ such that
\[
I_0 \text{ has no critical values in the interval } [c_0, c_0 + \alpha]
\]
then there exists $\epsilon_0 = \epsilon_0(f_0) > 0$ with the following property: if $f$ satisfies
\[
|f(x, q) - f_0(q)| < \epsilon_0|f_0(q)|
\]
for all $x \in \mathbb{R}^N$, $q \in \mathbb{R}$, then $(1.2)$ has a nontrivial solution $u \not\equiv 0$ with $u(x) \to 0$ as $|x| \to \infty$.

As shown in [9], $(1.12)$ holds in a wide variety of situations.
The missing monotonicity assumption

One interesting aspect of Theorem 1.1 is a condition that is not assumed. We do not assume

For all \( q \in \mathbb{R} \) and \( x \in \mathbb{R}^N \), \( F_0(q)/q^2 \) is

- a nondecreasing function of \( q \) for \( q > 0 \);
- \( F_0(q)/q^2 \) is a nonincreasing function of \( q \) for \( q < 0 \);  \hspace{1cm} (1.13)
- \( F(x, q)/q^2 \) is a nondecreasing function of \( q \) for \( q > 0 \); or
- \( F(x, q)/q^2 \) is a nonincreasing function of \( q \) for \( q < 0 \).

This condition holds in the power case, \( F_0(q) = |q|^{\alpha}/\alpha, \alpha > 2 \). The condition is due to Nehari.

If (1.13) were case, then for any \( u \in W^{1,2}(\mathbb{R}^N, \mathbb{R}) \setminus \{0\} \), the mapping \( s \mapsto I(su) \) would begin at 0 at \( s = 0 \), increase to a positive maximum, then decrease to \(-\infty\) as \( s \to \infty \). Defining

\[
\mathcal{S} = \{ u \in W^{1,2}(\mathbb{R}^N, \mathbb{R}) \setminus \{0\} \mid I'(u)u = 0 \},
\]

\( \mathcal{S} \) would be a codimension-one submanifold of \( E \), homeomorphic to the unit sphere in \( W^{1,2}(\mathbb{R}^N, \mathbb{R}) \) via radial projection. \( \mathcal{S} \) is known as the Nehari manifold in the literature. Any ray of the form \( \{ su \mid s \geq 0 \} \) \( (u \neq 0) \) intersects \( \mathcal{S} \) exactly once. All nonzero critical points of \( I \) are on \( \mathcal{S} \). Conversely, under suitable smoothness assumptions on \( F \), any critical point of \( I \) constrained to \( \mathcal{S} \) would be a critical point of \( I \) (in the large) (see [17]).

Therefore, one could work with \( \mathcal{S} \) instead of \( W^{1,2}(\mathbb{R}^N, \mathbb{R}) \), and look for, say, a local minimum of \( I \) constrained to \( \mathcal{S} \) (which may be easier than looking for a saddle point of \( I \)). There is another way to use (1.13): for any \( u \in \mathcal{S} \), the ray from 0 passing through \( u \) can be used (after rescaling in \( \theta \)) as a mountain-pass curve along which the maximum value of \( I \) is \( I(u) \). Conversely, any mountain-pass curve \( \gamma \in \Gamma \) intersects \( \mathcal{S} \) at least once [6].

Therefore, one may work with points on \( \mathcal{S} \) instead of paths in \( \Gamma \). Since assumption (1.13) is so helpful, it is found in many papers, such as [1,5,20], and [18].

In the paper [17], a result similar to Theorem 1.1 was proven for the \( N = 1 \) (ODE) case. The proof of Theorem 1.1 is similar except that a simple connectivity argument must be replaced by a degree theory argument [18], proves a version of Theorem 1.1 under the assumption (1.13). Without 1.13, the manifold \( \mathcal{S} \) must be replaced by a set with similar properties.

Define \( B_1(0) = \{ x \in \mathbb{R}^N \mid |x| < 1 \} \), and \( \overline{\Omega} \) and \( \partial \Omega \) to be, respectively, the topological closure and topological boundary of \( \Omega \). It is a simple consequence of the Brouwer degree [7] that for any continuous function \( h : B_1(0) \to \mathbb{R}^N \) with \( h(x) = x \) for all \( x \in \partial B_1(0) \), there exists \( x \in B_1(0) \) with \( h(x) = 0 \). We will need the following generalization:

**Lemma 1.2.** Let \( h \in C(B_1(0) \times [0, 1], \mathbb{R}^N) \) with, for all \( x \in B_1(0) \) and \( t \in [0, 1] \),

- (i) \( h(x, 0) = x = h(x, 1) \).
- (ii) \( x \in \partial B_1(0) \Rightarrow h(x, t) = x \).

Then there exists a connected subset \( C_0 \subset \overline{B_1(0)} \times [0, 1] \) with \( (0, 0), (0, 1) \in C_0 \) and \( h(x, t) = 0 \) for all \( (x, t) \in C_0 \).

Using the Brouwer degree, it is clear that under the hypotheses of Lemma 1.2, for each “horizontal slice” \( B_1(0) \times \{ t \} \) of the cylinder \( B_1(0) \times [0, 1] \), there exists \( x \in B_1(0) \) with \( h(x, t) = 0 \). The conclusion of Lemma 1.2 does not follow from this observation. A generalization of Lemma 1.2 is known [16]: however, the reference may be difficult to find, so a proof is given here.

This paper is organized as follows: Section 2 contains the proof of Theorem 1.1. The proof of Lemma 1.2 is deferred until Section 3.
2. Proof of Theorem 1.1

It is fairly easy to show that
\[ c \leq c_0, \]  
(2.1)
where \( c \) and \( c_0 \) are from (1.9)–(1.10): it is proven in [11] that there exists \( \gamma_1 \in \Gamma_0 \) with \( \max_{\theta \in [0,1]} I_0(\gamma_1(\theta)) = c_0 \).

Define the translation operator \( \tau \) as follows: for a function \( u \) on \( \mathbb{R}^N \) and \( a \in \mathbb{R}^N \), define let \( \tau_a u \) be \( u \) shifted by \( a \), that is, \( (\tau_a u)(x) = u(x - a) \). Let \( \epsilon > 0 \). Let \( e_1 = 1, 0, 0, \ldots, 0 \in \mathbb{R}^N \) and define \( \tau_{e_1} \gamma_1 \) by \( \gamma_1(\theta) = \tau_{\gamma_1(\theta)}(\theta) \). Then for large \( R > 0 \), by \( (f_3) \), \( \gamma_1 \in \Gamma \) and \( \max_{\theta \in [0,1]} I((\tau_{e_1} \gamma_1)(\theta)) < c_0 + \epsilon \). Since \( \epsilon > 0 \) was arbitrary, \( c \leq c_0 \).

A Palais-Smale sequence for \( I \) is a sequence \( (u_m) \subset W^{1,2}(\mathbb{R}^N, \mathbb{R}) \) with \( (I(u_m)) \) convergent and \( \|I'(u_m)\| \to 0 \) as \( m \to \infty \). It is well-known that \( I \) fails the “Palais-Smale condition”. That is, a Palais-Smale sequence need not converge. However, the following proposition states that a Palais-Smale sequence “splits” into the sum of a critical point of \( I \) and translates of critical points of \( I_0 \):

**Proposition 2.1.** If \( (u_m) \subset W^{1,2}(\mathbb{R}^N, \mathbb{R}) \) with \( I'(u_m) \to 0 \) and \( I(u_m) \to a > 0 \), then there exist \( k \geq 0 \), \( v_0, v_1, \ldots, v_k \in W^{1,2}(\mathbb{R}^N, \mathbb{R}) \), and sequences \( (x_m^i)_{m \geq 1} \subset \mathbb{R}^N \), such that

(i) \( I'(v_0) = 0 \);

(ii) \( I_0'(v_i) = 0 \) for all \( i = 1, \ldots, k \);

and along a subsequence (also denoted \( (u_m) \))

(iii) \( \|u_m - (v_0 + \sum_{i=1}^k \tau_{x_m^i} v_i)\| \to 0 \) as \( m \to \infty \);

(iv) \( |x_m^i| \to \infty \) as \( m \to \infty \) for \( i = 1, \ldots, k \);

(v) \( |x_m^i - x_m^j| \to \infty \) as \( m \to \infty \) for all \( i \neq j \);

(vi) \( I(v_0) + \sum_{i=1}^k I_0(v_i) = a \).

A proof for the case of \( x \)-periodic \( F \) is found in [6], and essentially the same proof works here. Similar propositions for nonperiodic coefficient functions, for both ODE and PDE, are found in [1, 5], and [19], for example. All are inspired by the “concentration-compactness” theorems of P.-L. Lions [12].

If \( c < c_0 \), then by standard deformation arguments [15], there exists a Palais-Smale sequence \( (u_m) \) with \( I(u_m) \to c \). By [11], the smallest nonzero critical value of \( I_0 \) is \( c_0 \). Applying Proposition 2.1, we obtain \( k = 0 \), and \( (u_m) \) has a convergent subsequence, proving Theorem 1.1. So assume from now on that

\[ c = c_0. \]  
(2.2)

For \( u \in L^2(\mathbb{R}^N, \mathbb{R}) \setminus \{0\} \) and \( i \in \{1, \ldots, N\} \), define \( L_i \), the \( i \)th component of the “location” of \( u \), by

\[ \int_{\mathbb{R}^N} u^2 \tan^{-1}(x_i - L_i(u)) \, dx = 0 \]  
(2.3)

and the “location” of \( u \) by

\[ L(u) = (L_1(u), \ldots, L_N(u)) \in \mathbb{R}^N. \]  
(2.4)

The following lemma establishes the existence and continuity of \( L \).

**Lemma 2.2.** \( L \) is well-defined and continuous on \( L^2(\mathbb{R}^N, \mathbb{R}) \) \setminus \( \{0\} \).

**Proof.** It suffices to show that \( L_1 \) is well-defined and continuous on \( L^2(\mathbb{R}^N, \mathbb{R}) \setminus \{0\} \). Let \( u \in L^2(\mathbb{R}^N, \mathbb{R}) \setminus \{0\} \). By Leibniz’s Theorem, the mapping \( \phi : s \mapsto \int_{\mathbb{R}^N} u^2 \tan^{-1}(x_1 - s) \, dx \) is continuous, differentiable, and strictly decreasing, with

\[ \phi'(s) = -\int_{\mathbb{R}^N} u^2(x)/(((x_1 - s)^2 + 1) \, dx < 0. \]  
(2.5)

\( \phi(s) \to \mp \infty \) as \( s \to \pm \infty \). Therefore \( L_1(u) \) is unique and well-defined. Let \( \epsilon > 0 \) and \( u_m \to u \). Now \( \int_{\mathbb{R}^N} u^2 \tan^{-1}(x_1 - (L_1(u) + \epsilon)) \, dx < 0 \). Since \( u^2_m \to u^2 \) in \( L_i^1(\mathbb{R}^N, \mathbb{R}) \), \( \int_{\mathbb{R}^N} u^2_m \tan^{-1}(x_1 - (L(u) + \epsilon)) \, dx < 0 \) for
large \( m \), so for large \( m \), \( \mathcal{L}_1(u_m) < \mathcal{L}_1(u) + \epsilon \). Similarly, for large \( m \), \( \mathcal{L}_1(u_m) > \mathcal{L}_1(u) - \epsilon \). Since \( \epsilon \) is arbitrary, \( \mathcal{L}_1(u_m) \to \mathcal{L}_1(u) \).

We are ready to begin the minimax argument. First we construct a mountain-pass curve \( \gamma_0 \) with some special properties:

**Lemma 2.3.** There exists \( \gamma_0 \in \Gamma \) such that for all \( \theta \in [0, 1] \),

(i) \( I_0(\gamma_0(\theta)) \leq c_0 \).

(ii) \( \theta > 0 \Rightarrow \gamma_0(\theta) \neq 0 \).

(iii) \( \theta \leq 1/2 \Rightarrow I_0(\gamma_0(\theta)) \leq c_0/2 \).

(iv) \( \theta > 0 \Rightarrow L(\gamma_0(\theta)) = 0 \).

**Proof.** By [10], there exists \( \gamma_1 \in \Gamma \) with \( \max_{\theta \in [0, 1]} I_0(\gamma_1(\theta)) = c_0 \). Assume without loss of generality that \( \gamma_1(\theta) \neq 0 \) for \( \theta > 0 \). By rescaling in \( \theta \) if necessary, assume that \( I_0(\gamma_1(\theta)) \leq c_0/2 \) for \( \theta \leq 1/2 \). Finally, define \( \gamma_0 \) by \( \gamma_0(0) = 0, \gamma_0(\theta) = \tau_{-L(\gamma_1(\theta))} \gamma_1(\theta) \) for \( \theta > 0 \).

Assume \( \epsilon_0 \) in (1.12) is small enough so that for all \( x \in \mathbb{R}^N \) and \( \theta \in [0, 1] \),

\[
I(\tau_x(\gamma_0(\theta))) < \min(2c_0, c_0 + \alpha) \text{ and } I(\tau_x(\gamma_0(1))) < 0, \tag{2.6}
\]

where \( \alpha \) is from (1.11).

**A substitute for \( S \)**

Using the mountain-pass geometry of \( I \) and the fact that Palais-Smale sequences of \( I \) are bounded in norm [6], we construct a set which has similar properties to \( S \), described in Section 1. Let \( \nabla I \) denote the gradient of \( I \), that is, \( \langle \nabla I(u), w \rangle = I'(u)w \) for all \( u, w \in W^{1,2}(\mathbb{R}^N, \mathbb{R}) \). Here, \( \langle \cdot, \cdot \rangle \) is the usual inner product defined by \( \langle u, w \rangle = \int_{\mathbb{R}^N} \nabla u \cdot \nabla w + uw \, dx \). Let \( \varphi : W^{1,2}(\mathbb{R}^N, \mathbb{R}) \to \mathbb{R} \) be locally Lipschitz, with \( I(u) \geq -1 \Rightarrow \varphi(u) = 1 \) and \( I(u) \leq -2 \Rightarrow \varphi(u) = 0 \). Let \( \eta \) be the solution of the initial value problem

\[
\frac{d\eta}{ds} = -\varphi(\eta) \nabla I(u), \quad \eta(0, u) = u. \tag{2.7}
\]

In [19] it is proven that \( \eta \) is well-defined on \( \mathbb{R}^+ \times W^{1,2}(\mathbb{R}^N) \). Let \( \mathcal{B} \) be the basin of attraction of 0 under the flow \( \eta \), that is,

\[
\mathcal{B} = \{ u \in W^{1,2}(\mathbb{R}^N, \mathbb{R}) \mid \eta(s, u) \to 0 \text{ as } s \to \infty \} \tag{2.8}
\]

\( \mathcal{B} \) is an open neighborhood of 0 [19]. Let \( \partial \mathcal{B} \) be the topological boundary of \( \mathcal{B} \) in \( W^{1,2}(\mathbb{R}^N, \mathbb{R}) \). \( \partial \mathcal{B} \) has some properties in common with \( S \). For example, for any \( \gamma \in \Gamma, \gamma([0, 1]) \) intersects \( \partial \mathcal{B} \) at least once.

A pseudo-gradient vector field for \( I' \) may be used in place of \( \nabla I \), in which case \( \mathcal{B} \) and \( \partial \mathcal{B} \) would be different, but the ensuing arguments would be the same.

Let

\[
c^+ = \inf \{ I(u) \mid u \in \partial \mathcal{B}, \, |\mathcal{L}(u)| \leq 1 \}. \tag{2.9}
\]

The reason for the label “\( c^+ \)” will become apparent in a moment. From now on, let us assume

\[
I \text{ has no critical values in } (0, c_0] = (0, c]. \tag{2.10}
\]

This will lead to the conclusion that \( I \) has a critical value greater than \( c_0 \).

We claim that under assumptions (2.2) and (2.10),

\[
c^+ > c_0. \tag{2.11}
\]

We use arguments that are sketched here and found in more detail in [19] and [5].
To prove the claim, suppose first that $c^+ < c_0$. Then there exists $u_0 \in \partial \mathcal{B}$ with $I(u_0) < c_0$. By arguments in [19], there exists a large positive constant $P$ with

$$I(u) \leq c_0 \quad \text{and} \quad \|u\| \geq 2P \Rightarrow I(\eta(s, u)) < 0 \quad \text{for some} \quad s > 0, \quad \text{and} \quad \|\eta(s, u)\| > P$$

\hspace{1cm} (2.12)

for all $s > 0$. Suppose $a > 0$ and $\|I'(\eta(s_m, U_0))\| \geq a$ for some sequence $(s_m)$ with $s_m \to \infty$. Since $u_0 \in \partial \mathcal{B}$, $\|\eta(u_0)\| < 2P$ for all $s > 0$. $I''$ is bounded on bounded subsets of $W^{1,2}(\mathbb{R})$, so $I'$ is Lipschitz on bounded subsets of $W^{1,2}(\mathbb{R})$. Therefore $I(\eta(s, u_0)) < 0$ for some $s > 0$. This is impossible since $u_0 \in \partial \mathcal{B}$. Therefore $I'(\eta(s, u_0)) \to 0$ as $s \to \infty$.

Define $u_m = \eta(n, u_0)$. Since $I'(u_n) \to 0$ and $u_n \in \partial \mathcal{B}$, there exists $b \in (0, c_0)$ with $I(u_n) \to b$. By [11], $I_0$ has no critical values between 0 and $c_0$. Therefore, Proposition 2.1, with $k = 0$, implies that $(u_n)$ converges along a subsequence to a critical point $w$ of $I$ with $0 < I(w) < c_0$. This contradicts assumption (2.10).

Next, suppose that $c^+ = c_0$. Then there exists a sequence $(u_n) \subset \partial \mathcal{B}$ with $|\mathcal{L}(u_n)| \leq 1$ for all $n$ and $I(u_n) \to c_0$ as $n \to \infty$. As above, $I'(u_n) \to 0$ as $n \to \infty$; to prove, suppose otherwise. Then there exist $a > 0$ and a subsequence of $(u_n)$ (also called $(u_n)$) along which $|I'(u_n)| > a$. Since $\partial \mathcal{B}$ is forward-$\eta$-invariant [19], $\eta(1, u_n) \in \partial \mathcal{B}$ for all $n$. Since $(\eta(1, u_n))_{n \geq 1}$ is bounded and $I'$ is Lipschitz on bounded subsets of $W^{1,2}(\mathbb{R}^N, \mathbb{R})$, for large $n$, $\eta(1, u_n) \in \partial \mathcal{B}$ with $I(\eta(1, u_n)) < c_0$. By the argument above, this implies that $I$ has a critical value in $(0, c_0)$, contradicting assumption (2.2). Thus $I'(u_n) \to 0$ as $n \to \infty$. Applying Proposition 2.1 and using the fact that $|\mathcal{L}(u_n)| \leq 1$ for all $n$, $(u_n)$ converges along a subsequence to a critical point of $I$, contradicting assumption (2.10). (2.11) is proven.

Let $R > 0$ be big enough so that for all $x \in \partial B_R(0) \subset \mathbb{R}^N$ and $\theta \in [0, 1]$,

$$I(\tau_x \gamma_0(\theta)) < c^+.$$  \hspace{1cm} (2.13)

This is possible by (1.12), (2.11), and Lemma 2.3(i). Define the minimax class

$$\mathcal{H} = \{h \in C(B_R(0) \times [0, 1], W^{1,2}(\mathbb{R}^N, \mathbb{R})) \mid \forall x \in B_R(0) \text{ and } t \in [0, 1],$$

$$t > 0 \Rightarrow h(x, t) \neq 0,$$

$$0 \leq t \leq 1/2 \Rightarrow h(x, t) = \tau_x \gamma_0(t)$$

$$x \in \partial B_R(0) \Rightarrow h(x, t) = \tau_x \gamma_0(t)$$

$$h(x, 1) = \tau_x \gamma_0(1)\}$$

and the minimax value

$$h_0 = \inf_{h \in \mathcal{H}} \max_{(x, t) \in B_R(0) \times [0, 1]} I(h(x, t)).$$  \hspace{1cm} (2.14)

We claim

$$c_0 < c^+ \leq h_0 < \min(2c_0, c_0 + \alpha).$$  \hspace{1cm} (2.15)

**Proof of Claim.** Define $h \in \mathcal{H}$ by $h(x, t) = \tau_x \gamma_0(t)$. Then $h \in \mathcal{H}$ and by (2.6), $\max_{(x, t) \in B_R(0) \times [0, 1]} \hat{h}(x, t) < \min(2c_0, c_0 + \alpha)$. Therefore $h_0 < \min(2c_0, c_0 + \alpha)$.

Next, let $h \in \mathcal{H}$. By Lemma 1.2, and a suitable rescaling of $x$ and $t$, there exists a connected set $C_2 \subset B_R(0) \times [1/2, 1]$ with $(0, 1/2), (0, 1) \in C_2$ and along which for all $(x, t) \in C_2$,

$$\mathcal{L}(h(x, t)) = 0.$$  \hspace{1cm} (2.16)

Joining $C_2$ with the segment $\{0\} \times [0, 1/2]$, we obtain a connected set $C_3 \subset B_R(0) \times [0, 1]$ such that $(0, 0), (0, 1) \in C_3$ and for all $(x, t) \in C_3$, $\mathcal{L}(h(x, t)) = 0$. $C_3$ is not necessarily path-connected, so let $r > 0$ be small enough so
that for all
\[(x, t) \in N_r(C_3) \equiv \{(y, s) \in B_R(0) \times [0, 1] \mid \exists (x', t') \in B_R(0) \times [0, 1] \text{ with } |y - x'|^2 + (s - t')^2 < r^2\}, \tag{2.17}\]
\[|\mathcal{L}(h(x, t))| < 1.\]

\(N_r(C_3)\) is path-connected [21], so there exists a path \(g \in C([0, 1], N_r(C_3))\) with \(g(0) = (0, 0), g(1) = (0, 1), \) and \(g(\theta) \in N_r(C_3)\) for all \(\theta \in [0, 1].\) If we define \(\tilde{\gamma} \in \Gamma\) by \(\tilde{\gamma}(\theta) = h(g(\theta)),\) then \(|\mathcal{L}(\tilde{\gamma}(\theta))| < 1\) for all \(\theta \in [0, 1].\) Since \(\tilde{\gamma}(0) = 0\) and \(I(\tilde{\gamma}(1)) < 0,\) there exists \(\theta^* \in [0, 1]\) with \(\tilde{\gamma}(\theta^*) \in \partial\mathcal{B}.\) By the definition of \(c^+ (2.9), I(\tilde{\gamma}(\theta^*)) \geq c^+\).

Since \(h\) was an arbitrary element of \(\mathcal{H}, h_0 \geq c^+.\)

By standard deformation arguments, such as described in [15], there exists a Palais-Smale sequence \((u_n) \subset W^{1,2}(\mathbb{R}^N, \mathbb{R})\) with \(I'(u_n) \rightarrow 0\) and \(I(u_n) \rightarrow h_0\) as \(n \rightarrow \infty, c_0 < h_0 < \min(2\alpha, \alpha + \alpha).\) Apply Proposition 2.1 to \((u_n)\). Since \(I_0\) has no positive critical values smaller than \(c_0 [11], k \leq 1.\) By (2.10), \((u_n)\) converges along a subsequence to a critical point \(z\) of \(I,\) with \(I(z) = h_0.\) Theorem 1.1 is proven.

3. A DEGREE-THEORETIC LEMMA

Here, we prove Lemma 1.2. Let \(h\) be as in the hypotheses of the lemma. For \(l > 0,\) define \(A_l \subset B_1(0) \times [0, 1]\) by
\[A_l = \{(x, t) \in B_1(0) \times [0, 1] \mid |f(x, t)| < l\}. \tag{3.1}\]

\(A_l\) is an open neighborhood of \((0, 0).\) Let \(C_1\) be the component of \(A_l\) containing \((0, 0).\) We will prove the following claim:

For all \(\epsilon > 0, (0, 1) \in C_\epsilon.\) \tag{3.2}

Then we will use the \(C_\epsilon\)'s to construct \(C_\cdot.\) For \(l > 0\) and \(\epsilon > 0,\) define
\[C_\epsilon^l = \{x \in B_1(0) \mid (x, t) \in C_1\}. \tag{3.3}\]

Fix \(\epsilon \in (0, 1).\) Define \(\phi: [0, 1] \rightarrow \mathbb{Z}\) by
\[\phi(t) = d(h(\cdot, t), C_\epsilon^l, 0), \tag{3.4}\]
where \(d\) is the topological Brouwer degree [7]. We will prove \(\phi(t) = 1\) for all \(t \in [0, 1],\) in particular \(\phi(1) = 1,\) so (3.2) is satisfied.

\(f\) is continuous on a compact domain, so \(f\) is uniformly continuous. Let \(\rho > 0\) be small enough so that for all \(x \in B_1(0)\) and \(t_1, t_2 \in [0, 1],\)
\[|t_1 - t_2| < \rho \Rightarrow |h(x, t_1) - h(x, t_2)| < \epsilon/4. \tag{3.5}\]

Clearly
\[\phi(0) = d(id, B_1(0), 0) = 1. \tag{3.6}\]

Let \(0 \leq t_1 < t_2 \leq 1\) with \(t_2 - t_1 < \rho.\) We will show \(\phi(t_1) = \phi(t_2),\) proving that \(\phi\) is constant, which by (3.6), implies (3.2).

\(\Omega\) is nonempty. For all \(x \in \partial C_{t_1},|h(x, t_1)| = \epsilon,\) so by (3.5),
\[x \in \partial C_{t_1} \Rightarrow |h(x, t_1)| \geq \frac{3}{4}r. \tag{3.7}\]

By the additivity property of \(d\) [7],
\[\phi(t_2) = d(f(\cdot, t_2), C^l_{t_2}, 0) \tag{3.8}\]
\[= d(f(\cdot, t_2), C^l_{t_2} \setminus C^l_{t_1}, 0) + d(f(\cdot, t_2), C^l_{t_1} \cap C^l_{t_2}, 0).\]
We will show:

There does not exist $x \in C_{t_2} \setminus C_{t_1}$ with $h(x, t_2) = 0$. \hfill (3.9)

Suppose such an $x$ exists. Then by (3.5), $|h| < \epsilon/4$ on the segment $\{x\} \times [t_1, t_2]$. $x \in C_{t_2}$, so $(x, t_2) \in C_{t_1}$, and by the definition of $C_\epsilon, (x, t_1) \in C_\epsilon$, and $x \in C_{t_2} \setminus C_{t_1}$, contradicting $x \in C_{t_2} \setminus C_{t_1}$. So (3.9) is true. Therefore by (3.8),

$$\phi(t_2) = d(f(\cdot, t_2), C_{t_1} \cap C_{t_2}, 0).$$ \hfill (3.10)

By the same argument, switching the roles of $t_1$ and $t_2$,

$$\phi(t_1) = d(f(\cdot, t_1), C_{t_1} \cap C_{t_2}, 0).$$ \hfill (3.11)

For all $t \in [t_1, t_2]$ and $x \in \partial C_{t_2} \cup \partial C_{t_1}$, (3.5) gives $|h(x, t_1)| > 3\epsilon/4$ and $|h(x, t) - h(x, t_1)| < \epsilon/4$. Therefore by the homotopy invariance property of the degree [7],

$$\phi(t_1) = d(f(\cdot, t_1), C_{t_1} \cap C_{t_2}, 0)$$

$$= d(f(\cdot, t_2), C_{t_1} \cap C_{t_2}, 0) = \phi(t_2).$$ \hfill (3.12)

$\phi(0) = 1$ and $\phi(t_1) = \phi(t_2)$ for any $t_1 < t_2$ with $t_1, t_2 \in [0, 1]$ and $t_2 - t_1 < \rho$. Therefore $\phi$ is constant, and $\phi(1) = 1$. Therefore $(0, 1) \in C_\epsilon$.

Now let

$$C_\epsilon = \bigcap_{\epsilon > 0} C_\epsilon.$$ \hfill (3.13)

Each $C_\epsilon$ is a connected set containing $(0, 0)$ and $(0, 1)$, so it is easy to show that $C_0$ is a connected set containing $(0, 0)$ and $(0, 1)$, and clearly for all $(x, t) \in C_0, h(x, t) = 0$.

REFERENCES


