HAMILTON-JACOBI EQUATIONS FOR CONTROL PROBLEMS OF PARABOLIC EQUATIONS

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Abstract. We study Hamilton-Jacobi equations related to the boundary (or internal) control of semilinear parabolic equations, including the case of a control acting in a nonlinear boundary condition, or the case of a nonlinearity of Burgers' type in 2D. To deal with a control acting in a boundary condition a fractional power \((-A)^{\beta}\) – where \(A, D(A)\) is an unbounded operator in a Hilbert space \(X\) – is contained in the Hamiltonian functional appearing in the Hamilton-Jacobi equation. This situation has already been studied in the literature. But, due to the nonlinear term in the state equation, the same fractional power \((-A)^{\beta}\) appears in another nonlinear term whose behavior is different from the one of the Hamiltonian functional. We also consider cost functionals which are not bounded in bounded subsets in \(X\), but only in bounded subsets in a space \(Y \hookrightarrow X\). To treat these new difficulties, we show that the value function of control problems we consider is equal in bounded sets in \(Y\) to the unique viscosity solution of some Hamilton-Jacobi-Bellman equation. We look for viscosity solutions in classes of functions which are Hölder continuous with respect to the time variable.

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1. INTRODUCTION

In this paper we study the uniqueness and existence of viscosity solution of the Hamilton-Jacobi-Bellman equation

\[
\begin{align*}
-\frac{\partial v}{\partial t}(t, x) &- \left(D_x v(t, x) \mid Ax\right)_X + \left(D_x v(t, x) \mid (-A)^{\beta} F(t, Ax)\right)_X \\
+ H(t, x, (-A)^{\beta} D_x v(t, x)) & = 0 \quad \text{in } (0, T) \times X, \\
v(T, x) & = g(x) \quad \text{in } X.
\end{align*}
\]

(1.1)

In this setting \(X\) is a real Hilbert equipped with the inner product \((\cdot \mid \cdot)_X\) and the norm \(\|\cdot\|_X\), \(A\) is an unbounded operator with domain \(D(A)\) in \(X\), it is supposed to be self-adjoint and strictly dissipative in \(X\), \((-A)^{\beta}\) is the \(\beta\)-fractional power of \((-A)\), and \(0 \leq \beta \leq \frac{1}{2}\). \(A\) is a bounded linear operator from \(D((-A)^{\alpha})\) into \(X_0\) with \(0 \leq \alpha < \frac{1}{2}\). \(X_0\) is another real Hilbert space equipped with the inner product \((\cdot \mid \cdot)_{X_0}\) and the norm \(\|\cdot\|_{X_0}\).

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$F \in C([0, T] \times X_0; X)$, the Hamiltonian functional $H$ is continuous in $[0, T] \times X \times X$, and the mapping $g$ is Lipschitz continuous in $X$. Precise assumptions are stated in Section 2.

Equation (1.1) is related to optimal control problems of semilinear parabolic equations (including in particular the case where the control acts in a nonlinear boundary condition). More precisely, for all $t \in [0, T]$ and $x \in X$, consider an optimal control problem of the form

$$\min \left\{ J(t, u, y) \mid u \in \mathcal{M}(t, T; U) \text{ and } (y, u) \text{ is solution of equation (1.2)} \right\},$$

where the cost functional $J$ is defined by

$$J(t, y, u) = \int_t^T L(r, y(r), u(r)) \, dr + g(y(T)),$$

and the state equation is

$$y' = Ay + (-A)^{\beta} [Bu - F(y, \Lambda x)], \quad y(t) = x. \quad (1.2)$$

The control space $\mathcal{M}(t, T; U)$ is a set of bounded measurable functions with values in $U$, and $U$ is a bounded subset in $X_T$, $X_T$ is a Banach space, $B \in L(X_T, X)$. In Section 3 we prove that equation (1.1) admits at most one viscosity solution. In Section 4, we prove that the value function of problem $(\mathcal{P}_{t,x})$ is the unique viscosity solution of equation (1.1), when $H$ is defined by

$$H(t, x, p) = \sup_{u \in U} \left[ - (p \mid Bu)_X - L(t, x, u) \right].$$

Applications are discussed in Section 5. Before presenting what is new in the present paper, observe that by setting

$$\mathbf{H}(t, x, (-A)^{\beta} D_x v(t, x)) = H(t, x, (-A)^{\beta} D_x v(t, x)) + \left( (-A)^{\beta} D_x v(t, x) \mid F(t, x, \Lambda x) \right)_X,$$

equation (1.1) can be written in the form

$$\frac{\partial v}{\partial t}(t, x) - \left( D_x v(t, x) \mid Ax \right)_X + \mathbf{H} \left( t, x, (-A)^{\beta} D_x v(t, x) \right) = 0 \quad \text{in } (0, T) \times X. \quad (1.3)$$

Equation (1.3) seems to be simpler to handle than equation (1.1). However assumptions on $F(t, \Lambda x)$ and on $H(t, x, p)$ are different and we cannot simplify the presentation of the paper by considering equation (1.3) (see e.g. the estimates involving $H$ and $F$ in the proof of Th. 3.5).

During the eighties and the nineties several fundamental advances have been made in the study of Hamilton-Jacobi equation in infinite dimension. These equations were first studied by Barbu and Da Prato (see e.g. [2]), mainly in classes of convex functions. The method of viscosity solutions has been extended to infinite dimension by Crandall and Lions in a series of papers [10–14]. All these papers correspond to the case when $\beta = 0$. Other contributions are due to Cannarsa and Frankowska [5], Ishii [20], Soner [27], Tataru [28, 29], Crandall and Lions [15, 16], Cannarsa and Tessitore [6–9] in order to deal with boundary controls. In particular equations of the form (1.3) with $0 < \beta < \frac{1}{2}$ are studied in [6], [8] to treat Neumann boundary controls. The case of Dirichlet controls is considered in [7, 9], it corresponds to the situation when $\frac{1}{2} < \beta < 1$ and has to be studied independently. More recently the case of the Navier-Stokes equations has been studied in [18, 26].

The main motivation of the present paper is to characterize the value function of control problems governed by semilinear parabolic equations, including the case of equations with a nonlinear boundary condition, or the case of nonlinearity of Burgers’ type in two dimension, and with cost functionals whose growth is quadratic or even higher than quadratic. For example we study the case of partial differential equations with nonlinear boundary conditions of the form:

$$\frac{\partial y}{\partial t} - \Delta y + y = f \quad \text{in } [t, T[ \times \Omega, \quad \frac{\partial y}{\partial n} + \tilde{h}(y) = u \quad \text{on } [t, T[ \times \Gamma, \quad y(t) = x \quad \text{in } \Omega, \quad (1.4)$$
with cost functionals of the type
\[ \hat{J}(t, y, u) = \int_0^T \hat{L}(r, y(r), u(r)) \, dr + \hat{g}(y(T)), \]
where \( \hat{h} \) is any regular nondecreasing function obeying \( \hat{h}(0) = 0 \), and where \( \hat{L} \) and \( \hat{g} \) may be quadratic cost functionals. Many thermal processes lead to the kind of model corresponding to equation (1.4) (see [23]). The papers mentioned above do not include this model in their possible applications. If the initial condition \( x \) belong to \( X = L^2(\Omega) \), equation (1.4) is well posed and it admits a unique weak solution belonging to \( C([0, T]; X) \) (the solution also belongs to \( L^2(0, T; H^1(\Omega)) \)). We can write equation (1.4) in the form
\[ y' = Ay + (-A)^{\beta} \left[ Bu - \hat{F}(\cdot, \Lambda y) \right], \quad y(t) = x, \quad (1.5) \]
by defining \( \Lambda \) as the trace mapping on \( \Gamma \):
\[ \Lambda : y \mapsto y|_{\Gamma}. \]
In this example \( \Lambda \) is bounded from \( H^{2\alpha}(\Omega) = D((-A)^{\alpha}) \) into \( X_0 = L^2(\Gamma) \) for all \( \frac{1}{2} < \alpha < 1 \), \( D(A) = \{ y \in H^2(\Omega) \mid \partial_n y = 0 \} \), \( Ay = \Delta y \), and we have to take \( \frac{1}{2} < \beta < \frac{1}{2} \). For a parabolic equation with a linearity of Burgers' type we can take \( \beta = \frac{1}{2} \). Let us denote by \( y_{t,x,u} \) the solution to equation (1.5). To characterize the value function \( \hat{v}(t, x) \) of the problem
\[ (\hat{P}_{t,x}) \min \{ \hat{J}(t, u, y) \mid u \in M(t, T; U) \text{ and } (y, u) \text{ is solution of equation (1.5)} \}, \]
we have to study the dependence of \( y_{t,x,u} \) and of \( \Lambda y_{t,x,u} \) with respect to \( t \) and to \( x \). Due to the nonlinear term in equation (1.5), we can prove continuity properties for \( y_{t,x,u} \) and \( Ay_{t,x,u} \) and Lipschitz properties for \( y_{t,x,u} \) and \( Ay_{t,x,u} \) when the initial condition \( x \) stays in bounded subsets in \( X \), for a space \( Y \hookrightarrow X \), but these properties are not true if we consider only bounded subsets in \( X \). Therefore it is natural to study the properties of the value function \( \hat{v}(t, x) \) when \( x \) remains in bounded subsets of \( Y \), and to look for solutions to equation (1.4) in a space of the type \( C([0, T]; Y) \) or at least \( L^\infty_0(0, T; Y) \) (the space of bounded and weakly measurable functions from \( (0, T) \) into \( Y \)).

Another difficulty comes from the cost functional. In the literature on Hamilton-Jacobi-Bellman equations, it is often assumed that the cost functionals either are bounded or satisfy a linear growth condition [6,8,18,20]. Thus the case of quadratic cost functionals is not treated in these papers.

To overcome the two difficulties mentioned above, the one coming from the nonlinearity in the state equation and the other one due to the growth condition of the cost functional, we suggest to proceed as follows. First, we show that, for an initial condition in \( B_Y(M_0) \) (the ball in \( Y \) centered at the origin and with radius \( M_0 \)), the solution \( y \) of equation (1.2) satisfies \( y(\cdot) \in B_Y(R_T) \) in \( (t, T) \) for some \( R_T = R(M_0, T) \) which can be explicitly estimated independently of \( t \in (0, T) \). Next, we associate with the mappings \( \hat{L}(t, \cdot, u), \hat{g} \) and \( \hat{F}(t, \Lambda) \), other mappings \( L(t, \cdot, u), g \) and \( F(t, \Lambda) \) which are identical to the previous ones in the ball \( B_Y(R_T) \), but which satisfies some global boundedness and Lipschitz properties. Let us consider the problem \( (P_{t,x}) \) – the one introduced at the beginning of the introduction – defined with \( L(t, \cdot, u), g \) and \( F(t, \Lambda) \). We are able to show that value function \( v(t, x) \) of problem \( (P_{t,x}) \) obeys \( \hat{v}(t, x) = v(t, x) \) for \( t \in (0, T) \) and \( x \in B_Y(R_T) \). We show that \( v \) is the unique viscosity solution of the Hamilton-Jacobi equation (1.1). Thus \( \hat{v} \) is not the viscosity solution to equation (1.1), but it is equal to the viscosity solution of equation (1.1) in bounded sets in \( (0, T) \times Y \).

Sections 2, 3 and 4 are devoted to the study of equation (1.2), equation (1.1), and the value function of problem \( (P_{t,x}) \). In these sections, only the mappings \( L(t, \cdot, u), g \) and \( F(t, \Lambda) \) intervene. The assumptions are precisely stated in Section 2. The definition of the mappings \( L(t, \cdot, u), g \) and \( F(t, \Lambda) \) from \( \hat{L}(t, \cdot, u), \hat{g} \) and \( \hat{F}(t, \Lambda) \) is treated in examples of Section 5 by using projection operators. Three examples are considered. The first one is a control problem for the state equation (1.4), and the two others correspond to problems for a two dimensional scalar equation of Burgers' type. The interest of the third example is to show that the method
using a projection operator in the cost functional and the state equation is flexible enough to involve different kind of projections adapted to the nonlinearity and to the functionals we have to deal with.

Let us finally mention that the definition of viscosity solutions that we take is not totally standard. Indeed we consider viscosity solutions which are Hölder continuous with respect to the time variable. This Hölder continuity condition, which is a new argument in the definition of viscosity solutions – see Definition 3.2 – plays a major role in the proof of uniqueness to estimate the nonlinear term $F$. A preliminary version of the present paper corresponds to a part of the Ph.D. thesis by the first author [17].

2. Preliminaries on the Evolution Equation

In this section we want to study properties of solutions of the evolution equation

$$y' = Ay + (-A)^{\beta} [Bu - F(t, Ay)] \quad \text{in} \ (t, T), \quad y(t) = x,$$

(2.1)

where $t \in [0, T)$.

2.1. Assumptions

Throughout the paper we make the following assumptions.

(i) The unbounded operator $A$, with domain $D(A)$ in $X$, is a closed and densely defined selfadjoint operator in $X$, such that $(Ax, x)_X \leq -\omega |x|^2_X$ for all $x \in D(A)$, where $\omega > 0$.

(ii) $B \in \mathcal{L}(X, X)$.

(iii) The linear operator $\Lambda$ is bounded from $D((-A)^\alpha)$ into $X_0$ for some $\alpha \in [0, \frac{1}{2}]$, that is:

$$|Ax|_{X_0} \leq C_\alpha |(-A)^\alpha x|_X \quad \text{for all} \ x \in D((-A)^\alpha).$$

(2.2)

The exponent $\beta \in [0, \frac{1}{2}]$ is given fixed.

(iv) $F$ is a continuous mapping from $[0, T] \times X_0$ into $X$, which satisfies:

$$|F(t, x) - F(t, y)|_X \leq K_F |x - y|_{X_0}, \quad \text{and} \quad |F(t, x)|_X \leq M_F,$$

(2.3)

for all $t \in [0, T]$, and all $x, y \in X_0$. Moreover, there exists $\eta_1 \in [0, 1]$ such that:

$$|F(t, x) - F(s, x)|_X \leq M_{1, F} (1 + |x|_{X_0}) |t - s|^{\eta_1}.$$

(2.4)

In addition, we assume that either $\beta < \frac{1}{4}$, or $\beta = \frac{1}{4}$ and

$$D((-A)^{\frac{1}{4}}) \hookrightarrow X_0,$$

$$|F(t, x)|_{D((-A)^{\beta_0})} \leq M \left( \beta_0, |x|_{D((-A)^{\frac{1}{2}})} \right) \quad \text{for all} \ t \in [0, T] \quad \text{and all} \ x \in D((-A)^{\frac{1}{2}}),$$

(2.5)

$$(-A)^{\beta_0} B \in \mathcal{L}(X, X), \quad \text{for some} \ 0 < \beta_0 < \beta = \frac{1}{2},$$

where $M \left( \beta_0, |x|_{D((-A)^{\frac{1}{2}})} \right) > 0$ only depends on $\beta_0$ and $|x|_{D((-A)^{\frac{1}{2}})}$.

(v) The control $u$ belongs to $\mathcal{M}(t, T; U)$, the space of measurable functions from $(t, T)$ into $U$, where $U$ is a nonempty, bounded and closed subset of $X_T$, such that

$$|u|_{X_T} \leq M_U \quad \text{for all} \ u \in U.$$

(2.6)

We now state assumptions needed in Section 3 to study equation (1.1).
The mapping \( g \in C(X) \) is Lipschitz continuous and bounded in \( X \), i.e.:
\[
|g(x) - g(y)| \leq K_g |x - y|_X \quad \text{and} \quad |g(x)| \leq M_g, \quad \text{for all} \ x, y \in X.
\]

The Hamiltonian functional \( H \) satisfies
\[
|H(t, x, p) - H(s, y, q)| \leq K_H (|t - s|^{\eta_2} + |x - y|_X + |p - q|_X).
\]  
(2.7)

In Section 4, we make the following additional assumption.

The Hamiltonian functional \( H : [0, T] \times X \times X \to \mathbb{R} \) is defined by:
\[
H(t, x, p) = \sup_{u \in U} [- (p \cdot Bu)_X - L(t, x, u)],
\]  
(2.8)

where the functional \( L \in C([0, T] \times X \times U) \) satisfies:
\[
|L(t, x, u) - L(s, y, u)| \leq K_L (|t - s|^{\eta_2} + |x - y|_X) \quad \text{and} \quad |L(t, x, u)| \leq M_L,
\]

for all \( t, s \in [0, T] \), all \( x, y \in X \), and all \( u \in U \), with \( 0 < \eta_2 \leq 1 \).

Observe that if \( H \) is defined by (2.8) and if \( L \) satisfies the estimate stated in (viii), then
\[
|H(t, x, p) - H(s, y, q)| \leq K_H (|t - s|^{\eta_2} + |x - y|_X + |p - q|_X),
\]
with \( K_H = \max(K_L, \|B\| \cdot M_U) \). Thus assumption (vii) is automatically satisfied in that case.

Due to assumption (i), \((A, D(A))\) is the infinitesimal generator of a strongly continuous analytic semigroup of contractions on \( X \) which satisfies
\[
\|e^{tA}\|_{L(X)} \leq e^{-\omega t}.
\]  
(2.9)

Moreover (see [19], Th. 1.4.3, Chap. 1 and [3], Prop. 5.1, Chap. 1), for all \( \delta \geq 0 \), there exists a constant \( M_\delta \) such that, for all \( t > 0 \):
\[
\left\| (-A)^{\delta} e^{tA} \right\|_{L(X)} \leq M_\delta t^{-\delta}.
\]  
(2.10)

If \( 0 < \delta \leq 1 \), and \( x \in D((-A)^{\delta}) \), we have:
\[
\left\| (e^{tA} - I) x \right\|_X \leq \frac{1}{\delta} M_{1-\delta} t^\delta \left\| (-A)^{\delta} x \right\|_X.
\]  
(2.11)

Besides, for all \( \delta < \gamma \) and all \( x \in D((-A)^{\gamma}) \), one has:
\[
\left\| (-A)^\delta x \right\|_X \leq M_{\delta, \gamma} \left\| (-A)^\gamma x \right\|_X^{\frac{\delta}{\gamma}} \left\| x \right\|_X^{1 - \frac{\delta}{\gamma}}.
\]  
(2.12)

With Young’s inequality the last estimate implies that, for all \( \delta \in \left]0, \frac{1}{2}\right[ \), and all \( \sigma > 0 \), there exists a constant \( C_{\delta, \sigma} \) such that:
\[
\left\| (-A)^\delta x \right\|_X \leq \sigma \left\| (-A)^{\frac{\delta}{2}} x \right\|_X + C_{\delta, \sigma} \left\| x \right\|_X \quad \text{for} \ x \in D((-A)^{\frac{\delta}{2}}).
\]  
(2.13)
2.2. Properties and regularities of mild solutions of equation (2.1)

**Theorem 2.1.** For all \( x \in X \) and all \( u \in \mathcal{M}(t, T; U) \), equation (2.1) admits a unique solution \( y_{t,x,u} \) in \( L^1(t, T; D((-A)^\alpha)) \), which belongs to \( C([t, T]; X) \) and satisfies formula (2.14). □

\[
y_{t,x,u}(s) = e^{(s-t)A}x + (-A)^\beta \int_t^s e^{(s-r)A} [Bu(r) - F(r, \Lambda y_{t,x,u}(r))] \, dr, \tag{2.14}
\]

for all \( s \in [t, T] \). Moreover \( y_{t,x,u} \) belongs to \( C([t, T]; X) \) and satisfies the estimate

\[
\|y_{t,x,u}\|_{C([t, T]; X)} \leq C(1 + |x|_X + \|u\|_{L^\infty(t, T; U)}).
\]

**Proof.** Let \( t_1 \in (t, T] \) be such that \( C_\alpha K_F M_{\alpha+\beta} \frac{(t_1-t)^{1-(\alpha+\beta)}}{1-(\alpha+\beta)} \leq 1/2 \). Let us set \( E = L^1(t, t_1; D((-A)^\alpha)) \), and let us show that the mapping

\[
y \mapsto (\Psi y)(s) = e^{(s-t)A}x + (-A)^\beta \int_t^s e^{(s-r)A} [Bu(r) - F(r, \Lambda y(r))] \, dr,
\]

is a contraction in \( E \). First we have:

\[
\int_t^{t_1} \|(-A)^\alpha (\Psi y)(s)\|_X \, ds \leq \int_t^{t_1} \|(-A)^\alpha e^{(s-t)A}x\|_X \, ds + \int_t^{t_1} \int_t^s \|(-A)^{\alpha+\beta} e^{(s-r)A} [Bu(r) - F(r, \Lambda y(r))]\|_X \, dr \, ds
\]

\[
\leq M_\alpha |x|_X \frac{(t_1-t)^{1-\alpha}}{1-\alpha} + M_{\alpha+\beta} \int_t^{t_1} \int_t^s \frac{1}{(s-r)^{\alpha+\beta}} \|Bu(r) - F(r, \Lambda y(r))\|_X \, ds \, ds
\]

\[
\leq M_\alpha |x|_X \frac{(t_1-t)^{1-\alpha}}{1-\alpha} + M_{\alpha+\beta} \|Bu\|_{L^\infty(t, T; U)} \|M_U + M_F\|_X \frac{(t_1-t)^{2-(\alpha+\beta)}}{1-(\alpha+\beta)}
\]

Thus, if \( y \in E \), \( \Psi y \) belongs to \( E \). Moreover if \( y_1, y_2 \in E \), we can write

\[
\int_t^{t_1} \|(-A)^\alpha (\Psi y_1)(s) - (-A)^\alpha (\Psi y_2)(s)\|_X \, ds
\]

\[
\leq \int_t^{t_1} \int_t^s \|(-A)^{\alpha+\beta} e^{(s-r)A} [F(r, \Lambda y_1(r)) - F(r, \Lambda y_2(r))]\|_X \, ds \, dr
\]

\[
\leq \int_t^{t_1} \int_t^s \frac{M_{\alpha+\beta}}{(s-r)^{\alpha+\beta}} K_F \|\Lambda y_1(r) - \Lambda y_2(r)\|_X \, ds \, dr
\]

\[
= K_F \int_t^{t_1} \|\Lambda y_1(r) - \Lambda y_2(r)\|_X \int_t^s \frac{M_{\alpha+\beta}}{(s-r)^{\alpha+\beta}} \, dr \, ds
\]

\[
\leq C_\alpha K_F M_{\alpha+\beta} \int_t^{t_1} \|(-A)^\alpha y_1(r) - (-A)^\alpha y_2(r)\|_X \left(\frac{(t_1-t)^{1-(\alpha+\beta)}}{1-(\alpha+\beta)}\right) \, dr
\]

\[
\leq C_\alpha K_F M_{\alpha+\beta} \frac{(t_1-t)^{1-(\alpha+\beta)}}{1-(\alpha+\beta)} \int_t^{t_1} \|(-A)^\alpha y_1(r) - (-A)^\alpha y_2(r)\|_X \, dr.
\]

Thus \( \Psi \) is a contraction in \( E \), and it admits a unique fixed point in \( E \), which is the unique solution \( y \) in \( E \) to equation (2.1). In addition \( y \) belongs to \( C([t, t_1]; X) \) and formula (2.14) is satisfied for all \( s \in [t, t_1] \). We can repeat this process on the interval \( [t_1, 2t_1] \), and step by step, we prove that equation (2.1) admits a unique solution in \( L^1(t, T; D((-A)^\alpha)) \), which belongs to \( C([t, T]; X) \) and satisfies formula (2.14). □
Proposition 2.2. Assume that \( x \in D((-A)^\alpha) \). Then the solution \( y_{t,x,u} \) of (2.1) satisfies:

\[
\| Ay_{t,x,u}(s) - Ax \|_{X_0} \to 0 \quad \text{uniformly with respect to } u \in \mathcal{M}(t,T;U) \quad \text{when } s \searrow t.
\]

(2.15)

Proof. Let \( x \) be in \( D((-A)^\alpha) \). With inequality (2.2) we have

\[
\| A(y_{t,x,u}(s) - x) \|_{X_0} \leq C_\alpha \| (-A)^\alpha (y_{t,x,u}(s) - x) \|_X.
\]

Due to (2.14), we can write

\[
\| (-A)^\alpha (y_{t,x,u}(s) - x) \|_X \leq \left( (-A)^\alpha \left( e^{(s-t)A}x - x \right) \right)_X \tag{2.16}
\]

\[
+ \left( (-A)^{\beta+\alpha} \int_t^s e^{(s-r)A} [Bu(r) - F(r, Ay_{t,x,u}(r))] \, dr \right)_X.
\]

We can estimate the two terms in the right hand side of (2.16) as follows:

\[
\left( (-A)^\alpha \left( e^{(s-t)A}x - x \right) \right)_X = \left| \left( e^{(s-t)A} - I \right) (s-t)^\alpha x \right|_X,
\]

and

\[
\left( (-A)^{\beta+\alpha} \int_t^s e^{(s-r)A} [Bu(r) - F(r, Ay_{t,x,u}(r))] \, dr \right)_X \leq M_{\alpha+\beta} \frac{(s-t)^{1-(\alpha+\beta)}}{1-(\alpha+\beta)} (M_F + \| B \| M_U).
\]

The two terms (2.17) and (2.18) go to 0 uniformly with respect to \( u \in \mathcal{M}(t,T;U) \) when \( s \searrow t \), because \( \alpha + \beta < 1 \).

Proposition 2.3. Let \( y_{t,x,u} \) be the weak solution of (2.1). There exists a constant \( C_1(\beta) \), independent of \( u \), such that

\[
\left| y_{t,x,u}(s) - e^{(s-t)A}x \right|_X \leq C_1(\beta) (s-t)^{1-\beta} \quad \text{for all } x \in X.
\]

(2.19)

Proof. Let \( y_{t,x,u} \) be the weak solution of equation (2.1). With the integral formulation (2.14), we have

\[
\left| y_{t,x,u}(s) - e^{(s-t)A}x \right|_X \leq \left( (-A)^\beta \int_t^s e^{(s-r)A} [Bu(r) - F(r, Ay(r))] \, dr \right)_X.
\]

From (2.3) and (2.10) it follows that

\[
\left| (-A)^\beta \int_t^s e^{(s-r)A} [Bu(r) - F(r, Ay(r))] \, dr \right|_X \leq \int_t^s \frac{M_\beta}{(s-r)^\beta} [\| B \| M_U + M_F] \, dr
\]

\[
\leq M_\beta \| B \| M_U + M_F \frac{(s-t)^{1-\beta}}{1-\beta}.
\]

The proof is complete.

Proposition 2.4. We assume that \( \beta = \frac{1}{2} \) and that the corresponding additional conditions of assumption (iv) is satisfied. There exists a constant \( C_1 \left( \beta_0, |x|_{D((-A)^\frac{1}{2})} \right) \), independent of \( u \), such that

\[
\left| y_{t,x,u}(s) - e^{(s-t)A}x \right|_X \leq C_1 \left( \beta_0, |x|_{D((-A)^\frac{1}{2})} \right) (s-t)^{1+\beta_0} \quad \text{for all } x \in D((-A)^\frac{1}{2}).
\]

(2.20)
Proof. Assume that \( x \in D((-A)^{1/2}) \). Let \( y_{t,x,u} \) be the weak solution of equation (2.1). By using the integral formulation (2.14), we have
\[
\left| y_{t,x,u}(s) - e^{(s-t)A}x \right|_X \leq \left| (-A)^{3-\beta_0} \int_t^s e^{(s-r)A} \left[ (-A)^{\beta_0} B_u(r) - (-A)^{\beta_0} F(r, Ay(r)) \right] \, dr \right|_X.
\]

With (2.5) and (2.10) we have
\[
\left| (-A)^{3-\beta_0} \int_t^s e^{(s-r)A} \left[ (-A)^{\beta_0} B_u(r) - (-A)^{\beta_0} F(r, Ay(r)) \right] \, dr \right|_X \\
\leq \left| (-A)^{3-\beta_0} \int_t^s \frac{M^{3-\beta_0}}{(s-r)^{3-\beta_0}} \left[ \left\| (-A)^{\beta_0} B \right\|_U + M \left( \beta_0, |x| \right) \right] \, dr \right|_X \\
\leq M_{\beta - \beta_0} \left[ \left\| (-A)^{\beta_0} B \right\|_U + M \left( \beta_0, |x| \right) \right] \left( s - t \right)^{1-\beta + \beta_0} \frac{1}{1 - \beta + \beta_0}.
\]

The proof is complete. \( \square \)

To prove the other propositions, we need the following theorem.

**Theorem 2.5** [1 Th. 3.3.1, Chap. 2]. Let \( \delta \) and \( \gamma \) be in \([0,1] \) and \( \epsilon > 0 \). If the mapping
\[
t \in J \subset \mathbb{R}^+ \mapsto t^\delta u(t),
\]
belongs to \( L^\infty_{loc}(J;\mathbb{R}) \), and if there exist two positive constants \( a, b \) such that
\[
u(t) \leq a t^{-\delta} + b \int_t^t (t - \tau)^{-\gamma} \nu(\tau) \, d\tau, \text{ for a.e. } t \in J^* = J \setminus \{0\},
\]
then there exists a positive constant \( c := c(\delta, \gamma, \epsilon) \) independent of \( a \) and \( b \) such that
\[
u(t) \leq a t^{-\delta} \left( 1 + cb t^{1-\gamma} \epsilon^{1-k(\gamma, \delta)t} \right) \text{ for a.e. } t \in J^*,
\]
where \( k(\gamma, b) := (\Gamma(1-\gamma) - b)^{1/(1-\gamma)}. \)

**Proposition 2.6.** Let \( x \) and \( x_0 \) be in \( X \), \( u \in \mathcal{M}(t,T;U) \), and let \( y_{t,x,u} \) and \( y_{t,x_0,u} \) be the corresponding solutions to equation (2.1). Then, for all \( \theta \in [0,1 - \alpha[ \), there exists a constant \( C_2(\alpha, \beta, \theta) \) such that:
\[
\left| \Lambda y_{t,x,u}(r) - \Lambda y_{t,x_0,u}(r) \right|_{X_0} \leq C_2(\alpha, \beta, \theta) \left( -A \right)^{-\theta} \left( x - x_0 \right) \Big| X, \tag{2.21}
\]
for all \( r \in (t,T] \). (The constant \( C_2(\alpha, \beta, \theta) \) is explicitly given in (2.24).)

**Proof.** Using the integral formulation (2.14) for \( y_{t,x,u} \) and \( y_{t,x_0,u} \), and (2.2), we obtain
\[
\left| \Lambda y_{t,x,u}(r) - \Lambda y_{t,x_0,u}(r) \right|_{X_0} \leq C_\alpha \left| -A \right|^{\alpha+\theta} e^{(r-t)A} \left( -A \right)^{-\theta} \left( x - x_0 \right) \Big| X
\]
\[
+ C_\alpha \left| -A \right|^{\beta+\alpha} \int_t^r e^{(r-s)A} \left[ F(s, Ay_{t,x,u}(s)) - F(s, Ay_{t,x_0,u}(s)) \right] ds \Big| X. \tag{2.22}
\]

Setting \( \theta = 0 \) in this estimate, we first obtain
\[
\left| \Lambda y_{t,x,u}(r) - \Lambda y_{t,x_0,u}(r) \right|_{X_0} \leq C_\alpha M_\alpha \left( r - t \right)^{\frac{\alpha}{\alpha + \beta}} \left| x - x_0 \right|_X + \frac{2C_\alpha M_\alpha + M_F}{1 - (\alpha + \beta)} \left( r - t \right)^{1-(\alpha + \beta)}.
\]
Multiplying both sides by \((r - t)^{\alpha + \theta}\) we have

\[
(r - t)^{\alpha + \theta} |Ay_{t,x,u}(r) - Ay_{t,x_0,u}(r)|_{X_0} \leq (T - t)^{\theta} C_{\alpha} M_{\alpha} |x - x_0|_X + \frac{2C_{\alpha} M_{\alpha + \beta} M_F}{1 - (\alpha + \beta)} (T - t)^{1 + \theta - \beta} \in L^\infty (t, T; \mathbb{R}).
\] (2.23)

Next with (2.22) we write

\[
|Ay_{t,x,u}(r) - Ay_{t,x_0,u}(r)|_{X_0} \leq \frac{C_{\alpha} M_{\alpha + \theta}}{(r - t)^{\alpha + \theta}} \bigg( -A \bigg)^{-\theta} (x - x_0) \bigg|_{X} + \int_t^r \frac{C_{\alpha} M_{\beta + \alpha}}{(r - s)^{\alpha + \theta}} K_F |Ay_{t,x,u}(s) - Ay_{t,x_0,u}(s)|_{X_0} ds.
\]

Since the function \(r \mapsto (r - t)^{\alpha + \theta} |Ay_{t,x,u}(r) - Ay_{t,x_0,u}(r)|_{X_0}\) belongs to \(L^\infty (t, T)\), we can use Theorem 2.5 with for example \(\varepsilon = 1\), and we obtain (2.21) by setting

\[
C_2(\alpha, \beta, \theta) = C_{\alpha} M_{\alpha + \theta} \left( 1 + C_{\beta + \alpha, \alpha + \theta} \left( c C_{\alpha} M_{\beta} K_F T \right)^{1 - \beta + \alpha} e^{2k(\beta + \alpha, K_F C_{\alpha} M_{\beta + \alpha} T)} \right)
\] (2.24)

where \(c\) is the constant appearing in Theorem 2.5.

\[\square\]

**Proposition 2.7.** Let \(x\) and \(x_0\) be in \(X\), \(u \in \mathcal{M}(t, T; U)\), and let \(y_{t,x,u}\) and \(y_{t,x_0,u}\) be the corresponding solutions to equation (2.1). Then, for all \(\theta \in [0, 1 - \alpha]\), there exists a constant \(C_3(\alpha, \beta, \theta)\) such that

\[
|y_{t,x,u}(r) - y_{t,x_0,u}(r)|_{X} \leq \frac{M_\theta}{(r - t)^{\theta}} \bigg( C_{\alpha} M_{\alpha + \theta} \bigg( -A \bigg)^{-\theta} (x - x_0) \bigg|_{X} \bigg) \leq \frac{M_\theta}{(r - t)^{\theta}} \bigg( -A \bigg)^{-\theta} (x - x_0) \bigg|_{X} \leq \frac{M_\theta}{(r - t)^{\theta}} + K_F C_2(\alpha, \beta, \theta) 1 + \frac{M_\theta}{(r - s)^{\alpha + \theta}} C_3(\alpha, \beta, \theta) 1 + \frac{M_\theta}{(r - s)^{\alpha + \theta}} C_3(\alpha, \beta, \theta) 1
\] (2.25)

(The constant \(C_3(\alpha, \beta, \theta)\) is explicitly given in (2.27).)

**Proof.** With Proposition 2.6, we have:

\[
|y_{t,x,u}(r) - y_{t,x_0,u}(r)|_{X} \leq \frac{M_\theta}{(r - t)^{\theta}} \bigg( -A \bigg)^{-\theta} (x - x_0) \bigg|_{X} + \int_t^r \frac{M_\theta}{(r - s)^{\alpha + \theta}} K_F |y_{t,x,u}(s) - y_{t,x_0,u}(s)|_{X_0} ds
\]

\[
\leq \frac{M_\theta}{(r - t)^{\theta}} \bigg( -A \bigg)^{-\theta} (x - x_0) \bigg|_{X} + \frac{M_\theta}{(r - s)^{\alpha + \theta}} \bigg( -A \bigg)^{-\theta} (x - x_0) \bigg|_{X} \int_t^r \frac{1}{(r - s)^{\alpha + \theta}} \frac{1}{(s - t)^{\alpha + \theta}} ds
\]

\[
\leq \frac{M_\theta}{(r - t)^{\theta}} + K_F C_2(\alpha, \beta, \theta) \int_t^r \frac{1}{(r - s)^{\alpha + \theta}} \frac{1}{(s - t)^{\alpha + \theta}} ds \bigg( -A \bigg)^{-\theta} (x - x_0) \bigg|_{X}.
\]

By using the integral formula of the beta function we have:

\[
\int_t^r \frac{1}{(r - s)^{\alpha + \theta}} \frac{1}{(s - t)^{\alpha + \theta}} ds = \frac{\Gamma(1 - \beta) \Gamma(1 - (\alpha + \theta))}{\Gamma(2 - (\alpha + \theta + \beta))} (r - t)^{1 - (\alpha + \theta + \beta)}.
\] (2.26)

By setting

\[
C_3(\alpha, \beta, \theta) = \frac{\Gamma(1 - \beta) \Gamma(1 - (\alpha + \theta))}{\Gamma(2 - (\alpha + \theta + \beta))} M_\beta K_F C_2(\alpha, \beta, \theta),
\] (2.27)

and

\[
C_4(\alpha, \beta, \theta, t; r) = \frac{M_\theta}{(r - t)^{\theta}} + \frac{C_3(\alpha, \beta, \theta)}{1 - (\alpha + \theta)} (r - t)^{1 - (\alpha + \theta + \beta)}.
\] (2.28)
we obtain
\[ |y_{t,x,u}(r) - y_{t,x_0,u}(r)|_X \leq C_4(\alpha, \beta, \theta, t; r) \left| (A)^{-\theta} (x - x_0) \right|_X. \]

\[ \square \]

**Proposition 2.8.** Let \( x \) be in \( X, s, t \in [0,T) \), and \( u \in \mathcal{M}(\min(t,s), T; U) \). Let us denote by \( y_{t,x,u} \) and \( y_{s,x,u} \) the solutions of equation (2.1) respectively corresponding to the initial data \((t,x)\) and \((s,x)\). Then there exist a constant \( C_5(\alpha, \beta) \) and a continuous mapping \( \bar{a}(t,s,x) \) (independent of \( u \)) such that, for all \( r \in [\max(s,t), T] \), we have:
\[
|\Lambda y_{t,x,u}(r) - \Lambda y_{s,x,u}(\sigma)|_X \leq C_5(\alpha, \beta) (r - t)^\alpha \int_s^r (r - \sigma)^{\beta} |\Lambda y_{t,x,u}(\sigma) - \Lambda y_{s,x,u}(\sigma)|_X d\sigma.
\]

(2.29)

The function \( s \mapsto \bar{a}(t,s,x) \) goes to 0 when \( s \) goes to \( t \), for all \( x \in X \). (The constant \( C_5(\alpha, \beta) \) and the mapping \( \bar{a} \) are explicitly defined in (2.34) and (2.35).)

**Proposition 2.9.** With the same assumptions and notation as in the previous proposition, there exists a constant \( C_6(\alpha, \beta) \) such that, for all \( r \in [\max(s,t), T] \), we have:
\[
|y_{t,x,u}(r) - y_{s,x,u}(r)|_X \leq C_6(\alpha, \beta) \bar{a}(t,s,x).
\]

(2.30)

(The constant \( C_6(\alpha, \beta) \) is explicitly defined in (2.36).)

**Proof of Proposition 2.8.** Consider the case where \( s < t \). The case \( t < s \) can be treated in a similar way. Let be \( r > t > s \), with estimate (2.2) and with (2.14), we have
\[
|\Lambda y_{t,x,u}(r) - \Lambda y_{s,x,u}(r)|_X \leq C_\alpha \left( (A)^{\alpha} e^{(r-t)A} \left( e^{(t-s)A} x - x \right) \right)_X
\]
\[
+ C_\alpha \left| \int_s^r (A)^{\alpha+\beta} e^{(r-\sigma)A}Bu(\sigma) - F(\sigma, \Lambda y_{s,x,u}(\sigma)) \right|_X d\sigma
\]
\[
+ C_\alpha \left| \int_r^t (A)^{\alpha+\beta} e^{(r-\sigma)A}F(\sigma, \Lambda y_{t,x,u}(\sigma)) - F(\sigma, \Lambda y_{s,x,u}(\sigma)) \right|_X d\sigma.
\]

(2.31)

Now we can write
\[
(2.31) \leq \frac{C_\alpha M_\alpha}{(r-t)^{\alpha}} \left( e^{(t-s)A} x - x \right)_X.
\]

Since \((r-t)^{\alpha} \leq (r-\sigma)^{\alpha}\) for all \( \sigma \in (s,t) \), we have
\[
(2.32) \leq C_\alpha M_{\alpha+\beta} (M_F + \|B\|_U) \int_s^r \frac{d\sigma}{(r-\sigma)^{\alpha+\beta}} \leq \frac{C_\alpha M_{\alpha+\beta} (M_F + \|B\|_U)}{(r-t)^{\alpha}} \int_s^r \frac{d\sigma}{(r-\sigma)^{\beta}}.
\]

Similarly, for all \( \sigma \in (s,t) \), we have \((r-\sigma)^{\beta} \geq (t-\sigma)^{\beta}\). Then
\[
\int_s^t \frac{d\sigma}{(r-\sigma)^{\beta}} \leq \int_s^t \frac{d\sigma}{(t-\sigma)^{\beta}} = \frac{(t-s)^{1-\beta}}{1-\beta},
\]
and therefore we obtain
\[
(2.32) \leq \frac{C_\alpha M_{\alpha+\beta} (M_F + \|B\|_U)}{(r-t)^{\alpha}} |t-s|^{1-\beta}.
\]

The last term can be estimated as follows
\[
(2.33) \leq \frac{C_\alpha M_{\alpha+\beta} K_F}{(r-t)^{\alpha}} \int_s^t \frac{1}{(r-\sigma)^{\beta}} |\Lambda y_{t,x,u}(\sigma) - \Lambda y_{s,x,u}(\sigma)| d\sigma
\]
\[
\leq \frac{C_\alpha M_{\alpha+\beta} K_F}{(r-t)^{\alpha}} \int_s^t \frac{1}{(r-\sigma)^{\beta}} |\Lambda y_{t,x,u}(\sigma) - \Lambda y_{s,x,u}(\sigma)| d\sigma.
\]
From the estimates obtained for (2.31), (2.32), (2.33), we deduce that the function \( r \mapsto (r-t)\alpha |\Lambda y_{t,x,u}(r) - \Lambda y_{s,x,u}(r)|_{X_0} \) belongs to \( L^\infty(t,T) \). Applying Theorem 2.5, we obtain

\[
|\Lambda y_{t,x,u}(r) - \Lambda y_{s,x,u}(r)|_{X_0} \leq C_5(\alpha, \beta) \left( r-t \right)^\alpha \tilde{a}(t, s, x),
\]

with

\[
C_5(\alpha, \beta) = 2C_\alpha \max \left( \frac{M_\alpha}{1-\alpha}, \frac{M_{\alpha+\beta}(M_F+\|B\|M_U)}{1-\beta} \right) \left( 1 + cC_\alpha M_{\beta+\alpha}K_FT^{1-\beta}e^{CT} \right),
\]

where \( c \) and \( C \) are given in Theorem 2.5, and

\[
\tilde{a}(t, s, x) = \left| e^{(t-s)A}x - x \right|_X + |t-s|^{1-\beta}.
\]

From (2.26) with \( \theta = 0 \), it yields:

\[
\frac{1}{(r-\sigma)^\beta \max(\beta, (r-t)^\beta)} \leq \frac{\Gamma(1-\beta)\Gamma(1-\alpha)}{\Gamma(2-(\alpha+\beta))}T^{1-(\alpha+\beta)} \leq 4T^{1-(\alpha+\beta)}.
\]

Hence

\[
|y_{t,x,u}(r) - y_{s,x,u}(r)|_{X} \leq \left( 1 + \frac{M_\beta(M_F+\|B\|M_U)}{1-\beta} + C_5(\alpha, \beta)K_FT^{1-(\alpha+\beta)} \right) \tilde{a}(t, s, x).
\]

The proof is complete.

**Proof of Proposition 2.9.** Consider the case where \( 0 \leq s < t \). We have:

\[
|y_{t,x,u}(r) - y_{s,x,u}(r)|_{X} \leq \left( 1 + \frac{M_\beta(M_F+\|B\|M_U)}{1-\beta} + C_5(\alpha, \beta)K_FT^{1-(\alpha+\beta)} \right) \tilde{a}(t, s, x).
\]

The proof is complete.

**Proposition 2.10.** Let \( x \) and \( x_0 \) be in \( X \), \( t \in [0,T) \), and \( u \in \mathcal{M}(t,T;U) \). Let us denote by \( y_{t,x,u} \) and \( y_{t,x_0,u} \) the solutions of equation (2.1) respectively corresponding to the initial data \( (t,x) \) and \( (t,x_0) \). Then, for all \( r \in [t,T] \) and all \( s \in [t,T] \), we have:

\[
|\Lambda y_{t,x,u}(s) - \Lambda y_{t,x_0,u}(s)|_{X_0} \leq C_7(\alpha, \beta) |x-x_0|_{X},
\]

and

\[
|y_{t,x,u}(r) - y_{t,x_0,u}(r)|_{X} \leq C_8(\alpha, \beta) |x-x_0|_{X}.
\]

**Proof.** The function \( w = y_{t,x,u} - y_{t,x_0,u} \) satisfies

\[
|\Lambda w(s)|_{X_0} \leq \frac{M_\alpha}{(s-t)^\alpha} |x-x_0|_X + \mathcal{H}(r, \Lambda y_{t,x,u}(r) - F(r, x_0, u)) \left[ F(r, \Lambda y_{t,x,u}(r)) - F(r, \Lambda y_{t,x_0,u}(r)) \right] \left( r \right) |x-x_0|_X.
\]
Estimate (2.37) now follows from Theorem 2.5. We can also obtain the estimate

$$|w(r)|_X \leq |x - x_0|_X + K_F \int_t^r \frac{1}{(r-s)\delta (s-t)\delta} |x - x_0|_X \, ds.$$  

By the same calculation as in the proof of Proposition 2.7, we have

$$|w(r)|_X \leq \left[ 1 + K_F C_\alpha T^{1-(\alpha+\beta)} \right] |x - x_0|_X,$$

and (2.38) is established. \(\square\)

3. Viscosity solutions and uniqueness result

In this section we study the uniqueness of solution to equation (1.1). It is well known that, by a change of variable in time, the terminal value problem (1.1) is equivalent the Hamilton-Jacobi-Bellman equation

$$\frac{\partial v}{\partial t}(t,x) - \left(D_x v(t,x) \big| Ax\right)_X + \left(( - A)^{2} D_x v(t,x) \big| F(t,\Lambda x)\right)_X + H(t,x,(-A)^{2} D_x v(t,x)) = 0, \quad \forall (t,x) \in [0,T] \times X, \quad (3.1)$$

$$v(0,x) = g(x).$$

Let \(C_A^1([0,T] \times X)\) be the set of all functions \(\Phi\) (called test functions) satisfying the following conditions:

(\(\alpha\)) \(\Phi \in C^1([0,T] \times X)\).

(\(\beta\)) \(D_x \Phi(\cdot,x)\) is constant (in \(t\)) and \(D_x \Phi(t,\cdot)\) is Lipschitz on \(X\), i.e.:

$$|D_x \Phi(t,x) - D_x \Phi(t,y)|_X \leq K_\Phi |x - y|_X.$$

(\(\gamma\)) For all \(\theta \in [0,1-\alpha[,\) \(D_x \Phi(t,x)\) belongs to \(D((-A)^{\theta})\) if and only if \(x \in D((-A)^{\theta})\).

(\(\delta\)) The mapping \(x \mapsto D_x \Phi(t,x)\) is continuous from \(D((-A)^{\frac{\theta}{2}})\) into itself.

Remark 3.1. Since \(D_x \Phi(t,x)\) does not depend on \(t\), from the last condition we can infer that the mapping \((t,x) \mapsto D_x \Phi(t,x)\) is continuous from \([0,T] \times D((-A)^{\frac{\theta}{2}})\) into \(D((-A)^{\frac{\theta}{2}})\).

Definition 3.2. Consider functions \(w\) satisfying:

(i) \(w \in C([0,T] \times X)\) and \(|w(t,x)| \leq M_w, \) for all \((t,x) \in [0,T] \times X\).

(ii) \(|w(t,x) - w(t,y)| \leq K_w \|x - y\|_X\) for all \(t \in [0,T]\), and all \(x, y \in X\).

(iii) \(|w(t,x) - w(t,y)| \leq C_{t,\theta} \left|(-A)^{-\theta} (x - y)\right|_X\), for all \(t \in [0,T]\), all \(x, y \in X\), and all \(\theta \in [0,1-\alpha[,\) where the constant \(C_{t,\theta}\) is bounded on all compact subset of \([0,T]\).

(iv) \(w(\cdot,x)\) is Hölder continuous in time of exponent \(0 < \eta \leq 1\), for all \(x \in D((-A)^{\frac{\theta}{2}})\). More precisely there exists a constant \(M_{1,w}\) such that:

$$|w(t,x) - w(s,x)| \leq M_{1,w} \left(1 + \left|(-A)^{\frac{\theta}{2}} x\right|_X\right) |t - s|^{\eta}.\$$

We say that a function \(w\) satisfying (i)-(iv) is a viscosity subsolution of (3.1) on \([0,T]\) if, for every \(\Phi \in C_A^1([0,T] \times X)\), the conditions (\(\alpha_1\)) and (\(\beta_1\)) are satisfied, where:

$$\frac{\partial \Phi}{\partial t}(t,x) + H \left(t,x,(-A)^{\beta} D_x \Phi(t,x)\right) + \left((-A)^{\frac{\theta}{2}} D_x \Phi(t,x) \big| (-A)^{\beta} \right)_X + \left((-A)^{\beta} D_x \Phi(t,x) \big| F(t,\Lambda x)\right)_X \leq 0$$

for all \((t,x) \in \left(0,T\right] \times D((-A)^{\frac{\theta}{2}})\) \(\cap \arg \max (w - \Phi),\) \(\square\)
(β₁) \[ \lim_{\ell \searrow 0} \sup_{x \in X} \left[ w(t, x) - g(\ell A x) \right]^+ = 0. \]

(Recall that \([f]^+ = \max(f, 0)\) and \([f]^− = \min(f, 0)\).)

We say that a function \(w\) satisfying (i)-(iv) is a viscosity supersolution of (3.1) on \([0, T]\) if, for every \(\Phi \in C_A^1([0, T] \times X)\), the two conditions (α₂) and (β₂) are satisfied, where:

\[
\frac{\partial \Phi}{\partial t}(t, x) + H(t, x, (-A)^\beta D_x \Phi(t, x)) + \left((-A)^\beta D_x \Phi(t, x) | F(t, \Lambda x)\right)_{X} \geq 0
\]

for all \((t, x) \in (0, T[ \times D((-A)^\beta)) \cap \arg \min (w - \Phi), \]

(β₂) \[ \lim_{\ell \searrow 0} \sup_{x \in X} \left[ w(t, x) - g(\ell A x) \right]^− = 0. \]

Finally, \(w\) is a viscosity solution of (3.1) if it is both a subsolution and a supersolution of equation (3.1).

**Remark 3.3.** If \((t, x)\) belongs to \([0, T[ \times D((-A)^\beta)),\) then \(\Lambda x\) is well defined and \(F(t, \Lambda x)\) is meaningful.

**Remark 3.4.** If in place of equation (3.1) we consider equation (1.1), the conditions (α₁), (α₂), (β₁), and (β₂) have to be modified accordingly (see Sect. 4).

**Theorem 3.5.** Assume that (i) – (vii) of Section 2 hold. Let \(w\) be a viscosity subsolution and \(v\) be a viscosity supersolution of the Hamilton-Jacobi-Bellman equation (3.1). Then

\[ w(t, x) \leq v(t, x) \quad \text{for all } (t, x) \in [0, T[ \times X. \]

(3.2)

Before proving this theorem let us state a useful lemma.

**Lemma 3.6.** Assume that \(\varphi\) and \(\psi \in C_A^1([0, T[ \times X)\), and let \(w\) and \(v\) be two continuous functions in \([0, T[ \times X\). If

\[
(t_0, x_0) \in \arg \max_\mathcal{O}(w - \varphi) \quad \text{and} \quad (s_0, y_0) \in \arg \min_\mathcal{O}(v - \psi),
\]

where \(\mathcal{O}\) is an open set of \([0, T[ \times X\), then

\[
D_x \varphi(t_0, x_0) \subset D_x^+ w(t_0, x_0) \quad \text{and} \quad D_x \psi(s_0, y_0) \subset D_x^+ v(s_0, y_0).
\]

(3.3)

**Proof.** We establish the result only for the function \(\varphi\). Due to (3.3), for all \(x \in X\), we have:

\[ w(t_0, x) - w(t_0, x_0) - [\varphi(t_0, x) - \varphi(t_0, x_0)] \leq 0. \]

With the condition (β) in the definition of \(C_A^1([0, T[ \times X)\), we have:

\[ |\varphi(t_0, x) - \varphi(t_0, x_0) - (D_x \varphi(t_0, x_0) \mid x - x_0)_X| \leq K_\varphi|x - x_0|^2. \]

Combining this estimate with the previous inequality, we obtain:

\[ \limsup_{|x - x_0| \to 0} \frac{w(t_0, x) - w(t_0, x_0) - (D_x \varphi(t_0, x_0) \mid x - x_0)_X}{|x - x_0|} \leq 0. \]

Let us recall Young’s inequality. For all \(p, q > 1\) such that \(\frac{1}{p} + \frac{1}{q} = 1\), we have:

\[ ab \leq \frac{\lambda^p}{p} a^p + \frac{1}{q \lambda^q} b^q \quad \text{for all } \lambda > 0, \quad \text{and all } a, b \geq 0. \]

(3.5)
Proof of Theorem 3.5. We are going to use the same kind of proof as in [6]. The proof is divided in five steps.

Step 1. Since \( w, v \in C([0,T] \times X) \), it is enough to prove that

\[
w_{\sigma}(t, x) \leq v_{\sigma}(t, x) \text{ for all } (t, x) \in [0, T[ \times X \text{ and all } \sigma > 0, \quad (3.6)
\]

where

\[
w_{\sigma}(t, x) = w(t, x) - \frac{\sigma}{T-t} \quad \text{and} \quad v_{\sigma}(t, x) = v(t, x) + \frac{\sigma}{T-t}.
\]

As \( w \) is a subsolution of (3.1), then \( w_{\sigma} \) is a subsolution of

\[
\frac{\partial w_{\sigma}}{\partial t}(t, x) - (D_x w_{\sigma}(t, x) \mid Ax)_X + ((-A)^2 D_x w_{\sigma}(t, x) \mid F(t, Ax))_X \\
+ H(t, x, (-A)^2 D_x w_{\sigma}(t, x)) = -\frac{\sigma}{(T-t)^2} \leq -\frac{\sigma}{T^2},
\]

(3.7)

Similarly, \( v_{\sigma} \) is a supersolution of

\[
\frac{\partial v_{\sigma}}{\partial t}(t, x) - (D_x v_{\sigma}(t, x) \mid Ax)_X + ((-A)^2 D_x v_{\sigma}(t, x) \mid F(t, Ax))_X \\
+ H(t, x, (-A)^2 D_x v_{\sigma}(t, x)) = \frac{\sigma}{(T-t)^2} \geq \frac{\sigma}{T^2},
\]

(3.8)

Step 2. Let \( 0 < \eta \leq 1 \) be an exponent such that \( v(\cdot, x) \) and \( w(\cdot, x) \) be Hölder continuous of exponent \( \eta \) (Condition (iv) in Def. 3.2). We set

\[
\bar{\eta} = \min(\eta, \eta_1, \eta_2). \quad (3.9)
\]

Let \( \varepsilon \) and \( \mu \) be in \([0, 1] \). For all \((t, x)\) and all \((s, y)\) in \([0, T[ \times X\), we define

\[
\Phi_{\varepsilon, \mu}(t, s, x, y) = w_{\sigma}(t, x) - v_{\sigma}(s, y) - \frac{1}{2\varepsilon} \left( (-A)^{-1}(x - y) \mid (x - y) \right)_X - \frac{(t-s)^2}{2\varepsilon^2} - \frac{\mu}{2} \left( |x|^2_X + |y|^2_X \right).
\]

Let \( \tau \) be a small parameter satisfying \( 0 < \tau < \frac{T}{2} \), and set \( Q_\tau = [\tau, T - \tau] \times X \). From condition (iii) in Definition 3.2, \( w \) and \( v \) are weakly continuous in \( Q_\tau \). Thus \( v_{\sigma} \) and \( w_{\sigma} \) are weakly continuous in \( Q_\tau \). Moreover the mapping

\[
(x, y) \mapsto \frac{1}{2\varepsilon} \left( (-A)^{-1}(x - y) \mid (x - y) \right)_X + \frac{\mu}{2} |x|^2_X + \frac{\mu}{2} |y|^2_X
\]

is convex, and continuous (for the strong topology of \( X \times X \)). Therefore \( \Phi_{\varepsilon, \mu} \) is weakly lower semicontinuous in \( Q_\tau^2 \). Besides for all couples \((t, x), (s, y) \in Q_\tau \)

\[
\Phi_{\varepsilon, \mu}(t, s, x, y) \leq M_w + M_v - \frac{\mu}{2} \left( |x|^2_X + |y|^2_X \right) \rightarrow -\infty \text{ when } \max(|x|_X, |y|_X) \rightarrow +\infty.
\]

Thus there exists \((t_{\varepsilon, \mu}, x_{\varepsilon, \mu}, s_{\varepsilon, \mu}, y_{\varepsilon, \mu}) \in Q_\tau^2 \) such that

\[
\Phi_{\varepsilon, \mu}(t_{\varepsilon, \mu}, s_{\varepsilon, \mu}, x_{\varepsilon, \mu}, y_{\varepsilon, \mu}) = \max_{Q_\tau \times Q_\tau} \Phi_{\varepsilon, \mu}.
\]

Let us verify that if \( \tau \) is small enough \((0 < \tau < \tau_\sigma) \), then \( t_{\varepsilon, \mu}, s_{\varepsilon, \mu}, < T - \tau \). Indeed, from the inequality

\[
w_{\sigma}(t, x) - v_{\sigma}(s, y) \leq M_w + M_v - \frac{\sigma}{T-t} - \frac{\sigma}{T-s},
\]
it follows that
\[
\lim_{\max(s,t)\to T} (w_\sigma(t,x) - v_\sigma(s,y)) = -\infty \text{ uniformly w.r. to } x \text{ and } y.
\]

Hence we have shown that there exists \((t_{\varepsilon,\mu}, x_{\varepsilon,\mu}, s_{\varepsilon,\mu}, y_{\varepsilon,\mu}) \in ([\tau, T - \tau] \times X)^2\) such that
\[
\Phi_{\varepsilon,\mu} (t_{\varepsilon,\mu}, s_{\varepsilon,\mu}, x_{\varepsilon,\mu}, y_{\varepsilon,\mu}) = \max_{Q_{\varepsilon,\tau}} \Phi_{\varepsilon,\mu}.
\]

**Step 3.** We are going to obtain some a priori estimates on \(t_{\varepsilon,\mu}, s_{\varepsilon,\mu}, x_{\varepsilon,\mu}, y_{\varepsilon,\mu}\). Since
\[
\Phi_{\varepsilon,\mu} (t_{\varepsilon,\mu}, t_{\varepsilon,\mu}, x_{\varepsilon,\mu}, x_{\varepsilon,\mu}) + \Phi_{\varepsilon,\mu} (s_{\varepsilon,\mu}, s_{\varepsilon,\mu}, y_{\varepsilon,\mu}, y_{\varepsilon,\mu}) \leq 2 \Phi_{\varepsilon,\mu} (t_{\varepsilon,\mu}, s_{\varepsilon,\mu}, x_{\varepsilon,\mu}, y_{\varepsilon,\mu}),
\]
then
\[
w_\sigma (t_{\varepsilon,\mu}, x_{\varepsilon,\mu}) - v_\sigma (t_{\varepsilon,\mu}, x_{\varepsilon,\mu}) + w_\sigma (s_{\varepsilon,\mu}, y_{\varepsilon,\mu}) - v_\sigma (s_{\varepsilon,\mu}, y_{\varepsilon,\mu}) \leq 2 \left( w_\sigma (t_{\varepsilon,\mu}, x_{\varepsilon,\mu}) - v_\sigma (s_{\varepsilon,\mu}, y_{\varepsilon,\mu}) \right) - \frac{1}{\varepsilon} \left( (-A)^{-1} (x_{\varepsilon,\mu} - y_{\varepsilon,\mu}) \right)_X - \frac{(t_{\varepsilon,\mu} - s_{\varepsilon,\mu})^2}{\varepsilon^2}.
\]

Consequently we obtain
\[
\frac{1}{\varepsilon} \left( (-A)^{-1} (x_{\varepsilon,\mu} - y_{\varepsilon,\mu}) \right)_X + \frac{(t_{\varepsilon,\mu} - s_{\varepsilon,\mu})^2}{\varepsilon^2} \leq w_\sigma (t_{\varepsilon,\mu}, x_{\varepsilon,\mu}) - v_\sigma (s_{\varepsilon,\mu}, y_{\varepsilon,\mu}) + v_\sigma (t_{\varepsilon,\mu}, x_{\varepsilon,\mu}) - v_\sigma (s_{\varepsilon,\mu}, y_{\varepsilon,\mu}) \leq 2 (M_w + M_v) = C.
\]

We deduce that
\[
|t_{\varepsilon,\mu} - s_{\varepsilon,\mu}| \leq C \varepsilon^\frac{1}{4} \quad (3.11)
\]
\[
\left( (-A)^{-1} (x_{\varepsilon,\mu} - y_{\varepsilon,\mu}) \right)_X \leq C \sqrt{\varepsilon}. \quad (3.12)
\]

Now let us show that
\[
\lim_{\mu \to 0} \mu \left( \frac{|x_{\varepsilon,\mu}|^2}{X} + \frac{|y_{\varepsilon,\mu}|^2}{X} \right) = 0, \quad (3.13)
\]
where \(0 < \varepsilon(\mu) < 1\) is any function of \(\mu\). For all \(x \in X\), we have:
\[
\Phi_{\varepsilon,\mu} (t_{\varepsilon,\mu}, t_{\varepsilon,\mu}, x, x) \leq \Phi_{\varepsilon,\mu} (t_{\varepsilon,\mu}, s_{\varepsilon,\mu}, x_{\varepsilon,\mu}, y_{\varepsilon,\mu}),
\]
i.e.
\[
w_\sigma (t_{\varepsilon,\mu}, x) - v_\sigma (t_{\varepsilon,\mu}, x) - \mu |x|^2_X \leq w_\sigma (t_{\varepsilon,\mu}, x_{\varepsilon,\mu}) - v_\sigma (s_{\varepsilon,\mu}, y_{\varepsilon,\mu}) - \frac{1}{2\varepsilon} \left( (-A)^{-1} (x_{\varepsilon,\mu} - y_{\varepsilon,\mu}) \right)_X - \frac{(t_{\varepsilon,\mu} - s_{\varepsilon,\mu})^2}{\varepsilon^2} - \frac{\mu}{2} \left( \frac{|x_{\varepsilon,\mu}|^2}{X} + \frac{|y_{\varepsilon,\mu}|^2}{X} \right).
\]

By taking \(x = 0\), we deduce that
\[
\frac{\mu}{2} \left( \frac{|x_{\varepsilon,\mu}|^2}{X} + \frac{|y_{\varepsilon,\mu}|^2}{X} \right) \leq 2 (M_w + M_v).
\]

Thus, \(\frac{\mu}{2} \left( \frac{|x_{\varepsilon,\mu}|^2}{X} + \frac{|y_{\varepsilon,\mu}|^2}{X} \right)\) is bounded independently of \(\varepsilon \in [0, 1]\) and \(\mu \in [0, 1]\). We are going to use this property to prove (3.13).
We define now $\Psi_\varepsilon$ on $Q_\tau \times Q_\tau$ by:

$$
\Psi_\varepsilon (t, s, x, y) = w_\sigma (t, x) - v_\sigma (s, y) + \frac{1}{2\varepsilon} \left( A^{-1} (x - y) \right)_X - \frac{(t - s)^2}{2\varepsilon^4}.
$$

The function $\Psi_\varepsilon$ is bounded from above (independently of $\varepsilon$):

$$
\Psi_\varepsilon \leq M_w + M_v,
$$

and upper semicontinuous. So, for every $\delta > 0$, there exists $(t, s, x, y) \in Q_\tau^2$ such that

$$
\Psi_\varepsilon (t, s, x, y) \geq \Psi_\varepsilon (t, s, x, y) - \delta \quad \forall (t, s, x, y) \in Q_\tau.
$$

The point $(t, s, x, y)$ corresponds to a supremum (and a priori not a maximum), because we have not proved that $\Psi_\varepsilon \to -\infty$ when $\max (|x|_X, |y|_X) \to \infty$. Starting from the inequality

$$
\Phi_{\varepsilon, \mu} (t, s, x, y, \mu, \epsilon) \geq \Phi_{\varepsilon, \mu} (t, s, x, y, \mu, \epsilon),
$$

it follows that

$$
\Psi_\varepsilon (t, s, x, y) - \frac{\mu}{2} \left( |x_{\varepsilon, \mu}|_X^2 + |y_{\varepsilon, \mu}|_X^2 \right) \geq \Psi_\varepsilon (t, s, x, y, \mu) - \frac{\mu}{2} \left( |x_{\varepsilon, \delta}|_X^2 + |y_{\varepsilon, \delta}|_X^2 \right) \geq \Psi_\varepsilon (t, s, x, y, \mu) - \delta - \frac{\mu}{2} \left( |x_{\varepsilon, \delta}|_X^2 + |y_{\varepsilon, \delta}|_X^2 \right).
$$

Hence we have

$$
\frac{\mu}{2} \left( |x_{\varepsilon, \delta}|_X^2 + |y_{\varepsilon, \delta}|_X^2 \right) \leq \frac{\mu}{2} \left( |x_{\varepsilon, \delta}|_X^2 + |y_{\varepsilon, \delta}|_X^2 \right) + \delta \quad \text{for all } \delta > 0.
$$

This inequality is satisfied for all $\varepsilon > 0$. In particular if $0 < \varepsilon (\mu) < 1$ is a function of $\mu$, we can write

$$
\frac{\mu}{2} \left( |x_{\varepsilon (\mu), \delta}|_X^2 + |y_{\varepsilon (\mu), \delta}|_X^2 \right) \leq \frac{\mu}{2} \left( |x_{\varepsilon (\mu), \delta}|_X^2 + |y_{\varepsilon (\mu), \delta}|_X^2 \right) + \delta \leq \mu^2 \delta (M_w + M_v) + \delta.
$$

Thus

$$
\limsup_{\mu \to 0} \frac{\mu}{2} \left( |x_{\varepsilon (\mu), \mu}|_X^2 + |y_{\varepsilon (\mu), \mu}|_X^2 \right) = \delta.
$$

We take the limit when $\delta \to 0$ to obtain (3.13).

Let us prove that

$$
\frac{1}{2\varepsilon} \left( (A)^{-1/2} (x_{\varepsilon, \mu} - y_{\varepsilon, \mu}) \right)_X^2 \leq \frac{|x_{\varepsilon, \mu} - y_{\varepsilon, \mu}|_X^2}{8\varepsilon \lambda} + 2M^2 \varepsilon \lambda + \frac{\mu}{2} |x_{\varepsilon, \mu}|_X^2,
$$

for all constant $\lambda > 0$. From the inequality

$$
\Phi_{\varepsilon, \mu} (t, s, x, y, \mu, \epsilon) \leq \Phi_{\varepsilon, \mu} (t, s, x, y, \mu, \epsilon),
$$

we deduce

$$
\frac{1}{2\varepsilon} \left( (A)^{-1} (x_{\varepsilon, \mu} - y_{\varepsilon, \mu}) \right)_X \leq \varepsilon \sigma (s_{\varepsilon, \mu}, x_{\varepsilon, \mu}) - \varepsilon \sigma (s_{\varepsilon, \mu}, x_{\varepsilon, \mu}) + \frac{\mu}{2} |x_{\varepsilon, \mu}|_X^2.
$$
With (ii) in Definition 3.2 we have:

$$\frac{1}{\varepsilon} \left| (-A)\frac{1}{\varepsilon} (x_{\varepsilon,\mu} - y_{\varepsilon,\mu}) \right|^2_X \leq 2M_v |x_{\varepsilon,\mu} - y_{\varepsilon,\mu}| + \mu |x_{\varepsilon,\mu}|^2_X.$$ 

Estimate (3.14) follows from Young’s inequality.

**Step 4.** We are going to show that \( \min(t_{\varepsilon,\mu}, s_{\varepsilon,\mu}) = \tau. \) If it is not true, then \( t_{\varepsilon,\mu} > \tau \) and \( s_{\varepsilon,\mu} > \tau. \) Let \( \varphi \) and \( \psi \) be two mappings defined by:

\[
\begin{align*}
\varphi(t, x) &= v_\sigma(s_{\varepsilon,\mu}, y_{\varepsilon,\mu}) + \frac{1}{2\varepsilon} \left((-A)^{-1} (x - y_{\varepsilon,\mu}) \right)_X + \frac{(t - s_{\varepsilon,\mu})^2}{2\varepsilon^2} + \frac{\mu}{2} \left( |x|^2_X + |y_{\varepsilon,\mu}|^2_X \right) \\
\psi(t, y) &= w_\sigma(t_{\varepsilon,\mu}, x_{\varepsilon,\mu}) - \frac{1}{2\varepsilon} \left((-A)^{-1} (x_{\varepsilon,\mu} - y) \right)_X - \frac{(t_{\varepsilon,\mu} - s)^2}{2\varepsilon^2} - \frac{\mu}{2} \left( |x_{\varepsilon,\mu}|^2_X + |y|^2_X \right).
\end{align*}
\]

The mappings \( \varphi, \psi \) belong to \( C^1([0, T] \times X). \) Indeed

(i) \( \varphi \in C^1([0, T] \times X) \) and \( D_x \varphi(t, x) = \frac{1}{\varepsilon} (-A)^{-1} (x - y_{\varepsilon,\mu}) + \mu x. \)

(ii) \( D_x \varphi(\cdot, x) \) is constant in \( t \), and \( x \mapsto D_x \varphi(t, x) \) is Lipschitz from \( X \) into \( X \) because \( (-A)^{-1} \) is a linear and continuous operator from \( X \) into \( X \):

\[
|D_x \varphi(t, x) - D_x \varphi(t, y)|^2_X \leq 2 \left( \frac{1}{\varepsilon} |A^{-1} (x - y)|^2_X + \mu |x - y|^2_X \right) \leq 2 \left( \mu + \frac{C}{\varepsilon} \right) |x - y|^2_X.
\]

(iii) It is clear that for all \( \theta \in [0, 1 - \alpha], \) \( D_x \varphi(t, x) \in D((-A)^\theta) \Leftrightarrow x \in D((-A)\theta). \)

(iv) Moreover the mapping \( x \mapsto D_x \varphi(t, x) \) is continuous from \( D((-A)^{1/2}) \) into itself.

The mappings \( \varphi, \psi \) have been chosen to satisfy:

\[
(t_{\varepsilon,\mu}, x_{\varepsilon,\mu}) \in \arg \max_{Q_{\varepsilon}} (w_\sigma - \varphi), \quad \text{and} \quad (s_{\varepsilon,\mu}, y_{\varepsilon,\mu}) \in \arg \min_{Q_{\varepsilon}} (v_\sigma - \psi).
\]

Since \( v_\sigma \) and \( w_\sigma \) satisfy (iii) in Definition 3.2, with [5, Cor. 3.4], \( D_x^+ v_\sigma(t_{\varepsilon,\mu}, x_{\varepsilon,\mu}) \) and \( D_x^- v_\sigma(s_{\varepsilon,\mu}, y_{\varepsilon,\mu}) \) are included in \( D((-A)^\theta) \) for all \( \theta \in [0, \frac{1}{2}]. \) Due to Lemma 3.6, \( D_x^+ \varphi(t_{\varepsilon,\mu}, x_{\varepsilon,\mu}) \subset D_x^+ w_\sigma(t_{\varepsilon,\mu}, x_{\varepsilon,\mu}) \) and \( D_x^- \psi(s_{\varepsilon,\mu}, y_{\varepsilon,\mu}) \subset D_x^- v_\sigma(s_{\varepsilon,\mu}, y_{\varepsilon,\mu}) \). Hence, we have

\[
D_x^+ \varphi(t_{\varepsilon,\mu}, x_{\varepsilon,\mu}), \quad D_x^- \psi(s_{\varepsilon,\mu}, y_{\varepsilon,\mu}) \subset D((-A)^\theta) \text{ for all } \theta \in [0, 1 - \alpha].
\]

Therefore \( x_{\varepsilon,\mu} \) and \( y_{\varepsilon,\mu} \) belong to \( D((-A)^\theta) \) for all \( \theta \in [0, 1 - \alpha], \) and in particular \( x_{\varepsilon,\mu}, y_{\varepsilon,\mu} \subset D((-A)^{\frac{1}{2}}). \)

Since \( w_\sigma \) is a viscosity subsolution of (3.7), \( (t_{\varepsilon,\mu}, x_{\varepsilon,\mu}) \) belongs to \( \left( [0, T] \times D((-A)^{1/2}) \right) \cap \arg \max (w_\sigma - \varphi), \) we have:

\[
\begin{align*}
\frac{t_{\varepsilon,\mu} - s_{\varepsilon,\mu}}{\varepsilon} + H \left( t_{\varepsilon,\mu}, x_{\varepsilon,\mu}, (-A)^\beta \left( \frac{1}{\varepsilon} (-A)^{-1} (x_{\varepsilon,\mu} - y_{\varepsilon,\mu}) + \mu x_{\varepsilon,\mu} \right) \right) \\
+ \left((-A)^{\frac{1}{2}} \left( \frac{1}{\varepsilon} (-A)^{-1} (x_{\varepsilon,\mu} - y_{\varepsilon,\mu}) + \mu x_{\varepsilon,\mu} \right) \right)_X \\
+ \left((-A)^\beta \left( \frac{1}{\varepsilon} (-A)^{-1} (x_{\varepsilon,\mu} - y_{\varepsilon,\mu}) + \mu x_{\varepsilon,\mu} \right) \right)_X
\end{align*}
\]

\[
\leq \frac{\sigma}{T^2}.
\]
In the same way, since \( v_\sigma \) is a viscosity supersolution of (3.8), with (3.15) we have:

\[
\frac{t_{\varepsilon,\mu} - s_{\varepsilon,\mu}}{\varepsilon^2} + H \left( s_{\varepsilon,\mu}, y_{\varepsilon,\mu}, (-A)^{\beta} \left( \frac{1}{\varepsilon} (-A)^{-1} (x_{\varepsilon,\mu} - y_{\varepsilon,\mu}) - \mu y_{\varepsilon,\mu} \right) \right) 
+ \left( (-A)^{\frac{1}{\varepsilon}} \left( \frac{1}{\varepsilon} (-A)^{-1} (x_{\varepsilon,\mu} - y_{\varepsilon,\mu}) - \mu y_{\varepsilon,\mu} \right) \right) X 
+ \left( (-A)^{\beta} \left( \frac{1}{\varepsilon} (-A)^{-1} (x_{\varepsilon,\mu} - y_{\varepsilon,\mu}) + \mu y_{\varepsilon,\mu} \right) \right) X 
\geq \frac{\sigma}{T^2}.
\]

Subtracting the previous two inequalities we obtain

\[
H \left( t_{\varepsilon,\mu}, x_{\varepsilon,\mu}, (-A)^{\beta} \left( \frac{1}{\varepsilon} (-A)^{-1} (x_{\varepsilon,\mu} - y_{\varepsilon,\mu}) + \mu x_{\varepsilon,\mu} \right) \right) 
- H \left( s_{\varepsilon,\mu}, y_{\varepsilon,\mu}, (-A)^{\beta} \left( \frac{1}{\varepsilon} (-A)^{-1} (x_{\varepsilon,\mu} - y_{\varepsilon,\mu}) - \mu y_{\varepsilon,\mu} \right) \right) 
+ \left( (-A)^{\frac{1}{\varepsilon}} \left( \frac{1}{\varepsilon} (-A)^{-1} (x_{\varepsilon,\mu} - y_{\varepsilon,\mu}) - \mu y_{\varepsilon,\mu} \right) \right) X 
+ \mu \left( (-A)^{\beta} \frac{1}{\varepsilon} (-A)^{-1} (x_{\varepsilon,\mu} - y_{\varepsilon,\mu}) \right) X 
+ \mu \left( (-A)^{\beta} x_{\varepsilon,\mu} \right) X 
- \mu \left( (-A)^{\beta} y_{\varepsilon,\mu} \right) X 
\leq -\frac{2\sigma}{T^2}.
\]

Thus we have

\[
\mu \left( (-A)^{\frac{1}{\varepsilon}} x_{\varepsilon,\mu} \right)^2 _X + \mu \left( (-A)^{\frac{1}{2}} y_{\varepsilon,\mu} \right)^2 _X + \frac{1}{\varepsilon} \left( x_{\varepsilon,\mu} - y_{\varepsilon,\mu} \right)^2 _X \leq \frac{2\sigma}{T^2} + K_H \left( \|t_{\varepsilon,\mu} - s_{\varepsilon,\mu}\|_2 + \|x_{\varepsilon,\mu} - y_{\varepsilon,\mu}\|_X + \mu \left( (-A)^{\beta} (x_{\varepsilon,\mu} + y_{\varepsilon,\mu}) \right) \right) \|X \right) \]

\[
+ \frac{1}{\varepsilon} \left( \left| (-A)^{\beta-1} (x_{\varepsilon,\mu} - y_{\varepsilon,\mu}) \right|_X \right) \left| F (t_{\varepsilon,\mu}, \Lambda x_{\varepsilon,\mu}) - F (s_{\varepsilon,\mu}, \Lambda y_{\varepsilon,\mu}) \right|_X \right) \]

\[
+ \mu \left( (-A)^{\beta} x_{\varepsilon,\mu} \right) X \left| F (t_{\varepsilon,\mu}, \Lambda x_{\varepsilon,\mu}) \right|_X + \mu \left( (-A)^{\beta} y_{\varepsilon,\mu} \right) X \left| F (s_{\varepsilon,\mu}, \Lambda y_{\varepsilon,\mu}) \right|_X. \]

Estimates of (3.17)–(3.19):

Estimate of (3.17). With (3.11) and Young’s inequality we can write

\[
|t_{\varepsilon,\mu} - s_{\varepsilon,\mu}|^2 \leq C \varepsilon^2 \leq C \varepsilon \quad \text{and} \quad K_H \left| x_{\varepsilon,\mu} - y_{\varepsilon,\mu} \right|_X \leq 2\varepsilon K_H^2 + \frac{\left| x_{\varepsilon,\mu} - y_{\varepsilon,\mu} \right|_X^2}{8\varepsilon}. \]
For all $\sigma > 0$, with (2.13), we have:

$$
\mu K_H \left| (-A)^{\beta} (x_{\varepsilon,\mu} + y_{\varepsilon,\mu}) \right|_X \leq \mu K_H \left[ \left| (-A)^{\beta} x_{\varepsilon,\mu} \right|_X + \left| (-A)^{\beta} y_{\varepsilon,\mu} \right|_X \right]
$$

$$
\leq \mu K_H \left[ \sigma \left( \left| (-A)^{\frac{1}{2}} x_{\varepsilon,\mu} \right|_X + \left| (-A)^{\frac{1}{2}} y_{\varepsilon,\mu} \right|_X \right) + C_{\beta,\sigma} \left( \left| x_{\varepsilon,\mu} \right|_X + \left| y_{\varepsilon,\mu} \right|_X \right) \right]
$$

$$
\leq \mu K_H \left[ \frac{1}{2} \sigma \left( \left| (-A)^{\frac{1}{2}} x_{\varepsilon,\mu} \right|_X^2 + \left| (-A)^{\frac{1}{2}} y_{\varepsilon,\mu} \right|_X^2 + 2 \right) + C_{\beta,\sigma} \left( \left| x_{\varepsilon,\mu} \right|_X + \left| y_{\varepsilon,\mu} \right|_X \right) \right].
$$

Choosing $\sigma = \frac{1}{K_H}$ we have:

$$
\mu K_H \left| (-A)^{\beta} (x_{\varepsilon,\mu} + y_{\varepsilon,\mu}) \right|_X \leq \mu \left[ \frac{1}{2} \left| (-A)^{\frac{1}{2}} x_{\varepsilon,\mu} \right|_X^2 + \frac{1}{2} \left| (-A)^{\frac{1}{2}} y_{\varepsilon,\mu} \right|_X^2 + 1 + K_H C_{\beta,1/K_H} \left( \left| x_{\varepsilon,\mu} \right|_X + \left| y_{\varepsilon,\mu} \right|_X \right) \right].
$$

With (3.20) and (3.21), we have:

$$
(3.17) \leq K_H C \varepsilon + 2 \varepsilon K_H^2 + \frac{|x_{\varepsilon,\mu} - y_{\varepsilon,\mu}|^2}{8 \varepsilon} \varepsilon / \varepsilon \varepsilon \varepsilon \varepsilon \varepsilon
$$

$$
+ \mu \left[ \frac{1}{2} \left| (-A)^{\frac{1}{2}} x_{\varepsilon,\mu} \right|_X^2 + \frac{1}{2} \left| (-A)^{\frac{1}{2}} y_{\varepsilon,\mu} \right|_X^2 + 1 + K_H C_{\beta,1/K_H} \left( \left| x_{\varepsilon,\mu} \right|_X + \left| y_{\varepsilon,\mu} \right|_X \right) \right].
$$

**Estimate of (3.18).** We first write

$$
(3.18) \leq \frac{1}{\varepsilon} \left( (-A)^{\beta - \frac{1}{2}} \left| (-A)^{\frac{1}{2}} (x_{\varepsilon,\mu} - y_{\varepsilon,\mu}) \right|_X \left| F (t_{\varepsilon,\mu}, \Lambda x_{\varepsilon,\mu}) - F (s_{\varepsilon,\mu}, \Lambda y_{\varepsilon,\mu}) \right|_X. \right)
$$

From (2.2) and (2.4), it yields

$$
|F (t_{\varepsilon,\mu}, \Lambda x_{\varepsilon,\mu}) - F (s_{\varepsilon,\mu}, \Lambda y_{\varepsilon,\mu})|_X \leq M_1 F (1 + |\Lambda x_{\varepsilon,\mu}|_X^0) \left| t_{\varepsilon,\mu} - s_{\varepsilon,\mu} \right|_{\eta_1}^1 + K_F |\Lambda x_{\varepsilon,\mu} - \Lambda y_{\varepsilon,\mu}|_X^0
$$

$$
\leq M_1 F \left( 1 + C_{1/2} \left| (-A)^{\frac{1}{2}} x_{\varepsilon,\mu} \right|_X \right) \left| t_{\varepsilon,\mu} - s_{\varepsilon,\mu} \right|_{\eta_1}^1 + K_F C_{\alpha} \left| (-A)^{\alpha} (x_{\varepsilon,\mu} - y_{\varepsilon,\mu}) \right|_X.
$$

Then, choosing $\alpha_0 > 0$ such that $\alpha + \alpha_0 < \frac{1}{2}$, we obtain

$$
(3.18) \leq C \frac{\left| (-A)^{-\frac{1}{2}} (x_{\varepsilon,\mu} - y_{\varepsilon,\mu}) \right|_X \left| (-A)^{\alpha} (x_{\varepsilon,\mu} - y_{\varepsilon,\mu}) \right|_X}{\varepsilon^{1-\alpha_0}}
$$

$$
+ C \frac{\left| (-A)^{-\frac{1}{2}} (x_{\varepsilon,\mu} - y_{\varepsilon,\mu}) \right|_X \left( 1 + \left| (-A)^{\frac{1}{2}} x_{\varepsilon,\mu} \right|_X \right) \left| t_{\varepsilon,\mu} - s_{\varepsilon,\mu} \right|_{\eta_1}^1}{\varepsilon^{1-\alpha_0}}.
$$

**Estimate of (3.23).** We first estimate the factor $\left| (-A)^{-\frac{1}{2}} (x_{\varepsilon,\mu} - y_{\varepsilon,\mu}) \right|_X / \varepsilon^{1-\alpha_0}$. From inequality (3.10) it follows that

$$
\frac{1}{\varepsilon} \left| (-A)^{-\frac{1}{2}} (x_{\varepsilon,\mu} - y_{\varepsilon,\mu}) \right|_X^2 \leq w \left( t_{\varepsilon,\mu}, x_{\varepsilon,\mu} \right) - w \left( s_{\varepsilon,\mu}, y_{\varepsilon,\mu} \right) + v \left( t_{\varepsilon,\mu}, x_{\varepsilon,\mu} \right) - v \left( s_{\varepsilon,\mu}, y_{\varepsilon,\mu} \right).
$$

As $t_{\varepsilon,\mu}, s_{\varepsilon,\mu} \in [\tau, T - \tau]$ and $[\tau, T - \tau]$ is compact, with properties (iii) and (iv) in Definition 3.2, setting $C(v, w) = M_1 + M_1 + C(\tau)$ and $C(\tau) = \sup_{t \in [\tau, T]} C_{t,1/2,v} + C_{t,1/2,w}$, we have

$$
\frac{1}{\varepsilon} \left| (-A)^{-\frac{1}{2}} (x_{\varepsilon,\mu} - y_{\varepsilon,\mu}) \right|_X^2 \leq 2 C(v, w) \left( 1 + \left| (-A)^{\frac{1}{2}} x_{\varepsilon,\mu} \right|_X \right) \left| t_{\varepsilon,\mu} - s_{\varepsilon,\mu} \right|_{\eta_1}^1 + C(\tau) \left| (-A)^{-\frac{1}{2}} (x_{\varepsilon,\mu} - y_{\varepsilon,\mu}) \right|_X.
$$
As
\[ C(\tau) \left| \left( -A \right)^{\frac{\nu}{2}} (x_{\varepsilon,\mu} - y_{\varepsilon,\mu}) \right|_X \leq \frac{\left| \left( -A \right)^{\frac{\nu}{2}} (x_{\varepsilon,\mu} - y_{\varepsilon,\mu}) \right|_X^2}{2\varepsilon} + \varepsilon C(\tau)^2, \]
then
\[ \frac{1}{\varepsilon} \left| \left( -A \right)^{\frac{\nu}{2}} (x_{\varepsilon,\mu} - y_{\varepsilon,\mu}) \right|_X^2 \leq 4C(v, w) \left( 1 + \left| \left( -A \right)^{\frac{\nu}{2}} x_{\varepsilon,\mu} \right|_X \right) |t_{\varepsilon,\mu} - s_{\varepsilon,\mu}|^\eta + \varepsilon C(\tau)^2. \]

Hence, we obtain:
\[ \frac{1}{\varepsilon^{1+\alpha_0}} \left| \left( -A \right)^{\frac{\nu}{2}} (x_{\varepsilon,\mu} - y_{\varepsilon,\mu}) \right|_X \leq 2 \sqrt{C(v, w)} \left( 1 + \left| \left( -A \right)^{\frac{\nu}{2}} x_{\varepsilon,\mu} \right|_X \right) \frac{|t_{\varepsilon,\mu} - s_{\varepsilon,\mu}|^\eta}{\varepsilon^{\frac{1}{2} - \alpha_0}} + \varepsilon^{\alpha_0} C(\tau). \]

We now estimate the factor \(|(-A)^\alpha (x_{\varepsilon,\mu} - y_{\varepsilon,\mu})|_X / \varepsilon^{\alpha_0} \). With (2.12), we have:
\[ \frac{1}{\varepsilon^{\alpha_0}} \left| (-A)^\alpha (x_{\varepsilon,\mu} - y_{\varepsilon,\mu}) \right|_X \leq M_{\alpha,1/2} \left| (-A)^{\frac{\nu}{2}} (x_{\varepsilon,\mu} - y_{\varepsilon,\mu}) \right|_X \left| x_{\varepsilon,\mu} - y_{\varepsilon,\mu} \right|^{1-2\alpha}/\varepsilon^{\alpha_0}. \]

Applying Young’s inequality (3.5) to the left hand side and taking \( q = 1/(1-2\alpha) \), \( p = 1/2\alpha \), \( q\lambda^\alpha = \varepsilon^{\frac{1}{2} - \frac{1}{4\alpha}} \), then \( \frac{p^\alpha}{\lambda^\alpha} = C\varepsilon^{(1-2\alpha - 2\alpha_0)/4\alpha} \) and we obtain:
\[ \frac{1}{\varepsilon^{\alpha_0}} \left| (-A)^\alpha (x_{\varepsilon,\mu} - y_{\varepsilon,\mu}) \right|_X \leq \frac{|x_{\varepsilon,\mu} - y_{\varepsilon,\mu}|_X}{\varepsilon^{1/2 - \alpha_0}} + C\varepsilon^{(1-2\alpha - 2\alpha_0)/4\alpha} \left| (-A)^{\frac{\nu}{2}} (x_{\varepsilon,\mu} - y_{\varepsilon,\mu}) \right|_X. \]

The exponent of \( (1-2\alpha - 2\alpha_0)/4\alpha \) of \( \varepsilon \) is positive because \( \alpha + \alpha_0 < \frac{1}{2} \).

With (3.25), (3.26), and with Young’s inequality we can write
\[ (3.23) \leq \left[ C \left( 1 + \left| (-A)^{\frac{\nu}{2}} x_{\varepsilon,\mu} \right|_X \right) \frac{|t_{\varepsilon,\mu} - s_{\varepsilon,\mu}|^\eta}{\varepsilon^{1-2\alpha_0}} + \varepsilon^{2\alpha_0} C(\tau) \right] + \frac{|x_{\varepsilon,\mu} - y_{\varepsilon,\mu}|_X^2}{8\varepsilon} + C\varepsilon^{(1-2\alpha - 2\alpha_0)/2\alpha} \left| (-A)^{\frac{\nu}{2}} (x_{\varepsilon,\mu} - y_{\varepsilon,\mu}) \right|_X^2. \]

With (3.11) one has
\[ \frac{|t_{\varepsilon,\mu} - s_{\varepsilon,\mu}|^\eta}{\varepsilon^{1-2\alpha_0}} \leq C \varepsilon^{\frac{\eta}{2}} \leq C\varepsilon^{2\alpha_0}, \]
which gives
\[ (3.23) \leq C \left( \varepsilon^{2\alpha_0} + \varepsilon^{\frac{1}{2} - \frac{1}{4\alpha} - 2\alpha_0/4\alpha} \right) \left( 1 + \left| (-A)^{\frac{\nu}{2}} x_{\varepsilon,\mu} \right|_X^2 + \left| (-A)^{\frac{\nu}{2}} y_{\varepsilon,\mu} \right|_X^2 + \varepsilon^{2\alpha_0} C(\tau) + \frac{|x_{\varepsilon,\mu} - y_{\varepsilon,\mu}|_X^2}{8\varepsilon} \right). \]

**Estimate of (3.24).** We finally estimate (3.24) with (3.25), (3.26) and (3.11), and we have
\[ (3.24) \leq C(\tau) \left( 1 + \left| (-A)^{\frac{\nu}{2}} x_{\varepsilon,\mu} \right|_X \right)^{3/2} \left( \frac{|t_{\varepsilon,\mu} - s_{\varepsilon,\mu}|^\eta \varepsilon^{\frac{\eta}{2}}}{\varepsilon^{\frac{\eta}{2}}} + \frac{|t_{\varepsilon,\mu} - s_{\varepsilon,\mu}|^\eta}{\varepsilon^{\frac{\eta}{2}}} \right) \leq C(\tau) \varepsilon \left( 1 + \left| (-A)^{\frac{\nu}{2}} x_{\varepsilon,\mu} \right|_X^2 \right). \]
We complete the estimate of (3.18) with (3.27) and (3.28):

\[ (3.18) \leq C \left( \varepsilon + \varepsilon^{2\alpha_0} + \varepsilon^{1-2\alpha-2\alpha_0/2}\right) \left(1 + \left|\left(-A\right)^{\frac{1}{2}}x_{\varepsilon,\mu}\right|^2_X + \left|\left(-A\right)^{\frac{1}{2}}y_{\varepsilon,\mu}\right|^2_X\right) \]

\[ + C(\varepsilon)e^{2\alpha_0} + \frac{|x_{\varepsilon,\mu} - y_{\varepsilon,\mu}|^2_X}{8\varepsilon}. \]

**Estimate of (3.19).** Since \( F \) is bounded, as in (3.17), and with (3.21) we obtain

\[ (3.19) \leq \mu \left[ \frac{1}{8} \left|\left(-A\right)^{\frac{1}{2}}x_{\varepsilon,\mu}\right|^2_X + \frac{1}{2} + \frac{1}{8} \left|\left(-A\right)^{\frac{1}{2}}y_{\varepsilon,\mu}\right|^2_X + \left|\left|x_{\varepsilon,\mu}\right|^2_X + \left|y_{\varepsilon,\mu}\right|^2_X\right| + C \right]. \]

**End of step 4**

Collecting the different estimates of the terms in the right hand side of (3.16) we obtain

\[ \left(\mu - \frac{\sigma}{2} \right) - C \left(\varepsilon^{2\alpha_0} + \varepsilon^{1-2\alpha-2\alpha_0/2}\right) - \frac{\delta_0}{8} \left[ \left|\left(-A\right)^{\frac{1}{2}}x_{\varepsilon,\mu}\right|^2_X + \left|\left(-A\right)^{\frac{1}{2}}y_{\varepsilon,\mu}\right|^2_X \right] + \frac{1}{\varepsilon} \left(1 - \frac{1}{8} - \frac{1}{2} - \frac{1}{3}\right) \left|x_{\varepsilon,\mu} - y_{\varepsilon,\mu}\right|^2_X \]

\[ \leq -\frac{\delta_0}{8} + \mu C \left[1 + \left|\left|x_{\varepsilon,\mu}\right|^2_X + \left|y_{\varepsilon,\mu}\right|^2_X\right| + C(\varepsilon)e^{2\alpha_0}\right]. \]

We take \( \varepsilon(\mu) \) small enough to have \( C \left(\varepsilon^{2\alpha_0} + \varepsilon^{1-2\alpha-2\alpha_0/2}\right) < \frac{3\delta_0}{8} \), and we take the limit when \( \mu \) tends to zero. We obtain the contradiction \( 0 \leq -\frac{\delta_0}{8} \). Thus the equality \( \min(t_{\varepsilon,\mu}, \varepsilon, \mu) = \tau \) is established.

**Step 5.** We are going to conclude with the initial data. We argue by contradiction. If (3.6) does not hold, then there exists \((t_0, x_0) \in [0, T] \times X\) such that

\[ 0 < w_{\sigma}(t_0, x_0) - v_{\sigma}(t_0, x_0) = \delta_0. \]  

We choose \( \tau \) and \( \varepsilon \) small enough to have \((t_0, x_0) \in Q_{\tau, \varepsilon}\), and

\[ [w(t, x) - g(e^{\varepsilon A}x)]^+ \leq \frac{\delta_0}{8} \quad \text{and} \quad [v(t, x) - g(e^{\varepsilon A}x)]^- \leq \frac{\delta_0}{8}, \]

for all \( x \in X \) and all \( t \in [0, \tau + C \varepsilon^{-1/\alpha}] \), where \( C \) is the constant in (3.11). One has

\[ \Phi_{\varepsilon,\mu}(t_0, x_0, x_0) \leq \Phi_{\varepsilon,\mu}(t_{\varepsilon,\mu}, s_{\varepsilon,\mu}, x_{\varepsilon,\mu}, y_{\varepsilon,\mu}) \leq w_{\sigma}(t_{\varepsilon,\mu}, x_{\varepsilon,\mu}) - v_{\sigma}(s_{\varepsilon,\mu}, y_{\varepsilon,\mu}), \]

i.e.

\[ w_{\sigma}(t_0, x_0) - v_{\sigma}(t_0, x_0) - \mu \left|x_0\right|_X^2 \leq w_{\sigma}(t_{\varepsilon,\mu}, x_{\varepsilon,\mu}) - v_{\sigma}(s_{\varepsilon,\mu}, y_{\varepsilon,\mu}) \]

\[ \leq \left[w_{\sigma}(t_{\varepsilon,\mu}, x_{\varepsilon,\mu}) + \frac{\sigma}{T - t_{\varepsilon,\mu}} - g(e^{\varepsilon A} x_{\varepsilon,\mu})\right] + g(e^{\varepsilon A}x_{\varepsilon,\mu}) - g(e^{\varepsilon A}y_{\varepsilon,\mu}) \]

\[ + \left[g(e^{e_{\varepsilon} A} y_{\varepsilon,\mu}) + \frac{\sigma}{T - s_{\varepsilon,\mu}} - v_{\sigma}(s_{\varepsilon,\mu}, y_{\varepsilon,\mu})\right] - \left[\frac{\sigma}{T - s_{\varepsilon,\mu}} + \frac{\sigma}{T - t_{\varepsilon,\mu}}\right] \]

\[ \leq \left[w(t_{\varepsilon,\mu}, x_{\varepsilon,\mu}) - g(e^{e_{\varepsilon} A} x_{\varepsilon,\mu})\right]^- + K g(e^{e_{\varepsilon} A} x_{\varepsilon,\mu}) - g(e^{e_{\varepsilon} A} y_{\varepsilon,\mu}) \]

\[ + \left[v(s_{\varepsilon,\mu}, y_{\varepsilon,\mu}) - g(e^{e_{\varepsilon} A} y_{\varepsilon,\mu})\right]^- \leq \frac{\delta_0}{8}. \]

Since \( 0 < \tau = \min(t_{\varepsilon,\mu}, s_{\varepsilon,\mu}) \) and \( |t_{\varepsilon,\mu} - s_{\varepsilon,\mu}| \leq C \varepsilon^{1/\alpha} \), we have

\[ [w(t_{\varepsilon,\mu} x_{\varepsilon,\mu}) - g(e^{e_{\varepsilon} A} x_{\varepsilon,\mu})]^+ \leq \frac{\delta_0}{8} \quad \text{and} \quad [v(s_{\varepsilon,\mu}, y_{\varepsilon,\mu}) - g(e^{e_{\varepsilon} A} y_{\varepsilon,\mu})]^- \leq \frac{\delta_0}{8}. \]
Still using $\tau = \min(t_{\varepsilon, \mu}, s_{\varepsilon, \mu})$, with (2.11), we can write
\[ |e^{\varepsilon_{t, \mu}A}x_{t, \mu} - e^{s_{t, \mu}A}y_{t, \mu}| \leq \left( |e^{\varepsilon_{t, \mu}A} - e^{s_{t, \mu}A}| \right)_{X} + |e^{s_{t, \mu}A} (x_{t, \mu} - y_{t, \mu})|_{X} \]
\[ \leq \left( |e^{\varepsilon_{t, \mu}A} - e^{s_{t, \mu}A}| \right)_{X} + \left( (-A)^{\frac{1}{2}} e^{s_{t, \mu}A} (-A)^{-\frac{1}{2}} (x_{t, \mu} - y_{t, \mu}) \right)_{X} \]
\[ \leq M_{0} |t_{\varepsilon, \mu} - s_{\varepsilon, \mu}| + \frac{C}{\sqrt{\tau}} \left| (-A)^{-\frac{1}{2}} (x_{t, \mu} - y_{t, \mu}) \right|_{X} \]
\[ \leq \frac{C}{\tau} |x_{t, \mu}|_{X} + \frac{C}{\sqrt{\tau}} \sqrt{\varepsilon} \leq \frac{C \varepsilon^{\frac{1}{2}}}{\tau \mu^{1/2}} + \frac{C}{\sqrt{\tau}} \sqrt{\varepsilon}. \]

(The last inequality is obtained with (3.12).) We choose $\mu \leq \frac{\delta_{0}}{4K_{g}}$, and next $\varepsilon$ such that
\[ \frac{C \varepsilon^{\frac{1}{2}}}{\tau \mu^{1/2}} + \frac{C}{\sqrt{\tau}} \sqrt{\varepsilon} \leq \mu. \]

We obtain:
\[ |e^{\varepsilon_{t, \mu}A}x_{t, \mu} - e^{s_{t, \mu}A}y_{t, \mu}| \leq \mu \leq \frac{\delta_{0}}{4K_{g}}. \]

Then with (3.32), (3.31) and this inequality we have:
\[ \delta_{0} - \mu |x_{0}|_{X}^{2} = w_{\sigma} (t_{0}, x_{0}) - v_{\sigma} (t_{0}, x_{0}) - \mu |x_{0}|_{X}^{2} \leq \delta_{0} + K_{g} \left( \frac{\delta_{0}}{4K_{g}} \right) + \frac{\delta_{0}}{8} = \frac{\delta_{0}}{2} \]

By passing to the limit when $\mu$ tends to zero, we obtain a contradiction. Therefore we have proved that
\[ w(t, x) \leq v(t, x) \quad \text{for all} \quad (t, x) \in Q_{T}. \]

\[ \square \]

4. PROPERTIES OF THE VALUE FUNCTION AND EXISTENCE RESULTS

For all $t \in [0, T]$ and $x \in X$, we consider the optimal control problem
\[
(P_{t, x}) \quad \min \left\{ J(t, y, u) \mid u \in M(t, T; U) \right\},
\]
where the cost functional $J$ is defined by
\[ J(t, y, u) = \int_{t}^{T} L(r, y(r), u(r)) \, dr + g(y(T)). \]

We assume that assumptions (i)-(viii) of Section 2 are satisfied. Let $v(t, x)$ be the value function of problem $(P_{t, x})$, that is
\[ v(t, x) = \inf_{u \in M(t, T; U)} J(t, y_{t, x, u}, u). \]

In the following it will be convenient to use the notation
\[ I_{t, x}(u) = J(t, y_{t, x, u}, u). \]
4.1. Properties of the value function

**Proposition 4.1.** For all \( x, x_0 \in X \), and all \( t \in [0,T] \), the value function \( v \) satisfies:

\[
|v(t,x) - v(t,x_0)| \leq K_v |x - x_0|_X,
\]

with \( K_v \) independent of \( t \).

**Proof.** With estimate (2.38), we have

\[
v(t,x) - v(t,x_0) = \inf_{u \in \mathcal{M}(t,T;U)} I_{t,x}(u) - \inf_{u \in \mathcal{M}(t,T;U)} I_{t,x_0}(u) \\
\leq \sup_{u \in \mathcal{M}(t,T;U)} \left\{ \int_t^T |L(\cdot,y_{t,x,u},u) - L(\cdot,y_{t,x_0,u},u)| \, dr + |g(y_{t,x,u}(T)) - g(y_{t,x_0,u}(T))| \right\} \\
\leq \sup_{u \in \mathcal{M}(t,T;U)} \left\{ \int_t^T K_L |y_{t,x,u}(r) - y_{t,x_0,u}(r)|_X \, dr + K_g |y_{t,x,u}(T) - y_{t,x_0,u}(T)|_X \right\} \\
\leq \sup_{u \in \mathcal{M}(t,T;U)} \left( K_L T + K_g \right) \|y_{t,x,u} - y_{t,x_0,u}\|_{L^\infty(0,T;X)} \\
\leq \left( K_L T + K_g \right) C_8(\alpha,\beta) |x - x_0|_X.
\]

By permuting \( x \) and \( x_0 \), we obtain estimate (4.1) with \( K_v = (K_L T + K_g) C_8(\alpha,\beta) \). \( \square \)

**Proposition 4.2.** The value function \( v \) is continuous and bounded in \([0,T] \times X\).

**Proof.** Let us show that \( v \) is bounded. As \( L \) and \( g \) are bounded, we have:

\[
v(t,x) = \inf_{u \in \mathcal{M}(t,T;U)} I_{t,x}(u) \leq M_g + TM_L = M_v.
\]

Moreover,

\[
v(t,x) \geq -\sup_{u \in \mathcal{M}(t,T;U)} |I_{t,x}(u())| \geq -(M_g + TM_L) = -M_v.
\]

Hence we have

\[
|v(t,x)| \leq M_v \quad \text{for all} \quad (t,x) \in [0,T] \times X.
\]

Let \((t,x) \in [0,T] \times X\) be fixed, and first show that the function \( t \mapsto v(t,x) \) is continuous. Let \( 0 \leq s < t \), we have:

\[
v(t,x) - v(s,x) \leq \sup_{u \in \mathcal{M}(t,T;U)} \left( \int_t^T |L(\cdot,y_{t,x,u},u) - L(\cdot,y_{s,x,u},u)| \, dr + |g(y_{t,x,u}(T)) - g(y_{s,x,u}(T))| \right) \\
+ \sup_{u \in \mathcal{M}(t,T;U)} \int_s^t |L(\cdot,y_{s,x,u},u)| \, dr \\
\leq \sup_{u \in \mathcal{M}(t,T;U)} \left( K_L T + K_g \right) \|y_{t,x,u} - y_{s,x,u}\|_{L^\infty(0,T;X)} + |t - s| M_L.
\]

By permuting \( s \) and \( t \), and with Proposition 2.9 we have

\[
|v(t,x) - v(s,x)| \leq C_6(\alpha,\beta) \left( K_L T + K_g \right) \bar{a}(t,s,x) + |t - s| M_L.
\]
Hence with (4.1) we obtain
\[ |v(t, x) - v(s, x_0)| \leq \left| C_0(\alpha, \beta)(KL + K_g) \bar{a}(t, s, x) + K_v |x - x_0|_X + |t - s| M_L. \]

The proof is complete. \(\square\)

**Proposition 4.3.** For all \( t \in [0, T[ \), and all \( x, x_0 \in X \), the value function satisfies
\[ |v(t, x) - v(t, x_0)| \leq C(\alpha, \beta, \theta, t) \left| (-A)^{-\theta} (x - x_0) \right|_X \quad \text{for all } \theta \in [0, 1 - \alpha[. \] (4.5)

The constant \( C(\alpha, \beta, \theta, t) \) is explicitly given in (4.7), it blows up when \( t \to T \) and when \( \alpha + \theta \to 1 \), but it stays bounded on all compact subset of \([0, T[\).

**Proof.** We have
\[ v(t, x) - v(t, x_0) \leq \sup_{u \in M(t, T; U)} \left\{ \int_t^T K_L |y_{t, x, u}(r) - y_{t, x_0, u}(r)|_X \, dr + K_g |y_{t, x, u}(T) - y_{t, x_0, u}(T)|_X \right\}. \]

With estimate (2.25), we obtain
\[ v(t, x) - v(t, x_0) \leq \int_t^T K_L C_4(\alpha, \beta, \theta, t; r) \, dr + K_g C_4(\alpha, \beta, \theta, t; T) \right\} \left| (-A)^{-\theta} (x - x_0) \right|_X. \]

where \( C_4(\alpha, \beta, \theta, t; r) \) is given in (2.28). As \( 1 - (\alpha + \beta + \theta) > -1 \), then function \( r \mapsto (r - t)^{1-(\alpha + \beta + \theta)} \) is integrable over \((t, T)\). So, one has
\[ \int_t^T K_L C_4(\alpha, \beta, \theta, t; r) \, dr = K_L \int_t^T \left( \frac{M_\theta}{(r - t)^{\theta}} + \frac{C_3(\alpha, \beta, \theta)}{1 - (\alpha + \theta)} (r - t)^{1-(\alpha + \beta + \theta)} \right) \, d \theta \]
\[ = K_L \left[ \frac{M_\theta}{1 - \theta} (T - t)^{1-\theta} + \frac{C_3(\alpha, \beta, \theta)}{2 - (\alpha + \beta + \theta)} \frac{T - t)^{2-(\alpha + \beta + \theta)}}{[1 - (\alpha + \theta)]^2} \right]. \] (4.6)

Setting
\[ C(\alpha, \beta, \theta, t) = (4.6) + K_g C_4(\alpha, \beta, \theta, t; T), \] (4.7)
we can write
\[ v(t, x) - v(t, x_0) \leq C(\alpha, \beta, \theta, t) \left| (-A)^{-\theta} (x - x_0) \right|_X. \]

And permuting \( x \) and \( x_0 \) we obtain (4.5). \(\square\)

**Proposition 4.4.** For all \( x \in D((-A)^{\frac{\beta}{2}}) \) and all \( s, t \in [0, T[ \) there exists a constant \( C \), independent of \( x, t \) and \( s \), such that
\[ |v(t, x) - v(s, x)| \leq C \left( 1 + \left| (-A)^{\frac{\beta}{2}} x \right|_X \right) |t - s|^{\frac{\beta}{2}}. \] (4.8)

**Proof.** Let us recall (4.4):
\[ |v(t, x) - v(s, x)| \leq C_0(\alpha, \beta) (KL + K_g) \bar{a}(t, s, x) + |t - s| M_L, \]
where \( \bar{a}(t, s, x) = \left| e^{(t-s)A} x \left|_X + |t - s|^{1-\beta} \right. \). With (2.11), if \( x \in D((-A)^{\frac{\beta}{2}}) \), we get:
\[ \left| e^{(t-s)A} x \right|_X \leq C |t - s|^{\frac{\beta}{2}} \left| (-A)^{\frac{\beta}{2}} x \right|_X. \]
Since $\beta \leq 1/2$, we have
\[ a(t, s, x) \leq C(\beta, T) \left( 1 + \left| (-A)^{\frac{\beta}{2}} x \right|_X \right) |t - s|^{\frac{\beta}{2}}. \]
Hence
\[ |v(t, x) - v(s, x)| \leq \left[ C_6(\alpha, \beta) \left( K_\alpha T + K_\gamma + C(\beta, T) \left( 1 + \left| (-A)^{\frac{\beta}{2}} x \right|_X \right) \right) + M_\gamma T^{\frac{\beta}{2}} \right] |t - s|^{\frac{\beta}{2}}. \]

The proof is complete. \(\square\)

4.2. Existence results

**Theorem 4.5.** Assume that assumptions (i)–(viii) of Section 2 hold. Then the value function $v$ is a viscosity solution of the Hamilton-Jacobi-Bellman equation (1.1) in the sense of Definition 3.2.

The proof is split into three steps:

**Step 1.** We show that $v$ satisfies the condition $(\alpha_1)$ in the definition of subsolutions.

**Step 2.** We show that $v$ satisfies condition $(\alpha_2)$ in the definition of supersolutions.

**Step 3.** We show that $v$ satisfies both terminal conditions $(\beta_1)$ and $(\beta_2)$.

**Proof.** We only treat the case $0 \leq \beta < \frac{1}{2}$. The case $\beta = \frac{1}{2}$ can be treated with obvious modifications by using estimate (2.20) in place of (2.19).

**Step 1.** Let $\Phi \in C^1_\alpha([0, T] \times X)$, and $(t, x) \in ([0, T] \times D((-A)^{\frac{\beta}{2}})) \cap \arg\max_{[0,T] \times X} (v - \Phi).$ Let $u(\cdot) = u \in U$ be a constant control. For all $s > t$ we have
\[ v(t, x) - \Phi(t, x) = \max (v - \Phi) \geq v(s, y_{t,x,u}(s)) - \Phi(s, y_{t,x,u}(s)). \tag{4.9} \]

By the dynamic programming principle it yields
\[ v(t, x) \leq \int_t^s L(r, y_{t,x,u}(r), u) dr + v(s, y_{t,x,u}(s)), \quad \text{for all } \tilde{u} \in U. \]

This inequality holds true in particular for $u$. From (4.9) we deduce
\[ \Phi(t, x) - \Phi(s, y_{t,x,u}(s)) \leq v(t, x) - v(s, y_{t,x,u}(s)) \leq \int_t^s L(r, y_{t,x,u}(r), u) dr, \]
hence
\[ \Phi(t, x) - \Phi(s, x) + \Phi(s, x) - \Phi(s, y_{t,x,u}(s)) - \int_t^s L(r, y_{t,x,u}(r), u) dr \leq 0. \tag{4.10} \]

With assumption (viii) of Section 2 satisfied by $L$ and a classical calculation we obtain:
\[ \lim_{s \searrow t} \frac{\Phi(t, x) - \Phi(t, x)}{s - t} - \frac{1}{s - t} \int_t^s L(r, y_{t,x,u}(r), u) dr \right) = - \frac{\partial \Phi}{\partial t}(t, x) - L(t, x, u). \tag{4.11} \]

On the other hand,
\[ \frac{\Phi(s, x) - \Phi(s, y_{t,x,u}(s))}{s - t} = \frac{\Phi(s, x) - \Phi(s, e^{(s-t)A}x)}{s - t} + \frac{\Phi(s, e^{(s-t)A}x) - \Phi(s, y_{t,x,u}(s))}{s - t}. \tag{4.12} \]
So,
\[
\Phi \left( s, e^{(s-t)A}x \right) - \Phi \left( s, y_{t,x,u}(s) \right) = \frac{1}{s-t} \left( D_x \Phi \left( s, \xi(s) \right) | e^{(s-t)A}x - y_{t,x,u}(s) \right)_X
\]
\[
= \frac{1}{s-t} \left( D_x \Phi \left( s, e^{(s-t)A}x \right) | e^{(s-t)A}x - y_{t,x,u}(s) \right)_X
\]
\[
+ \frac{1}{s-t} \left( D_x \Phi \left( s, \xi(s) \right) - D_x \Phi \left( s, e^{(s-t)A}x \right) | e^{(s-t)A}x - y_{t,x,u}(s) \right)_X,
\]
where
\[
\xi(s) = \lambda(s) e^{(s-t)A}x + (1 - \lambda(s)) y_{t,x,u}(s),
\]
for a function \( \lambda : [t, T] \to [0, 1] \). We denote by \( K_\Phi \) the Lipschitz constant of \( D_x \Phi (t, \cdot) \) in \( X \) (condition (\( \beta \)) in the definition of \( C_1^\beta([0, T] \times X) \)). With estimate (2.19), we have
\[
\left| \frac{1}{s-t} \left( D_x \Phi \left( s, \xi(s) \right) - D_x \Phi \left( s, e^{(s-t)A}x \right) | e^{(s-t)A}x - y_{t,x,u}(s) \right)_X \right|
\]
\[
\leq \frac{1}{s-t} K_\Phi \left| \xi(s) - e^{(s-t)A}x \right|_X \left| e^{(s-t)A}x - y_{t,x,u}(s) \right|_X
\]
\[
= \frac{1}{s-t} K_\Phi \left( 1 - \lambda(s) \right) \left| e^{(s-t)A}x - y_{t,x,u}(s) \right|^2_X
\]
\[
\leq \frac{1}{s-t} K_\Phi \left| e^{(s-t)A}x - y_{t,x,u}(s) \right|^2_X
\]
\[
\leq C_1(\beta) K_\Phi \frac{|s-t|^{2\beta}}{s-t} \to 0 \text{ when } s \searrow t, \text{ as } \beta < 1/2,
\]
and \( C \) is independent of \( u \) (\( C \) depends only on \( M_U \)). In the case when \( \beta = \frac{1}{2} \), using (2.20) we obtain
\[
C_1 \left( \beta, |x|_{D((-A)^{\frac{1}{2}})} \right) K_\Phi |s-t|^{2\beta_0} \text{ in place of } C|s-t|^{1-2\beta}.
\]
For the term (4.13), we have
\[
\frac{1}{s-t} \left( D_x \Phi \left( s, e^{(s-t)A}x \right) | e^{(s-t)A}x - y_{t,x,u}(s) \right)_X
\]
\[
= \left( (-A)^{\beta} D_x \Phi \left( s, e^{(s-t)A}x \right) | -\frac{1}{s-t} \int_t^s e^{(s-r)A} \left[ B u - F (r, \Lambda y_{t,x,u}(r)) \right] dr \right)_X. \quad (4.15)
\]
We know that \( e^{(s-t)A}x \) tends to \( x \) in \( D((-A)^{\frac{1}{2}}) \) when \( s \searrow t \). Thus using condition (\( \delta \)) in the definition of \( C_1^\delta([0, T] \times X) \) and Proposition 2.2, we can pass to the limit in (4.15), when \( s \searrow t \), and we obtain
\[
\lim_{s \searrow t} \Phi \left( s, e^{(s-t)A}x \right) - \Phi \left( s, y_{t,x,u}(s) \right) = -\left( (-A)^{\beta} D_x \Phi (t, x) | B u - F (t, \Lambda x) \right)_X. \quad (4.16)
\]
Moreover
\[
\frac{\Phi \left( s, e^{(s-t)A}x \right) - \Phi \left( s, e^{(s-t)A}x \right)}{s-t} = \left( D_x \Phi \left( s, \eta(s) \right) | \frac{x - e^{(s-t)A}x}{s-t} \right)_X,
\]
where
\[
\eta(s) = \lambda(s) e^{(s-t)A}x + (1 - \lambda(s)) x,
\]
for a function $\lambda : [t, T] \to ]0, 1[$. We remark that $\eta(s) \in D((-A)^{\frac{1}{2}})$ as $x \in D((-A)^{\frac{1}{2}})$. Thus $D_x\Phi(s, \eta(s)) \in D((-A)^{\frac{1}{2}})$ due to $(\gamma)$ in the definition of $C^1_A([0, T] \times X)$. The right hand side of (4.17) can be written as follows

$$
\left( D_x\Phi(s, \eta(s)) \mid \frac{-1}{s-t} \int_0^{s-t} A e^{rA} x dr \right)_X = \frac{1}{s-t} \int_0^{s-t} \left( (-A)^{\frac{1}{2}} D_x\Phi(s, \eta(s)) \mid (-A)^{\frac{1}{2}} e^{rA} x \right)_X dr
$$

$$
\to \left( (-A)^{\frac{1}{2}} D_x\Phi(t, x) \mid (-A)^{\frac{1}{2}} x \right)_X \quad \text{when } s \downarrow t,
$$

(4.19)

because the mapping $(s, x) \mapsto D_x\Phi(s, x)$ is continuous from $[0, T] \times D((-A)^{\frac{1}{2}})$ into $D((-A)^{\frac{1}{2}})$. Hence with (4.10), (4.11), (4.12), (4.16) and (4.19), we conclude that

$$
0 \geq -\partial \Phi/\partial t(t, x) - L(t, x, u) + \left( (-A)^{\beta} D_x\Phi(t, x) \mid (-A)^{\frac{1}{2}} x \right)_X
$$

$$
- \left( (-A)^{\beta} D_x\Phi(t, x) \mid B u \right)_X + \left( (-A)^{\frac{1}{2}} D_x\Phi(t, x) \mid (-A)^{\frac{1}{2}} x \right)_X.
$$

By passing to the supremum with respect to $u \in U$ we have:

$$
0 \geq -\partial \Phi/\partial t(t, x) + \left( (-A)^{\frac{1}{2}} D_x\Phi(t, x) \mid (-A)^{\frac{1}{2}} x \right)_X
$$

$$
+ \left( (-A)^{\beta} D_x\Phi(t, x) \mid F(t, A x) \right)_X + \left( t, x (-A)^{\beta} D_x\Phi(t, x) \right)_X.
$$

Thus the condition $(a_1)$ in the definition of subsolutions of equation (1.1) is satisfied.

**Step 2.** Let $\Phi$ be in $C^1_A([0, T] \times X)$, and $(t, x) \in ]0, T[ \times D((-A)^{\frac{1}{2}}) \cap \text{arg min } (v - \Phi)$. For all $u(\cdot) \in \mathcal{M}(0, T; U)$ we have

$$
v(t, x) - v(s, y_{t, x, u}(s)) \leq \Phi(t, x) - \Phi(s, y_{t, x, u}(s)).
$$

(4.20)

Thanks to the dynamic programming principle, for all $\varepsilon > 0$, there exists a control $u_\varepsilon(\cdot)$, $\varepsilon$-optimal, such that

$$
\varepsilon(s - t) + v(t, x) \geq \int_t^s L(r, y_{t, x, u}(r), u_\varepsilon(r)) dr + v(s, y_{t, x, u}(s)).
$$

(4.21)

Setting $u = u_\varepsilon$ in (4.20), and substracting (4.21), we obtain:

$$
\Phi(t, x) - \Phi(s, y_{t, x, u_\varepsilon}(s)) + \varepsilon(s - t) - \int_t^s L(r, y_{t, x, u_\varepsilon}(r), u_\varepsilon(r)) dr \geq 0.
$$

That is:

$$
\Phi(t, x) - \Phi(s, y_{t, x, u_\varepsilon}(s)) + \varepsilon(s - t) - \int_t^s L(r, y_{t, x, u_\varepsilon}(r), u_\varepsilon(r)) dr \geq 0.
$$

(4.22)

There exist a function $\eta_1 : [0, T-t] \mapsto \mathbb{R}$, a function $\lambda_2 : [t, T] \mapsto [0, 1]$, and a function $\lambda_3 : [t, T] \mapsto [0, 1]$ such that

$$
\Phi(t, x) - \Phi(s, y_{t, x, u_\varepsilon}(s))
$$

$$
\begin{align*}
\frac{s - t}{s - t} &= \Phi(t, x) - \Phi(s, x) + \Phi(s, x) - \Phi(s, e^{(s-t)A}x) + \Phi(s, e^{(s-t)A}x) - \Phi(s, y_{t, x, u_\varepsilon}(s)) \\
&= -\frac{\partial \Phi}{\partial t}(t, x) + \eta_1(s - t) + \left( D_x\Phi(s, \eta_2(s)) \mid \frac{x - e^{(s-t)A}x}{s - t} \right)_X
\end{align*}
$$

$$
+ \frac{1}{s - t} \left( D_x\Phi(s, e^{(s-t)A}x) \mid e^{(s-t)A}x - y_{t, x, u_\varepsilon}(s) \right)_X
$$

$$
+ \frac{1}{s - t} \left( D_x\Phi(s, \eta_3(s)) - D_x\Phi(s, e^{(s-t)A}x) \mid e^{(s-t)A}x - y_{t, x, u_\varepsilon}(s) \right)_X
$$

(4.23)
where
\[ \eta_1 (s - t) \to 0 \quad \text{when} \quad s \searrow t, \quad \eta_2(s) = \lambda_2(s)e^{(s-t)Ax} + (1 - \lambda_2(s))x, \]
\[ \eta_3(s) = \lambda_3(s)e^{(s-t)Ax} + (1 - \lambda_3(s))y_{t,x,u}(s). \]

As in (4.14) we have
\[ \frac{1}{s - t} \left| \left( D_x \Phi(s, \eta_3(s)) - D_x \Phi(s, e^{(s-t)Ax}) \right) \mid e^{(s-t)Ax} - y_{t,x,u}(s) \right|_X \leq C_1(\beta)K_\Phi |s - t|^{1-2\beta} \to 0 \quad \text{when} \quad s \searrow t. \]

In (4.19) we have shown that
\[ \left( D_x \Phi(s, \eta_2(s)) \mid \frac{x - e^{(s-t)Ax}}{s - t} \right)_X = \left( (-A)^{\frac{1}{2}} D_x \Phi(t, x) \mid (-A)^{\frac{1}{2}} x \right)_X + \eta_4(s - t), \]

where \( \eta_4(s - t) \to 0 \quad \text{when} \quad s \searrow t. \)

Due to the definition of \( H \), we can use the inequality
\[ - \left( (-A)^{\beta} e^{(s-r)Ax} D_x \Phi(s, e^{(s-t)Ax}) \right) \mid Bu_x(r) - L(r, y_{t,x,u}(r), u_x(r)) \]
\[ \leq H \left( r, y_{t,x,u}(r), (-A)^{\beta} e^{(s-r)Ax} D_x \Phi(s, e^{(s-t)Ax}) \right). \]

Thus
\[ \frac{1}{s - t} \left( D_x \Phi(s, e^{(s-t)Ax}) \mid e^{(s-t)Ax} - y_{t,x,u}(s) \right) \]
\[ = \left( D_x \Phi(s, e^{(s-t)Ax}) \mid -1 \frac{1}{s - t} \int_t^s (-A)^{\beta} e^{(s-r)Ax}[Bu_x(r) - F(r, Ay_{t,x,u}(r))] \right) dr \]
\[ \leq \frac{1}{s - t} \int_t^s \left[ H \left( r, y_{t,x,u}(r), (-A)^{\beta} e^{(s-r)Ax} D_x \Phi(s, e^{(s-t)Ax}) \right) + L(r, y_{t,x,u}(r), u_x(r)) \right] dr \]
\[ + \frac{1}{s - t} \int_t^s \left( (-A)^{\beta} e^{(s-r)Ax} D_x \Phi(s, e^{(s-t)Ax}) \mid F(r, Ay_{t,x,u}(r)) \right) dr. \]

By collecting together all the terms in (4.22), we obtain
\[ \varepsilon + \frac{1}{s - t} \int_t^s H \left( r, y_{t,x,u}(r), (-A)^{\beta} e^{(s-r)Ax} D_x \Phi(s, e^{(s-t)Ax}) \right) \]
\[ = -\frac{\partial \Phi}{\partial t}(t, x) + \eta_1(s - t) + \eta_4(s - t) + \left( (-A)^{\frac{1}{2}} D_x \Phi(t, x) \mid (-A)^{\frac{1}{2}} x \right)_X \]
\[ + \frac{1}{s - t} \int_t^s \left( (-A)^{\beta} e^{(s-r)Ax} D_x \Phi(s, e^{(s-t)Ax}) \mid F(r, Ay_{t,x,u}(r)) \right) dr \]
\[ \geq -C_1(\beta)K_\Phi |s - t|^{1-2\beta}. \]

(If \( \beta = \frac{1}{2} \), we have to replace the last line by \( -C_1(\beta, |x|_{D((-A)^{\frac{1}{2}})}} K_\Phi |s - t|^{1-2\beta} \). With Proposition 2.2 we have
\[ Ay_{t,x,u}(r) \to Ax \quad \text{uniformly w.r. to} \quad u_x \quad \text{when} \quad r \to t. \]

By passing to the limit in the previous inequality when \( s \to t \), and after when \( \varepsilon \to 0 \), we obtain:
\[ 0 \leq -\frac{\partial \Phi}{\partial t}(t, x) + \left( (-A)^{\frac{1}{2}} D_x \Phi(t, x) \mid (-A)^{\frac{1}{2}} x \right)_X \]
\[ + \frac{1}{s - t} \int_t^s \left( (-A)^{\beta} e^{(s-r)Ax} D_x \Phi(s, e^{(s-t)Ax}) \mid F(t, Ax) \right) dr. \]
Step 3. Let us show that
\[
\limsup_{t \to 0} v(T-t,x) - g(e^{tA}x) = 0.
\]
From the definition of \(v(T-t,x)\), it follows that
\[
v(T-t,x) - g(e^{tA}x) \leq \int_{T-t}^{T} L(r, y_{T-t,x,u}(r), u(r)) \, dr + K_{g} \left| y_{T-t,x,u}(T) - e^{tA}x \right|_{X},
\]
for all \(u \in \mathcal{M}(T-t;T;U)\). Due to estimate (2.19), there exists a constant \(C\) independent of \(t\) and \(u\) such that
\[
\left| y_{T-t,x,u}(T) - e^{tA}x \right| \leq Ct^{1-\beta}.
\]
Then we have
\[
v(T-t,x) - g(e^{tA}x) \leq \int_{T-t}^{T} L(r, y_{T-t,x,u}(r), u(r)) \, dr + K_{g}Ct^{1-\beta}
\]
for all \(u (\cdot) \in \mathcal{M}(T-t;T;U)\). Since \(L\) is bounded we obtain
\[
\limsup_{t \to 0, \delta \to 0} \left[ v(T-t,x) - g(e^{tA}x) \right] \leq 0.
\]
(4.25)
For the opposite inequality, we choose a control \(u_\delta (\cdot) \in \mathcal{M}(T-t;T;U)\), \(\delta\)-optimal, such that
\[
v(T-t,x) + \delta > \int_{T-t}^{T} L(r, y_\delta(r), u_\delta(r)) \, dr + g(y_\delta(T)),
\]
where \(y_\delta = y_{T-t,x,u_\delta}\). As \(L\) is bounded, for \(t\) small enough we have
\[
v(T-t,x) + \delta > -\delta + g(y_\delta(T)).
\]
We can write
\[
g(y_\delta(T)) \geq -\left[ g(y_\delta(T)) - g(e^{tA}x) \right] + g(e^{tA}x),
\]
and
\[
v(T-t,x) - g(e^{tA}x) > -\delta - \left( 2\delta - K_{g} \right) \left| y_{T-t,x,u}(T) - e^{tA}x \right|_{X} \geq -2\delta - K_{g}t^{1-\beta}.
\]
Hence
\[
\limsup_{t \to 0, \delta \to 0} \left[ v(T-t,x) - g(e^{tA}x) \right] \geq -2\delta.
\]
(4.26)
We can pass to the limit when \(\delta\) tends to zero, and we have shown that conditions \((\beta_1)\) and \((\beta_2)\) of Definition 3.2 are satisfied by \(v\).

\[\square\]

5. Examples of Optimal Control Problems

In this section we study the value function of problems of the form
\[
(P_{t,x}) \quad \min \left\{ \hat{J}(t,u,y) \mid u \in \mathcal{M}(t,T;U) \text{ and } (y,u) \text{ is solution of equation } (5.1) \right\}.
\]
where
\[
y' = Ay + (-A)^{\beta} \left[ Bu - \hat{F}(\cdot, Ay) \right] \quad \text{in } (t,T), \quad y(t) = x,
\]
and
\[
\hat{J}(t,y,u) = \int_{t}^{T} \hat{L}(r,y(r),u(r)) \, dr + \hat{g}(y(T)).
\]
(5.1)
We do not assume that $\hat{F}$, $\hat{L}$ and $\hat{g}$ obeys the assumptions of Section 2. But we consider examples such that, for all $u \in \mathcal{M}(t; T; U)$, and all $x \in Y$ – where $Y$ is a suitably chosen Banach space – equation (5.1) admits a unique solution which satisfies

$$
\|y\|_{L^\infty(0, T; Y)} \leq R(M_0, T) \quad \text{and} \quad \|Ay\|_{L^\infty(0, T; Y_0)} \leq R(M_0, T) \quad \text{if } |x|_Y \leq M_0,
$$

for all $M_0 > 0$, where $R(M_0, T)$ is a function of $M_0$ and $T$, $Y_0$ is a subspace of $X_0$ such that $\Lambda$ is continuous from $D((-A)^\alpha) \cap Y$ into $Y_0$. We denote by $\hat{v}(t, x)$ the value function of problem $P_{t,x}$. We introduce a projection operator $P_{M_0}$ from $X$ on the ball $B_Y(R(M_0, T))$ in $Y$, centered at the origin and with radius $R(M_0, T)$, and a projection operator $P_{M_0}^0$ from $X_0$ on the ball $B_{Y_0}(R(M_0, T))$ in $Y_0$, centered at the origin and with radius $R(M_0, T)$. We set

$$
F_{M_0}(t, y) = \hat{F}(t, P_{M_0}^0 Ay), \quad L_{M_0}(t, y, u) = \hat{L}(t, P_{M_0}^0 y, u), \quad \text{and} \quad g_{M_0}(y) = \hat{g}(P_{M_0}^0 y).
$$

In the different examples we verify that $F_{M_0}$, $L_{M_0}$, and $g_{M_0}$ obeys the assumptions of Section 2 (with constants depending on $M_0$). We denote by $(\mathcal{P}^{M_0}_{t,x})$ the problem $(\mathcal{P}_{t,x})$ of Section 1, corresponding to $F_{M_0}$, $L_{M_0}$, and $g_{M_0}$, and by $v_{M_0}(t, x)$ its value function. We verify that

$$
\hat{v}(t, x) = v_{M_0}(t, x) \quad \text{for all } x \in Y \quad \text{such that } |x|_Y \leq M_0.
$$

Due to Theorems 3.5 and 4.5, we know that $v_{M_0}$ is the unique viscosity solution of equation (1.1) corresponding to $F_{M_0}$, $L_{M_0}$, and $g_{M_0}$. Due to the definition of problem $(\mathcal{P}^{M_0}_{t,x})$, it is obvious that

$$
v_{M_1}(t, x) = v_{M_0}(t, x) \quad \text{for all } x \in Y \quad \text{such that } |x|_Y \leq M_0,
$$

if $M_1 \geq M_0$. Thus, to characterize $\hat{v}(t, x)$ when $x \in Y$, it is enough to characterize $v_{M_0}(t, x)$ for all $M_0$. And $v_{M_0}(t, x)$ is characterized as the unique viscosity solution of equation (1.1) corresponding to $F_{M_0}$, $L_{M_0}$, and $g_{M_0}$.

In the following $\Omega$ is a bounded open subset in $\mathbb{R}^N$, with a regular boundary $\Gamma$, and we set $\Omega_{t,T} = ]t, T[ \times \Omega$ and $\Sigma_{t,T} = ]t, T[ \times \Gamma$.

**5.1. State equation of example 1**

Consider the equation

$$
\frac{\partial y}{\partial t} - \Delta y + y = f \quad \text{in } \Omega_{t,T}, \quad \frac{\partial y}{\partial n} + \hat{h}(y) = u \quad \text{on } \Sigma_{t,T}, \quad y(t) = x_0 \quad \text{in } \Omega, \quad (5.2)
$$

where $\hat{h}$ is a regular nondecreasing function satisfying $h(y) = 0$ (e.g. the well known ‘Stefan-Boltzmann radiation condition’ corresponds to $\hat{h}(y) = k_r |y|^3 + k_c y$, $k_r$ is the radiation coefficient and $k_c$ the convection coefficient [23]). We make the following assumptions.

(A1) $U$ is a closed bounded convex and nonempty subset in $L^q(\Gamma)$ for some $q \geq 2$, and it obeys the condition

$$
q > N - 1, \quad |u|_{L^2(\Gamma)} \leq M_U \quad \text{and} \quad |u|_{L^q(\Gamma)} \leq M_U \quad \text{for all } u \in U.
$$

(A2) The function $f$ belongs to $C^{0,\eta_1}([0, T]; L^p(\Omega))$ for some $p \geq 2$, and it obeys the condition

$$
p \geq N/2, \quad \|f\|_{C([0, T]; L^p(\Omega))} \leq M_f, \quad \text{and} \quad \|f\|_{C([0, T]; L^2(\Omega))} \leq M_f.
$$
Let be $X = L^2(\Omega)$, $X_\Gamma = L^2(\Gamma)$, $X_0 = L^2(\Gamma)$, and let us define the unbounded operator $A$ in $X$ by

$$D(A) = \left\{ x \in H^2(\Omega) \mid \frac{\partial x}{\partial n} = 0 \text{ on } \Gamma \right\} \quad \text{and} \quad Ax = \Delta x - x \text{ for all } x \in D(A).$$

Assumption (i) of Section 2 is clearly satisfied. We define the Neumann operator $N \in \mathcal{L}(X_\Gamma; X)$ by $Nu = z$, where $z$ is the solution of the boundary value problem

$$-\Delta z + z = 0 \quad \text{in } \Omega, \quad \frac{\partial z}{\partial n} = u \quad \text{on } \Gamma.$$ 

The operator $N$ is also bounded from $X_\Gamma$ into $H^{3/2}(\Omega)$, and from $X_\Gamma$ into $D((-A)^\alpha)$ for all $0 < \alpha < \frac{3}{4}$ (see [21]). With the extrapolation method, the semigroup $(e^{tA})_{t \geq 0}$ can be extended to $(D(A^*))'$ = $(D(A))'$. Denoting by $(e^{\beta A})_{t \geq 0}$ the corresponding semigroup, $(\hat{A}, D(\hat{A}))$, its infinitesimal generator, is an unbounded operator in $(D(A^*))'$ with domain $D(\hat{A}) = X$. Thus the operator $(-\hat{A})N$ is bounded from $X_\Gamma$ into $(D(A^*))'$. The operator $\Lambda$ is the trace operator. It satisfies assumption (iii) of Section 2 for all $\alpha \in \left[\frac{1}{4}, \frac{1}{2}\right[$. It is well known – see e.g. [4] – that equation (5.2) can be rewritten in the form

$$y' = \hat{\Lambda}y + f(-\hat{\Lambda})\left[u - \hat{h}(Ay)\right] \quad \text{in } (t,T), \quad y(t) = x_0. \quad (5.3)$$

We define the operator $B$ by $B = (-\hat{\Lambda})^{1-\beta}N$ for some $\beta$ given fixed in $]1/4, 1/2[$. Due to the regularizing properties of $N$, mentioned above, we can also verify that $B \in \mathcal{L}(X_\Gamma, X)$. We set $\hat{F}(t, y) = (-A)^{-\beta}f(t) + B\hat{h}(y)$. Equation (5.3) is nothing else than

$$y' = \hat{\Lambda}y + (-\hat{\Lambda})^\beta \left[Bu - \hat{F}(Ay)\right] \quad \text{in } (t,T), \quad y(t) = x_0. \quad (5.4)$$

As in [24], Theorem 3.1, we can prove the following result.

**Theorem 5.1.** For all $x_0 \in L^\infty(\Omega)$ and all $u \in \mathcal{M}(t,T;U)$, equation (5.2) admits a unique weak solution in $C([t,T];L^2(\Omega))$. This solution belongs to $C_b([t,T] \times \overline{\Omega})$ and it satisfies the estimate

$$\|y\|_{L^\infty(\Omega, t,T)} + \|\eta\|_{L^\infty(\Omega, t,T)} \leq \|x_0\|_{L^\infty(\Omega)} + C(p, \tilde{p}, q, \tilde{q}, \Omega, T)\left(\|f\|_{L^p([t,T;L^p(\Omega)])} + \|u\|_{L^q([t,T;L^q(\Gamma)])}\right), \quad (5.5)$$

where the exponents $1 < \tilde{p} < \infty$ and $1 < \tilde{q} < \infty$ obeys

$$\frac{N}{2p} + \frac{1}{p} < 1 \quad \text{and} \quad \frac{N - 1}{2q} + \frac{1}{q} < \frac{1}{2},$$

and the constant $C(p, \tilde{p}, q, \tilde{q}, \Omega, T)$ depends on $N$, $p$, $\tilde{p}$, $q$, $\tilde{q}$, $\Omega$, $T$, but is independent of $t$.

From Theorem 5.1, and assumptions $(A_1)$ and $(A_2)$, we deduce

$$\|y\|_{L^\infty(\Omega, t,T)} \leq \|x_0\|_{L^\infty(\Omega)} + C(p, \tilde{p}, q, \tilde{q}, \Omega, T)\left((T-t)^{1\tilde{p}}M_f + (T-t)^{1\tilde{q}}M_u\right).$$

**5.2. Setting of the control problem for example 1**

For all $t \in [0, T]$, and all $x_0 \in L^\infty(\Omega)$, we study the following control problem

$$\hat{P}_{t,x_0} \min \left\{ \hat{J}(t,u,y) \mid u \in \mathcal{M}(t,T;U) \text{ and } (y,u) \text{ is solution of equation (5.2)} \right\}. $$
where the cost function $\hat{J}$ is of the form
\[
\hat{J}(t, u, y) = \int_t^T \int_{\Omega} \hat{G}(s, \xi, y(s, \xi))d\xi ds + \int_t^T \int_{\Gamma} \hat{K}(s, \sigma, u(s, \sigma))d\sigma ds + \int_{\Omega} \hat{k}(\xi, y(T, \xi))d\xi.
\] (5.6)

We make the following assumptions on the data of problem $(\hat{P}_{t,x_0})$.

(A3) For all $(t, y) \in [0, T] \times \mathbb{R}$, $\hat{G}(t, \cdot, y)$ is measurable in $\Omega$. For almost all $\xi \in \Omega$, $\hat{G}(\cdot, \xi, \cdot)$ is continuous in $[0, T] \times \mathbb{R}$, and we have the estimates:
\[
C_{\hat{G}} \leq \hat{G}(t, \xi, y) \leq \hat{G}_1(\xi) \times \eta(\|y\|),
\]
\[
\left| \hat{G}(t, \xi, y) - \hat{G}(s, \xi, z) \right| \leq \hat{G}_2(\xi)(\eta(\|y\|) + \eta(\|z\|))(|t-s|^{\alpha_2} + |y-z|),
\]
where $\hat{G}_1 \in L^1(\Omega)$, $\hat{G}_2 \in L^2(\Omega)$, and $\eta$ an increasing function from $\mathbb{R}^+$ in $\mathbb{R}^+$.

(A4) For all $(t, u) \in \mathbb{R}^2$, $\hat{K}(t, \cdot, u)$ is measurable in $\Gamma$. For a.e. $(t, \sigma) \in \Sigma$, $\hat{K}(t, \cdot, \cdot)$ is convex. For a.e. $\sigma \in \Gamma$, $\hat{K}(\cdot, \sigma, \cdot)$ is continuous in $\mathbb{R}^2$ and we have the estimates:
\[
C_{\hat{K}} \leq \hat{K}(t, \sigma, u) \leq \hat{K}_1(\sigma) + c_0 |u|^q \text{ and } \left| \hat{K}(t, \sigma, u) - \hat{K}(s, \sigma, u) \right| \leq \left( \hat{K}_1(\sigma) + c_0 |u|^q \right)|t-s|^{\alpha_2},
\]
where $\hat{K}_1 \in L^1(\Gamma)$.

(A5) For all $y \in \mathbb{R}$, $\hat{\kappa}(\cdot, y)$ is measurable in $\Omega$. For a.e. $\xi \in \Omega$, $\hat{\kappa}(\cdot, \cdot)$ is continuous in $\mathbb{R}$ and we have the estimates:
\[
C_{\hat{\kappa}} \leq \hat{\kappa}(\xi, y) \leq \hat{k}_1(\xi) \times \eta(\|y\|) \text{ and } \left| \hat{\kappa}(\xi, y) - \hat{\kappa}(\xi, z) \right| \leq \hat{\kappa}_2(\xi)(\eta(\|y\|) + \eta(\|z\|))|y-z|,
\]
where $\hat{k}_1 \in L^1(\Omega)$, $\hat{k}_2 \in L^2(\Omega)$, and $\eta$ as in (A1).

The value function of problem $(\hat{P}_{t,x_0})$ is defined by
\[
\hat{v}(t, x_0) = \inf_{\mathcal{A}(t,T)} \hat{J}(t, \hat{y}_{t,x_0,u}, u),
\] (5.7)
where $\hat{y}_{t,x_0,u}$ is the solution of equation (5.2).

5.3. Existence of solution to $(\hat{P}_{t,x_0})$

We have the following theorem [23, Th. 6.1].

**Theorem 5.2.** For all $t \in [0, T]$, all $x_0 \in L^{\infty}(\Omega)$, problem $(\hat{P}_{t,x_0})$ admits at least one solution.

By setting
\[
\hat{L}(t, y, u) = \int_{\Omega} \hat{G}(t, \xi, y(\xi))d\xi + \int_{\Gamma} \hat{K}(t, \sigma, u(\sigma))d\sigma,
\]
\[
\hat{g}(y) = \int_{\Omega} \hat{k}(\xi, y(\xi))d\xi,
\]
we notice that problem $(\hat{P}_{t,x_0})$ is an optimal control problem of the form of problems studied in Section 4. However $\hat{L}$, $\hat{g}$, and $\hat{F}$ do not satisfy assumptions of Section 2. Thus we cannot apply Theorem 4.5 to the value function $\hat{v}$, and we cannot claim that $\hat{v}$ is the viscosity solution to Hamilton-Jacobi-Bellman equation corresponding to $\hat{L}$, $\hat{g}$, and $\hat{F}$. To overcome this drawback, we introduce in the next section a family of problems whose value function is locally equal to $\hat{v}$ and is the unique viscosity solution to a Hamilton-Jacobi-Bellman equation.
5.4. Definition of a problem \((P_{t,x_0})\) equivalent to \((\hat{P}_{t,x_0})\)

For all \(M_0 > 0\), we set \(R(M_0, T) = M_0 + C(p, \bar{p}, q, \bar{q}, \Omega, T)(T^{1/\bar{q}}M_U + T^{1/\bar{p}}M_I)\), where \(C(p, \bar{p}, q, \bar{q}, \Omega, T)\) is the constant appearing in (5.5). We set

\[
T_{R(M_0, T)}(y) = \min \left( R(M_0, T), \max(-R(M_0, T), y) \right) \quad \text{for all } y \in \mathbb{R}.
\]

In this example \(Y = L^\infty(\Omega), \ Y_0 = L^\infty(\Gamma), \ P_{M_0}\) (respectively \(P^0_{M_0}\)) is the projection operator from \(L^2(\Omega)\) (respectively \(L^2(\Gamma)\)) into the ball in \(L^\infty(\Omega)\) (respectively \(L^\infty(\Gamma)\)), centered at the origin and with radius \(R(M_0, T)\), defined by

\[
P_{M_0}(\xi) = T_{R(M_0, T)}(\xi) \quad \text{for a.e. } \xi \in \Omega \quad \text{(resp. } P^0_{M_0}(\xi) = T_{R(M_0, T)}(\xi) \quad \text{for a.e. } \xi \in \Gamma)\).
\]

We set \(G(t, \xi, y(\xi)) = \hat{G}(t, \xi, P_{M_0}(y(\xi)))\) and \(k(\xi, y(\xi)) = \hat{k}(\xi, P_{M_0}(y(\xi)))\) for all \(y \in L^2(\Omega)\), and \(h(y(\xi)) = \hat{h}(P^0_{M_0}(y(\xi)))\) for all \(y \in L^2(\Gamma)\). The mappings \(h, G, k\) clearly depend on \(M_0\). We have not noticed this dependence in order not to load the notation. We set

\[
L(t, y, u) = \int_{\Omega} G(t, \xi, y(\xi)) \, d\xi + \int_{\Gamma} \hat{K}(t, \sigma, u(\sigma)) \, d\sigma, \quad g(y) = \int_{\Omega} k(\xi, y(\xi)) \, d\xi, \quad \text{and } F(y) = h(y).
\]

**Proposition 5.3.** The mappings \(L, g\) and \(F\) satisfy the assumptions of Section 2.

**Proof.** With \((A_3), (A_4)\), and the definition of \(G\), for all \((t, y, u) \in [0, T] \times L^2(\Omega) \times L^2(\Gamma)\), we have:

\[
|L(t, y, u)| \leq \int_{\Omega} |G(t, \xi, y(\xi))| \, d\xi + \int_{\Gamma} |\hat{K}(t, \sigma, u(\sigma))| \, d\sigma
\]

\[
\leq \eta(R(M_0, T))|\hat{G}_1|_{L^1(\Omega)} + |\hat{K}_1|_{L^1(\Gamma)} + c_0 M^2_U = M_L.
\]

With \((A_3)\) and Cauchy-Schwarz inequality, we have

\[
|L(t, y, u) - L(s, z, u)| = \left| \int_{\Omega} (G(s, \xi, y(\xi)) - G(t, \xi, z(\xi))) \, d\xi + \int_{\Gamma} \left( \hat{K}(t, \sigma, u(\sigma)) - \hat{K}(s, \sigma, u(\sigma)) \right) \, d\sigma \right|
\]

\[
\leq 2\eta(R(M_0, T))(|\hat{G}_2|_{L^\infty(\Omega)} + |\hat{G}_2|_{L^1(\Gamma)})(|y - z|_{L^1(\Omega)} + |t - s|_{L^1(\Gamma)}) + (|\hat{K}_1|_{L^1(\Gamma)} + c_0 M^2_U)(|t - s|_{L^1(\Gamma)}),
\]

that is

\[
|L(t, y, u) - L(s, z, u)| \leq K_L(|y - z|_{L^1(\Omega)} + |t - s|_{L^1(\Gamma)}).
\]

The estimates for \(g\) can be obtained in a similar way with \((A_5)\). The estimates for \(F\) directly follows from the definition of \(h\).

For all \(t \in [0, T]\) and all \(x_0 \in L^2(\Omega)\), we consider the optimal control problem

\[\min \left\{ J(t, u, y) \mid u \in \mathcal{M}(t, T; U) \text{ and } (y, u) \text{ is solution of equation (5.8)} \right\},\]

where the cost function \(J\) is defined by

\[
J(t, u, y) = \int_t^T L(s, y(s), u(s)) \, ds + g(y(T)),
\]

and the state equation is

\[
\frac{\partial y}{\partial t} - \Delta y + y = f \quad \text{in } Q_{t,T}, \quad \frac{\partial y}{\partial n} + h(\Lambda y) = u \quad \text{on } \Sigma_{t,T}, \quad y(t) = x_0 \quad \text{in } \Omega, \quad (5.8)
\]
Equation (5.8) can be written in the form
\[ y' = \hat{A}y + (-\hat{A})^\beta [Bu - F(\Lambda y)] \quad \text{in } (t, T), \quad y(t) = x_0, \]
where \( \hat{F}(t, y) = (-A)^{-\beta}f(t) + Bh(y) \) satisfies assumption (iv) of Section 2. The value function of problem \((\mathcal{P}_{t,x_0})\) is defined by
\[ v(t, x_0) = \inf_{u \in \mathcal{M}(t, T; U)} \overline{J}(t, y_{t,x_0,u}, u), \quad (5.9) \]
where \( y_{t,x_0,u} \) is solution of equation (5.8). From Theorems 3.5 and 4.5, we deduce that for all \( u \in \mathcal{M}(t, T; U) \), the proof. Assume that \( (y, u, \bar{u}) \) is a solution of \((\mathcal{P}_{t,x_0})\), then for all \( u \in \mathcal{M}(t, T; U) \), we have
\[ J(t, y_{t,x_0,u}, \bar{u}) = \overline{J}(t, y_{t,x_0,u}, u) \leq \overline{J}(t, y_{t,x_0,u}, u) = J(t, y_{t,x_0,u}, u), \]
that is \( (y_{t,x_0,u}, \bar{u}) \) is a solution of \((\mathcal{P}_{t,x_0})\). We prove that any solution of \((\mathcal{P}_{t,x_0})\) is a solution of \((\mathcal{P}_{t,x_0})\) in a similar way. \( \square \)

**Corollary 5.5.** For all \( t \in [0, T] \) and all \( x_0 \) satisfying \( |x_0|_{L^\infty(\Omega)} \leq M_0 \), we have \( \bar{v}(t, x_0) = v(t, x_0) \).

5.5. State equation of example 2

Consider the following Burgers type equation in 2-D:
\[ \frac{\partial y}{\partial t} - \Delta y + \partial_{x_1}(y^2) = u \chi_\omega \quad \text{in } \Omega_{t,T}, \quad y = 0 \quad \text{on } \Sigma_{t,T}, \quad y(t) = x_0 \quad \text{in } \Omega. \quad (5.10) \]

In this example, \( \omega \) is an open subset in \( \Omega \), \( \chi_\omega \) is the characteristic function of \( \omega \), and \((\mathcal{A}_1) \) is replaced by
\((\mathcal{A}_1') \) \( U \) is a closed bounded convex and nonempty subset in \( L^{10}(\omega) \) and it obeys the condition
\[ |u|_{L^{10}(\omega)} \leq MU \quad \text{for all } u \in U. \]

Set \( X = L^2(\Omega), \ X_\Gamma = L^2(\omega), \ X_0 = L^2(\Omega), \) \( \alpha = 0 \), and let \( \Lambda \) be the identity in \( X \). We now define the unbounded operator \( A \) in \( X \) by
\[ D(A) = H^2(\Omega) \cap H_0^1(\Omega) \quad \text{and} \quad Ax = \Delta x \quad \text{for all } x \in D(A). \]

Equation (5.10) can be rewritten in the form
\[ y' = \hat{A}y + (-A)^{\frac{1}{2}} \left[ Bu - \hat{F}(\Lambda y) \right] \quad \text{in } (t, T), \quad y(t) = x_0, \quad (5.11) \]
where
\[ \hat{F}(y) = 2(-A)^{-\frac{1}{2}}(y \partial_{x_1} y) \quad \text{and} \quad Bu = (-A)^{-\frac{1}{2}}(u \chi_\omega). \]

In this example we take \( \beta = \frac{1}{2} \).
Theorem 5.6. For all \( x_0 \in L^\infty(\Omega) \) and all \( u \in \mathcal{M}(t, T; U) \), equation (5.10) admits a unique weak solution in \( C([t, T]; L^2(\Omega)) \cap L^2(t, T; H^1_0(\Omega)) \) and that
\[
\|y\|_{L^\infty([t, T]; L^p(\Omega))} + \|y\|_{L^2([t, T]; H^1_0(\Omega))} \leq 2|x_0|_{L^\infty(\Omega)} + C(\Omega, p)\|u\|_{L^2((t, T) \times \omega)}.
\]

Proof. It is well known that equation (5.10) admits a unique weak solution in \( C([t, T]; L^2(\Omega)) \cap L^2(t, T; H^1_0(\Omega)) \) and that
\[
\|y\|_{C([t, T]; L^2(\Omega))} + \|y\|_{L^2([t, T]; H^1_0(\Omega))} \leq 2|x_0|_{L^\infty(\Omega)} + C(\Omega, p)\|u\|_{L^2((t, T) \times \omega)}.
\]

If we multiply equation (5.10) by \( |y|^{2p-2}y \), and if we integrate over \((t, \tau) \times \Omega\), after integration by parts, we formally obtain:
\[
\frac{1}{2p} \int_\Omega |y(\tau)|^{2p} + \int_t^\tau \int_\Omega (2p - 1)|\nabla y|^2|y|^{2p-2} = \frac{1}{2p} \int_\Omega |x_0|^{2p} + \int_t^\tau \int_\omega |y|^{2p-2}y.
\]

This identity leads to the estimate:
\[
\|y\|_{L^\infty(t, T; L^{2p}(\Omega))} \leq C(\Omega, p)\left(|x_0|_{L^{2p}(\Omega)} + \|u\|_{L^{2p}((t, T) \times \omega)}\right) \quad \text{for all } 1 \leq p \leq 5.
\]

This formal estimate can be justified (see [22], Th. 5). Thus we have
\[
\|y^2\|_{L^\infty(t, T; L^p(\Omega))} \leq C\left(|x_0|_{L^{2p}(\Omega)}^2 + \|u\|_{L^{2p}((t, T) \times \omega)}^2\right).
\]

Passing the term \( \partial_{x_1}(y^2) \) in the right hand side of the equation and using regularity results for the heat equation, we obtain:
\[
\|y\|_{L^p(t, T; W^{1, p}(\Omega))} \leq C_1(\Omega, p, \bar{p})\left(|x_0|_{L^{p}(\Omega)} + \|u\|_{L^p((t, T) \times \omega)} + \|y^2\|_{L^p(t, T; L^p(\Omega))}\right)
\]
\[
\leq C_2(\Omega, p, \bar{p})\left(|x_0|_{L^{p}(\Omega)} + \|u\|_{L^p((t, T) \times \omega)} + |x_0|_{L^{2p}(\Omega)}^2 + \|u\|_{L^{2p}((t, T) \times \omega)}^2\right),
\]

for all \( 1 < \bar{p} < 2 \) and all \( 1 \leq p \leq 5 \). In addition we have:
\[
\|y \partial_{x_1} y\|_{L^p(t, T; L^p(\Omega))} \leq \|y\|_{L^\infty(t, T; L^p(\Omega))}\|\partial_{x_1} y\|_{L^p(t, T; L^p(\Omega))}.
\]

Using this estimate and regularity results for the heat equation we can write:
\[
\|y\|_{L^\infty(\Omega, \tau)} \leq |x_0|_{L^\infty(\Omega)} + C(\Omega, p, \bar{p})\left(\|u\|_{L^p((t, T) \times \omega)} + \|y \partial_{x_1} y\|_{L^p(t, T; L^p(\Omega))}\right),
\]

provided that \( \frac{2}{\bar{p}} + \frac{1}{p} < 1 \). Choosing \( p = 5 \), \( \bar{p} = \frac{15}{4} \), and combining the previous estimates we obtain the desired result. \( \square \)

From Theorem 5.6, and assumption \((A'_1)\), we deduce
\[
\|y\|_{L^\infty(\Omega, \tau)} \leq C(\Omega, T)\left(|x_0|_{L^\infty(\Omega)} + |x_0|_{L^\infty(\Omega)}^3 + (T-t)^{1/10}M_U + (T-t)^{1/3}M_U^3\right).
\]

5.6. Setting of the control problem for example 2

For all \( t \in [0, T] \), and all \( x_0 \in L^\infty(\Omega) \), we study the following control problem
\[
\hat{P}_{t, x_0} \min \left\{ \hat{J}(t, u, y) \mid u \in \mathcal{M}(t, T; U) \text{ and } (y, u) \text{ is solution of equation (5.10)} \right\}.
\]
where the cost function $\hat{J}$ is
\[
\hat{J}(t, u, y) = \int_{t}^{T} \int_{\Omega} \hat{G}(s, \xi, y(s, \xi)) d\xi ds + \int_{t}^{T} \int_{\omega} \hat{K}(s, \xi, u(s, \xi)) d\xi ds + \int_{\Omega} \hat{k}(\xi, y(T, \xi)) d\xi.
\]

We assume that $\hat{G}$ obeys $(A_3)$, $\hat{k}$ obeys $(A_5)$, and that $\hat{K}$ obeys
\[(A_4')\] For all $(t, u) \in \mathbb{R}^2$, $\hat{K}(t, \cdot, u)$ is measurable in $\omega$. For a.e. $(t, \xi) \in \mathbb{R} \times \omega$, $\hat{K}(t, \xi, \cdot)$ is convex. For a.e. $\xi \in \omega$, $\hat{K}(\cdot, \xi, \cdot)$ is continuous in $\mathbb{R}^2$ and we have the estimates:
\[
C_{\hat{K}} \leq \hat{K}(t, \xi, u) \leq \hat{K}_1(\xi) + c_0 |u|^q \quad \text{and} \quad |\hat{K}(t, \xi, u) - \hat{K}(s, \xi, u)| \leq \left(\hat{K}_1(\xi) + c_0 |u|^q\right)|t - s|^{q_2},
\]
where $q = 10$, $\hat{K}_1 \in L^1(\omega)$.

For all $M_0 > 0$, we set
\[
R(M_0, T) = C(\Omega, T) \left(M_0 + M_0^3 + T^{1/10} M_U + T^{3/10} M_U^3\right),
\]
and we define the truncated problem in a similar way as in example 1, with obvious modifications. More precisely, $Y$, $P_{M_0}$, $G$, $k$, and $y$ are defined as in Example 1,
\[
L(t, y, u) = \int_{\Omega} G(t, \xi, y(\xi)) d\xi + \int_{\omega} \hat{K}(t, \xi, u(\xi)) d\xi, \quad \text{and} \quad F(y) = 2(-A)^{-\frac{1}{2}} (P_{M_0} y \partial_{x_1} P_{M_0} y),
\]
for all $y \in L^2(\Omega)$. We can take any $\beta_0$ in $(0, 1/2)$. For example setting $\beta_0 = 1/4$, we have
\[
|F(y)|_{D((-A)^{\frac{1}{2}})} \leq C \left(P_{M_0} y\right)^2_{D((-A)^{\frac{1}{2}})} \leq C |y|^2_{D((-A)^{\frac{1}{2}})} \quad \text{for all} \quad y \in D((-A)^{\frac{1}{2}}) = H_0^1(\Omega),
\]
and the additional condition in assumption (iv) when $\beta = \frac{1}{2}$ is satisfied. The other conditions in assumption (iv) are also satisfied. We can define $(P_{t,x_0})$ similarly as in Example 1. In particular the state equation for $(P_{t,x_0})$ is
\[
y' = Ay + (-A)^{\frac{1}{2}} (Bu - F(Ay)) \quad \text{in} \quad (t, T), \quad y(t) = x_0.
\]

Denoting by $\hat{v}(t, x_0)$ the value function of problem $(\hat{P}_{t,x_0})$, and by $v(t, x_0)$ the value function of problem $(P_{t,x_0})$, as in example 1, we can prove the following Theorem.

**Theorem 5.7.** The value function $v(t, x_0)$ is the unique viscosity solution of the Hamilton-Jacobi-Bellman equation (1.1) associated with $(P_{t,x_0})$. For all $t \in [0, T]$ and all $x_0 \in X$ satisfying $|x_0|_{L^\infty(\Omega)} \leq M_0$, we have $\hat{v}(t, x_0) = v(t, x_0)$.

#### 5.7. Example 3

We consider the same equation as in Example 2, and now $(A_1)$ is replaced by $(A_4')$ $U$ is a closed bounded convex and nonempty subset in $L^8(\omega)$ and it obeys the condition
\[
|u|_{L^8(\omega)} \leq M_U \quad \text{for all} \quad u \in U.
\]

Set $X = L^2(\Omega)$, $X_T = L^2(\omega)$, $X_0 = H_0^{3/4}(\Omega)$, $\alpha = \frac{3}{2}$, and let $\Lambda$ be the identity in $X_0$ (thus $\Lambda$ is considered as an unbounded operator in $X$). We define the unbounded operator $(A, D(A))$ in $X$ as in Example 2. Equation (5.10) can be rewritten in the form
\[
y' = Ay + (-A)^{\frac{1}{2}} [Bu - \tilde{F}(y)] \quad \text{in} \quad (t, T), \quad y(t) = x_0,
\]
(5.13)
where
\[ \hat{F}(y) = 2(-A)\frac{d}{dy}(y \partial_x y) \quad \text{and} \quad Bu = (-A)\frac{d}{dy}(u \chi_\omega), \quad \text{with} \ 0 < \epsilon' \leq 1. \]

In this example, we take \( \beta = \frac{5}{2} \) (below we choose \( \epsilon' = 5/8 \)).

**Theorem 5.8.** For all \( x_0 \in H^{3/4}_0(\Omega) \) and all \( u \in \mathcal{M}(t,T;U) \), equation (5.10) admits a unique weak solution in \( C([t,T];H^{3/4}_0(\Omega)) \), and it satisfies the estimate

\[
\|y\|_{C([t,T];H^{3/4}_0(\Omega))} \leq C(\Omega,T)(|x_0|_{H^{3/4}_0(\Omega)} + |x_0|^3_{H^{3/4}_0(\Omega)} + \|u\|_{L^4((t,T)\times\omega)} + \|u\|^3_{L^4((t,T)\times\omega)}). \tag{5.14}
\]

**Proof.** Observe that \( H^{3/4}_0(\Omega) \hookrightarrow L^8(\Omega) \). As in Example 2, we have

\[
\|y\|^2_{L^\infty(t,T;L^8(\Omega))} \leq C(\Omega,\bar{p})(|x_0|^2_{L^8(\Omega)} + \|u\|^2_{L^8((t,T)\times\omega)}),
\]

\[
\|y\|_{L^p(t,T;W^{1,p}(\Omega))} \leq C(p,p\bar{p})(|x_0|_{L^p(\Omega)} + \|u\|_{L^p((t,T)\times\omega)} + |x_0|^2_{L^p(\Omega)} + \|u\|^2_{L^p((t,T)\times\omega)}),
\]

for all \( 1 < \bar{p} < 2 \) and all \( 1 \leq p \leq 4 \), and

\[
\|y\|_{L^p(t,T;L^p(\Omega))} \leq \|y\|_{L^\infty(t,T;L^p(\Omega))}\|\partial_y y\|_{L^p(t,T;L^p(\Omega))} \quad \text{if in addition} \ 2 \leq p \leq 4.
\]

Using this estimate and regularity results for the heat equation we can write:

\[
\|y\|_{C([t,T];H^{3/4}_0(\Omega))} \leq C(\Omega,\bar{p})(|x_0|_{H^{3/4}_0(\Omega)} + \|u\|_{L^2((t,T)\times\omega)} + \|\partial_y y\|_{L^2(t,T;L^2(\Omega))}), \tag{5.15}
\]

if \( \frac{1}{p} < \frac{5}{8} \). Thus, choosing \( p = 4, \bar{p} = \frac{32}{11} \), and combining the above estimates, we obtain:

\[
\|y\|_{C([t,T];H^{3/4}_0(\Omega))} \leq C(\Omega,T)(|x_0|_{H^{3/4}_0(\Omega)} + |x_0|^3_{H^{3/4}_0(\Omega)} + \|u\|_{L^8((t,T)\times\omega)} + \|u\|^3_{L^8((t,T)\times\omega)}). \tag{5.16}
\]

From Theorem 5.8, and assumptions \((A'_7)\) and \((A_5)\), we deduce

\[
\|y\|_{C([t,T];H^{3/4}_0(\Omega))} \leq C(\Omega,T)(|x_0|_{H^{3/4}_0(\Omega)} + |x_0|^3_{H^{3/4}_0(\Omega)} + (T-t)^{1/8}M_U + (T-t)^{3/8}M_U^3).
\]

Now we set

\[ R(M_0, T) = C(\Omega,T)(M_0 + M_0^{1/8} + T^{1/8}M_U + T^{3/8}M_U^3). \]

Let us denote by \( P^{\delta}_{M_0} \) the orthogonal projection in \( H^{3/4}_0(\Omega) \) on the ball centered at zero and with radius \( R(M_0,T) \). Let us set

\[ F(y) = 2(-A)\frac{d}{dy}(P^{\delta}_{M_0} y \partial_x (P^{\delta}_{M_0} y)) \quad \text{for all} \ y \in H^{3/4}_0(\Omega). \]

Let us show that \( F \) is bounded and Lipschitz from \( H^{3/4}_0(\Omega) \) into \( X \). We have

\[ |F(y)|_{L^2(\Omega)} = 2|P^{\delta}_{M_0} y \partial_x (P^{\delta}_{M_0} y)|_{H^{-\epsilon'}(\Omega)} = 2|P^{\delta}_{M_0} y \partial_x (P^{\delta}_{M_0} y)|_{H^{2\epsilon}(-\Omega)}. \]

Using the product estimate [25], page 171

\[ B^{s_1}_{p_1,q_1}(\Omega) \cdot B^{s_2}_{p_2,q_2}(\Omega) \hookrightarrow B^{s_1}_{p,q}(\Omega), \]

with

\[ s_1 = -\epsilon' = -\frac{5}{8}, \quad s_2 = \epsilon = \frac{3}{4}, \quad p_1 = \frac{32}{11}, \quad p_2 = p = 2, \quad q_1 = q_2 = q = 2. \]
we have
\[ |F(y)|_{L^2(\Omega)} \leq C|P^0_{M_0}y|_{B_{2,2}^\epsilon(\Omega)}|\partial_x_1\left(P^0_{M_0}y\right)|_{B_{2,2}^{\epsilon'}(\Omega)}.\]

Recall that
\[ B_{2,2}^\epsilon(\Omega) = H^\epsilon(\Omega), \quad B_{2,2}^{\epsilon'}(\Omega) = H^{-\epsilon'}(\Omega), \quad \text{and} \quad B_{1,2}^{\epsilon'}(\Omega) = \left(B_{1,2}^{\epsilon'}(\Omega)\right)^\epsilon.\]

where \( p'_1 = p_1/(p_1 - 1) \). Moreover
\[ B_{1,2}^{\epsilon'}(\Omega) \hookrightarrow B_{p_1,2}^{\epsilon'}(\Omega) = W^{\epsilon',p'_1}(\Omega) \hookrightarrow H^\epsilon(\Omega), \]

for all \( \epsilon < \epsilon'' < \epsilon' \), and if \( 2 = \frac{2\epsilon'}{2-(\epsilon''-\epsilon')}p'_1 \). We choose \( \epsilon = \frac{1}{4} \) and \( \epsilon'' = \frac{\alpha}{16} \), we have \( H^{-\epsilon}(\Omega) \hookrightarrow B_{p_1,2}^{\epsilon'}(\Omega) \), and we obtain
\[ |F(y)|_{L^2(\Omega)} \leq C|P^0_{M_0}y|_{H^{3/4}(\Omega)}|\partial_x_1\left(P^0_{M_0}y\right)|_{H^{-3/4}(\Omega)} \leq C|P^0_{M_0}y|_{H^{3/4}(\Omega)}|P^0_{M_0}y|_{H^{3/4}(\Omega)}.\]

Thus \( F \) is bounded from \( X_0 \) into \( X \). Let us verify that \( F \) is Lipschitz. We have
\[
|F(y) - F(z)|_{L^2(\Omega)} \leq 2|P^0_{M_0}y_1\partial_x_1\left(P^0_{M_0}y\right) - P^0_{M_0}z_1\partial_x_1\left(P^0_{M_0}z\right)|_{H^{-3/4}(\Omega)}
\]
\[
\leq 2|P^0_{M_0}y_1\partial_x_1\left(P^0_{M_0}y - P^0_{M_0}z\right)|_{H^{-3/4}(\Omega)} + 2|P^0_{M_0}z_1\partial_x_1\left(P^0_{M_0}y - P^0_{M_0}z\right)|_{H^{-3/4}(\Omega)}
\]
\[
\leq C \left(|P^0_{M_0}y|_{H^{3/4}(\Omega)} + |P^0_{M_0}y|_{H^{3/4}(\Omega)}\right)|y - z|_{H^{3/4}(\Omega)}
\]

To obtain the last inequality we have used the Lipschitz continuity of \( P^0_{M_0} \) from \( H^{3/4}(\Omega) \) into itself. We define \((\tilde{T}_{t,x_0})\) as in Example 2, and we assume that \( \tilde{G} \) obeys \((\tilde{A}^0_t)\), \( \tilde{\eta} \) obeys \((\tilde{A}^0_t)\), and \( \tilde{K} \) obeys \((\tilde{A}^0_t)\), where \((\tilde{A}^0_t)\), \((\tilde{A}^0_t)\), and \((\tilde{A}^0_t)\) respectively correspond to \((A^0_t)\), \((A^0_t)\), and \((A^0_t)\), where \( \eta(|y|) = |y|^r \), \( 1 \leq r \leq 4 \), \( \tilde{G} \in L^3(\Omega) \), \( \tilde{G}_2 \in L^{3/2}(\Omega) \), \( \tilde{k}_1 \in L^3(\Omega) \), \( \tilde{k}_2 \in L^{3/2}(\Omega) \), and \( q = 8 \), with \( \frac{4}{5} + \frac{1}{p_1} = 1 \) and \( \frac{4}{5} + \frac{1}{p_2} = \frac{1}{2} \). We define \((\tilde{T}_{t,x_0})\) as in Example 2. Next we define \((P_{t,x_0})\) with the state equation
\[
y'(t) = Ay + (\tilde{A}^0_t)^\frac{1}{2} [Bu - F(Ay)] \quad \text{in} (t,T), \quad y(t) = x_0,
\]

and with \( G, k, g, \) and \( L \) defined as in Example 2 but where \( P_{M_0} \) is the projection in \( L^2(\Omega) \) on the ball in \( H^{3/4}(\Omega) \) centered at zero and with radius \( R(M_0,T) \). Recall that \( P_{M_0} \) is Lipschitz continuous from \( L^2(\Omega) \) into itself. Since the embedding from \( H^{3/4}(\Omega) \) into \( L^3(\Omega) \) is continuous, with assumptions \((\tilde{A}^0_t)\), \((\tilde{A}^0_t)\), and \((\tilde{A}^0_t)\), we easily verify that assumptions of Section 2 are satisfied by \( L, g, \) and \( F \). Denoting by \( \hat{v}(t, x_0) \) the value function of problem \((\tilde{T}_{t,x_0})\), and by \( v(t, x_0) \) the value function of problem \((P_{t,x_0})\), we can prove the following Theorem.

**Theorem 5.9.** The value function \( v(t, x_0) \) is the unique viscosity solution of the Hamilton-Jacobi-Bellman equation (1.1) associated with \((P_{t,x_0})\). For all \( t \in [0,T] \) and all \( x_0 \) satisfying \( |x_0|_{H^{3/4}(\Omega)} \leq M_0 \), we have \( \hat{v}(t, x_0) = v(t, x_0) \).

**References**


