CARLEMAN ESTIMATES FOR THE NON-STATIONARY LAMÉ SYSTEM AND THE APPLICATION TO AN INVERSE PROBLEM

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Abstract. In this paper, we establish Carleman estimates for the two dimensional isotropic non-stationary Lamé system with the zero Dirichlet boundary conditions. Using this estimate, we prove the uniqueness and the stability in determining spatially varying density and two Lamé coefficients by a single measurement of solution over \((0,T) \times \omega\), where \(T > 0\) is a sufficiently large time interval and a subdomain \(\omega\) satisfies a non-trapping condition.

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1. Introduction

This paper is concerned with Carleman estimates for the two dimensional non-stationary isotropic Lamé system with the zero Dirichlet boundary condition and an application to an inverse problem of determining spatially varying density and the Lamé coefficients by a single interior measurement of the solution. The Carleman estimate is a weighted \(L^2\)-estimate of the solution to a partial differential equation and it has been fundamental for proving the uniqueness in a Cauchy problem for the partial differential equation or the unique continuation.

More precisely, we consider the two dimensional isotropic non-stationary Lamé system:

\[
(Pu)(x_0, x') \equiv \rho(x') \partial_{x_0}^2 u(x_0, x') - (L_{\lambda, \mu} u)(x_0, x') = f(x_0, x'),
\]

\(x \equiv (x_0, x') \in Q \equiv (0, T) \times \Omega,\) (1.1)

where

\[
(L_{\lambda, \mu} v)(x') \equiv \mu(x') \Delta v(x') + (\mu(x') + \lambda(x')) \nabla_{x'} \text{div} v(x')
+ (\text{div} v(x')) \nabla_{x'} \lambda(x') + (\nabla_{x'} v + (\nabla_{x'} v)^T) \nabla_{x'} \mu(x'), \quad x' \in \Omega.
\] (1.2)

Throughout this paper, \(\Omega \subset \mathbb{R}^2\) is a bounded domain whose boundary \(\partial \Omega\) is of class \(C^3\), \(x_0\) and \(x' = (x_1, x_2)\) denote the time variable and the spatial variable respectively, and \(u = (u_1, u_2)^T\) where \(\cdot^T\) denotes the transpose.

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of matrices, $E_k$ is the identity matrix of the size $k \times k$,

$$\partial_{x_j} \varphi = \varphi_{x_j} = \frac{\partial \varphi}{\partial x_j}, \quad j = 0, 1, 2.$$ 

We set $\nabla_x \varphi = (\partial_{x_k} v_j)_{1 \leq j, k \leq 2}$ for a vector function $\varphi = (v_1, v_2)^T$ and $\nabla_x \phi = (\partial_{x_k} \phi, \partial_{x_k} \phi)^T$ for a scalar function $\phi$. Henceforth $\nabla$ means $\nabla_x = (\partial_{x_k}, \partial_{x_1}, \partial_{x_2})$ if we do not specify.

Moreover the coefficients $\rho, \lambda, \mu$ satisfy

$$\rho, \lambda, \mu \in C^2(\bar{\Omega}), \quad \rho(x') > 0, \quad \mu(x') > 0, \quad \lambda(x') + \mu(x') > 0 \quad \text{for} \ x' \in \bar{\Omega}. \quad (1.3)$$

As for more details for the Lamé system, see for example, Chapter III of Duvaut and Lions [11] or Gurtin [14].

The Carleman estimate is an essential technique not only for the unique continuation, but also for solving the exact controllability and stabilizability (e.g., Bellassoued [2–4], Imanuvilov [17], Imanuvilov and Yamamoto [25], Kazemi and Klibanov [32], Tataru [44], Zhang [51], and Lasiecka and Triggiani [37] as a related book) and the inverse problems (e.g., Bukhgeim [6], Bukhgeim and Klibanov [8], Klibanov [35]). Thus the first main purpose of this paper is to establish Carleman estimates for system (1.1). Our method works, in principle, also for the three dimensional case but the arguments are more complicated and independent consideration is required. Thus in this paper, we will exclusively discuss the spatially two dimensional case. In a forthcoming paper, we will treat the three dimensional case.

Since the pioneering work [9] by Carleman, the theory of inequalities of Carleman’s type has been rapidly developed and now many general results are available for a single partial differential equation (see [12, 15, 29, 30, 44]), while for strongly coupled systems of partial differential equations, the situation is more complicated and much less understood. To our best knowledge, the most general result for systems of partial differential equations is Calderon’s uniqueness theorem (see e.g., [12, 52]). The technique developed by Calderon, reduces the system of partial differential equations to a system of pseudo-differential operators of the first order:

$$\frac{dU}{dx} = M(x, D_{x_0}, D_{x_1})U + F,$$

where $M(x, D_{x_0}, D_{x_1})$ is a matrix pseudo-differential operator. Then by some change of variables $U = S(x, D_{x_0}, D_{x_1}) \tilde{U}$, this matrix pseudo-differential operator $M$ is reduced to $S^{-1}MS$ such that $S^{-1}MS$ consists of blocks of a small size located on the main diagonal and that in each block the principal symbols of all the operators located below the main diagonal are zero. In order to construct the matrix $S$, the eigenvalues and eigenvectors of the matrix $M(x, \xi_0, \xi_1)$ should be smooth functions of the variables $x$ and $\xi_0, \xi_1 \in \mathbb{R}^1$ and each eigenvalue should not change the multiplicity. This condition is restrictive, especially in the case where we are looking for a Carleman estimate near boundary, and therefore the choice for a variable $x_2$ is limited. For example the non-stationary Lamé system does not satisfy this condition, in general. On the other hand, for the stationary Lamé system, this method works well and produces the unique continuation result from an arbitrary open subset (see [19]). See also Imanuvilov and Yamamoto [27] as for a Carleman estimate for the stationary Lamé system.

As long as the non-stationary Lamé system is concerned, it is known that thanks to the special structure of the system, the functions $\text{div} \ u$ and $\text{rot} \ u$ satisfy scalar wave equations (modulo lower order terms). The system of partial differential equations for the functions $u, \text{div} u, \text{rot} \ u$, is coupled via only first order terms. This allows us to apply the Carleman estimate for a scalar hyperbolic equation in the case where the function $u$ has a compact support (see [13, 16, 19]).

The structure of our proof is in principle similar to Yamamoto [49]. That is, we work mainly with two hyperbolic equations depending on a parameter $s > 0$ for the functions $z_{\lambda + 2\mu} \equiv e^{s\phi}\text{div} \ u$ and $z_{\mu} \equiv e^{s\phi}\text{rot} \ u$: $P_{\lambda + 2\mu}(x, D, s)z_{\lambda + 2\mu} = (\text{div} \ f)e^{s\phi}$ and $P_{\mu}(x, D, s)z_{\mu} = (\text{rot} \ f)e^{s\phi}$. The main difficulty one should overcome, is that there are no boundary conditions for these functions. This problem is solved in the following way: outside
an exceptional set in the contangent bundle $T^*(Q)$, the operators $P_{\lambda+2\mu}$ and $P_\mu$ can be microlocally factorized as the product of some function $\tilde{\beta}(x)$ and two pseudo-differential operators of the first order:

$$P_\beta(x, D, s) = \tilde{\beta}P_{\lambda+2\mu}(x, D, s)P_{\mu}(x, D, s),$$

where $\beta = \lambda + 2\mu$ or $= \mu$, $P_{\lambda+2\mu} = D_{x_2} - \Gamma_{\lambda+2\mu}^+(x, D_{x_0}, D_{x_1}, s)$, and $x_2$ is normal to the boundary $\partial \Omega$. Since the principal symbol of the operator $\Gamma_{\lambda+2\mu}^+(x, \xi_0, \xi_1, s)$ satisfies the inequality

$$-\text{Im} \Gamma_{\lambda+2\mu}^-(x, \xi_0, \xi_1, s) \geq C|s|$$

with a constant $C > 0$, we have a priori estimates for $P_{\lambda+2\mu}(x, D, s)z_{\beta}|_{x_2=0}$ in an $L^2$-space. These estimates and the zero Dirichlet boundary condition yield the $H^1$-boundary estimates for $z_\beta$. The set on which we cannot factorize both the operators $P_{\lambda+2\mu}(x, D, s)$ into a product of the first order operators, has to be discussed separately.

Next we will prove a Carleman estimate with the $H^{-1}(Q)$ norm of the force $\mathbf{f}$ in the right hand side. The Carleman estimate with right hand side in $H^{-1}(Q)$-space was proved by Imanuvilov [18], Ruiz [43], for a scalar hyperbolic equation and by Imanuvilov and Yamamoto [26] for a parabolic equation. In this paper, by a method in [26], we will derive an $H^{-1}(Q)$-Carleman estimate (Th. 2.3) for (1.1) from a Carleman estimate (Th. 2.1) with $H^1$-norm.

Finally we consider an inverse problem of determining the coefficients $\lambda$, $\mu$ and $\rho$ from one single measurement of the solution $u$ in $(0, T) \times \omega$, where $\omega \subset \Omega$ is a suitable subdomain and $T > 0$ is sufficiently large. By our $H^{-1}(Q)$-Carleman estimate for the Lamé system, we will establish the uniqueness and the stability result for the inverse problem.

This paper is composed of nine sections and two appendices. In Section 2, we state Carleman estimates (Ths. 2.1–2.3) for functions which do not have compact supports but satisfy the zero Dirichlet boundary condition on $(0, T) \times \partial \Omega$. Theorem 2.1 is a Carleman estimate whose right hand side is estimated in $H^1$-space. Theorems 2.2 and 2.3 are Carleman estimates respectively with right hand sides in $L^2$-space and in $H^{-1}$-space. In Section 3, we will apply the $H^{-1}$-Carleman estimate (Th. 2.3), and prove the uniqueness and the conditional stability in the inverse problem with a single interior measurement. In Sections 4–8, we prove Theorem 2.1; In Section 4, we will reduce Theorem 2.1 to Lemma 4.1, and in Section 5, we further localize Lemma 4.1 by means of pseudo-differential operators. Dividing all the possible cases into three cases, in Sections 6–8, we will complete the proof of the localized estimate separately in those three cases. Finally Theorems 2.2 and 2.3 are proved in Section 9.

2. CARLEMAN ESTIMATES FOR THE TWO DIMENSIONAL NON-STATIONARY LAMÉ SYSTEM

Let us consider the two dimensional Lamé system

$$P\mathbf{u}(x_0, x') \equiv \rho(x') \partial_{x_0}^2 \mathbf{u}(x_0, x') - (L_{\lambda, \mu} \mathbf{u})(x_0, x') = \mathbf{f}(x_0, x') \quad \text{in} \quad Q,$$  

$$\mathbf{u}|_{(0, T) \times \partial \Omega} = 0, \quad \mathbf{u}(T, x') = \partial_{x_0} \mathbf{u}(T, x') = \mathbf{u}(0, x') = \partial_{x_0} \mathbf{u}(0, x') = 0,$$  

where $\mathbf{u} = (u_1, u_2)^T$, $\mathbf{f} = (f_1, f_2)^T$ are vector-valued functions, and the partial differential operator $L_{\lambda, \mu}$ is defined by (1.2). The coefficients $\rho, \lambda, \mu \in C^2(\overline{\Omega})$ are assumed to satisfy (1.3).

Let $\omega \subset \Omega$ be an arbitrarily fixed subdomain (not necessary connected). Denote by $\hat{n}(x') = (n_1(x'), n_2(x'))$ and $\hat{\tau}(x')$ the outward unit normal vector and a unit tangential vector to $\partial \Omega$ at $x'$ respectively, and set $\frac{\partial}{\partial n} = \nabla_{x'} v \cdot \hat{n}$ and $\frac{\partial}{\partial \tau} = \nabla_{x'} v \cdot \hat{\tau}$.

We set

$$Q_\omega = (0, T) \times \omega.$$
Let $\xi = (\xi_0, \xi') = (\xi_0, \xi_1, \xi_2)$. We set
\begin{equation}
\begin{cases}
p_1(x, \xi) = \rho(x')|\xi_0^2 - \mu(x')(|\xi_1|^2 + |\xi_2|^2), \\
p_2(x, \xi) = \rho(x')|\xi_0^2 - (\lambda(x') + 2\mu(x'))(|\xi_1|^2 + |\xi_2|^2)
\end{cases}
\tag{2.3}
\end{equation}
and $\nabla_\xi = (\partial_{\xi_0}, \partial_{\xi_1}, \partial_{\xi_2})$. For arbitrary smooth functions $\varphi(x, \xi)$ and $\psi(x, \xi)$, we define the Poisson bracket by the formula
\begin{equation}
\{\varphi, \psi\} = \sum_{j=0}^{2} (\partial_{\xi_j} \varphi)(\partial_{\xi_j} \psi) - (\partial_{\xi_j} \psi)(\partial_{\xi_j} \varphi).
\end{equation}
We set $i = \sqrt{-1}$ and $(a, b) = \sum_{k=1}^{3} a_k b_k$ for $a = (a_1, a_2), b = (b_1, b_2, b_3) \in \mathbb{C}^3$.

We assume that the density $\rho$, the Lamé coefficients $\lambda, \mu$ and the domains $\Omega, \omega$ satisfy the following condition (cf. [15]).

**Condition 2.1.** There exists a function $\psi \in C^3(Q)$ such that $|\nabla_\xi \psi| \neq 0$ on $Q \setminus Q_\omega$ and
\begin{equation}
\{p_k, \{p_k, \psi\}\}(x, \xi) > 0, \quad \forall k \in \{1, 2\}
\end{equation}
if $(x, \xi) \in (Q \setminus Q_\omega) \times (\mathbb{R}^3 \setminus \{0\})$ satisfies $p_k(x, \xi) = \{p_k, \psi\}(x, \xi) = 0$,
\begin{equation}
\{p_k(x, \xi - is\nabla \psi(x)), p_k(x, \xi + is\nabla \psi(x))\}/2is > 0, \quad \forall k \in \{1, 2\}
\end{equation}
if $(x, \xi, s) \in (Q \setminus Q_\omega) \times (\mathbb{R}^3 \setminus \{0\}) \times (\mathbb{R} \setminus \{0\})$ satisfies
\begin{equation}
\{p_k(x, \xi + is\nabla \psi(x)) = (\nabla_\xi p_k(x, \xi + is\nabla \psi(x)), \nabla \psi(x)) = 0.
\end{equation}

On the lateral boundary, we assume
\begin{equation}
\sqrt{p} |\psi_{x_0}| < \frac{\mu}{\sqrt{\lambda + 2\mu}} |\frac{\partial \psi}{\partial x'}| + \frac{\sqrt{\mu \lambda + \mu}}{\sqrt{\lambda + 2\mu}} |\frac{\partial \psi}{\partial n}|, \quad p_1(x, \nabla \psi) < 0, \quad \forall x \in (0, T) \times \partial \Omega, \quad \text{and} \quad \frac{\partial \psi}{\partial n} |_{(0, T) \times (\partial \Omega \setminus \partial \omega)} < 0.
\end{equation}

Let $\psi(x)$ be the weight function in Condition 2.1. Using this function, we introduce the function $\phi(x)$ by
\begin{equation}
\phi(x) = e^{\tau \psi(x)}, \quad \tau > 1,
\end{equation}
where the parameter $\tau > 0$ will be fixed below. Denote
\begin{equation}
\|u\|^2_{E(\phi, Q)} = \int_Q \left( \sum_{|\alpha| \leq 2} s^{4-2|\alpha|} |\partial_\alpha u|^2 + s|\nabla \nabla u|^2 + s^3 |\rot u|^2 \right)e^{2\phi} dx,
\end{equation}
where $\alpha = (\alpha_0, \alpha_1, \alpha_2), \alpha_j \in \mathbb{N}_+ \cup \{0\}, \partial_\alpha = \partial_{x_0}^{\alpha_0} \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2}$.

Now we formulate our Carleman estimates as main results.
Theorem 2.1. Let \( f \in (H^1(Q))^2 \), and let the function \( \psi \) satisfy Condition 2.1. Then there exists \( \bar{\tau} > 0 \) such that for any \( \tau > \bar{\tau} \), there exists \( s_0 = s_0(\tau) > 0 \) such that for any solution \( u \in (H^1(Q))^2 \cap L^2(0,T; (H^2(\Omega))^2) \) to problem (2.1)–(2.2), the following estimate holds true:

\[
\|u\|_{Y(\phi,Q)}^2 \triangleq \|u\|_{B(\phi,Q)}^2 + s \|\frac{\partial u}{\partial \hat{n}}\|^2_{(H^1((0,T) \times \partial \Omega))^2} + \|\frac{\partial^2 u}{\partial n^2}e^{s\phi}\|^2_{(L^2((0,T) \times \partial \Omega))^2} + s^3 \|\frac{\partial u}{\partial \hat{n}} e^{s\phi}\|^2_{(L^2((0,T) \times \partial \Omega))^2} \leq C_1(s^2\|f_e^{s\phi}\|^2_{(L^2(Q))^2} + \|\langle \nabla f \rangle e^{s\phi}\|^2_{(L^2(Q))^2} + \|u\|_{B(\phi,Q,\omega)}^2), \quad \forall s \geq s_0(\tau), \tag{2.9}
\]

where the constant \( C_1 = C_1(\tau) > 0 \) is independent of \( s \).

Remark. In Carleman estimate (2.9), the weights which correspond to \( \text{rot} u \) and \( \text{div} u \) are better than the weights which correspond to \( \nabla u \). This is a result of the special structure of the Lamé system which allows us to decouple into two wave equations for \( \text{rot} u \) and \( \text{div} u \) (see (4.1)).

Next we formulate other two Carleman estimates where norms of the function \( f \) are taken in \( (L^2(Q))^2 \) and \( H^{-1}(Q) \). In particular, the second of these two Carleman estimate is essential for obtaining our sharp uniqueness result in the inverse problem.

In addition to Condition 2.1, we assume

\[
\partial_{x_0} \psi(T, x') < 0, \quad \partial_{x_0} \psi(0, x') > 0, \quad \forall x' \in \overline{\Omega}. \tag{2.10}
\]

Theorem 2.2. Let \( f \in (L^2(Q))^2 \) and let the function \( \psi \) satisfy (2.10) and Condition 2.1 and let function \( \phi \) be given by (2.7). Then there exists \( \bar{\tau} > 0 \) such that for any \( \tau > \bar{\tau} \), there exists \( s_0 = s_0(\tau) > 0 \) such that for any solution \( u \in (H^1(Q))^2 \) to problem (2.1)–(2.2), the following estimate holds true:

\[
\int_Q \left( \|\nabla u\|^2 + s^2\|u\|^2 \right) e^{2s\phi} \, dx \leq C_1 \left( \|fe^{s\phi}\|^2_{(L^2(Q))^2} + \int_{Q_\omega} (\|\nabla u\|^2 + s^2\|u\|^2) e^{2s\phi} \, dx \right), \quad \forall s \geq s_0(\tau), \tag{2.11}
\]

where the constant \( C_1 = C_1(\tau) > 0 \) is independent of \( s \).

Theorem 2.3. Let \( f = f_{-1} + \sum_{j=0}^2 \partial_{x_j} f_j \) with \( f_{-1} \in (H^{-1}(Q))^2 \) and \( f_0, f_1, f_2 \in (L^2(Q))^2 \), and let the function \( \psi \) satisfy (2.10) and Condition 2.1 and let the function \( \phi \) be given by (2.7). Then there exists \( \bar{\tau} > 0 \) such that for any \( \tau > \bar{\tau} \), there exists \( s_0 = s_0(\tau) > 0 \) such that for any solution \( u \in (L^2(Q))^2 \) to problem (2.1)–(2.2), the following estimate holds true:

\[
\int_Q |u|^2 e^{2s\phi} \, dx \leq C_1 \left( \|f_{-1} e^{s\phi}\|^2_{(H^{-1}(Q))^2} + \sum_{j=0}^2 \|f_j e^{s\phi}\|^2_{(L^2(Q))^2} + \int_{Q_\omega} |u|^2 e^{2s\phi} \, dx \right), \quad \forall s \geq s_0(\tau), \tag{2.12}
\]

where the constant \( C_1 = C_1(\tau) > 0 \) is independent of \( s \).

3. Determination of the density and the Lamé coefficients by a single measurement

Recall that the differential operator \( L_{\lambda,\mu} \) is defined by (1.2). We assume (1.3) for \( \rho, \lambda, \mu \). By \( u = u(\lambda, \mu, \rho, p, q, \eta)(x) \), we denote the sufficiently smooth solution to
\[ \rho(x')(\partial_{x_0}^2 u)(x) = (L_{\lambda,\mu} u)(x), \quad x \in Q, \]  
(3.1)

\[ u(x) = \eta(x), \quad x \in (0,T) \times \partial \Omega, \]  
(3.2)

\[ u(T/2, x') = p(x'), \quad (\partial_{x_0} u)(T/2, x') = q(x'), \quad x' \in \Omega, \]  
(3.3)

where \( \eta, p \) and \( q \) are suitably given functions.

Let \( \omega \subset \Omega \) be a suitably given subdomain. We consider

**Inverse Problem.** Let \( p_j, q_j, \eta_j, 1 \leq j \leq N, \) be appropriately given. Then determine \( \lambda(x'), \mu(x'), \rho(x'), \) \( x' \in \Omega, \) by

\[ u(\lambda, \mu, \rho, p_j, q_j, \eta_j)(x), \quad x \in Q_{\omega} \equiv (0,T) \times \omega. \]  
(3.4)

Our formulation of the inverse problem is one with a finite number of observations (i.e., \( N < \infty \)). For inverse problems for the non-stationary Lamé equation by infinitely many boundary observations (i.e., Dirichlet-to-Neumann map), we refer to Rachele [42], for example. A monograph of Yahkno [48] is concerned with the inverse problems for the Lamé system.

For the formulation with a finite number of observations, Bukhgeim and Klibanov [8] proposed a remarkable method based on a Carleman estimate and established the uniqueness for similar inverse problems for scalar partial differential equations. As works after [8], see:

2. Imanuvilov and Yamamoto [21], Isakov [29, 30], Klibanov [35] for the corresponding inverse problems for parabolic equations;
3. Bukhgeim, Cheng, Isakov and Yamamoto [7], Imanuvilov and Yamamoto [22–24], Isakov [28–30], Isakov and Yamamoto [31], Khaıdarov [33, 34], Klibanov [35], Puel and Yamamoto [40, 41], Yamamoto [50] for inverse problems of determining potentials, damping coefficients or the principal terms in scalar hyperbolic equations.

In particular, for inverse hyperbolic equations, we have to assume that the observation subdomain \( \omega \) should satisfy a geometric condition and the observation time \( T \) has to be sufficiently large, which is a natural consequence of the hyperbolicity of the governing partial differential equations. Such situations are similar for our inverse problem for the Lamé system.

The Lamé estimate for the non-stationary Lamé equation was obtained for functions with compact supports, by Eller, Isakov, Nakamura and Tataru [13], Ikehata, Nakamura and Yamamoto [16], Imanuvilov, Isakov and Yamamoto [19], Isakov [28], and, by the methodology by [8] or [22], several uniqueness results are available for the inverse problem for Lamé system (3.1)–(3.3): [28] established the uniqueness in determining a single coefficient \( \rho(x') \), using four measurements (i.e., \( N = 4 \)). Later [16] reduced the number of measurements to three (i.e., \( N = 3 \)) for determining \( \rho \).

Recently [19] proved conditional stability and the uniqueness in the determination of the three functions \( \lambda(x'), \mu(x'), \rho(x'), \) \( x' \in \Omega, \) with only two measurements (i.e., \( N = 2 \)).

In the papers [16,19,28], the authors have to assume that \( \partial \omega \supset \partial \Omega \) because the basic Carleman estimates require that solutions under consideration have compact supports in \( Q \).

In [28] and [16], the key is a Carleman estimate where the right hand side is estimated in an \( L^2 \)-space with the divergence and the estimate is proved via a system of hyperbolic equations of \( u \) and \( \text{div} u \) with the same principal terms. On the other hand, in [19], the key is a Carleman estimate with \( L^2 \)-right hand side where \( \|e^\phi \text{div} u\|_{L^2(Q)}^2 \) is reduced to \( \|ue^\phi\|_{L^2(Q)}^2 \) by means of an \( H^{-1} \)-Carleman estimate for a scalar hyperbolic equation. In [19], as its consequence, we can reduce \( N \) to take \( N = 2 \) for simultaneous determination of all the three functions \( \lambda, \mu, \rho \).

In this section, we will further apply a Carleman estimate (Th. 2.3) whose right hand side is estimated in \( H^{-1} \) space to prove the conditional stability and the uniqueness with a single measurement: \( N = 1 \). Thus the main achievements are...
the reduction of the number of observations, i.e., \( N = 1 \). The previous paper [19] requires \( N = 2 \);
(2) the relaxation of the assumptions on the observation subdomain \( \omega \).
We will be able to prove similar results on the uniqueness and the stability in the three dimensional case on the basis of the corresponding Carleman estimate, and in a forthcoming paper, we will discuss the details.
In order to formulate our main result, we will introduce notations and an admissible set of unknown parameters \( \lambda, \mu, \rho \). Henceforth we set \((x',y') = \sum_{j=1}^{2} x_j y_j \) for \( x' = (x_1,x_2) \) and \( y' = (y_1,y_2) \). Let a subdomain \( \omega \subset \Omega \) satisfy
\[
\partial \omega \supset \{ x' \in \partial \Omega; \{(x' - y'), \eta(x') \} \geq 0 \} \equiv \Gamma
\]
with some \( y' \not\in \overline{\Omega} \).

**Remark.** Under Condition (3.5) on \( \omega \), we can prove the observability inequality for the wave equation \( \partial_{x_0}^2 \Delta - \Delta \) if the observation time \( T \) is larger than \( 2 \sup_{x' \in \Omega} |x' - y'| \) (e.g., [39]). If (3.5) holds and \( T > 0 \) is sufficiently large, then \( \omega \) and \( T \) satisfy the geometric optics condition in [1], so that we can prove observability inequalities. On the other hand, for solving inverse problems, a Carleman estimate is essential and observability inequalities are not directly applicable. If for other \( \omega \) and \( T > 0 \), we will be able to verify Condition 2.1 similarly to Lemma 3.1 or [24], then we can establish similar results to Theorem 3.1 below. However searches for other \( \omega \) and \( T \) are omitted here because those are lengthy.

Denote
\[
d = \left( \sup_{x' \in \overline{\Omega}} |x' - y'|^2 - \inf_{x' \in \overline{\Omega}} |x' - y'|^2 \right)^{\frac{1}{2}}.
\]
Next we define an admissible set of unknown coefficients \( \lambda, \mu, \rho \). Let \( M_0 \geq 0, \ 0 < \theta_0 \leq 1 \) and \( \theta_1 > 0 \) be arbitrarily fixed and let us introduce the conditions on a function \( \beta \):
\[
\begin{align*}
\beta(x') \geq \theta_1 & > 0, \quad x' \in \overline{\Omega}, \\
\|\beta\|_{C^1(\overline{\Omega})} & \leq M_0, \quad \frac{\nabla_{x'} \beta(x')(x' - y')}{2\beta(x')} \leq 1 - \theta_0, \quad x' \in \overline{\Omega \setminus \omega}.
\end{align*}
\]
(3.7)

For fixed functions \( a, b, \eta \) on \( \partial \Omega \) and \( p, q \) in \( \Omega \), we set
\[
\mathcal{W} = \mathcal{W}_{M_0, M_1, \theta_0, \theta_1, a, b} = \left\{ (\lambda, \mu, \rho) \in (C^3(\overline{\Omega}))^3; \lambda = a, \mu = b \quad \text{on} \ \partial \Omega, \right. \\
\frac{\lambda + 2\mu}{\rho}, \frac{\mu}{\rho} \quad \text{satisfy (3.7)}, \left. \frac{\min\{\mu^2(x'), \mu(x')(\lambda + \mu)(x')\}}{\rho(x')(\lambda + 2\mu)(x')} \geq \theta_1 > 0, \ x' \in \overline{\Omega}, \ \|u(\lambda, \mu, \rho, \mathbf{p}, \mathbf{q}, \eta)\|_{W^{7, \infty}(\overline{Q})} \leq M_1 \right\}
\]
(3.8)

where the constant \( M_1 \) is given.

**Remark.** The admissible set \( \mathcal{W} \) is restrictive, but contains sufficiently many \((\lambda, \mu, \rho)\). We here give a subset of \( \mathcal{W} \) which suggests that the set \( \mathcal{W} \) is not very small. Let \( \mathbf{p}, \mathbf{q} \in C^\infty(\overline{\Omega}) \) be given arbitrarily and let us choose arbitrary positive constants \( a, b, \rho_0 \). Then, for the Dirichlet boundary data \( \eta \in C^\infty([0,T] \times \partial \Omega) \), we assume
\[
(\partial_{x_0}^{2j} \eta)(T/2, x') = \left( \frac{1}{\rho_0} L_{a,b} \right)^j \mathbf{p}(x'), \quad (\partial_{x_0}^{2j+1} \eta)(T/2, x') = \left( \frac{1}{\rho_0} L_{a,b} \right)^j \mathbf{q}(x'),
\]
x' \in \partial \Omega, \ 0 \leq j \leq N_0.

Here \( N_0 \) is a sufficiently large natural number.
Theorem 3.1. We assume that 
\[
W \subseteq W_0, \quad \text{a neighbourhood of } \partial \Omega,
\]
with sufficient orders with \(\lambda = a, \mu = b, \rho = \rho_0\) in a neighbourhood of \(\partial \Omega\),
\[
\frac{\lambda + 2\mu}{\rho}(x') > \theta_1, \quad \frac{\mu(x')}{\rho} > \theta_1, \quad \min\left\{ \frac{\mu^2(x')}{\rho(x')(\lambda + \mu)(x')} > \theta_1, \quad x' \in \Omega,
\]
\[
\|\rho\|_{C^\infty(\Omega)}, \|\lambda\|_{C^\infty(\Omega)}, \|\mu\|_{C^\infty(\Omega)} < M_0,
\]
\[
\left\| \frac{\rho}{2(\lambda + 2\mu)} \nabla \left( \frac{\lambda + 2\mu}{\rho} \right) \right\|_{C^\infty(\Omega)}, \quad \left\| \frac{\rho}{2\mu} \nabla \left( \frac{\mu}{\rho} \right) \right\|_{C^\infty(\Omega)} < \frac{1 - \theta_0}{\sup_{x' \in \Omega \setminus \{0\}} |x' - y'|},
\]
The set \(W_0\) is not empty and is not “thin”. Then, since the conditions on \(\eta\) yield compatibility conditions of sufficient orders with \(p, q\) at \((T/2) \times \partial \Omega\) for any \((\lambda, \mu, \rho) \in W_0\), we can prove by an argument similar to [pp. 1369–1370, 19] that \(u(\lambda, \mu, \rho, p, q, \eta) \in C^\infty(\overline{\Omega})\) and there exists a constant \(M_1 = M_1(a, b, \rho_0, \theta_1, M_0, p, q, \eta) > 0\) such that
\[
\|u(\lambda, \mu, \rho, p, q, \eta)\|_{C^\infty(\overline{\Omega})} \leq M_1
\]
for all \((\lambda, \mu, \rho) \in W_0\). Therefore we see that \(W_0\) is a subset of \(W = W_{M_0,M_1,\theta_0,\theta_1,a,b}\) defined by (3.8). Thus, after a suitable choice of \(\eta\), we can conclude that the admissible set \(W\) can contain sufficiently many elements.

It is rather restrictive that \(\frac{\lambda + 2\mu}{\rho}\) and \(\frac{\mu}{\rho}\) should satisfy (3.7), which is one possible sufficient condition for the pseudoconvexity (i.e., Condition (2.1)). We can relax Condition (3.7) to a more generous condition which can be related with a necessary condition enabling us to establish a Carleman estimate. See Imanuvilov, Isakov and Yamamoto [20], where a scalar hyperbolic equation is discussed but the modification to the Lamé system is straightforward. Such a relaxed condition guarantees that the geodesics which are generated by the hyperbolic equations defined by (2.3), cannot remain on the level sets given by the weight function \(\phi\). In particular, by [20], we can replace the condition that \(\frac{\lambda + 2\mu}{\rho}\) and \(\frac{\mu}{\rho}\) satisfy (3.7) by one that the Hessians
\[
\partial_{x_j} \partial_{x_k} \left( \frac{\rho}{\mu} \right)_{1 \leq j, k \leq 2}, \quad \partial_{x_j} \partial_{x_k} \left( \frac{\mu}{\lambda + 2\mu} \right)_{1 \leq j, k \leq 2}
\]
are non-negative and \(|\nabla \left( \frac{\rho}{\mu} \right)| \neq 0\) and \(|\nabla \left( \frac{\mu}{\lambda + 2\mu} \right)| \neq 0\) on \(\overline{\Omega}\).

We choose \(\theta > 0\) such that
\[
\theta + \frac{M_0d}{\sqrt{\theta_1}} \sqrt{\theta} < \theta_0 \theta_1, \quad \theta_1 \inf_{x' \in \Omega} |x' - y'|^2 - \theta d^2 > 0. \quad (3.9)
\]
Here we note that since \(y' \not\in \overline{\Omega}\), such \(\theta > 0\) exists.

Let \([\cdot]\) be the first component of the vector under consideration and let \(E_2\) the \(2 \times 2\) identity matrix. We note that \((L_{\lambda,\mu}p)(x'), \text{ etc.}, \) are 2-column vectors for 2-column vectors \(p\). Let \((\lambda, \mu, \rho)\) be an arbitrary element of \(W\).

Now we are ready to state

**Theorem 3.1.** We assume that
\[
\Omega = \{(x_1, x_2); \gamma_0(x_2) < x_1 < \gamma_1(x_2), x_2 \in I\}
\]
with some open interval \(I\) and \(\gamma_0, \gamma_1 \in C(\overline{I})\). Moreover we assume that the functions \(p = (p_1, p_2)^T\) and \(q = (q_1, q_2)^T\) satisfy
\[
\det \left( \begin{pmatrix} (L_{\lambda,\mu}p)(x') & (\text{div}p(x')) & (\text{div}q(x')) E_2 \\ (L_{\lambda,\mu}q)(x') & (\text{div}q(x')) & (\text{div}q(x')) E_2 \end{pmatrix} \right) \neq 0, \forall x' \in \overline{\Omega}, \quad (3.11)
\]
Then there exist constants \( \kappa = \kappa(\mathcal{W}, \omega, \Omega, T, \lambda, \mu, \rho) \in (0, 1) \) and \( C_1 = C_1(\mathcal{W}, \omega, \Omega, T, \lambda, \mu, \rho) > 0 \) such that

\[
\| \tilde{\lambda} - \lambda \|_{L^2(\Omega)} + \| \tilde{\mu} - \mu \|_{L^2(\Omega)} + \| \tilde{\rho} - \rho \|_{H^{-1}(\Omega)} \\
\leq C_1 \| u(\lambda, \mu, \rho, p, q, \eta) - u(\tilde{\lambda}, \tilde{\mu}, \tilde{\rho}, p, q, \eta) \|_{H^1(0, T; L^2(\Omega)^2)}
\]

for any \( (\tilde{\lambda}, \tilde{\mu}, \tilde{\rho}) \in \mathcal{W} \).

As for the corresponding results on the stability for inverse problems for scalar hyperbolic equations, we refer to [22–24] for example.

Our stability and uniqueness result requires only one measurement: \( N = 1 \). For the determination of the three coefficients by a single measurement, we have to choose initial data which satisfy stronger Conditions (3.11)–(3.13) than in the case of \( N \geq 2 \). Thus Conditions (3.11)–(3.13) are not generic properties and should be realized in a non-physical way by us. Moreover, as the following example shows, we can take \( p \) and \( q \) satisfying those.

**Example of \( \Omega, p, q \) meeting (3.11)–(3.13).** We assume that \( \lambda, \mu \) are positive constants and that \( \{(x_1, x_2) \in \Omega; x_2 = y_2\} \) and \( \{(x_1, x_2) \in \Omega; x_1 = y_1\} \) are empty. Noting that the fourth columns of the matrices in (3.11) and (3.12) have \( x' - y' \) as factors, we will take quadratic functions in \( x' \). For example, we take

\[
p(x') = \begin{pmatrix} 0 \\ (x_1 - y_1)(x_2 - y_2) \end{pmatrix}, \quad q(x') = \begin{pmatrix} (x_2 - y_2)^2 \\ 0 \end{pmatrix}.
\]

Then we can verify that (3.11)–(3.13) are all satisfied.

**Remark 3.1.** In place of (3.10), let us assume

\[
\Omega = \left\{ (x_1, x_2); \gamma_0(x_1) < x_2 < \gamma_1(x_1), x_1 \in \tilde{I} \right\}
\]

with some open interval \( \tilde{I} \). Then, after replacing (3.13) by

\[
[x_2 - y_2 = 0, \quad [L_{\lambda, \mu} q_1](\partial_{x_1} p_2 + \partial_{x_2} p_1)(x') \neq [L_{\lambda, \mu} p_1](\partial_{x_1} q_2 + \partial_{x_2} q_1)(x'), \quad x' \in \Omega,
\]

the conclusion of Theorem 3.1 holds under Conditions (3.11), (3.12) and (3.14). Moreover in the case when \( \Omega \) is a more general smooth domain, we can prove the conditional stability in our inverse problem under other conditions on \( \omega \subset \Omega \). We will omit the details, for the sake of compact description of the proof.

We set

\[
\psi(x) = |x' - y'|^2 - \theta \left( x_0 - \frac{T}{2} \right)^2, \quad \phi(x) = e^{\tau \psi(x)}, \quad x = (x_0, x') \in Q.
\]
First we show

**Lemma 3.1.** Let \((\lambda, \mu, \rho) \in \mathcal{W}\), and let us assume (3.9) and (3.14). Then, for sufficiently large \(\tau > 0\), the function \(\psi\) given by (3.15) satisfies Conditions 2.1 and (2.10). Therefore the conclusion of Theorem 2.3 holds and the constants \(C_1(\tau), \tilde{\tau}\) and \(s_0(\tau)\) in (2.12) can be taken independently of \((\lambda, \mu, \rho) \in \mathcal{W}\).

**Proof.** Conditions (2.10) and the third condition in (2.6) are directly verified by means of (3.5). Conditions (2.4) and (2.5) can be verified by the same way as in Imanuvilov and Yamamoto [24], for example. Finally we have

\[
\text{Proof.}
\]

Let \(\tilde{\lambda}, \tilde{\mu}, \tilde{\rho}\) in terms of (3.8), it suffices to verify

\[
\left( \left| \frac{\partial \psi}{\partial \tilde{\tau}} \right|^2 + \left| \frac{\partial \psi}{\partial \tilde{\eta}} \right|^2 \right)^{\frac{1}{2}} = |\nabla_x \psi|
\]

and that the right hand side of the first inequality in (2.6) is greater than or equal to

\[
\min \left\{ \frac{\mu(x')}{\sqrt{\lambda + 2\mu}(x')} \frac{\sqrt{\mu(\lambda + \mu)(x')}}{\sqrt{\lambda + 2\mu}(x')} \right\} \left( \left| \frac{\partial \psi}{\partial \tilde{\tau}} \right|^2 + \left| \frac{\partial \psi}{\partial \tilde{\eta}} \right|^2 \right)^{\frac{1}{2}}
\]

in terms of (3.8), it suffices to verify

\[-(\theta(x_0 - T/2))^2 + \theta_1 |x' - y'|^2 > 0\]

for \(x \in [0, T] \times \partial \Omega\). In fact, by means of the second inequality in (3.9), we have

\[
4\theta_1 |x' - y'|^2 - 4\theta^2 \left( x_0 - \frac{T}{2} \right)^2 \geq 4\theta_1 \inf_{x' \in \Omega} |x' - y'|^2 - \theta(T^2)
\]
\[
\geq 4\theta_1 \inf_{x' \in \Omega} |x' - y'|^2 - \theta(2d + \varepsilon \sqrt{\theta})^2
\]
\[
> 0
\]

because \(\varepsilon > 0\) is sufficiently small. The uniformity of the constants \(C_1(\tau), \tilde{\tau}\) and \(s_0(\tau)\) follows similarly to [19]. Thus the proof of Lemma 3.1 is complete. \(\square\)

Next we prove a Carleman estimate for a first order partial differential operator

\[
(P_0 g)(x') = \sum_{j=1}^{2} p_{0,j}(x') \partial_{x_j} g(x'),
\]

where \(p_{0,j} \in C^1(\overline{\Omega}), j = 1, 2\).

**Lemma 3.2.** We assume

\[
\sum_{j=1}^{2} p_{0,j}(x') \partial_{x_j} \phi(T/2, x') > 0, \quad x' \in \overline{\Omega},
\]

(3.16)

Then there exists a constant \(\tau_0 > 0\) such that for all \(\tau > \tau_0\), there exist \(s_0 = s_0(\tau) > 0\) and \(C_2 = C_2(s_0, \tau_0, \Omega, \omega) > 0\) such that

\[
\int_{\Omega} s^2 |g|^2 e^{2s\phi(T/2, x')} dx' \leq C_2 \int_{\Omega} |P_0 g|^2 e^{2s\phi(T/2, x')} dx'
\]
for all \( s > s_0 \) and \( g \in H^1(\Omega) \) satisfying

\[
g = 0 \quad \text{on} \quad \left\{ x' \in \partial \Omega : \sum_{j=1}^{2} p_{0,j}(x') n_j(x') \geq 0 \right\}.
\]

**Lemma 3.3.** We assume

\[
\sum_{j=1}^{2} p_{0,j}(x') \partial_{x_j} \phi(T/2, x') \neq 0, \quad x' \in \overline{\Omega}.
\]

Then the conclusion of Lemma 3.2 is true for all \( s > s_0 \) and \( g \in H^1_0(\Omega) \).

**Proof of Lemma 3.2.** For simplicity, we set \( \phi_0(x') = \phi(T/2, x') \) and \( w = e^{s\phi_0} g, \; Q_0 w = e^{s\phi_0} P_0 (e^{-s\phi_0} w) \). Then

\[
\int_{\Omega} |P_0 g|^2 e^{2s\phi(T/2, x')} \, dx' = \int_{\Omega} |Q_0 w|^2 \, dx'.
\]

We have

\[
Q_0 w = P_0 w - sq_0 w,
\]

where \( q_0(x') = \sum_{j=1}^{2} p_{0,j}(x') \partial_{x_j} \phi_0(x') \). Therefore, by (3.16) and integration by parts, we obtain

\[
\|Q_0 w\|_{L^2(\Omega)}^2 = \|P_0 w\|_{L^2(\Omega)}^2 + s^2 \|q_0 w\|_{L^2(\Omega)}^2 - 2s \int_{\Omega} \sum_{j=1}^{2} p_{0,j} (\partial_{x_j} w) q_0 w \, dx' \geq s^2 \int_{\Omega} q_0(x')^2 w^2(x') \, dx' - s \int_{\Omega} \sum_{j=1}^{2} p_{0,j} q_{0,j} (w^2) \, dx' \geq C_0 s^2 \int_{\Omega} w^2(x') \, dx' - s \int_{\partial \Omega} \sum_{j=1}^{2} p_{0,j} q_{0,j} w^2 n_j \, dS + s \int_{\Omega} \sum_{j=1}^{2} \partial_{x_j} (p_{0,j} q_{0,j}) w^2 \, dx' \geq (C_2 s^2 - C_3 s) \int_{\Omega} w^2 \, dx' \geq 0 \]

By (3.16), we have \( q_0 > 0 \) on \( \partial \Omega \), so that the right hand side is greater than or equal to \( (C_2 s^2 - C_3 s) \int_{\Omega} w^2 \, dx' \). Thus by taking \( s > 0 \) sufficiently large, the proof of Lemma 3.2 is complete. \( \square \)

The proof of Lemma 3.3 is similar, thanks to the fact that the integral on \( \partial \Omega \) vanishes for \( g \in H^1_0(\Omega) \).

Now we proceed to

**Proof of Theorem 3.1.** The proof is similar to Isakov, Imanuvilov and Yamamoto [19], Imanuvilov and Yamamoto [22–24] and the new ingredient is an \( H^{-1} \)-Carleman estimate (Lem. 3.1) . Henceforth, for simplicity, we set

\[
\mathbf{u} = \mathbf{u}(\lambda, \mu, \rho, \mathbf{p}, \mathbf{q}, \eta), \quad \mathbf{v} = \mathbf{u}(\tilde{\lambda}, \tilde{\mu}, \tilde{\rho}, \mathbf{p}, \mathbf{q}, \eta)
\]

and

\[
\mathbf{y} = \mathbf{u} - \mathbf{v}, \quad f = \rho - \tilde{\rho}, \quad g = \lambda - \tilde{\lambda}, \quad h = \mu - \tilde{\mu}.
\]

In (3.13), without loss of generality, we may assume that

\[
x_1 - y_1 > 0, \quad (x_1, x_2) \in \overline{\Omega}.
\]
Then we set
\[ F(x_1, x_2) = \int_{\gamma_1(x_2)} f(\xi, x_2) d\xi, \quad (x_1, x_2) \in \Omega. \]  
(3.17)

If \( x_1 - y_1 < 0 \) for \((x_1, x_2) \in \overline{\Omega}\), then it is sufficient to replace (3.17) by \( F(x_1, x_2) = \int_{\gamma_0(x_2)} f(\xi, x_2) d\xi, \quad (x_1, x_2) \in \Omega. \)

Then
\[ \hat{\rho} \bar{\partial}_{x_0}^2 y = L_{\lambda, \mu} y + G u \quad \text{in} \quad Q \]  
(3.18)

and
\[ y \left( \frac{T}{2}, x' \right) = \partial_{x_0} y \left( \frac{T}{2}, x' \right) = 0, \quad x' \in \Omega \]  
(3.19)

and
\[ y = 0 \quad \text{on} \quad (0, T) \times \partial \Omega. \]  
(3.20)

Here we set
\[ G u(x) = -\partial_{x_1} F(x') \bar{\partial}_{x_0}^2 u(x) + (g + h)(x') \nabla_{x'}(\text{div} u)(x) + h(x') \Delta u(x) \]
\[ + (\text{div} u)(x) \nabla_{x'} g(x') + (\nabla_{x'} u(x) + (\nabla_{x'} u(x))^T) \nabla_{x'} h(x'). \]  
(3.21)

By (3.14), we have the inequality \( \frac{\phi'^2}{T^2} > d'. \) Therefore, by (3.6) and Definition (3.15) of the function \( \phi \), we have
\[ \phi(T/2, x') \geq d_1, \quad \phi(0, x') \leq \phi(T, x') < d_1, \quad x' \in \overline{\Omega} \]
with \( d_1 = \exp(\tau \inf_{x' \in \Omega} |x' - y'|^2) \). Thus, for given \( \epsilon > 0 \), we can choose a sufficiently small \( \delta = \delta(\epsilon) > 0 \) such that
\[ \phi(x) \geq d_1 - \epsilon, \quad x \in \left[ \frac{T}{2} - \delta, \frac{T}{2} + \delta \right] \times \overline{\Omega} \]  
(3.22)

and
\[ \phi(x) \leq d_1 - 2 \epsilon, \quad x \in ([0, 2 \delta] \cup [T - 2 \delta, T]) \times \overline{\Omega}. \]  
(3.23)

In order to apply Lemma 3.1, it is necessary to introduce a cut-off function \( \chi \) satisfying \( 0 \leq \chi \leq 1, \chi \in C^\infty(\mathbb{R}) \)
and
\[ \chi = \begin{cases} 0 & \text{on} \ [0, \delta] \cup [T - \delta, T], \\ 1 & \text{on} \ [2 \delta, T - 2 \delta]. \end{cases} \]  
(3.24)

In the sequel, \( C_J > 0 \) denote generic constants depending on \( s_0, T, M_0, M_1, \theta_0, \theta_1, \eta, T, y', \omega, \chi \) and \( p, q, \varepsilon, \delta \), but independent of \( s > s_0 \).

Setting \( z_1 = \chi \bar{\partial}_{x_0}^2 y, z_2 = \chi \bar{\partial}_{x_0}^4 y \) and \( z_3 = \chi \bar{\partial}_{x_0}^4 y \), we have
\[
\begin{cases}
\hat{\rho} \bar{\partial}_{x_0}^2 z_1 = L_{\lambda, \mu} z_1 + \chi G(\bar{\partial}_{x_0}^2 u) + 2\hat{\rho}(\bar{\partial}_{x_0}^2 \chi) \bar{\partial}_{x_0}^4 y + \hat{\rho}(\bar{\partial}_{x_0}^2 \chi) \bar{\partial}_{x_0}^2 y, \\
\hat{\rho} \bar{\partial}_{x_0}^2 z_2 = L_{\lambda, \mu} z_2 + \chi G(\bar{\partial}_{x_0}^4 u) + 2\hat{\rho}(\bar{\partial}_{x_0} \chi) \bar{\partial}_{x_0}^4 y + \hat{\rho}(\bar{\partial}_{x_0} \chi) \bar{\partial}_{x_0}^2 y, \\
\hat{\rho} \bar{\partial}_{x_0}^2 z_3 = L_{\lambda, \mu} z_3 + \chi G(\bar{\partial}_{x_0}^4 u) + 2\hat{\rho}(\bar{\partial}_{x_0} \chi) \bar{\partial}_{x_0}^4 y + \hat{\rho}(\bar{\partial}_{x_0} \chi) \bar{\partial}_{x_0}^4 y \quad \text{in} \quad Q.
\end{cases}
\]  
(3.25)

Henceforth we set
\[ \mathcal{E} = \int_{Q} (|\bar{\partial}_{x_0}^2 y|^2 + |\bar{\partial}_{x_0}^4 y|^2 + |\bar{\partial}_{x_0}^4 y|^2) e^{2s_0^2} dx. \]
Noting that $u \in W^{7,\infty}(Q)$, in view of (3.24) and Lemma 3.1, we can apply Theorem 2.3 to (3.25), so that

\[
\sum_{j=2}^{4} \int_{Q} |\partial_{x_{0}}^{j}y|^{2} \chi^{2} e^{2s\phi} dx \leq C_{5}(\|Fe^{s\phi}\|_{L^{2}(Q)}^{2} + \|ge^{s\phi}\|_{L^{2}(Q)}^{2} + \|he^{s\phi}\|_{L^{2}(Q)}^{2})
\]

\[
+ C_{5} \sum_{j=4}^{5} \|\partial_{x_{0}}^{j}\chi\partial_{x_{0}}^{j}y)e^{s\phi}\|_{L^{2}(0,T;\mathcal{H}^{-1}(\Omega))}^{2}
\]

\[
+ C_{5} \sum_{j=2}^{4} \|\partial_{x_{0}}^{2}\chi\partial_{x_{0}}^{2}y)e^{s\phi}\|_{L^{2}(0,T;\mathcal{H}^{-1}(\Omega))}^{2} + C_{5}\epsilon
\]

\[
\leq C_{6}(\|Fe^{s\phi}\|_{L^{2}(Q)}^{2} + \|ge^{s\phi}\|_{L^{2}(Q)}^{2} + \|he^{s\phi}\|_{L^{2}(Q)}^{2}) + C_{6}e^{2s(d_{1} - 2\epsilon)} + C_{7}\epsilon
\]

(3.26)

for all large $s > 0$.

On the other hand,

\[
\int_{\Omega} |(\partial_{x_{0}}^{2}y)(T/2, x')|^{2} e^{2s\phi(T/2, x')} dx'
\]

\[
= \int_{0}^{T/2} \frac{\partial}{\partial x_{0}} \left( \int_{\Omega} |(\partial_{x_{0}}^{2}y)(x_{0}, x')|^{2} \chi(x_{0})^{2} e^{2s\phi} dx' \right) dx_{0}
\]

\[
= \int_{0}^{T/2} \int_{\Omega} 2(\partial_{x_{0}}^{3}y) \cdot (\partial_{x_{0}}^{2}y)\chi^{2} e^{2s\phi} dx
\]

\[
+ 2s \int_{0}^{T/2} \int_{\Omega} |\partial_{x_{0}}^{2}y|^{2} \chi \partial_{x_{0}}(\chi) e^{2s\phi} dx + \int_{0}^{T/2} \int_{\Omega} |\partial_{x_{0}}^{3}y|^{2} (\partial_{x_{0}}(\chi)) e^{2s\phi} dx
\]

\[
\leq C_{7} \int_{Q} s\chi^{2}(|\partial_{x_{0}}^{1}y|^{2} + |\partial_{x_{0}}^{2}y|^{2}) e^{2s\phi} dx + C_{7} e^{2s(d_{1} - 2\epsilon)}.
\]

Therefore (3.26) yields

\[
\int_{\Omega} |(\partial_{x_{0}}^{2}y)(T/2, x')|^{2} e^{2s\phi(T/2, x')} dx' \leq C_{8}s \int_{Q} \left( |F|^{2} + |g|^{2} + |h|^{2} \right) e^{2s\phi} dx + C_{8} e^{2s(d_{1} - 2\epsilon)} + C_{8}s\epsilon
\]

(3.27)

for all large $s > 0$. Similarly we can estimate $\int_{\Omega} |(\partial_{x_{0}}^{3}y)(T/2, x')|^{2} e^{2s\phi(T/2, x')} dx'$ to obtain

\[
\int_{\Omega} |((\partial_{x_{0}}^{2}y)(T/2, x')|^{2} + |(\partial_{x_{0}}^{3}y)(T/2, x')|^{2}) e^{2s\phi(T/2, x')} dx'
\]

\[
\leq C_{9}s \int_{Q} \left( |F|^{2} + |g|^{2} + |h|^{2} \right) e^{2s\phi} dx + C_{9} e^{2s(d_{1} - 2\epsilon)} + C_{9}s\epsilon
\]

(3.28)

for all large $s > 0$.

Now first order partial differential equations satisfied by $h$, $g$ and $F$ are going to be considered. That is, by (3.18), (3.19) and $u, v \in W^{7,\infty}(Q)$, we have

\[
\tilde{\rho} \partial_{x_{0}}^{2}y \left( \frac{T}{2}, x' \right) = Gu \left( \frac{T}{2}, x' \right), \quad \tilde{\rho} \partial_{x_{0}}^{3}y \left( \frac{T}{2}, x' \right) = G \partial_{x_{0}}u \left( \frac{T}{2}, x' \right).
\]

(3.29)
Then, setting
\[
\begin{align*}
-\frac{1}{\rho} L_{\lambda, \mu} p &= \begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix}, \quad -\frac{1}{\rho} L_{\lambda, \mu} q = \begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix}, \\
\text{div } p &= b_1, \quad \text{div } q = b_2,
\end{align*}
\]
we rewrite (3.29) as
\[
\begin{align*}
\nabla x' p + (\nabla x' p)^T &= \begin{pmatrix} e_1 \\ d_1 \\ e_1 \end{pmatrix}, \quad \nabla x' q + (\nabla x' q)^T = \begin{pmatrix} c_2 \\ d_2 \\ c_2 \end{pmatrix},
\end{align*}
\]
Because linear system (3.31) possesses a solution \((\partial_x F, \partial_x g, \partial_x g)\), the coefficient matrix must satisfy
\[
\det \begin{pmatrix} a_{11} & b_1 & 0 & c_1 \\ a_{21} & 0 & b_1 & d_1 \\ a_{12} & b_2 & 0 & c_2 \\ a_{22} & 0 & b_2 & d_2 \end{pmatrix} = 0,
\]
that is,
\[
(\partial_x, h) \det \begin{pmatrix} a_{11} & b_1 & 0 & c_1 \\ a_{21} & 0 & b_1 & d_1 \\ a_{12} & b_2 & 0 & c_2 \\ a_{22} & 0 & b_2 & d_2 \end{pmatrix} + (\partial_x, h) \det \begin{pmatrix} a_{11} & b_1 & 0 & d_1 \\ a_{21} & 0 & b_1 & e_1 \\ a_{12} & b_2 & 0 & d_2 \\ a_{22} & 0 & b_2 & e_2 \end{pmatrix} = \det \begin{pmatrix} a_{11} & b_1 & 0 & G_{11} \\ a_{21} & 0 & b_1 & G_{21} \\ a_{12} & b_2 & 0 & G_{12} \\ a_{22} & 0 & b_2 & G_{22} \end{pmatrix},
\]
by the linearity of the determinant. Under Condition (3.11), taking into consideration \(h = \mu - \tilde{\mu} = 0\) on \(\partial \Omega\) and considering (3.32) as a first order partial differential operator in \(h\), we apply Lemma 3.3, so that
\[
s^2 \int_{\Omega} |h|^2 e^{2s\phi(T/2, x')} dx' \leq C_{10} \left\| \begin{pmatrix} a_{11} & b_1 & 0 & G_{11} \\ a_{21} & 0 & b_1 & G_{21} \\ a_{12} & b_2 & 0 & G_{12} \\ a_{22} & 0 & b_2 & G_{22} \end{pmatrix} \right\|_{L^2(\Omega)}^2
\]
\[
\leq C_{11} \int_{\Omega} \left( \left| \partial_{x_0} y \left( \frac{T}{2}, x' \right) \right|^2 + \left| \partial_{x_0} y \left( \frac{T}{2}, x' \right) \right|^2 \right) e^{2s\phi(T/2, x')} dx' + C_{11} \int_{\Omega} (|g|^2 + |h|^2) e^{2s\phi(T/2, x')} dx',
\]
in view of (3.30). We rewrite (3.29) as
\[
\begin{pmatrix} a_{11} & c_1 & d_1 \\ a_{21} & d_1 & e_1 \\ a_{12} & e_2 & d_2 \\ a_{22} & d_2 & e_2 \end{pmatrix} \begin{pmatrix} \partial_x F \\ \partial_x g \\ \partial_x h \end{pmatrix} = \begin{pmatrix} G_{11} - b_1 \partial_x g \\ G_{21} - b_1 \partial_x g \\ G_{12} - b_2 \partial_x g \\ G_{22} - b_2 \partial_x g \end{pmatrix},
\]
\[ s^2 \int_\Omega |g|^2 e^{2s\phi(T/2,x')} dx' \leq C_{12} \int_\Omega \left( |\partial_{x_0}^2 y(T/2,x')|^2 + |\partial_{x_0}^4 y(T/2,x')|^2 \right) e^{2s\phi(T/2,x')} dx' \]
\[ + C_{12} \int_\Omega (|g|^2 + |h|^2) e^{2s\phi(T/2,x')} dx' \]  
(3.34)
for all large \( s > 0 \). By (3.33) and (3.34), for sufficiently large \( s > 0 \), we have
\[ s^2 \int_\Omega (|g|^2 + |h|^2) e^{2s\phi(T/2,x')} dx' \leq C_{13} \int_\Omega \left( |\partial_{x_0}^2 y(T/2,x')|^2 + |\partial_{x_0}^4 y(T/2,x')|^2 \right) e^{2s\phi(T/2,x')} dx'. \]  
(3.35)
Moreover, eliminating \( \partial_{x_2} h \) in the first and the third rows in (3.31) and using (3.13), we have
\[ \partial_{x_1} \left( F + \frac{d_2 b_1 - d_1 b_2}{d_2 a_{11} - d_1 a_{12}} g + \frac{d_2 c_1 - d_1 c_2}{d_2 a_{11} - d_1 a_{12}} h \right) = \frac{d_2 G_{11} - d_1 G_{12}}{d_2 a_{11} - d_1 a_{12}} + g \partial_{x_1} \left( \frac{d_2 b_1 - d_1 b_2}{d_2 a_{11} - d_1 a_{12}} \right) + h \partial_{x_1} \left( \frac{d_2 c_1 - d_1 c_2}{d_2 a_{11} - d_1 a_{12}} \right) . \]
By (3.10) and (3.17), if \( n_1(x') \geq 0 \), then \( x_1 = \gamma_1(x_2) \), that is, we have: \( F(x_1, x_2) = 0 \) for \( n_1(x') \geq 0 \). Therefore, noting \( g = h = 0 \) on \( \partial \Omega \) and setting \( p_{0,1} = 1 \), \( p_{0,2} = 0 \) in Lemma 3.2, we can apply the lemma. Thus, in view of (3.35) and (3.30), we obtain
\[ s^2 \int_\Omega |F|^2 e^{2s\phi(T/2,x')} dx' \leq C_{14} \int_\Omega \left( |\partial_{x_0}^2 y(T/2,x')|^2 + |\partial_{x_0}^4 y(T/2,x')|^2 \right) e^{2s\phi(T/2,x')} dx' \]  
(3.36)
for all large \( s > 0 \). Consequently, substituting (3.35) and (3.36) into (3.28) and using \( \phi(T/2,x') \geq \phi(x_0, x') \) for \((x_0, x') \in Q\), we obtain
\[ \int_\Omega (|F|^2 + |g|^2 + |h|^2) e^{2s\phi(T/2,x')} dx' \leq \frac{C_{15} T}{s} \int_\Omega (|F|^2 + |g|^2 + |h|^2) e^{2s\phi(T/2,x')} dx' + \frac{C_{15}}{s} e^{2s(d_1 - 2\epsilon)} + \frac{C_{15}}{s} \epsilon \]
for all large \( s > 0 \). Taking \( s > 0 \) sufficiently large and noting \( e^{2s\phi(T/2,x')} \geq e^{2ad_1} \) for \( x' \in \Omega \), we obtain
\[ \int_\Omega (|F|^2 + |g|^2 + |h|^2) dx' \leq C_{16} e^{-4\epsilon} + C_{17} e^{2sC_{18} \epsilon} \]  
(3.37)
for all large \( s > s_0 \): a constant which is dependent on \( \tau \), but independent of \( s \). Next we take in (3.37) instead of the constant \( C_{17} \) the constant \( C_{17} e^{2sC_{18} \epsilon} \). Now this inequality holds true for all \( s > 0 \).

Now we choose \( s > 0 \) such that
\[ e^{2sC_{18} \epsilon} = e^{-4\epsilon}, \]
that is,
\[ s = -\frac{1}{4\epsilon + 2C_{16}} \ln \epsilon. \]
Here we may assume that \( \epsilon < 1 \) and so \( s > 0 \). Then it follows from (3.37) that
\[ \int_\Omega (|F|^2 + |g|^2 + |h|^2) dx' \leq 2C_{15} e^{\frac{2s}{\epsilon} + \frac{2sC_{18} \epsilon}{\epsilon}}. \]
By Definition (3.17) of $F$, we have
\[ \int_\Omega f r dx_1 dx_2 = \int_\Omega (\partial_{x_2} F) r dx_1 dx_2 = \int_\Omega F(\partial_{x_2} r) dx_1 dx_2 \]
for all $r \in H_0^1(\Omega)$ by integration by parts. Hence we can directly verify that $\|f\|_{H^{-1}(\Omega)} \leq C \|F\|_{L^2(\Omega)}$, so that the proof of Theorem 3.1 is complete. 

4. PROOF OF THEOREM 2.1

Without loss of generality, we may assume that $\rho \equiv 1$. Otherwise we introduce new coefficients $\mu_1 = \mu/\rho, \lambda_1 = \lambda/\rho$ to argue similarly. We can directly verify that the functions $\text{rot } u \equiv \partial_{x_2} u_2 - \partial_{x_1} u_1$ and $\text{div } u$ satisfy the equations
\[ \partial_{x_0}^2 \text{rot } u - \mu \Delta \text{rot } u = m_1, \quad \partial_{x_0}^2 \text{div } u - (\lambda + 2\mu) \Delta \text{div } u = m_2 \quad \text{in } Q, \tag{4.1} \]
where
\[ m_1 = K_1 \text{rot } u + K_2 \text{div } u + K_1 u + \text{rot } f, \quad m_2 = K_3 \text{rot } u + K_4 \text{div } u + K_2 u + \text{div } f \]
and $K_1, K_2$ are first order differential operators with $L^\infty$ coefficients.

Thanks to Condition 2.1 on the weight function $\psi$, there exists $\hat{\tau}$ such that for all $\tau > \hat{\tau}$, the Carleman estimate for equations (4.1) (see e.g., [45]) yields the inequality
\[ \|((\nabla \text{rot } u) e^{s\phi})^2_{(L^2(Q))^2} + s\|((\text{div } u) e^{s\phi})^2_{(L^2(Q))^2} + s^3\|((\text{rot } u) e^{s\phi})^2_{(L^2(Q))^2} \leq C_1 \left( s^2\|e^{s\phi}\|_{L^2(Q)}^2 + s\|\frac{\partial u}{\partial n} e^{s\phi}\|_{(H^1((0,T) \times \partial\Omega))^2}^2 \right) \]
\[ + s\left( \frac{\partial^2 u}{\partial n^2} e^{s\phi}\right)^2_{(L^2((0,T) \times \partial\Omega))^2} + s^3\left( \frac{\partial u}{\partial n} e^{s\phi}\right)^2_{(L^2((0,T) \times \partial\Omega))^2} + \|u\|^2_{L^2(Q)} \right), \quad \forall s \geq s_0(\tau), \tag{4.2} \]
where the constant $C_1$ is independent of $s$.

In order to estimate the $H^1(Q)$-norm of the function $u$, we need the following proposition.

**Proposition 4.1.** There exists $\hat{\tau} > 1$ such that for any $\tau > \hat{\tau}$, there exists $s_0(\tau)$ such that
\[ \int_{Q_\omega} \left( \frac{1}{s} \sum_{j,k=1}^2 |\partial_{x_j} \partial_{x_k} u|^2 + s|\nabla u|^2 + s^3|u|^2 \right) e^{2s\phi} dx \]
\[ \leq C_2 \left( \|\text{rot } u\|^2_{L^2(Q)} + \|\text{div } u\|^2_{L^2(Q)} + \int_{Q_\omega} (s|\nabla u|^2 + s^3|u|^2) e^{2s\phi} dx \right), \quad \forall s \geq s_0(\tau), \quad u \in (H_0^1(Q))^2. \tag{4.3} \]

**Proof of Proposition 4.1.** Denote $\text{rot } u = y$ and $\text{div } u = w$ and let $\text{rot }^* v = \left( \frac{\partial w}{\partial x_2}, -\frac{\partial w}{\partial x_1} \right)$. Using a well-known formula: $\text{rot } \text{rot } = -\Delta_y + \nabla_y \text{div }$, we obtain
\[ -\Delta_y u = -\text{rot }^* y - \nabla_y w \quad \text{in } Q, \quad u|_{\partial Q} = 0. \]

Then (4.3) follows from the Carleman estimate for an elliptic equations obtained by the first author in [17].
By (4.2) and (4.3), we estimate \( \sum_{|\alpha|=0,\alpha=(0,1,2)} \| \partial_\alpha^2 \mathbf{u} e^{\phi} \|_{L^2(\Omega))^2} \) via the right hand side of inequality (4.2). Next using this estimate and equation (1.1), we obtain the estimate for the norm \( \| \partial_\alpha^2 \mathbf{u} e^{\phi} \|_{L^2(\Omega))^2} \) via the right hand side of (4.2). Finally we obtain the estimate for \( \| \partial_\alpha^2 \mathbf{u} e^{\phi} \|_{L^2(\Omega))^2} \) and \( s^2 \| \partial_\alpha \mathbf{u} e^{\phi} \|_{L^2(\Omega))^2} \) by the interpolation argument. Therefore, combining these estimates with (4.2), we have

\[
\| \mathbf{u} \|_{Y(\phi,\Omega)}^2 \leq C_3 \left( s^2 \| \mathbf{f} e^{\phi} \|_{L^2(\Omega))^2} + \| \nabla \mathbf{f} e^{\phi} \|_{L^2(\Omega))^2} \right) + s \| \partial_\alpha \mathbf{u} e^{\phi} \|_{(H^1(0,T) \times \partial \Omega))^2} + s \| \partial_\alpha^2 \mathbf{u} e^{\phi} \|_{(L^2(0,T) \times \partial \Omega))^2} + \| \mathbf{u} \|_{(\phi,Q_\omega)}^2 , \quad \forall s \geq s_0(\tau),
\]

(4.4)

where the constant \( C_3 \) is independent of \( s \). Here we recall definition (2.8) of \( \| \mathbf{u} \|_{(\phi,Q_\omega)}^2 \) and the definition of \( \| \mathbf{u} \|_{Y(\phi,\Omega)}^2 \) in (2.9).

Now we need to estimate the boundary integrals at the right hand side of (4.4). In order to do that, it is convenient to use another weight function \( \varphi \) such that \( \varphi|_{\partial \Omega} = \phi|_{\partial \Omega} \) and \( \varphi(x) < \phi(x) \) for all \( x \) in small neigbourhood of \( (0,T) \times \partial \Omega \). We introduce the function \( \varphi \) by formulae:

\[
\varphi(x) = e^{\varphi(x)}, \quad \tilde{\varphi}(x) = \varphi(x) - \hat{\epsilon} \ell_1(x') + N \ell_1^2(x'),
\]

where \( \hat{\epsilon} > 0 \) is a small positive parameter, \( N > 0 \) is a large positive parameter, and \( \ell_1 \in C^1(\overline{\Omega}) \) is a function such that

\[
\ell_1(x') > 0, \quad \forall x' \in \Omega, \quad \ell_1|_{\partial \Omega} = 0, \quad \nabla x' \ell_1|_{\partial \Omega} \neq 0.
\]

Denote \( \Omega_{1/N^2} = \{ x' \in \Omega; \text{dist}(x',\partial \Omega) \leq \frac{1}{N^2} \} \). Obviously for any fixed \( \hat{\epsilon} > 0 \), there exists \( N_0(\hat{\epsilon}) \) such that

\[
\varphi(x) < \phi(x), \quad \forall x \in [0,T] \times \Omega_{1/N^2}, \quad N \in (N_0(\hat{\epsilon}),\infty).
\]

Now we will prove the following estimate:

**Lemma 4.1.** Under conditions of Theorem 2.1, there exist \( \hat{\tau} > 0 \) and \( N_0 > 1 \) such that for all \( \tau > \hat{\tau} \), there exists \( s_0(\tau,N) \) such that

\[
\| \mathbf{u} \|_{Y(\varphi,\Omega)}^2 + N \sum_{|\alpha|=0}^2 s^{\delta-2|\alpha|} \| \partial_\alpha^2 \mathbf{u} e^{\phi \varphi} \|_{L^2(\Omega))^2} \leq C_4 \left( s^2 \| \mathbf{f} e^{\phi \varphi} \|_{L^2(\Omega))^2} + \| \nabla \mathbf{f} e^{\phi \varphi} \|_{L^2(\Omega))^2} + \| \mathbf{u} \|_{(\phi,Q_\omega)}^2 \right) , \quad \forall s \geq s_0(\tau,N), \quad N > N_0, \quad \text{supp} \mathbf{u} \subset [0,T] \times \Omega_{1/N^2},
\]

(4.5)

where the constant \( C_4 \) is independent of \( s \) and \( N \).

The proof of Lemma 4.1 is given in Sections 5–8. Now, using this result of the lemma, we finish the proof of Theorem 2.1. Let us fix the parameter \( N \) such that (4.5) holds true. We take \( \delta \in \left(0, \frac{1}{4T}\right) \) sufficiently small such that

\[
\phi(x) > \varphi(x), \quad \forall x \in \Omega_{\delta/2} \setminus \Omega_{\delta/2}.
\]

(4.6)

We consider a cut off function \( \vec{\vartheta} \in C^1(\overline{\Omega}) \) such that \( \vec{\vartheta}|_{\Omega_{\delta}} = 1 \) and \( \vec{\vartheta}|_{\Omega_{\delta}} \cap T_{\omega/\delta} = 0 \). The function \( \vec{\vartheta} \mathbf{u} \) satisfies the equation

\[
P(\vec{\vartheta} \mathbf{u}) = \vec{\vartheta} \mathbf{f} + [P, \vec{\vartheta}] \mathbf{u}, \quad \mathbf{u}|_{(0,T) \times \partial \Omega} = 0, \quad \mathbf{u}(0,\cdot) = \mathbf{u}_{\omega_0}(0,\cdot) = \mathbf{u}(T,\cdot) = \mathbf{u}_{\omega_0}(T,\cdot) = 0.
\]
Applying Carleman estimate (4.5) to this equation, we obtain

\[ s^2 \frac{\partial u}{\partial n} e^{s\varphi} \leq C_b \left( s^2 \| e^{s\varphi} \|_{L^2(Q)}^2 + \| (\nabla e^{s\varphi}) \|_{L^2(Q)}^2 \right) + C_3 \left( \| u \|_{B((\varphi,Q_0))}^2 \right), \quad \forall s \geq s_0(\tau). \]  

(4.7)

Since the supports of the coefficients of the commutator \([P, \tilde{\theta}]\) are in \(\Omega_2 \setminus \Omega_{\delta/2}\) by (4.6), we have

\[ s^2 \| (P, \tilde{\theta}) u e^{s\varphi} \|_{L^2(Q)}^2 \leq C_9 \left( \sum_{|\alpha| = 0}^2 s^{3-2|\alpha|} \| (\partial_x^\alpha u) e^{s\varphi} \|_{L^2(Q)}^2 + \| u \|_{B((\varphi,Q_0))}^2 \right). \]  

(4.8)

Combining (4.7) and (4.8), we obtain

\[ s^2 \left( \sum_{|\alpha| = 0}^2 s^{3-2|\alpha|} \| (\partial_x^\alpha u) e^{s\varphi} \|_{L^2(Q)}^2 + \| u \|_{B((\varphi,Q_0))}^2 \right), \quad \forall s \geq s_0(\tau). \]  

(4.9)

Finally we will estimate the surface integrals at the right hand side of (4.4) by the right hand side of (4.9). In the new inequality, the term

\[ \sum_{|\alpha| = 0}^2 s^{3-2|\alpha|} \| (\partial_x^\alpha u) e^{s\varphi} \|_{L^2(Q)}^2 \]

which appears at the right hand side, can be absorbed by \(\| u \|_{\nabla(\varphi,Q)}^2\). Thus the proof of Theorem 2.1 is complete.

\[ \square \]

5. PROOF OF LEMMA 4.1

In this section, we will prove Lemma 4.1. Following the standard technique, we reduce the proof of estimate (4.5) to subelliptic estimate (5.13) for the operator \(P_{\sigma}\). Next show that we can act microlocally in this case. Namely we reduce estimate (5.13) to estimate (5.15). In the situation with the Lamé system this reduction is not trivial, since we have the subelliptic estimate with loss of one derivative. This difficulty is overcome with the help of the second large parameter \(N\) inserted into the function \(\varphi\). Finally we formulate several lemmata on factorization of pseudo-differential operators, \textit{a priori} estimates of Cauchy problem for pseudo-differential operators, and Carleman estimate for a second order scalar hyperbolic equation, which are used in Sections 6–8.

\textbf{Proof of Lemma 4.1.} First we note that, thanks to the large parameter \(N\), it suffices to prove (4.5) only locally by assuming

\[ \text{supp } u \subset B_3 \cap [(0, T] \times \Omega_{1/N^2}], \]

where \(B_3\) is the ball of the radius \(\delta > 0\) centered at some point \(y^*\). In the case of \(B_3 \cap ((0, T) \times \partial \Omega) = \emptyset\), we can prove (4.5) in a usual way for a function with compact support (see e.g., [15]). Without loss of generality, we may assume that \(y^* = (y_0, 0, 0)\). Moreover the parameter \(\delta > 0\) can be chosen arbitrarily small. Assume that near \((0, 0, 0)\), the boundary \(\partial \Omega\) is locally given by the equation \(x_2 - \ell(x_1) = 0\). Furthermore, since the function
\( \bar{u} = O \mathbf{u}(x_0, \mathcal{O}^{-1}x') \) satisfies system (2.1) and (2.2) with \( \bar{f} = O \mathbf{f}(x_0, \mathcal{O}^{-1}x') \) for any orthogonal matrix \( \mathcal{O} \), we may assume that

\[
\ell'(0) \equiv \frac{d \ell}{dx_1}(0) = 0. \tag{5.1}
\]

Making the change of variables \( y_1 = x_1 \) and \( y_2 = x_2 - \ell(x_1) \), we reduce equation (2.1) to the form

\[
\begin{cases}
\mathbb{P}_1 \mathbf{u} = \frac{\partial^2 u_1}{\partial y_0^2} - \mu \left( \frac{\partial^2 u_1}{\partial y_1^2} - 2 \ell'(y_1) \frac{\partial^2 u_1}{\partial y_1 \partial y_2} + (1 + |\ell'(y_1)|^2) \frac{\partial^2 u_1}{\partial y_2^2} \right) + \mu \ell''(y_1) \frac{\partial u_1}{\partial y_2}, \\
-(\lambda + \mu) \frac{\partial}{\partial y_1} \left( \text{div} \mathbf{u} - \frac{\partial u_1}{\partial y_2} \ell' \right) + (\lambda + \mu) \frac{\partial}{\partial y_2} \left( \text{div} \mathbf{u} - \frac{\partial u_1}{\partial y_2} \ell' \right) \ell' + \mathcal{K}_1 \mathbf{u} = f_1,
\end{cases}
\]

\[
\begin{cases}
\mathbb{P}_2 \mathbf{u} = \frac{\partial^2 u_2}{\partial y_0^2} - \mu \left( \frac{\partial^2 u_2}{\partial y_1^2} - 2 \ell'(y_1) \frac{\partial^2 u_2}{\partial y_1 \partial y_2} + (1 + |\ell'(y_1)|^2) \frac{\partial^2 u_2}{\partial y_2^2} \right) + \mu \ell''(y_1) \frac{\partial u_2}{\partial y_2}, \\
-(\lambda + \mu) \frac{\partial}{\partial y_1} \left( \text{div} \mathbf{u} - \frac{\partial u_2}{\partial y_2} \ell' \right) + \mathcal{K}_2 \mathbf{u} = f_2,
\end{cases} \tag{5.2}
\]

where we use the same notations \( \mathbf{u}, \mathbf{f} \) after the change of variables and \( \mathcal{K}_1, \mathcal{K}_2 \) are partial differential operators of the first order. We set \( \mathbb{P} = (\mathbb{P}_1, \mathbb{P}_2) \) and

\[
z_1 = \frac{\partial u_2}{\partial y_1} - \frac{\partial u_2}{\partial y_2} \ell'(y_1) - \frac{\partial u_1}{\partial y_2}, \quad z_2 = \frac{\partial u_1}{\partial y_1} + \frac{\partial u_2}{\partial y_2} - \frac{\partial u_1}{\partial y_2} \ell'(y_1).
\]

After the change of variables, equations (4.1) have the form

\[
P_\mu z_1 = \frac{\partial^2 z_1}{\partial y_0^2} - \mu \left( \frac{\partial^2 z_1}{\partial y_1^2} - 2 \ell'(y_1) \frac{\partial^2 z_1}{\partial y_1 \partial y_2} + (1 + |\ell'(y_1)|^2) \frac{\partial^2 z_1}{\partial y_2^2} \right) + \mu \ell''(y_1) \frac{\partial z_1}{\partial y_2} = m_1 \quad \text{in} \quad \mathcal{G}_N \equiv \mathbb{R}^2 \times \left[ 0, \frac{\kappa}{\mathcal{N}^2} \right],
\]

\[
P_{\lambda + 2\mu} z_2 = \frac{\partial^2 z_2}{\partial y_0^2} - \lambda + 2\mu \left( \frac{\partial^2 z_2}{\partial y_1^2} - 2 \ell'(y_1) \frac{\partial^2 z_2}{\partial y_1 \partial y_2} + (1 + |\ell'(y_1)|^2) \frac{\partial^2 z_2}{\partial y_2^2} \right) + (\lambda + 2\mu) \ell''(y_1) \frac{\partial z_2}{\partial y_2} = m_2 \quad \text{in} \quad \mathcal{G}_N. \tag{5.4}
\]

Here we use the same notations \( m_1, m_2 \) after the change of variables and the constant \( \kappa > 0 \) is chosen sufficiently large such that the image of \( (0, T) \times \Omega_1(\mathcal{N}) \cap \partial_\mathcal{B}(y') \) belongs to \( \mathcal{G}_N \). Henceforth we write \( (z_1, z_2) = R(y, D) \mathbf{u} \), where

\[
D = (D_{y_1}, D_{y_2}, D_{y_3}), \quad D_{y_j} = \frac{1}{i} \frac{\partial}{\partial y_j}, \quad j = 0, 1, 2, \text{etc.,}
\]

and \( \mathfrak{g} \) denotes the complex conjugate of \( c \in \mathbb{C} \).

Now we claim that in order to prove Lemma 4.1, it suffices to establish the following estimate for the function \( \mathbf{w} = (w_1, w_2) = e^{s \varphi}(z_1, z_2) = e^{s \varphi} R(y, D) \mathbf{u} \):

\[
\| \mathbf{w} \|_s^2 \leq s \| \mathbf{w} \|_{L^2(\mathcal{G}_N)}^2 + s^3 \| \mathbf{u} \|_{L^2(\partial \mathcal{G}_N)}^2 + s \| \frac{\partial \mathbf{w}}{\partial y_2} \|_{L^2(\partial \mathcal{G}_N)}^2 + s \| \mathbf{w} \|_{H^1(\partial \mathcal{G}_N)}^2
\]

\[
+ s^3 \| \mathbf{w} \|_{L^2(\partial \mathcal{G}_N)}^2 \leq C_s \| \mathbf{u} \|_{L^2(\mathcal{G}_N)}^2 + s \| \mathbf{w} \|_{L^2(\partial \mathcal{G}_N)}^2 + s \| \mathbf{w} \|_{L^2(\mathcal{G}_N)}^2 + \sum_{|\alpha| = 0}^2 s^{4-2|\alpha|} \| \partial_\mathcal{B}^{\alpha} \mathbf{u} \|_{L^2(\mathcal{G}_N)}^2, \quad \forall s \geq s_0(\tau, \mathcal{N}), \tag{5.5}
\]

where \( \mathcal{B}^{\alpha} \) and \( \partial_\mathcal{B}^{\alpha} \) denote the \( \alpha \)-th order partial derivatives of \( \mathbf{w} \) and \( \mathbf{u} \), respectively.
for all \( u \in (H^2(\mathcal{G}_N))^2 \) satisfying \( u|_{\partial\mathcal{G}_N} = 0 \) and \( \text{supp} u \subset B_\delta \cap \mathcal{G}_N \). Obviously the function \( w \) satisfies the boundary condition

\[
\frac{\partial w_1}{\partial y_2} = \frac{\lambda + 2\mu}{\mu} \frac{\partial w_2}{\partial y_1} + s\varphi_2(y^*) w_1 - \frac{\lambda + 2\mu}{\mu} \varphi_1(y^*) w_2 + g_1, \quad \text{on } \partial\mathcal{G}_N, \tag{5.6}
\]

\[
\frac{\partial w_2}{\partial y_2} = -\frac{\mu}{\lambda + 2\mu} \frac{\partial w_1}{\partial y_1} + s\varphi_2(y^*) w_2 + s\frac{\mu}{\lambda + 2\mu} \varphi_1(y^*) w_1 + g_2, \quad \text{on } \partial\mathcal{G}_N, \tag{5.7}
\]

where the function \( g = (g_1, g_2) \) satisfies the estimate

\[
s\|g\|^2_{L^2(\partial\mathcal{G}_N)^2} \leq c(\delta) \left( s \|\partial w\|^2_{L^2(\partial\mathcal{G}_N)^2} + s\|w\|^2_{H^1(\partial\mathcal{G}_N)^2} + s^3\|w\|^2_{L^2(\partial\mathcal{G}_N)^2} \right) + C_6 s\|\text{Pe}^{s\varphi}\|^2_{L^2(\partial\mathcal{G}_N)^2}, \tag{5.8}
\]

and \( \lim_{s \to 0} c(\delta) = 0 \).

Boundary Conditions (5.6) and (5.7) with property (5.8) follow from equation (5.2) and the zero Dirichlet boundary condition for \( u \).

In order to deduce (4.5) from estimate (5.5), it suffices to show

\[
\|u\|^2_{L^2(\varphi, \mathcal{G}_N)} \leq C_7 (\|w\|^2 + \|\text{Pe}^{s\varphi}\|^2_{H^1(\mathcal{G}_N)^2} + s^3\|\text{Pe}^{s\varphi}\|^2_{L^2(\mathcal{G}_N)^2}), \quad \forall s \geq s_0(\tau, N). \tag{5.9}
\]

For the proof of (5.9), we need

**Proposition 5.1.** There exist \( \tilde{\tau} > 1 \) and \( N_0 > 1 \) such that for any \( \tau > \tilde{\tau} \) and \( N > N_0(\tau) \), there exists \( s_0(\tau, N) \) such that

\[
N \int_\mathcal{G}_N \left( \frac{1}{s\varphi} \sum_{j,k=1}^2 |\partial_{y_j}\partial_{y_k} u|^2 + s\varphi|\partial_{y_j} u|^2 + s^3\varphi^3|u|^2 \right) e^{2s\varphi} \text{d}y \\
\leq C_5 (\|z_1 e^{s\varphi}\|^2_{H^1(\mathcal{G}_N)} + \|z_2 e^{s\varphi}\|^2_{H^1(\mathcal{G}_N)}), \quad \forall u \in (H^1_0(\mathcal{G}_N))^2, \quad \text{supp } u \subset B_\delta \cap \mathcal{G}_N, \forall s \geq s_0(\tau, N),
\]

where the constant \( C_5 \) is independent of \( N \).

We give the proof of Proposition 5.1 in Appendix I. Thanks to Proposition 5.1 and equations (5.2), we obtain

\[
N\|\partial_{y_0} u\|_{L^2(\mathcal{G}_N)^2} + s^2\|\partial_{y_0} u\|_{L^2(\mathcal{G}_N)^2} \leq C_9 \|w\|^2_{L^2(\mathcal{G}_N)^2}, \quad \forall s \geq s_0(\tau, N). \tag{5.10}
\]

By (5.5) and (5.8)–(5.10), we obtain

\[
N\|\partial_{y_0} u\|_{L^2(\mathcal{G}_N)^2} + s^2\|\partial_{y_0} u\|_{L^2(\mathcal{G}_N)^2} \leq \|w\|^2_{L^2(\mathcal{G}_N)^2}, \quad \forall s \geq \max\{s_0(\tau, N), N\}. \tag{5.11}
\]

Finally, combining (5.11) with the estimates

\[
s^2\|\partial_{y_0} u\|_{L^2(\mathcal{G}_N)^2} \leq C_{11} \left( \|\partial_{y_0} u\|_{L^2(\mathcal{G}_N)^2} + s^4\|u\|^2_{L^2(\mathcal{G}_N)^2} \right)
\]
and 
\[ \| (\partial_{y_0} \partial_{y_k} u)e^{sx}\|_{L^2(G_N)^2}^2 \leq C_{11} \sum_{j=0}^2 \| (\partial_{y_j} u)e^{sx}\|_{L^2(G_N)^2}^2, \quad k \in \{1, 2\}, \]

we obtain (5.9).

Now we will proceed to the proof of (5.5). We set \( P_{\mu, s} = e^{s|x|} P_{\mu} e^{-s|x|} \) and \( P_{\lambda + 2\mu, s} = e^{s|x|} P_{\lambda + 2\mu} e^{-s|x|} \). By \( p(y, \xi_0, \xi_1, \xi_2) \) and \( p(y, \xi_0, \xi_1, \xi_2) \) with \( \beta = \mu \) or \( \lambda + 2\mu \), we denote the principal symbols of the operators \( P \) and \( P_\beta \) respectively. In order to prove Carleman estimate (5.5), it is convenient for us to introduce a new variable \( \sigma \) and consider \( s \) as a dual variable to \( \sigma \). Following [46], Chapter 14, we consider the pseudo-differential operators defined by

\[
\begin{align*}
\mathbb{P}_\beta(y, D, D_{y_0}, D_{y_1}, D_{y_2})v &= \int_\mathbb{R}\mathbb{P}(y, \xi_0 + i\xi_1|\varphi_{y_0}, \xi_1 + i\xi_2, D_{y_2} + i\xi_2|\varphi_{y_2})\hat{\nu}(s, \xi_0, \xi_1, y_2) e^{i<y, \xi'>+\sigma s}\,d\sigma d\xi', \\
\mathbb{P}_\sigma(y, D, D_{y_0}, D_{y_1}, D_{y_2})v &= \int_\mathbb{R}\mathbb{P}(y, \xi_0 + i\xi_1|\varphi_{y_0}, \xi_1 + i\xi_2, D_{y_2} + i\xi_2|\varphi_{y_2})\hat{\nu}(s, \xi_0, \xi_1, y_2) e^{i<y, \xi'>+\sigma s}\,d\sigma d\xi',
\end{align*}
\]

where \( \xi' = (\xi_0, \xi_1) \), \( y' = (y_0, y_1) \) and \( \hat{\nu}(s, \xi_0, \xi_1, y_2) \) is the Fourier transform of \( v(\sigma, y_0, y_1, y_2) \) with respect to \( \sigma, y_0, y_1 \). Let \( v(\sigma, y) = (v_1(\sigma, y), v_2(\sigma, y)) \) be a function with the domain \( \mathbb{Q} = \mathbb{R}^2 \times \mathbb{R}_+^4 \). Henceforth \( \mathcal{F}_\sigma \) denotes the Fourier transform with respect to the variable \( \sigma \). Let \( h(s) = (1 + s^2)^\frac{1}{2}, \Sigma = \partial \mathbb{Q} \). Moreover we set \( g = (g_1, g_2) \),

\[
R_s(y, D)u = e^{s|x|} R(y, D)e^{-s|x|} u,
\]

and

\[
\begin{align*}
B_1 w &\triangleq \frac{\partial w_1}{\partial y_2} - \frac{\lambda + 2\mu}{\mu} \frac{\partial w_1}{\partial y_1} + |s|\varphi_{y_2}(y^*)w_1 + |s|\frac{\lambda + 2\mu}{\mu} \varphi_{y_1}(y^*)w_2, \\
B_2 w &\triangleq -\frac{\partial w_2}{\partial y_2} + \frac{\lambda + 2\mu}{\mu} \frac{\partial w_1}{\partial y_1} + |s|\varphi_{y_2}(y^*)w_2 + |s|\frac{\mu}{\lambda + 2\mu} \varphi_{y_1}(y^*)w_1 \quad \text{on } \Sigma.
\end{align*}
\]

for \( w = (w_1, w_2) \), provided that the right hand sides are well-defined.

Then we claim that in order to prove (5.5), it sufficest to establish the following estimate

\[
\|v\|^2 \triangleq \sum_{j=0}^1 \|h(D_j)\|^2_{L^2(\mathbb{R}^1; L^2(G_N)^2)} + \|h(D_j)\|^3_{L^2(G_N)^2} \leq C_{12} \left( \|\mathbb{P}_\sigma(y, D)\mathcal{F}_\sigma^{-1} u\|^2_{L^2(G_N)^2} + \|h(D_j)\mathcal{F}_\sigma^{-1} g\|^2_{L^2(G_N)^2} + \|\mathcal{F}_\sigma^{-1} u\|^2_{L^2(G_N)^2} \right),
\]

if \( u \) and \( v \) satisfy \( \text{supp} u \subset \mathbb{R}^1 \times (B_0 \cap G_N) \), \( \text{supp} \mathcal{F}_\sigma^{-1} u \subset (-\sigma_0, \sigma_0) \times (B_0 \cap G_N) \) with arbitrarily small parameter \( \sigma_0 > 0 \), and

\[
\begin{align*}
R_s(y, D)u &= \mathcal{F}_\sigma v, \quad u|_{\Sigma} = 0 \\
B_1(\mathcal{F}_\sigma v) &= g_1, \quad B_2(\mathcal{F}_\sigma v) = g_2 \quad \text{on } \Sigma.
\end{align*}
\]

We set \( \mathcal{F}_\sigma v = w \).

Then

\[
(\text{B}_1 w, B_2 w) = (g_1, g_2) \equiv g.
\]

This fact can be proved exactly in the same way as in [46], Chapter 14, Section 2.

Consider the finite covering of the unit sphere \( S^2 \equiv \{(s, \xi_0, \xi_1): s^2 + \xi_0^2 + \xi_1^2 = 1\} : S^2 \subset \bigcup_{\zeta \in S^2} \{ \zeta = (s, \xi_0, \xi_1) \in S^2; |\zeta - \zeta'| < \delta_1 \} \) and the partition of unity \( \chi_\nu(\zeta) : \sum_{\nu=1}^R(\chi_\nu(\zeta) = 1 \text{ for any } \zeta \in S^2 \text{ and } \text{supp } \chi_\nu \subset \{ \zeta \in S^2; |\zeta - \zeta'| < \delta_1 \} \).
We extend the function $\chi_\nu$ on the set $|\zeta| > 1$ as the homogeneous function of the order zero in such a way that

$$\text{supp } \chi_\nu \subset \mathcal{O}(\delta_1) \equiv \left\{ \zeta; \left| \frac{\zeta}{|\zeta|} - \zeta^* \right| < \delta_1 \right\},$$

and continue $\chi_\nu$ on the set $|\zeta| < 1$ up to a $C^\infty$ function.

We set $D' = (D_\sigma, D_{\gamma_0}, D_{\gamma_1})$, and consider the pseudo-differential operator $\chi_\nu(D')$ and the function $\chi_\nu(D')v$. Obviously equalities (5.14) hold true with $w$ and $g$ replaced by $w_\nu$ and $g_\nu = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \chi_\nu(D')\varphi e^{-is\sigma}d\sigma$.

Moreover we claim that instead of (5.13), it suffices to prove the following estimate

$$||\chi_\nu(D')v|| \leq C_{13} \left( ||\mathcal{P}_\sigma \chi_\nu(D')\mathcal{F}_\sigma^{-1}U||_{H^1(\mathbb{Q})^2} + ||h(D_\sigma)\chi_\nu(D')\mathcal{F}_\sigma^{-1}g||_{L^2(\Sigma)^2} + ||\mathcal{F}_\sigma^{-1}U||_{H^2(\mathbb{Q})^2} \right), \quad (5.15)$$

where

$$R_\nu(y, D')U = \mathcal{F}_\sigma v, \quad U|_\Sigma = 0, \quad \text{supp } \mathcal{F}_\sigma^{-1}U \subset (-\sigma_0, \sigma_0) \times (B_\delta \cap \mathcal{Q}_N),$$

$$B_1(w_{1,\nu}, w_{2,\nu}) = g_{1,\nu}, \quad B_2(w_{1,\nu}, w_{2,\nu}) = g_{2,\nu} \quad (5.16)$$

and $C_{13}$ is independent of $N$. In fact, assume that estimate (5.15) is already proved. Then

$$||v||^2 \leq \sum_{\nu=1}^{K(\delta_1)} ||\chi_\nu(D')v||^2 \leq C_{14} \sum_{\nu=1}^{K} \left( ||\mathcal{P}_\sigma(y, D)\chi_\nu\mathcal{F}_\sigma^{-1}U||_{H^1(\mathbb{Q})^2}^2 + ||h(s)g_\nu||_{L^2(\Sigma)^2}^2 + ||\chi_\nu(D')\mathcal{F}_\sigma^{-1}U||_{H^1(\mathbb{Q})^2}^2 \right)$$

$$\leq C_{15} \sum_{\nu=1}^{K} \left( ||\chi_\nu(D')\mathcal{P}_\sigma(y, D)\mathcal{F}_\sigma^{-1}U||_{H^1(\mathbb{Q})^2}^2 + ||[\chi_\nu(D'), \mathcal{P}_\sigma(y, D')][\mathcal{F}_\sigma^{-1}U||_{H^1(\mathbb{Q})^2}^2 + \right)$$

$$+ ||h(s)g_\nu||_{L^2(\Sigma)^2}^2 + ||\chi_\nu(D')\mathcal{F}_\sigma^{-1}U||_{H^1(\mathbb{Q})^2}^2 \right) \leq C_{16} \left( ||\mathcal{P}_\sigma(y, D)\mathcal{F}_\sigma^{-1}U||_{H^1(\mathbb{Q})^2}^2 + ||h(s)g||_{L^2(\Sigma)^2}^2 + ||\mathcal{F}_\sigma^{-1}U||_{H^2(\mathbb{Q})^2}^2 \right),$$

where $K = K(\delta_1)$ and $C_{16}$ are independent of $N$.

Estimate (5.15) follows from Lemmas 6.1, 7.1 and 8.1 which are proved in Sections 6–8.

Now we formulate some results and introduce some definitions which will be used in the proof of estimate (5.15).

The principal symbol of the operator $P_{\beta, s}$ has the form

$$p_{\beta}(y, s, \xi_0, \xi_1) = -\langle \xi_0 + is|\varphi_{y_0} \rangle^2 + \beta \langle (\xi_1 + is|\varphi_{y_1} \rangle^2 - 2\ell'(\xi_1 + is|\varphi_{y_1} \rangle \xi_2 + is|\varphi_{y_2} \rangle + (\xi_2 + is|\varphi_{y_2} \rangle^2 |G|^2 \rangle, \quad (5.17)$$

where $|G|^2 = 1 + (\ell'(y_1))^2$.

The roots of this polynomial with respect to the variable $\xi_2$, are

$$\Gamma_{\beta}^\pm(y, s, \xi_0, \xi_1) = -i|s|\varphi_{y_2}(y) + \alpha_{\beta}^\pm(y, s, \xi_0, \xi_1), \quad (5.18)$$

$$\alpha_{\beta}^\pm(y, s, \xi_0, \xi_1) = \frac{\langle \xi_1 + is|\varphi_{y_1}(y) \ell'(y_1) \rangle}{|G|^2} \pm \sqrt{r_{\beta}(y, s, \xi_0, \xi_1)}, \quad (5.19)$$

$$r_{\beta}(y, \zeta) = \frac{(\xi_0 + is|\varphi_{y_0}(y))^2 - \beta(\xi_1 + is|\varphi_{y_1}(y))^2 |G|^2 + \beta(\xi_2 + is|\varphi_{y_2} (y))^2 (\ell')^2}{|\beta|^4}, \quad (5.20)$$

where the function $\sqrt{r_{\beta}}$ is defined below.
Denote $\gamma = (y^*,\zeta^*) = (y^*,s^*,\xi_0^*,\zeta_1^*)$.

**Proposition 5.2.** Suppose that $|r_\beta(\gamma)| \geq 2\delta > 0$. Then there exists $\delta_0 = \delta_0(\delta) > 0$ such that for all $\delta, \delta_1 \in (0, \delta_0)$, there exists a constant $C_{20} > 0$, independent of $s$, such that for one of the roots of polynomial (5.17), which we denote by $\Gamma^-_\beta$, we have

$$-\text{Im}\, \Gamma^-_\beta(y,s,\xi_0,\xi_1) \geq C_{20}|s|, \quad \forall (y,s,\xi_0,\xi_1) \in B_\delta \times \mathcal{O}(\delta_1). \quad (5.21)$$

**Proof of Proposition 5.2.** If $\text{Im}\sqrt{r_\beta(\gamma)} \neq 0$, then statement (5.21) is trivial. So it suffices to consider the case $\text{Im}\sqrt{r_\beta(\gamma)} = 0$. Let $\theta \in (0, \frac{1}{2})$ be a constant. Obviously there exists $\delta(\theta)$ such that for all $\delta, \delta_1 \in (0, \delta(\theta))$, we have

$$\text{Re}\, r_\beta(y,\zeta) \geq (1 - 2\theta)|r_\beta(y,\zeta)|, \quad \forall (y,s,\xi_0,\xi_1) \in B_\delta \times \mathcal{O}(\delta_1).$$

Then

$$|\text{Im}\, r_\beta(y,\zeta)| \leq \frac{2\theta}{1-2\theta} \text{Re}\, r_\beta(y,\zeta), \quad \forall (y,s,\xi_0,\xi_1) \in B_\delta \times \mathcal{O}(\delta_1).$$

We denote $b(y,\zeta) = \text{Im}\, r_\beta(y,\zeta)$ and $a(y,\zeta) = \text{Re}\, r_\beta(y,\zeta)$ with $\zeta = (s,\xi_0,\xi_1)$. First, if $\text{Im}\sqrt{r_\beta(\gamma)} = 0$, then we have $a(\gamma) > 0$ and $b(\gamma) = 0$. In that case we define the function $\sqrt{r_\beta(y,\zeta)}$ by the infinite series

$$(1 + x)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} c_n x^n, \quad |x| < 1,$$

where $c_n = \frac{1}{n!} \frac{1}{(\frac{1}{2}-1)(\frac{1}{2}-2)\ldots(\frac{1}{2}-(n-1))}$. That is, assuming that $|\frac{b}{a}| < \frac{2\theta}{1-2\theta} < \frac{1}{2}$ for all $(y,s,\xi_0,\xi_1) \in B_\delta \times \mathcal{O}(\delta_1)$, we set

$$\sqrt{r_\beta(y,\zeta)} = \sqrt{a} \sum_{n=0}^{\infty} c_n \left(\frac{ib}{a}\right)^n = \sqrt{a} + \frac{ib}{2|a|} \left(\frac{b}{s|a|}\right) - |s| \left(\frac{b}{a}\right) \frac{b}{s|a|} \sum_{n=0}^{\infty} c_n + 2 \left(\frac{ib}{a}\right)^n. \quad (5.22)$$

The first term in infinite series (5.22) is real, and the absolute value of the third term is $|s| \frac{b}{s|a|} O(\theta)$. The function $\frac{b}{|s|\sqrt{a}(\gamma)}$ is a continuous homogeneous function of the order zero in the variable $\zeta$.

If $\frac{b}{|s|\sqrt{a}(\gamma)} \leq 0$, then we take $\Gamma^-_\beta(y,\zeta) = -i|s| \frac{\partial \varphi''}{\partial y} + \alpha^-_\beta(y,\zeta)$ where $\alpha^-_\beta(y,\zeta)$ equals the right hand side of (5.22) plus $(\xi_1 + i|s|s)\ell''(y)/|\Gamma|^2$. Otherwise $\Gamma^-_\beta(y,\zeta) = -i|s| \frac{\partial \varphi''}{\partial y} + \alpha^+_\beta(y,\zeta)$ where $\alpha^+_\beta(y,\zeta)$ equals the right hand side of (5.22) multiplied by $-1$ plus $(\xi_1 + i|s|s)\ell''(y)/|\Gamma|^2$.

For $\frac{b}{|s|\sqrt{a}(\gamma)} \geq 0$, we obtain that $\frac{b}{|s|\sqrt{a}(\gamma)} - \frac{1}{2} \varphi''(y) < 0$ for all $(y,s,\xi_0,\xi_1) \in B_\delta \times \mathcal{O}(\delta_1)$ and for $\frac{b}{|s|\sqrt{a}(\gamma)} \leq 0$ we obtain that $-\frac{b}{|s|\sqrt{a}(\gamma)} - \frac{1}{2} \varphi''(y) < 0$ for all $(y,s,\xi_0,\xi_1) \in B_\delta \times \mathcal{O}(\delta_1)$. These inequalities imply (5.21) provided that $\delta_1$ is taken sufficiently small. The proof of Proposition 5.2 is finished.

Under some conditions, we can see that the operator $P_\beta$ can be factorized as a product of two first order pseudo-differential operators:

**Proposition 5.3.** Let $\beta \in \{\mu, \lambda + 2\mu\}$ and $|r_\beta(y,\zeta)| \geq \delta > 0$ for all $(y,\zeta) \in B_\delta \times \mathcal{O}(2\delta_1)$. Then we can factorize the operator $P_\beta$ as the product of two first order pseudo-differential operators:

$$P_\beta \chi_\nu(D')V = \beta |G|^2(D_{y_2} - \Gamma^-_\beta(y,D'))(D_{y_2} - \Gamma^+_\beta(y,D'))\chi_\nu(D')V + T_\beta V, \quad (5.23)$$

where $\text{supp}\, V \subset B_\delta \cap \mathcal{G}_N$ and $T_\beta$ is a continuous operator:

$$T_\beta : L^2(0,1; H^1(R^3)) \rightarrow L^2(0,1; L^2(R^3)).$$
Let us consider the equation
\[(D_{y_2} - \Gamma_\beta (y, D'))\chi_\nu(D')V = q, \quad V|_{y_2=\chi_{y_2}} = 0, \quad \text{supp } V \subset B_\delta \cap \mathcal{G}_N.\]

For the solutions to this problem, we have an \textit{a priori} estimate:

**Proposition 5.4.** Let \(\beta \in \{\mu, \lambda + 2\mu\}\) and \(|r_\beta (y, \zeta)| \geq \tilde{\delta} > 0\) for all \((y, \zeta) \in B_\delta \times \mathcal{O}(2\delta_1)\). Then there exists a constant \(C_{22} > 0\), which is independent of \(N\), such that
\[
\|h(D_\sigma)\chi_\nu(D')V|_{y_2=0}\|_{L^2(\Sigma)} \leq C_{22}\|q\|_{L^2(\Sigma)}. \tag{5.24}
\]

**Proof of Proposition 5.4.** Taking the scalar product of \(q\) and \(h^2(D_\sigma)\chi_\nu(D')V\) for fixed \(y_2\), we obtain
\[
2\text{Re}(q(y_2), h^2(D_\sigma)\chi_\nu(D')V(y_2))_{L^2(\Sigma)} e^{2\tilde{\delta} y_2} = \frac{\partial}{\partial y_2} \left( e^{2\tilde{\delta} y_2} \|h(D_\sigma)\chi_\nu(D')V(y_2)\|_{L^2(\Sigma)}^2 \right)
- 2\text{Re}(i\Gamma_\beta (y, D')h^2(D_\sigma)\chi_\nu(D')V + \tilde{\kappa}\chi_\nu(D')V, h^2(D_\sigma)\chi_\nu(D')V)_{L^2(\Sigma)} e^{2\tilde{\delta} y_2}.
\]

By (5.21) and Proposition 2.4.A in [47], for sufficiently large positive \(\tilde{\kappa}\), we have
\[
\text{Re}(i\Gamma_\beta (y, D')h^{-2}(D_\sigma)h^2(D_\sigma)\chi_\nu(D')V + \tilde{\kappa}\chi_\nu(D')V, h^2(D_\sigma)\chi_\nu(D')V)_{L^2(\Sigma)} \geq C_{23}\|h^2(D_\sigma)\chi_\nu(D')V\|_{L^2(\Sigma)}^2.
\]

Thus
\[
2\text{Re}(q(y_2), h^2(D_\sigma)\chi_\nu(D')V(y_2))_{L^2(\Sigma)} e^{2\tilde{\delta} y_2} \leq \frac{\partial}{\partial y_2} \left( e^{2\tilde{\delta} y_2} \|h(D_\sigma)\chi_\nu(D')V(y_2)\|_{L^2(\Sigma)}^2 \right) - C_{23}\|h^2(D_\sigma)\chi_\nu(D')V(y_2)\|_{L^2(\Sigma)}^2 e^{2\tilde{\delta} y_2},
\]
and (5.24) follows from Gronwall’s inequality. \(\square\)

Let \(\tilde{w}(s, y)\) satisfy a scalar second order hyperbolic equation
\[
P_{\beta,s}\tilde{w} = q \quad \text{in } \mathcal{G}_N, \quad \frac{\partial \tilde{w}}{\partial y_2}|_{y_2=1} = \tilde{w}|_{y_2=1} = 0, \quad \text{supp } \tilde{w} \subset \mathbb{R}^1 \times (B_\delta \cap \mathcal{G}_N)
\]
for almost all \(s \in \mathbb{R}^1\). Let \(P_{\beta,s}^*\) be the formally adjoint operator to \(P_{\beta,s}\), where \(\beta \in \{\mu, \lambda + 2\mu\}\). Set
\[
L_{+, \beta} = \frac{P_{\beta,s} + P_{\beta,s}^*}{2}, \quad L_{-, \beta} = \frac{P_{\beta,s} - P_{\beta,s}^*}{2}.
\]

One can easily check that the principal part operator \(L_{-, \beta}\) is given by formula
\[
L_{-, \beta}\tilde{w} = -2|s|\varphi_{y_0}\frac{\partial \tilde{w}}{\partial y_0} + \beta \left( 2|s|\varphi_{y_2}\frac{\partial \tilde{w}}{\partial y_1} - 2|s|\ell'(y_1) \left( \varphi_{y_2}\frac{\partial \tilde{w}}{\partial y_1} + \varphi_{y_1}\frac{\partial \tilde{w}}{\partial y_2} \right) + 2|s|(1 + (\ell'(y_1))^2)\varphi_{y_2}\frac{\partial \tilde{w}}{\partial y_2} \right).
\]

Obviously \(L_{+, \beta}\tilde{w} + L_{-, \beta}\tilde{w} = q\). For almost all \(s \in \mathbb{R}^1\), the following equality holds true:
\[
B_\beta + \|L_{-, \beta}\tilde{w}\|_{L^2(\mathcal{G}_N)}^2 + \|L_{+, \beta}\tilde{w}\|_{L^2(\mathcal{G}_N)}^2 + \text{Re} \int_{\mathcal{G}_N} (|L_{+, \beta}, L_{-, \beta}|\tilde{w}, \tilde{w}) dy = \|q\|_{L^2(\mathcal{G}_N)}^2. \tag{5.25}
\]
where
\[
B_\beta = \text{Re} \int_{\partial \Omega_N} \tilde{p}_\beta(y, \nabla \varphi, -\xi_0)(|s| \tilde{p}_\beta(y, \nabla \tilde{w}) - |s|^2 \tilde{p}_\beta(y, \nabla \varphi, \tilde{w})^2) dy_0 dy_1 + \text{Re} \int_{\partial \Omega_N} \tilde{p}_\beta(y, \nabla \tilde{w}, -\xi_0) L_{-\beta} \tilde{w} dy_0 dy_1.
\]
(5.26)
\[
\xi_3 = (0, 0, 1) \text{ and }
\]
\[
\tilde{p}_\beta(y, \xi_0, \xi_1, \xi_0) = \xi_0 \xi_0 - \beta (\xi_1 \xi_1 - \ell'(y)(\xi_1 \xi_2 + \xi_2 \xi_1)) + (1 + |\ell'(y)|^2 \xi_2 \xi_2).
\]

We note that \(\phi_{y_k}|_{\bar{\Sigma}} = \varphi_{y_k}|_{\bar{\Sigma}}\) for \(k \in \{0, 1\}\) and \(\varphi_{y_2}|_{\Sigma} = (\phi_{y_2} - \tilde{\tau}(\partial_{y_2} \ell_1) \varphi)|_{\Sigma}\). Therefore on \(\Sigma\) the function \(\nabla \varphi\) is independent of \(N\) and \(|\nabla \phi(y) - \nabla \varphi(y)| \leq C_{25}\) for all \(y \in \Sigma\) where \(C_{25} > 0\) is independent of \(\tilde{\tau}\) and \(N\). In particular, taking \(\tilde{\tau}\) sufficiently small, we have (2.6) for the function \(\varphi\). It is convenient for us to rewrite (5.26) in the form
\[
B_\beta = B_\beta^{(1)} + B_\beta^{(2)},
\]
where \(B_\beta^{(1)} \equiv \text{Re} \int_{y_2 = 0} 2|s| \beta \frac{\partial \tilde{w}}{\partial y_2}(\beta \frac{\partial \varphi_{y_2}}{\partial y_1}(y^*) + \beta \frac{\partial \tilde{w}}{\partial y_2} \varphi_{y_2}(y^*) - \beta \frac{\partial \tilde{w}}{\partial y_0} \varphi_{y_0}(y^*)) dy_0 dy_1
\]
\[
+ \int_{y_2 = 0} |s| \beta \varphi_{y_2}(y^*) \left( \frac{\partial \tilde{w}}{\partial y_0}^2 - \beta \left( \frac{\partial \tilde{w}}{\partial y_1}^2 + \frac{\partial \tilde{w}}{\partial y_2}^2 \right) \right)
\]
\[
- |s|^2 \varphi_{y_2}(y^*) \delta(y^*) \frac{\varphi_{y_2}(y^*)}{\varphi_{y_2}(y^*)} \left| \frac{\varphi_{y_2}(y^*)}{\varphi_{y_2}(y^*)} \right|^2 \right) dy_0 dy_1.
\]

Then
\[
|B_\beta^{(2)}| \leq \epsilon_0 \left( |s| \left| \frac{\partial \tilde{w}}{\partial y_2} \right|^2 + |s| \left| \frac{\varphi_{y_2}}{\partial y_0} \right|^2 \right),
\]
where \(\epsilon_0 = \epsilon_0(\delta) \to 0\) as \(|\delta| \to 0\). It is known (see e.g., [18]) that there exists a parameter \(\tilde{\tau} > 1\) such that for any \(\tau > \tilde{\tau}\), there exists \(s_0(\tau)\) such that
\[
\|L_{-\beta} \tilde{w}\|_{L^2(\Omega_N)}^2 + \|L_{+\beta} \tilde{w}\|_{L^2(\Omega_N)}^2 + \text{Re} \int_{\partial \Omega_N} \left( |L_{+\beta} + L_{-\beta}| \tilde{w}, \bar{w} \right) dy
\]
\[
+ C_{26} |s| \|\tilde{w}\|_{H^1(\Omega_N)}^2 \|\partial_{y_2} \tilde{w}\|_{L^2(\Omega_N)}^2 \geq C_{26} \left( |s| \|\tilde{w}\|_{H^1(\Omega_N)}^2 + |s|^2 \|\tilde{w}\|_{L^2(\Omega_N)}^2 \right), \quad \forall |s| \geq s_0(\tau),
\]
where \(C_{26} > 0\) is independent of \(s\). We also claim that the constant \(C_{26}\) is independent of \(N\). The proof of estimate (5.28) is given in Appendix II.

Set
\[
\Xi_\beta = \int_{-\infty}^{\infty} B_\beta ds, \quad \Xi_\beta^{(j)} = \int_{-\infty}^{\infty} B_\beta^{(j)} ds, \quad j = 1, 2.
\]

Therefore, integrating (5.28) with respect to \(s\) in \(R^1\), we have
\[
C_{27} \left( \|h(s) \tilde{w}\|_{H^1(\Omega)}^2 + \|h^3(s) \tilde{w}\|_{L^2(\Omega)}^2 \right) + \Xi_\beta \leq C_{26} |s| \int_{-\infty}^{\infty} \|\tilde{w}\|_{L^2(\Omega_N)}^2 \|\partial_{y_2} \tilde{w}\|_{L^2(\Omega_N)}^2 \|d s
\]
\[
+ \|q\|_{L^2(\Omega)}^2 + \|\tilde{w}\|_{H^1(\Omega)}^2, \quad \forall |s| \geq s_0(\tau),
\]
(5.29)

with some constant \(C_{27} > 0\) and by (5.27)
\[
|\Xi_\beta^{(2)}| + |s| \int_{-\infty}^{\infty} \|\tilde{w}\|_{L^2(\Omega_N)}^2 \|\partial_{y_2} \tilde{w}\|_{L^2(\Omega_N)}^2 \|ds \leq \epsilon(\delta) \left( \left( \frac{\partial \tilde{w}}{\partial y_2} \right) \right)^2,
\]
(5.30)
where we set
\[ \left\| \left( \frac{\partial \tilde{w}}{\partial y_2} \right) \right\|_X^2 = \left\| h(s) \frac{\partial \tilde{w}}{\partial y_2} \right\|_{L^2(\Sigma)}^2 + \left\| h(s) \tilde{w} \right\|_{L^2(\mathbb{R}, H^1(\mathbb{R}))}^2 + \left\| h^3(s) \tilde{w} \right\|_{L^2(\Sigma)}^2 \]
and the parameter \( \epsilon(\delta) \to +0 \) as \( \delta \to +0 \).

We set
\[ w_{1,\nu} = \mathcal{F}_\sigma \chi_{\nu}(D')v_1, \quad w_{2,\nu} = \mathcal{F}_\sigma \chi_{\nu}(D')v_2. \]

Later we will need to apply (5.29) and (5.30) to the functions \( w_{1,\nu} \) and \( w_{2,\nu} \), since we would like to take the advantage of (5.23). However it is directly impossible because the condition \( \text{supp} \chi_{\nu}(D') \subset B_2 \times \mathbb{R}^3 \) does not hold true, in general. On the other hand, using the fact that
\[ \sup \chi_{\nu}(D') \subset B_2 \times \mathbb{R}^3 \]
we can modify (5.29) and (5.30):
\[ C_{29} \left( \| h(s) w_j(\beta, \nu) \|_{H^2(\Omega)}^2 + \| h^3(s) w_j(\beta, \nu) \|_{L^2(\Omega)}^2 \right) + \Xi \]
\[ \leq \| P_{\beta, s} w_j(\beta, \nu) \|_{L^2(\Omega)}^2 + C_{30} \| v \|_{H^1(\Omega)}^2 + C_{30} |s| \int_{-\infty}^{\infty} \| w_j(\beta, \nu) \|_{L^2(\partial \Omega_N)} \| \partial_{y_2} w_j(\beta, \nu) \|_{L^2(\partial \Omega_N)} ds, \]
(5.31)
where \( C_{29} > 0 \) is independent of \( s, N \) and we set \( j(\beta) = 1 \) if \( \beta = \mu \) and \( j(\beta) = 2 \) if \( \beta = \lambda + 2\mu \), and
\[ |\Xi| \leq |s| \int_{-\infty}^{\infty} \| w_j(\beta, \nu) \|_{L^2(\partial \Omega_N)} \| \partial_{y_2} w_j(\beta, \nu) \|_{L^2(\partial \Omega_N)} ds \leq \| \left( \frac{\partial w_j(\beta, \nu)}{\partial y_2}, w_j(\beta, \nu) \right) \|_X^2 + C_{31} \| v \|_{H^1(\Omega)}^2. \]
(5.32)

Now we will prove (5.15) separately in the cases: \( r_\mu(\gamma) = 0 \) (Sect. 6), \( r_{\lambda + 2\mu}(\gamma) = 0 \) (Sect. 7) and \( r_\mu(\gamma) \neq 0 \), \( r_{\lambda + 2\mu}(\gamma) \neq 0 \) (Sect. 8).

6. The Case \( r_\mu(\gamma) = 0 \)

In this section, we treat the case where \( r_\mu(\gamma) = 0 \) with \( \gamma = (y^*, \zeta^*) \equiv (y^*, s^*, \zeta^*, \xi^*) \in \Sigma \times \mathbb{S}^2 \). Let \( \chi_{\nu} \) be a member of the partition of unity such that
\[ \text{supp} \chi_{\nu} \subset O(\delta_1) \equiv \left\{ \zeta = (s, \zeta_0, \xi_0) : \frac{\zeta}{|\zeta|} - \zeta^* < \delta \right\}. \]

We note that by (5.31) and (5.32), there exist \( C_1 > 0 \) and \( C_2 > 0 \) such that
\[ C_1 \left( \| h(s) w_{1,\nu} \|_{H^1(\Omega)}^2 + \| h^3(s) w_{1,\nu} \|_{L^2(\Omega)}^2 \right) + \Xi_{\mu}^{(1)} \]
\[ \leq C_2 \left( \| P_{\mu} v_{1,\nu} \|_{L^2(\Omega)}^2 + \| w_{1,\nu} \|_{H^1(\Omega)}^2 \right) + \epsilon(\delta) \left\| \left( \frac{\partial w_{1,\nu}}{\partial y_2}, w_{1,\nu} \right) \right\|_X^2, \]
(6.1)
and the parameter \( \epsilon \) can be taken sufficiently small, if we decrease \( \delta \). Note that \( \Xi_{\mu}^{(1)} \) can be written in the form

\[
\Xi_{\mu}^{(1)} = \int_{\Sigma} \left( |s|^2 \partial_{1,\nu} \varphi_{y_2}(y^*) \right)^2 + |s|^2 \partial_{1,\nu} \varphi_{y_2}(y^*) \left| \frac{\partial w_{1,\nu}}{\partial y_2} \right|^2 \right) d\Sigma
\]

\[
+ \operatorname{Re} \int_{\Sigma} 2|s| \mu \frac{\partial w_{1,\nu}}{\partial y_2} \left( \mu \varphi_{y_1}(y^*) \frac{\partial w_{1,\nu}}{\partial y_1} - \varphi_{y_1}(y^*) \frac{\partial w_{1,\nu}}{\partial y_0} \right) d\Sigma
\]

\[
+ \int_{\Sigma} |s| \mu \varphi_{y_2}(y^*) \left( \xi_0^2 - \mu \xi_1^2 - s^2 \varphi_{y_0}(y^*)^2 + s^2 \mu \varphi_{y_1}(y^*) \right) \nu_{1,\nu}^2 d\Sigma
\]

\[
\equiv J_1 + J_2 + J_3.
\]  

(6.2)

Let us introduce the set \( \mathcal{M} \) by formula

\[
\mathcal{M} = \left\{ \zeta = (s, \xi_0, \xi_1) \in \mathbb{S}^2; \frac{\mu}{2} \varphi_{y_2}(y^*) \mathcal{C} s^2 > 4 \frac{\varphi_{y_1}(y^*)}{|\varphi_{y_2}(y^*)|} \xi_0^2 + 4 \frac{\varphi_{y_0}(y^*)}{|\varphi_{y_2}(y^*)|} \xi_1^2 + 2 \mu \varphi_{y_2}(y^*) \right\},
\]  

(6.3)

where \( \mathcal{C} = -p_{\mu}(y^*, \nabla \varphi(y^*)) \). By (2.6), it follows that \( \mathcal{C} \) is positive.

Next we introduce the set \( \overline{\mathcal{M}} \) by the formula

\[
\overline{\mathcal{M}} = \left\{ \zeta = (s, \xi_0, \xi_1) \in \mathbb{S}^2; \frac{\mu}{2} \varphi_{y_2}(y^*) \mathcal{C} s^2 < 4 \frac{\varphi_{y_1}(y^*)}{|\varphi_{y_2}(y^*)|} \xi_0^2 + 4 \frac{\varphi_{y_0}(y^*)}{|\varphi_{y_2}(y^*)|} \xi_1^2 + 2 \mu \varphi_{y_2}(y^*) \right\}.
\]

Then we can see that \( \mathbb{S}^2 \subset \mathcal{M} \cup \overline{\mathcal{M}} \). Therefore, taking the parameter \( \delta_1 \) sufficiently small, we obtain either \( \mathcal{O}(\delta_1) \subset \mathcal{M} \) or \( \mathcal{O}(\delta_1) \subset \overline{\mathcal{M}} \). The main purpose of this section is the proof of the following lemma.

**Lemma 6.1.** If \( \gamma = (y^*, \zeta^*) \) is a point on \( \Sigma \times \mathbb{S}^2 \) such that \( r_{\mu}(\gamma) = 0 \) and \( \operatorname{supp} \chi_{\nu} \subset \mathcal{O}(\delta_1) \subset \overline{\mathcal{M}} \), then estimate (5.15) holds true. If \( \gamma \in (y^*, \zeta^*) \in \mathcal{M} \), then estimate (5.15) holds true also.

**Proof.** We consider two cases.

**Case A.** Assume that \( \operatorname{supp} \tilde{\chi}_{\nu} \subset \mathcal{O}(\delta_1) \subset \mathcal{M} \).

Applying the Cauchy-Bunyakovskii inequality and using (6.3) and (2.6), we see that there exists a constant \( C_3 > 0 \) such that

\[
\Xi_{\mu}^{(1)} \geq \int_{\Sigma} \left( |s|^2 \partial_{1,\nu} \varphi_{y_2}(y^*) \right)^2 + |s|^2 \partial_{1,\nu} \varphi_{y_2}(y^*) \left| \frac{\partial w_{1,\nu}}{\partial y_2} \right|^2 \right) d\Sigma
\]

\[
- \int_{\Sigma} \left( \frac{1}{2} |s|^2 \partial_{1,\nu} \varphi_{y_2}(y^*) \right)^2 + 4 |s|^2 \partial_{1,\nu} \varphi_{y_2}(y^*) \left| \frac{\partial w_{1,\nu}}{\partial y_1} \right|^2 + 4 |s|^2 \left| \frac{\partial w_{1,\nu}}{\partial y_0} \right|^2 \right) d\Sigma
\]

\[
- \int_{\Sigma} |s|^2 \partial_{1,\nu} \varphi_{y_2}(y^*) \xi_0^2 \nu_{1,\nu}^2 d\Sigma
\]

\[
\geq C_3 \int_{\Sigma} \left( \frac{1}{2} |s|^2 \partial_{1,\nu} \varphi_{y_2}(y^*) \right)^2 + |s|^2 \left| \frac{\partial w_{1,\nu}}{\partial y_1} \right|^2 + |s|^2 \left| \frac{\partial w_{1,\nu}}{\partial y_0} \right|^2 \right) d\Sigma. \tag{6.4}
\]

Similarly we have

\[
\Xi_{\mu}^{(1)} \geq C_4 \int_{\Sigma} \left( \left| \frac{\partial w_{2,\nu}}{\partial y_2} \right|^2 + \left| \frac{\partial w_{2,\nu}}{\partial y_1} \right|^2 + \left| \frac{\partial w_{2,\nu}}{\partial y_0} \right|^2 \right) d\Sigma. \tag{6.5}
\]
Combining (6.4) and (6.5), we obtain

\[
\Xi^{(1)}_{\mu} + \Xi^{(1)}_{\lambda + 2\mu} \geq C_5 \left\| \left( \frac{\partial w_{1,\nu}}{\partial y_2}, w_{1,\nu} \right) \right\|^2_{X^1}.
\]  

(6.6)

If we apply (5.31) with $\beta = \lambda + 2\mu$, then (6.1), (6.4) and (6.6) imply (5.15).

**Case B.** Assume that supp $\hat{\nu}_c \subset \hat{\mathcal{M}}$.

By (5.18)–(5.20), there exists $C_6 > 0$ such that

\[
|\xi_0^2 - s^2\varphi_0(y^*) - \mu \xi_1^2 + \mu s^2 \varphi_{\gamma_1}(y^*)| + |\xi_0 s \varphi_0(y^*) - \mu s \xi_1 \varphi_{\gamma_1}(y^*)| \leq \delta_1 C_6(\xi_1^2 + |\xi_0|^2 + s^2), \quad \forall \zeta \in \mathcal{O}(\delta_1).
\]

(6.7)

Now we suppose that the parameter $\delta_1$ is sufficiently small such that there exists a constant $C_7 > 0$ such that

\[
|\xi_0|^2 \leq C_7(\xi_1^2 + s^2), \quad \forall \zeta \in \mathcal{O}(\delta_1).
\]

(6.8)

Then, by (6.7), we have

\[
|J_3| \leq \delta_1 \mu \varphi_{\gamma_1}(y^*) \left( \left\| \frac{\partial w_{1,\nu}}{\partial y_2}, w_{1,\nu} \right\|^2_{X^1} \right).
\]

(6.9)

Moreover we claim that there exists $\delta_0 > 0$ such that if $\delta_1 \in (0, \delta_0)$, then there exists $C_8 > 0$ such that

\[
|\xi_0| \leq C_8 \xi_1^2, \quad \forall \zeta \in \mathcal{O}(\delta_1).
\]

(6.10)

Our proof is by contradiction. Suppose that (6.10) is not true. Then for the sequence $\delta_1(n) = \frac{1}{n}$, there exists a sequence $(\xi_0(n), \xi_1(n)) \rightarrow (\xi_0^*, \xi_1^*)$ such that $\xi_1(n)/\xi_0(n) \rightarrow 0$. Hence for $\zeta^*$ we have $r_{\mu}(y^*, \zeta^*) = 0$, and $\xi_1^* = 0, \xi_0^* \neq 0$ by the definition of the set $\hat{\mathcal{M}}$. Therefore $s^* \varphi_0(y^*) = 0$. If $s^* = 0$, then we obtain $(\xi_0^*)^2 = 0$ and if $\varphi_0(y^*) = 0$, then $(\xi_0^*)^2 + \mu s^2 \varphi_{\gamma_1}(y^*)^2 = 0$ by (5.19), (5.20). Therefore in the both cases, we have the equality $\xi_0^* = 0$ which leads us to a contradiction.

Note that if $r_{\lambda + 2\mu}(\gamma) = 0$, then

\[
\varphi_0(y^*) = 0, \quad \varphi_{\gamma_1}(y^*) = 0, \quad \xi_0^* = \xi_1^* = 0, \quad s^* = 1
\]

and the conic neighbourhood of $\zeta^*$ is in the set $\hat{\mathcal{M}}$ provided that the parameter $\delta_1$ is chosen sufficiently small. Therefore if $\gamma \in \mathcal{M}$ and $r_{\mu}(\gamma) = 0$, then we have $r_{\lambda + 2\mu}(\gamma) \neq 0$ and by Proposition 5.4, decomposition (5.23) holds true. We set $V_{\lambda + 2\mu} = (D_{y_2} - \Gamma^+_{\lambda + 2\mu}(y, D'))v_{2,\nu}$. Then

\[
P_{\lambda + 2\mu} = (\lambda + 2\mu)|G|^2(D_{y_2} - \Gamma^+_{\lambda + 2\mu}(y, D'))V^+_{\lambda + 2\mu} + T_{\lambda + 2\mu}v_{2,\nu},
\]

where $T_{\lambda + 2\mu} \in L(H^1(\mathcal{Q}), L^2(\mathcal{Q}))$. This decomposition and Proposition 5.4 immediately imply

\[
\|h(D_{\sigma})(D_{y_2} - \Gamma^+_{\lambda + 2\mu}(y, D'))v_{2,\nu}|_{y_2=0}\|_{L^2(\Sigma)} \leq C_9(\|P_{\lambda + 2\mu}w_{2,\nu}\|_{L^2(\mathcal{Q})} + \|v\|_{(H^1(\mathcal{Q}))^2}).
\]

(6.11)

Now we need again obtain the estimate of $\Xi^{(1)}_{\mu}$. We start from the term $J_2$. By (5.16), we have

\[
J_2 = \text{Re} \int_\Sigma 2|s|(|\lambda + 2\mu| \mu \frac{\partial w_{1,\nu}}{\partial y_1} - |s|\varphi_{\gamma_1}(y^*)w_{2,\nu}) \times \left( \mu \frac{\partial w_{1,\nu}}{\partial y_1} \varphi_{\gamma_1}(y^*) - \frac{\partial w_{1,\nu}}{\partial y_0} \varphi_0(y^*) \right) d\Sigma
\]

\[
+ \text{Re} \int_\Sigma 2|s|\mu(|s|\varphi_0(y^*)w_{1,\nu} - g_{1,\nu}) \left( \mu \frac{\partial w_{1,\nu}}{\partial y_1} \varphi_{\gamma_1}(y^*) - \frac{\partial w_{1,\nu}}{\partial y_0} \varphi_0(y^*) \right) d\Sigma.
\]

(6.12)
and denoting
\[ \alpha_{\lambda+2\mu}^+(y', D) = \alpha_{\lambda+2\mu}^+(y', D) + i|D_\sigma|(|\varphi_{y_1} - \varphi_{y_2}(y^*)|), \]
\[ -\frac{\mu}{\lambda + 2\mu} \left( \frac{\partial \varphi_{y_1}}{\partial y_1} - |D_\sigma|\varphi_{y_1}(y^*)v_{1,\nu} \right) - i\alpha_{\lambda+2\mu}^+(y, D')v_{2,\nu} = iV_{\lambda+2\mu}^+(\cdot, 0) - \frac{\mu}{\lambda + 2\mu} F^{-1}_\sigma g_{2,\nu}. \] (6.13)

Here and henceforth \(|D_\sigma|\) is the pseudo-differential operator with the symbol \(|s|\).

First, assume that \(s^* = 0\). Then we can see by \(|s|^2 + |\xi|^2 + |\xi_1|^2 = 1\) that \(|\alpha_{\lambda+2\mu}^+(\gamma)| = |r_{\lambda+2\mu}(\gamma)| \neq 0\).

Therefore, by Proposition 5.2.A from [47], p. 105, there exists a parametrix of the operator \(\alpha_{\lambda+2\mu}^+(y, D')\) which we denote by \((\tilde{\alpha}_{\lambda+2\mu}^+(y, D'))^{-1}\). From (6.13) we obtain
\[ v_{2,\nu} = -\frac{1}{i}(\tilde{\alpha}_{\lambda+2\mu}^+(y, D'))^{-1} \left( \frac{\mu}{\lambda + 2\mu} \left( \frac{\partial \varphi_{y_1}}{\partial y_1} - |D_\sigma|\varphi_{y_1}(y^*)v_{1,\nu} \right) + iV_{\lambda+2\mu}^+(\cdot, 0) - \frac{\mu}{\lambda + 2\mu} F^{-1}_\sigma g_{2,\nu} \right) + T_0v_{2,\nu}, \] (6.14)
where \(T_0 \in \mathcal{L}(L^2(\Sigma), H^1(\Sigma)).\) Using (6.14), we transform (6.12) to obtain
\[ J_2 = \text{Re} \int_{\Sigma} -\frac{2|D_\sigma|\mu}{i} \left( \frac{\partial}{\partial y_1} - |D_\sigma|\varphi_{y_1}(y^*) \right) (\tilde{\alpha}_{\lambda+2\mu}^+(y, D'))^{-1} \left( \frac{\partial \varphi_{y_1}}{\partial y_1} - |D_\sigma|\varphi_{y_1}(y^*)v_{1,\nu} \right) \left( \frac{\mu}{\lambda + 2\mu} \left( \frac{\partial \varphi_{y_1}}{\partial y_1} - \frac{\partial \varphi_{y_0}}{\partial y_0} \varphi_{y_0}(y^*) \right) \right) \cdot d\Sigma + \kappa_3, \] (6.15)
where
\[ \kappa_3 = \text{Re} \int_{\Sigma} 2|D_\sigma|\mu(|D_\sigma|\varphi_{y_2}(y^*)v_{1,\nu} + F^{-1}_\sigma g_{1,\nu}) \left( \frac{\partial \varphi_{y_1}}{\partial y_1} - |D_\sigma|\varphi_{y_1}(y^*) \right) \cdot \left( \frac{\mu}{\lambda + 2\mu} \left( \frac{\partial \varphi_{y_1}}{\partial y_1} - \frac{\partial \varphi_{y_0}}{\partial y_0} \varphi_{y_0}(y^*) \right) \right) \cdot d\Sigma \]
\[ \times \left[ -\frac{1}{i}(\tilde{\alpha}_{\lambda+2\mu}^+(y, D'))^{-1} \left( iV_{\lambda+2\mu}^+(\cdot, 0) - \frac{\mu}{\lambda + 2\mu} F^{-1}_\sigma g_{2,\nu} \right) + T_0v_{2,\nu} \right] \times \left( \frac{\mu}{\lambda + 2\mu} \left( \frac{\partial \varphi_{y_1}}{\partial y_1} - \frac{\partial \varphi_{y_0}}{\partial y_0} \varphi_{y_0}(y^*) \right) \right) \cdot d\Sigma. \]

Then we have
\[ |\kappa_3| \leq \varepsilon \left\| \left( \frac{\partial w_{2,\nu}}{\partial y_2}, w_{\nu} \right) \right\|^2_{L^2(\mathcal{Q})} + C_{10} \left( \|h(s)g\|_{L^2(\Sigma)}^2 + \|P_{\lambda+2\mu}s w_{2,\nu}\|_{L^2(\mathcal{Q})}^2 \right) \] (6.16)
and \(\varepsilon\) can be chosen arbitrarily small by taking \(\delta\) small enough.

Let us consider the pseudo-differential operator
\[ b(y, D') = \frac{1}{i} \left( \frac{\partial}{\partial y_1} - |s|\varphi_{y_1}(y^*) \right) (\tilde{\alpha}_{\lambda+2\mu}^+(y, D'))^{-1}. \]

By (6.7), for the principal symbol of this operator, we have
\[ b(y^*, \zeta) = \frac{1}{i} \left( i\xi_1 - |s|\varphi_{y_1}(y^*) \right) (\tilde{\alpha}_{\lambda+2\mu}^+(y^*, \zeta))^{-1} \]
\[ = -\text{sign}(\xi_1) \sqrt{\frac{\lambda + \mu}{\lambda + 2\mu}} \frac{|s|\varphi_{y_1}(y^*)}{\xi_1 + i|s|\varphi_{y_1}(y^*)} + \tilde{b}(y^*, \zeta) \]
\[ = \frac{1}{i} \left( \frac{\lambda + \mu}{\lambda + 2\mu} \right) (y^*) + \tilde{b}(y^*, \zeta), \] (6.17)
where \( \tilde{b}(y^*, \xi^*) = 0 \). Therefore the operator \( b(y, D') \) can be represented in the form

\[
b(y, D') = \frac{1}{i} \sqrt{\frac{\lambda + \mu}{\lambda + 2\mu}} (y) + \tilde{b}(y, D'),
\]

where \( \tilde{b}(y, D') \in \mathcal{L}(L^2(\Sigma), L^2(\Sigma)) \) and

\[
\| \tilde{b}(y, D') \|_{\mathcal{L}(L^2(\Sigma), L^2(\Sigma))} \leq \epsilon. \tag{6.18}
\]

Using (6.17) in (6.15), we obtain

\[
J_2 = \text{Re} \int_\Sigma -2|D_\sigma|\mu \left( \frac{\text{sign}(\xi_1^*)}{i} \sqrt{\frac{\lambda + \mu}{\lambda + 2\mu}} (y^*) + \tilde{b}(y, D') \right) \left( \frac{\partial v_{1,\nu}}{\partial y_1} - |D_\sigma|\varphi_{y_1}(y^*)v_{1,\nu} \right)
\]

\[
\times \left( \frac{\partial v_{1,\nu}}{\partial y_1} \varphi_{y_1}(y^*) - \frac{\partial v_{1,\nu}}{\partial y_0} \varphi_{y_0}(y^*) \right) d\Sigma + \kappa_3
\]

\[
= \text{Re} \int_\Sigma -2|D_\sigma|\mu \left( \frac{\text{sign}(\xi_1^*)}{i} \sqrt{\frac{\lambda + \mu}{\lambda + 2\mu}} (y^*) \right) \left( \frac{\partial v_{1,\nu}}{\partial y_1} - |D_\sigma|\varphi_{y_1}(y^*)v_{1,\nu} \right)
\]

\[
\times \left( \frac{\partial v_{1,\nu}}{\partial y_1} \varphi_{y_1}(y^*) - \frac{\partial v_{1,\nu}}{\partial y_0} \varphi_{y_0}(y^*) \right) d\Sigma + \text{Re} \kappa_3.
\]

By (6.7), (6.16) and (6.18), taking the parameters \( \delta, \delta_1 \) sufficiently small, we obtain

\[
|J_2| \leq \epsilon(\delta, \delta_1) \left\| \left( \frac{\partial w_{\nu}}{\partial y_2}, w_{\nu} \right) \right\|_X^2 + C_{11} \left( ||h(s)|\mathbf{g}\|_L^2(\Sigma))^2 + ||P_{\lambda+2\mu,\nu}w_{2,\nu}\|_L^2(Q) + \|\mathbf{v}\|_{H^1(Q))^2} \right),
\tag{6.19}
\]

and \( \epsilon(\delta, \delta_1) \to 0 \) as \( |\delta| + |\delta_1| \to 0 \).

Next assume that \( s^* \neq 0 \). Then by (6.7) we have

\[
|\mu(y^*)\varphi_{y_1}(y^*)\xi_1 - \varphi_{y_0}(y^*)\xi_0| \leq C\delta_1|\xi|, \quad \forall \xi \in \mathcal{O}(\delta_1)
\]

and (6.19) follows immediately. Therefore, for any \( s^* \in \mathbb{R}^1 \), by (6.1), (6.2), (6.9) and (6.19), we have

\[
\int_\Sigma \left( h^2(s)\mu^2\varphi_{y_2}(y^*) \left| \frac{\partial w_{1,\nu}}{\partial y_2} \right|^2 + h^3(s)\mu^2\varphi_{y_2}(y^*)|w_{2,\nu}|^2 \right) d\Sigma + C_{12} \left( ||h(s)|w_{1,\nu}\|_{H^1(Q))^2} + \|h^2(s)|w_{1,\nu}\|_{L^2(Q))^2} \right)
\]

\[
\leq C_{13} \left( ||P_{\lambda+2\mu,\nu}w_{2,\nu}\|_{L^2(Q))^2} + \|h(s)|\mathbf{g}\|_L^2(Q))^2 + \|h(s)|\mathbf{v}\|_{H^1(Q))^2} \right) + \epsilon \left( \left\| \frac{\partial w_{\nu}}{\partial y_2}, w_{\nu} \right\|_X \right)^2. \tag{6.20}
\]

From (5.16), we obtain

\[
\int_\Sigma \left( \left| \frac{\partial w_{2,\nu}}{\partial y_1} \right|^2 + |s|^3 \mu^2 \varphi_{y_2}^3(y^*)|w_{2,\nu}|^2 \right) d\Sigma
\]

\[
\leq C_{14} \int_\Sigma \left( \left| \frac{\partial w_{1,\nu}}{\partial y_2} \right|^2 + |s|^3 \mu^2 \varphi_{y_2}^3(y^*)|w_{1,\nu}|^2 \right) d\Sigma + C_{14} \|h(s)|\mathbf{g}\|^2_{L^2(\Sigma))^2}. \tag{6.21}
\]
Using (6.10), (6.21) and the definition of the set \( \mathcal{M} \), we obtain

\[
\int_{\Sigma} \left( h^2(s) \left| \frac{\partial w_{2,\nu}}{\partial y_1} \right|^2 + h^2(s) \left| \frac{\partial w_{2,\nu}}{\partial y_0} \right|^2 + h^6(s) |w_{2,\nu}|^2 \right) d\Sigma \\
\leq C_{15} \left\{ \int_{\Sigma} \left[ |s| \mu^2 \varphi_{y_2}(y^*) \left| \frac{\partial w_{3,\nu}}{\partial y_2} \right|^2 + |s| \mu^2 \varphi_{y_2}^2(y^*) |w_{1,\nu}|^2 \right] d\Sigma + \epsilon(\sigma_0) \left\| \left( \frac{\partial w_{\nu}}{\partial y_2}, w_\nu \right) \right\|^2_X + \|h(s)|g_\nu|^2_{(L^2(\Sigma))^2} \right\}.
\]

From (6.11) and (6.22), we have

\[
\int_{\Sigma} h^2(s) \left| \frac{\partial w_{2,\nu}}{\partial y_2} \right|^2 d\Sigma \\
\leq C_{16} \left\{ \int_{\Sigma} \left( h^2(s) \left| \frac{\partial w_{2,\nu}}{\partial y_1} \right|^2 + h^2(s) \left| \frac{\partial w_{2,\nu}}{\partial y_0} \right|^2 + h^6(s) |w_{2,\nu}|^2 \right) d\Sigma \\
+ \|V_{\lambda_{r_2}(s,0)}\|_{L^2(\Sigma)} + \epsilon(\sigma_0) \left\| \left( \frac{\partial w_{\nu}}{\partial y_2}, w_\nu \right) \right\|^2_X + \|h(s)|g_\nu|^2_{(L^2(\Sigma))^2} \right\} \\
\leq C_{17} \left\{ \int_{\Sigma} \left( h^2(s) \left| \frac{\partial w_{1,\nu}}{\partial y_2} \right|^2 + h^6(s) |w_{1,\nu}|^2 \right) d\Sigma + \|h(s)|g_\nu|^2_{(L^2(\Sigma))^2} \\
+ \|v\|^2_{(H^1(\Omega))^2} + \|P_{\lambda_{r_2}s}w_{2,\nu}\|^2_{L^2(\Omega)} + \epsilon(\sigma_0) \left\| \left( \frac{\partial w_{\nu}}{\partial y_2}, w_\nu \right) \right\|^2_X \right\}.
\]

Finally (5.16), (6.10), (6.20) and (6.23) imply

\[
\int_{\Sigma} h^2(s) \left( \left| \frac{\partial w_{1,\nu}}{\partial y_1} \right|^2 + \left| \frac{\partial w_{1,\nu}}{\partial y_0} \right|^2 \right) d\Sigma \\
\leq C_{18} \left\{ \int_{\Sigma} \left( h^2(s) \left| \frac{\partial w_{1,\nu}}{\partial y_2} \right|^2 + h^6(s) |w_{1,\nu}|^2 \right) d\Sigma + \|h(s)|g_\nu|^2_{(L^2(\Sigma))^2} \\
+ \|v\|^2_{(H^1(\Omega))^2} + \|P_{\lambda_{r_2}s}w_{2,\nu}\|^2_{L^2(\Omega)} + \epsilon(\sigma_0) \left\| \left( \frac{\partial w_{\nu}}{\partial y_2}, w_\nu \right) \right\|^2_X \right\}.
\]

Inequalities (6.1), (6.20)–(6.24) imply

\[
\left\| \left( \frac{\partial w_\nu}{\partial y_2}, w_\nu \right) \right\|^2_X + \|h(s)|w_{1,\nu}|^2_{H^1(\Omega)} + \|h^3(s)w_{1,\nu}|^2_{L^2(\Omega)} \leq \epsilon \left\| \left( \frac{\partial w_{\nu}}{\partial y_2}, w_\nu \right) \right\|^2_X \\
+ C_{19} \left\{ \|v\|^2_{(H^1(\Omega))^2} + \|h(s)|g_\nu|^2_{(L^2(\Sigma))^2} + \|P_{\lambda_{r_2}s}w_{2,\nu}\|^2_{L^2(\Omega)} + \|P_{\lambda_{r_2}s}w_{2,\nu}\|^2_{L^2(\Omega)} \right\}.
\]

From this inequality and (5.31), (5.32) with \( \beta = \lambda + 2\mu \), we obtain (5.15). Thus the proof of Lemma 6.1 is complete. \( \square \)
In this section, we will prove

**Lemma 7.1.** Let \( \gamma = (y^*, \zeta^*) \) be a point on \( \Sigma \times S^2 \) such that \( r_{\lambda+2\mu}(\gamma) = 0 \). If \( \text{supp} \chi_\nu \subset \mathcal{O}(\delta_1) \subset \mathcal{M} \), then estimate (5.15) holds true.

**Proof.** We note that if \( r_\mu(\gamma) = 0 \), then \( s^* \neq 0 \) and \( \xi_0^* = \xi_1^* = \psi_{y^*}(y^*) = \varphi_{y^*}(y^*) = 0 \). Consequently \( \zeta^* \in \mathcal{M} \) and this case was treated in the previous section. Therefore, taking the parameters \( \delta \) and \( \delta_1 \) sufficiently small, we may assume that there exists a constant \( \hat{C} > 0 \) such that

\[
|r_\mu(y, \zeta)| \geq \hat{C} |\zeta|^2, \quad \forall (y, \zeta) \in B_\delta \times \mathcal{O}(\delta_1), \quad |\zeta| \geq 1.
\]

By (5.19) and (5.20), there exist \( \delta_0 > 0 \) and \( C_1 > 0 \) such that for all \( \delta_1 \in (0, \delta_0) \) we have

\[
|\xi_0|^2 \leq C_1 (\xi_1^2 + s^2), \quad \forall \zeta \in \mathcal{O}(\delta_1).
\] (7.1)

We consider the following three cases.

**Case A.** Assume that \( s^* = 0 \) and

\[
\mu(y^*) \psi_{y^*}(y^*) > \frac{|\mu(y^*) \psi_{y^*}(y^*) \xi_1^* - \varphi_{y^*}(y^*) \xi_0^*|}{\sqrt{\lambda + \mu(y^*) |\xi_1^*|}}.
\]

In that case, there exists a constant \( C_2 > 0 \) such that

\[-\text{Im} \Gamma_{\mu}^\pm(y, \zeta) \geq C_2 |s|, \quad \forall (y, \zeta) \in B_\delta \times \mathcal{O}(\delta_1),\]

provided that \( |\delta| + |\delta_1| \) is sufficiently small. Since \( s^* = 0 \), we may assume that for some constant \( C_3 > 0 \),

\[
|\xi_0|^2 + s^2 \leq C_3 \xi_1^2, \quad \forall \zeta \in \mathcal{O}(\delta_1),
\] (7.2)

taking a sufficiently small \( \delta_1 \). We set \( V_\mu^\pm = (D_{y^2} - \Gamma_{\mu}^\pm(y, D'))v_{1, \nu} \). Then, by Proposition 5.3,

\[
P_\mu v_{1, \nu} = |G|^2 \mu(D_{y^2} - \Gamma_{\mu}^\pm(y, D'))V_\mu^\pm + T_\mu^\pm v_{1, \nu},
\] (7.3)

where \( T_\mu^\pm \in \mathcal{L}(H^1(Q), L^2(Q)) \). This decomposition and Proposition 5.4 imply

\[
\|h(D_\sigma)(D_{y^2} - \Gamma_{\mu}^\pm(y, D'))v_{1, \nu}|_{y^2=0}\|_{L^2(\Sigma)} \leq C_4 (\|P_\mu v_{1, \nu}\|_{L^2(Q)} + \|\psi\|_{(H^1(Q))^2}).
\] (7.4)

We have

\[
-V_\mu^+(\cdot, 0) + V_\mu^-(\cdot, 0) = (\alpha_\mu^+(y, D') - \alpha_\mu^-(y, D'))v_{1, \nu} \quad \text{on} \Sigma.
\] (7.5)

Since \( \alpha_\mu^+(y^*, \zeta^*) - \alpha_\mu^-(y^*, \zeta^*) = 2\sqrt{r_\mu(y^*, \zeta^*)} \neq 0 \), by (7.4), (7.5) and Gårding’s inequality, we have

\[
\int_\Sigma \left( h^2(s) \left( \left| \frac{\partial w_{1, \nu}}{\partial y_1} \right|^2 + \left| \frac{\partial w_{1, \nu}}{\partial y_0} \right|^2 \right) + h^0(s)|w_{1, \nu}|^2 \right) d\Sigma \leq C_5 (\|P_\mu s w_{1, \nu}\|_{L^2(Q)}^2 + \|\psi\|_{(H^1(Q))^2})^2.
\] (7.6)

By (7.6) and (7.4), we obtain

\[
\int_\Sigma h^2(s) \left| \frac{\partial w_{1, \nu}}{\partial y_2} \right|^2 d\Sigma \leq C_6 \left( \|P_\mu s w_{1, \nu}\|_{L^2(Q)}^2 + \|\psi\|_{(H^1(Q))^2} \right).
\] (7.7)
Finally, by (7.6), (7.7) combined with (5.16), we obtain
\[
\left\| \left( \frac{\partial w_{2,\nu}}{\partial y_2}, w_{2,\nu} \right) \right\|^2_X \leq C_T \left( \left\| P_{\mu,\nu} w_{1,\nu} \right\|_{L^2(Q)}^2 + \left\| v \right\|_{L^2(Q)}^2 + \left\| h(s) g \right\|_{L^2(Q)}^2 \right).
\tag{7.8}
\]
Since (7.6)–(7.8), (5.31) and (5.32), we obtain (5.15).

**Case B.** Assume that \( s^* = 0 \) and
\[
\mu(y^*) \varphi_{y^*}(y^*) \leq \frac{\left| \mu(y^*) \varphi_{y^*}(y^*) \xi^*_1 - \varphi_{y_0}(y^*) \xi^*_0 \right|}{\sqrt{\frac{\lambda + \mu(y^*)}{\mu(y^*)}}} \xi^*_1.
\tag{7.9}
\]
Then \( \lim_{\zeta \to -\zeta} \text{Im} r_{\mu}(y^*, \zeta)\big|_{|s| = 0} \neq 0 \). Since \( s^* = 0 \), we note that \( \text{Re} r_{\mu}(y^*, \zeta) > 0 \). Set \( I = \text{sign} \lim_{\zeta \to -\zeta} \text{Im} r_{\mu}(y^*, \zeta) \). Then we have
\[
\Gamma^+_\mu(y^*, \zeta^*) = I \sqrt{\text{Re} r_{\mu}(y^*, \zeta^*)}.
\]
Therefore for some \( C_8 > 0 \) we have
\[
-\Gamma^+_\mu(y^*, \zeta^*)(\mu \varphi_{y^*}(y^*) \xi^*_1 - \varphi_{y_0}(y^*) \xi^*_0) > C_8.
\]
Taking the parameters \( \delta > 0 \) and \( \delta_1 > 0 \) sufficiently small, we obtain
\[
-\text{Re} \Gamma^+_\mu(y, \zeta)(\mu \varphi_{y_1}(y) \xi^*_1 - \varphi_{y_0}(y) \xi^*_0) > 0, \quad \forall (y, \zeta) \in B_\delta \times O(\delta_1).
\tag{7.10}
\]
Let us consider estimate (6.1). Let us recall that \( J_1, J_2, J_3 \) are defined in (6.2). We have
\[
J_2 = \text{Re} \int_{\Sigma} 2|s| \mu \frac{\partial w_{1,\nu}}{\partial y_2} \left( \mu \frac{\partial w_{1,\nu}}{\partial y_1} \varphi_{y^*}(y^*) - \frac{\partial w_{1,\nu}}{\partial y_0} \varphi_{y_0}(y^*) \right) d\Sigma
\]
\[
= \text{Re} \int_{\Sigma} 2[D_{\sigma}] \text{Re} \left( \mu \frac{\partial w_{1,\nu}}{\partial y_1}(y, D') v_{1,\nu} \left( \mu \frac{\partial v_{1,\nu}}{\partial y_1} \varphi_{y^*}(y^*) - \frac{\partial v_{1,\nu}}{\partial y_0} \varphi_{y_0}(y^*) \right) \right) d\Sigma
\]
\[
+ \text{Re} \int_{\Sigma} 2[D_{\sigma}] \text{Im} \left( \mu \frac{\partial w_{1,\nu}}{\partial y_1}(y, D') v_{1,\nu} \left( \mu \frac{\partial v_{1,\nu}}{\partial y_1} \varphi_{y^*}(y^*) - \frac{\partial v_{1,\nu}}{\partial y_0} \varphi_{y_0}(y^*) \right) \right) d\Sigma
\]
\[
= \text{Re} \int_{\Sigma} 2[D_{\sigma}] \text{Re} \left( \mu \frac{\partial w_{1,\nu}}{\partial y_1}(y, D') v_{1,\nu} \left( \mu \frac{\partial v_{1,\nu}}{\partial y_1} \varphi_{y^*}(y^*) - \frac{\partial v_{1,\nu}}{\partial y_0} \varphi_{y_0}(y^*) \right) \right) d\Sigma
\]
\[
+ \text{Re} \int_{\Sigma} 2[D_{\sigma}] \text{Im} \left( \mu \frac{\partial w_{1,\nu}}{\partial y_1}(y, D') v_{1,\nu} \left( \mu \frac{\partial v_{1,\nu}}{\partial y_1} \varphi_{y^*}(y^*) - \frac{\partial v_{1,\nu}}{\partial y_0} \varphi_{y_0}(y^*) \right) \right) d\Sigma.
\tag{7.11}
\]
By (7.10) we obtain from Gårding’s inequality that the first integral in the right hand side of (7.11) is negative. Consider two cases. First let
\[
\varphi_{y_1}(y^*) \xi^*_1 \Gamma^+_\mu(y^*, \zeta^*) > 0.
\]
This inequality and (7.10) yield \( |\xi^*_0 \varphi_{y_0}(y^*)| > |\xi^*_1 \mu(y^*) \varphi_{y_1}(y^*)| \). If \( \xi^*_0 \varphi_{y_0}(y^*) > 0 \) then \( \Gamma^+_\mu(y^*, \zeta^*) = |\sqrt{\text{Re} r_{\mu}(\gamma)}| \) and \( \xi^*_1 \varphi_{y_1}(y^*) > 0 \). Hence \( \varphi_{y_2}(y^*) > \frac{|\varphi_{y_1}(y^*) \xi^*_1 - \frac{\beta y_1(y^*)}{\beta y_0(y^*)} \xi^*_0|}{\left( \frac{\beta y_1(y^*)}{\beta y_0(y^*)} \right)^2 \xi^*_1} = -\frac{\varphi_{y_1}(y^*) \xi^*_1 + \frac{\beta y_1(y^*)}{\beta y_0(y^*)} \xi^*_0}{\left( \frac{\beta y_1(y^*)}{\beta y_0(y^*)} \right)^2 \xi^*_1} \). By the first equation in (2.6), this contradicts (7.9).

If \( \xi^*_0 \varphi_{y_0}(y^*) < 0 \) then \( \Gamma^+_\mu(y^*, \zeta^*) = -|\sqrt{\text{Re} r_{\mu}(\gamma)}| \) and \( \xi^*_1 \varphi_{y_1}(y^*) < 0 \). Therefore \( \varphi_{y_2}(y^*) > \frac{|\varphi_{y_1}(y^*) \xi^*_1 - \frac{\beta y_1(y^*)}{\beta y_0(y^*)} \xi^*_0|}{\left( \frac{\beta y_1(y^*)}{\beta y_0(y^*)} \right)^2 \xi^*_1} = \frac{\varphi_{y_1}(y^*) \xi^*_1 - \frac{\beta y_1(y^*)}{\beta y_0(y^*)} \xi^*_0}{\left( \frac{\beta y_1(y^*)}{\beta y_0(y^*)} \right)^2 \xi^*_1} \). By (2.6) this again contradicts (7.9).
In the second case one have to consider \( \varphi_{y_1}(y^*)\xi_1^+ \xi_2^+(y^*, \zeta^*) < 0 \). By Gårding’s inequality we have
\[
\Re \int \sum 2|D_o| \mu^2 \Gamma_\mu^+(y, D') v_{1, \nu} \varphi(y^*) \frac{\partial v_{1, \nu}}{\partial y_1} d\Sigma < 0.
\]
This inequality and the fact that the second integral in the right hand side of \( J_2 \) is negative, imply that
\[
-\Re \int \sum 2|D_o| \mu^2 \Gamma_\mu^+(y, D') v_{1, \nu} \varphi(y^*) \left( \lambda + 2\mu \frac{\partial v_{1, \nu}}{\partial y_1} \varphi_{y_1}(y^*) - \frac{\partial v_{1, \nu}}{\partial y_0} \varphi_{y_0}(y^*) \right) d\Sigma > 0. \tag{7.12}
\]
Note that
\[
\Xi^{(1)}_{\lambda + 2\mu} = \int \sum \left( |s| (\lambda + 2\mu)^2 \varphi_{y_2}(y^*) \left| \frac{\partial w_{2, \nu}}{\partial y_2} \right|^2 + |s|^3 (\lambda + 2\mu)^2 \varphi_{y_2}(y^*) |w_{2, \nu}|^2 \right) d\Sigma
+ \Re \int \sum 2|s| (\lambda + 2\mu) \frac{\partial w_{2, \nu}}{\partial y_2} \left( \lambda + 2\mu \varphi_{y_1}(y^*) \frac{\partial w_{2, \nu}}{\partial y_1} - \varphi_{y_0}(y^*) \frac{\partial w_{2, \nu}}{\partial y_0} \right) d\Sigma
+ \int |s| (\lambda + 2\mu) \varphi_{y_2}(y^*) (\xi_0^2 - (\lambda + 2\mu) \xi_1^2 + s^2 \varphi_{y_0}^2(y^*) + s^2 (\lambda + 2\mu) \varphi_{y_1}^2(y^*)) |w_{2, \nu}|^2 d\Sigma
= \tilde{J}_1 + \tilde{J}_2 + \tilde{J}_3. \tag{7.13}
\]
Using equalities (5.14) we can transform \( \tilde{J}_2 \) as
\[
\tilde{J}_2 = -\Re \int \sum 2|s| \frac{\mu}{\lambda + 2\mu} \frac{\partial w_{1, \nu}}{\partial y_2} \left( \lambda + 2\mu \varphi_{y_1}(y^*) \frac{\partial w_{1, \nu}}{\partial y_1} - \varphi_{y_0}(y^*) \frac{\partial w_{1, \nu}}{\partial y_0} \right) d\Sigma + I,
\]
where
\[
|I| \leq \epsilon(\delta) \left\| \left( \frac{\partial w_{1, \nu}}{\partial y_2}, w_{1, \nu} \right) \right\|_X^2 + C_9 \| h(s) \|_{L^2(\Sigma)}^2.
\]
Then by (7.12) there exists \( C_{10} > 0 \) such that
\[
\tilde{J}_2 > C_{10} \int \Sigma \left( |s| \left| \frac{\partial w_{1, \nu}}{\partial y_1} \right|^2 + |s|^3 |w_{1, \nu}|^2 \right) d\Sigma. \tag{7.14}
\]
Since \( r_{\lambda + 2\mu}(\gamma) = 0 \), we have
\[
|\tilde{J}_3| \leq C'_{11} \delta \left\| \left( \frac{\partial w_{2, \nu}}{\partial y_2}, w_{2, \nu} \right) \right\|_X^2.
\]
This inequality and (7.14) imply
\[
\Xi^{(1)}_{\lambda + 2\mu} \geq C'_{11} \int \Sigma \left( |s| \left| \frac{\partial w_{2, \nu}}{\partial y_2} \right|^2 + |s|^3 |w_{2, \nu}|^2 + |s| \left| \frac{\partial w_{1, \nu}}{\partial y_1} \right|^2 + |s|^3 |w_{1, \nu}|^2 \right) d\Sigma
- \epsilon(\delta) \left\| \left( \frac{\partial w_{1, \nu}}{\partial y_2}, w_{1, \nu} \right) \right\|_X^2 + C_9 \| h(s) \|_{L^2(\Sigma)}^2. \tag{7.15}
\]
Now we will estimate \( J_3 \). By (5.18) and (5.19), there exists a constant \( C'_{12} > 0 \) such that
\[
|\xi_0^2 - s^2 \varphi_{y_0}^2(y^*) - (\lambda + 2\mu) \xi_1^2 + (\lambda + 2\mu) s^2 \varphi_{y_1}^2(y^*)|
\leq C'_{12} \delta \left| \xi_0^2 + |\xi_1|^2 + s^2 \right|^2, \quad \forall \zeta \in \mathcal{O}(\delta_1). \tag{7.16}
\]
Using this inequality we obtain
\[
\xi_0^2 - \mu \xi_1^2 - s^2 \varphi_{y_0}^2(y^*) + s^2 \varphi_{y_1}^2(y^*)
= (\lambda + \mu)(\xi_1^2 - s^2 \varphi_{y_1}^2(y^*)) + (\xi_0^2 - (\lambda + 2\mu)\xi_1^2 - s^2 \varphi_{y_0}^2(y^*) + s^2(\lambda + 2\mu)\varphi_{y_1}^2(y^*))
\ge (\lambda + \mu)(\xi_1^2 - s^2 \varphi_{y_1}^2(y^*)) - C_{12}\delta_1(|\xi_0|^2 + |\xi_1|^2 + s^2).
\]

Therefore, for all sufficiently small \( \delta_1 \), there exists \( C_{13} > 0 \) such that
\[
\xi_0^2 - \mu \xi_1^2 - s^2 \varphi_{y_0}^2(y^*) + s^2 \varphi_{y_1}^2(y^*) \ge C_{13}(|\xi_0|^2 + |\xi_1|^2 + s^2).
\] (7.17)

By (7.17), we see that \( J_3 \ge 0 \). Therefore by (7.15) and (6.1), there exist constants \( C'_{13} > 0 \), \( C_4 > 0 \) such that
\[
\mathcal{E}_{\lambda+2\mu}^{(1)} - C_{13}\mathcal{E}_{\lambda+2\mu} \ge C_{14}\left\| \left( \frac{\partial w_{1,\nu}}{\partial y_2}, w_{1,\nu} \right) \right\|_X^2 - C_{10}(\delta, \delta_1)(\|\mathcal{P}_\mu w_{1,\nu}\|_{L^2(Q)}^2 + \|v\|_{H^1(Q)}^2)
- \epsilon(\delta) \left\| \left( \frac{\partial w_{1,\nu}}{\partial y_2}, w_{1,\nu} \right) \right\|_X^2 + C_9\|h(s)\|_{L^2(\Sigma)}^2.
\]

This inequality, (5.16) and (7.4) with the sign +, imply
\[
\mathcal{E}_{\lambda+2\mu}^{(1)} \ge C_{15}\left\| \left( \frac{\partial w_{1,\nu}}{\partial y_2}, w_{1,\nu} \right) \right\|_X^2
- C_{16}(\delta, \delta_1)(\|\mathcal{P}_\mu w_{1,\nu}\|_{L^2(Q)}^2 + \|h(s)\|_{L^2(\Sigma)}^2 + \|v\|_{H^1(Q)}^2).
\] (7.18)

By (7.18), (5.31) and (5.32), we obtain (5.15).

**Case C.** Assume that \( s^* \neq 0 \). If \( \delta_1 > 0 \) is small enough, then there exists a constant \( C_{17} > 0 \) such that
\[
|\xi_0\varphi_{y_1}(y^*) - (\lambda + 2\mu)\xi_1\varphi_{y_1}(y^*)|^2 \le \delta_1^2C_{17}(|\xi_1|^2 + s^2).
\] (7.19)

By (5.31), there exists \( C_{18} > 0 \) such that
\[
\mathcal{E}_{\lambda+2\mu}^{(1)} + C_{18}\left( \|h(s)w_{2,\nu}\|_{H^1(Q)}^2 + \|h^3(s)w_{2,\nu}\|_{L^2(Q)}^2 \right)
\le C_{18}\left( \|\mathcal{P}_{\lambda+2\mu}w_{2,\nu}\|_{L^2(Q)}^2 + \|v\|_{H^1(Q)}^2 \right) + \epsilon\left\| \left( \frac{\partial w_{2,\nu}}{\partial y_2}, w_{2,\nu} \right) \right\|_X^2.
\] (7.20)

By (7.16) and (7.19), we have
\[
|\mathcal{J}_2 + \mathcal{J}_3| \le C_{10}\delta_1\left\| \left( \frac{\partial w_{2,\nu}}{\partial y_2}, w_{2,\nu} \right) \right\|_X^2.
\] (7.21)

By (7.21) we obtain from (7.13) that there exists a constant \( C_{20} > 0 \) such that
\[
\mathcal{E}_{\lambda+2\mu}^{(1)} \ge -\epsilon\left\| \left( \frac{\partial w_{2,\nu}}{\partial y_2}, w_{2,\nu} \right) \right\|_X^2
+ C_{20}\int_\Sigma \left( h^2(s)(\lambda + 2\mu)^2\varphi_{y_2}(y^*) \left| \frac{\partial w_{2,\nu}}{\partial y_2} \right|^2 + h^6(s)(\lambda + 2\mu)^3\varphi_{y_2}^6(y^*)|w_{2,\nu}|^2 \right) d\Sigma.
\] (7.22)
From (5.16), we easily obtain
\[
\left\| h(s) \left( \frac{\partial w_{2,\nu}}{\partial y_2} - |s| \varphi_{y_2}(y^*) w_{2,\nu} + g_{2,\nu} \right) \right\|_{L^2(\Sigma)}^2 = \frac{\mu^2}{(\lambda + 2\mu)^2} \left( \left\| h(s) \frac{\partial w_{1,\nu}}{\partial y_1} \right\|_{L^2(\Sigma)}^2 + \varphi_{y_1}(y^*) |h(s)|s|w_{1,\nu}\|_{L^2(\Sigma)}^2 \right).
\]

Hence (7.22) and this equality imply
\[
\Xi^{(1)}_{\lambda+2\mu} \geq C_{21} \int_\Sigma \left( h^2(s) \left( \left\| \frac{\partial w_{2,\nu}}{\partial y_2} \right\|^2 + \left\| \frac{\partial w_{1,\nu}}{\partial y_1} \right\|^2 + \left\| \frac{\partial w_{1,\nu}}{\partial y_0} \right\|^2 \right) + h^6(s) |w_{2,\nu}|^2 \right) d\Sigma - \epsilon \left\| \left( \frac{\partial w_{2,\nu}}{\partial y_2}, w_{2,\nu} \right) \right\|_X^2 - C_{22} |h(s)g|^2_{L^2(\Sigma)}.
\]

Now we claim that inequality (7.2) holds true for all sufficiently small \( \delta_1 \). First we may assume that for all \( \zeta \in O(\delta_1) \) we have \( s^2 \leq C_{23}(\zeta^2 + \zeta^4) \). In fact, if the last inequality is not true, then \( \zeta \in M \) and the case was treated in the previous section. Suppose that (7.2) is not true. In that case \( \zeta^1 = 0 \) and \( \zeta^2 \neq 0, s^2 \neq 0 \). Therefore \( \varphi_{y_0}(y^*) = 0 \) by (5.18). However, this implies \( (\zeta^2)^2 + (\lambda(y^*) + 2\mu(y^*)\varphi_{y_1}(y^*) s^2 = 0 \). Hence we arrived at a contradiction and the verification of (7.2) is complete.

Inequalities (7.2) and (7.23) imply that there exists a constant \( C_{24} \) such that
\[
\Xi^{(1)}_{\lambda+2\mu} \geq C_{24} \int_\Sigma \left( h^2(s) \left( \left\| \frac{\partial w_{2,\nu}}{\partial y_2} \right\|^2 + \left\| \frac{\partial w_{1,\nu}}{\partial y_1} \right\|^2 + \left\| \frac{\partial w_{1,\nu}}{\partial y_0} \right\|^2 \right) + h^6(s) |w_{2,\nu}|^2 \right) d\Sigma - \epsilon \left\| \left( \frac{\partial w_{2,\nu}}{\partial y_2}, w_{2,\nu} \right) \right\|_X^2 - C_{22} |h(s)g|^2_{L^2(\Sigma)}.
\]

By inequality (7.4) for \( V_\mu^+(\cdot, 0) \), we obtain the estimate
\[
\left\| h(s) \frac{\partial w_{2,\nu}}{\partial y_2} \right\|_{L^2(\Sigma)}^2 \leq C_{25} \left\{ \int_\Sigma \left( h^2(s) \left( \left\| \frac{\partial w_{1,\nu}}{\partial y_1} \right\|^2 + \left\| \frac{\partial w_{1,\nu}}{\partial y_0} \right\|^2 \right) + h^6(s) |w_{1,\nu}|^2 \right) d\Sigma + \|P_{\mu, w_{1,\nu}}\|_{L^2(\Sigma)}^2 + \|V\|_{H^1(\Sigma)}^2 \right\}.
\]

Inequalities (7.24) and (7.25) imply that there exists a constant \( C_{26} \) such that
\[
\Xi^{(1)}_{\lambda+2\mu} \geq C_{26} \left\{ \left( \frac{\partial w_{2,\nu}}{\partial y_2}, w_{2,\nu} \right) \right\|_X^2 - C_{27}(\delta, \delta_1) \left( \|P_{\mu, w_{1,\nu}}\|_{L^2(\Sigma)}^2 + \|h(s)g\|_{L^2(\Sigma)}^2 + \|V\|_{H^1(\Sigma)}^2 \right).
\]

By (7.26), (5.31) and (5.32), we obtain (5.15). The proof of Lemma 7.1 is finished.

8. The case \( r_\mu(\gamma) \neq 0 \) and \( r_{\lambda+2\mu}(\gamma) \neq 0 \)

In this section, we will prove

Lemma 8.1. Let \( \gamma = (y^*, \zeta^*) \in \Sigma \times S^2 \) be a point such that
\[
|r_\mu(y^*, \zeta^*)| \neq 0 \quad \text{and} \quad |r_{\lambda+2\mu}(y^*, \zeta^*)| \neq 0.
\]

If \( \text{supp} \chi_\nu \subset O(\delta_1) \) and \( \delta_1 > 0 \) is sufficiently small, then estimate (5.15) holds true.
Proof. Thanks to (8.1) and Proposition 5.3, decomposition (5.23) holds true for \( \beta = \mu \) and \( \beta = \lambda + 2\mu \). Therefore we have

\[
(D_{y_2} - \Gamma^+_\mu (y, D'))v_{1,\nu}|_{y_2=0} = V^+_{\mu}(\cdot, 0),
\]

\[
(D_{y_2} - \Gamma^+_{\lambda+2\mu}(y, D'))v_{2,\nu}|_{y_2=0} = V^+_{\lambda+2\mu}(\cdot, 0).
\]  

By Proposition 5.4, we have an a priori estimate

\[
||h(D_{\sigma})V^+_{\mu}(\cdot, 0)||_{L^2(\Sigma)}^2 + ||h(D_{\sigma})V^+_{\lambda+2\mu}(\cdot, 0)||_{L^2(\Sigma)}^2 \leq C_1 \left( ||P_{\lambda+2\mu}v_2||_{L^2(\Sigma)}^2 + ||P_\mu v_1||_{L^2(\Sigma)}^2 + ||v||_{H^1(\Sigma)}^2 \right).
\]  

Denote

\[
\tilde{\alpha}^+_\mu(y', D) = \alpha^+_\mu(y', D) + i|D_{\sigma}|(\varphi_{y_2} - \varphi_{y_2}(y^*)).
\]

Using (5.16), we may rewrite (8.2) and (8.3) as

\[
\frac{\lambda + 2\mu}{\mu} \left( \frac{\partial v_{1,\nu}}{\partial y_1} - |D_{\sigma}|\varphi_{y_1}(y^*)v_{2,\nu} \right) - i\tilde{\alpha}^+_\mu(y, D')v_{1,\nu} = i\nu^+_{\mu}(\cdot, 0) + i\nu^{-1}_\sigma g_{1,\nu},
\]

\[
\frac{\mu}{\lambda + 2\mu} \left( \frac{\partial v_{1,\nu}}{\partial y_1} + |D_{\sigma}|\varphi_{y_1}(y^*)v_{1,\nu} \right) - i\tilde{\alpha}^+_{\lambda+2\mu}(y, D')v_{2,\nu} = i\nu^+_{\lambda+2\mu}(\cdot, 0) + i\nu^{-1}_\sigma g_{2,\nu}.
\]

Let \( B(y, D') \) be the matrix pseudo-differential operator with the symbol

\[
B(y, \zeta) = \left( \begin{array}{cc} -i\tilde{\alpha}^+_\mu(y', \zeta) & \frac{\lambda + 2\mu}{\mu}(\xi_1 - |s|\varphi_{y_1}(y)) \\ -i\tilde{\alpha}^+_{\lambda+2\mu}(y', \zeta) & -i\tilde{\alpha}^+_\mu(y', \zeta) \end{array} \right).
\]

By (5.19) and (5.20), we see: if \( \det B(y^*, \zeta^*) = 0 \), then either \( \xi_0^* + i|s|\varphi_{y_0}(y^*) = 0 \) or

\[
\zeta^* \in \left\{ \zeta \in \mathbb{R}^2 : (\xi_1 + i|s|\varphi_{y_1}(y^*))^2 = \frac{(\xi_0^* + i|s|\varphi_{y_0}(y^*))^2}{(\lambda + 3\mu)(y^*)} \right\}.
\]  

In this case of (8.7), we have \( \varphi_{y_0}(y^*) = \varphi_{y_1}(y^*) = \xi_0^* = \xi_1^* = 0, \ s^* = 1 \). Now we consider two cases

**Case A.** \( \det B(\gamma) \neq 0 \).

In this case, there exists a parametrix of the operator \( B(y, D') \), which we denote by \( B^{-1}(y, D') \), such that

\[
(v_{1,\nu}, v_{2,\nu}) = B^{-1}(y, D')(V^+_{\mu}(\cdot, 0) - \nu^{-1}_\sigma g_{1,\nu}, V^+_{\lambda+2\mu}(\cdot, 0) - \nu^{-1}_\sigma g_{2,\nu})^T + K(v_{1,\nu}, v_{2,\nu}),
\]

where \( K : (L^2(\Omega))^2 \to (H^1(\Omega))^2 \). By (8.4) and (8.8),

\[
||\xi| + |\xi_{\lambda+2\mu}| \leq C_2 \left( ||P_\mu v_1||_{L^2(\Sigma)}^2 + ||P_{\lambda+2\mu}v_2||_{L^2(\Sigma)}^2 + ||h(s)g||_{L^2(\Sigma)}^2 + ||v||_{H^1(\Sigma)}^2 \right).
\]  

(Here and henceforth, for simplicity, we do not distinguish \( a^T \) from a vector \( a \).) By (8.9), (5.30) and (5.31), we obtain (5.15).

**Case B.** \( \det B(\gamma) = 0 \).

We claim that this situation is possible in the two cases:

\[
(i) \quad \varphi_{y_0}(y^*) = \varphi_{y_1}(y^*) = \xi_0^* = \xi_1^* = 0, \ s^* = 1;
\]

\[
(ii) \quad \xi_0^* = 0, \ s^* \varphi_{y_0}(y^*) = 0.
\]  

\( \text{(8.10)} \)
The first subcase was treated in Section 6. Let us consider the second subcase (8.10). Next we may assume that
\[ \zeta^* \in \bar{\mathcal{M}}. \]
Otherwise, \( \zeta^* \in \mathcal{M} \), so that the case was treated in Section 6. Next we may assume that
\[ \text{Im} \Gamma^+_{\mu}(\gamma) = \text{Im} \Gamma^+_{\lambda+2\mu}(\gamma) \geq 0. \] (8.11)
Really if
\[ \text{Im} \Gamma^+_{\mu}(\gamma) = \text{Im} \Gamma^+_{\lambda+2\mu}(\gamma) < 0, \] (8.12)
then the situation is simple since we have the decomposition
\[ P_{\beta v_{j(\beta)}}, v = \beta|G|^2 (D_{y_2} - \Gamma^+_\beta(y, D')) V^+_{\beta} + T^+_{\mu} v_{j(\beta)}, v, \]
where \( T^+_{\mu} \in \mathcal{L}(H^1(\mathcal{Q}), L^2(\mathcal{Q})) \), \( \beta \in \{\mu, \lambda + 2\mu\} \), \( j(\beta) = 1 \) for \( \beta = \mu \) and \( j(\beta) = 2 \) for \( \beta = \lambda + 2\mu \). This decomposition, (8.12) and Proposition 5.3 imply
\[ \|h(D_x)(D_{y_2} - \Gamma^+_\beta(y, D')) V^+_{\beta, \mu}\|_{L^2(\mathcal{Q})} \leq C_3 \left( \|P_{\beta v_{j(\beta)}, v}\|_{L^2(\mathcal{Q})} + \|v\|_{H^2(\mathcal{Q})}^2 \right). \] (8.13)
Obviously
\[ -V^+_{\beta}(-, 0) + V^-_{\beta}(-, 0) = (\alpha^+_{\beta}(y, D') - \alpha^-_{\beta}(y, D')) v_{1, \mu} \text{ on } \Sigma. \]
Since \( \alpha^+_{\mu}(y^*, \zeta^*) - \alpha^-_{\mu}(y^*, \zeta^*) = 2/\sqrt{r_\mu(y^*, \zeta^*)} \neq 0 \), we have
\[ \left\| \left( \frac{\partial w_{\mu}}{\partial y_2}, w_{\mu} \right) \right\|^2 \leq C_4 \left( \|P_{\lambda+2\mu, \mu} w_{2, \mu}\|^2_{L^2(\mathcal{Q})} + \|P_{\mu, \mu} w_{1, \mu}\|^2_{L^2(\mathcal{Q})} + \|v\|^2_{H^1(\mathcal{Q})} \right) \] (8.14)
by (8.13) and Gårding’s inequality.

By (8.14), (5.30) and (5.31), we obtain (5.15) under Condition (8.12).

In order to treat (8.10) under (8.11), we will use Calderon’s method. First we introduce the new variables \( U = (U_1, U_2, U_3, U_4) \) with four components, where
\[ (U_1, U_2) = \Lambda(D') \mathcal{F}_{\sigma}^{-1} \mathcal{U}, \quad (U_3, U_4) = (D_2 + i|D_\sigma|\varphi_{y_2}) \mathcal{F}_{\sigma}^{-1} \mathcal{U}, \]
and \( \Lambda \) is the pseudo-differential operator with the symbol \( (s^2 + \xi^2 + \xi_0^2 + 1)^{1/2} \). In the new notations, problem (5.1) and (5.2) can be written in the form
\[ D_{y_2} U = M(y, D') U + F \text{ in } \mathbb{R}^3 \times [0, 1], \quad (U_1, U_2)(y)|_{y_2=0} = 0, \] (8.15)
where \( F = (0, \mathcal{F}_{\sigma}^{-1} \mathcal{U}) \). Here \( M(y, D') \) is the matrix pseudo-differential operator whose principal symbol \( M_1(y, \zeta) \) is given by
\[ M_1(y, \zeta) = \begin{pmatrix} 0 & A_1^{-1} M_21^{-1} E_1 \end{pmatrix} \]
(see [49]). Here we set \( \tilde{\theta} = (\xi_1 + i|s|\varphi_{y_1}, 0), G(y_1) = (-d\ell(y_1)/dy_1, 1), A_1 = |\zeta|, M_21(y, \zeta_1 + i|s|\nabla_y \varphi(y)) = ((\xi_1 + i|s|\varphi_{y_1}(y))^2 - \mu(\xi_1 + i|s|\varphi_{y_1}(y))^2)E_2 - (\lambda + \mu)(y)\tilde{\theta} G + G^T \tilde{\theta} - 2\mu(\tilde{\theta}, G)E_2, \quad A = (\lambda + \mu)(y)G^T G + \mu(y)|G|^2 E_2. \]
The matrix \( M_1(\gamma) \) has only two eigenvalues \( M^\pm \) given by (5.18)–(5.20). Moreover it is known that the Jordan form of the matrix \( M_1(\gamma) \) has two Jordan blocks of the form
\[ M^\pm = \begin{pmatrix} \Gamma^+_{\mu}(\gamma) & 1 \\ 0 & \Gamma^+_{\mu}(\gamma) \end{pmatrix}. \]
Following [46] and using the change of variables $W = S^{-1}(y, D')U$ which is constructed below, we can reduce system (8.15) to the form

$$D_{y'2}W = \widetilde{M}(y, D')W + T(y, D')W + \tilde{F},$$

where the matrix $\widetilde{M}$ has the form

$$\widetilde{M}(y, \zeta) = \begin{pmatrix} M_{+}(y, \zeta) & 0 \\ M_{-}(y, \zeta) & \end{pmatrix}, \quad M_{\pm} = \begin{pmatrix} \Gamma_{\lambda+2\mu}^{\pm} y, \zeta \pm \mu y, \zeta \\ \Gamma_{\mu}^{\pm} y, \zeta \end{pmatrix},$$

the operator $T$ is in $L^\infty(0, 1; L((H^1(\Sigma))^4, (H^1(\Sigma))^4))$, $m_{12}(y, D')$ are first order operators and

$$\|\tilde{F}\|_{L^2(R^1; L((H^1(\Sigma))^2))} \leq C_5 \left( \|\mathcal{P}_\sigma \mathcal{F}_{\sigma}^{-1} U\|_{L^2(R^1; (H^1(\Sigma))^2)} + \|\mathcal{F}_{\sigma}^{-1} U\|_{L^2(R^1; (H^1(\Sigma))^2)} \right).$$

Now we describe the construction of the pseudo-differential operator $S$. We take the symbol $S$ in the form $S = (s_1^+, s_2^+, s_1^-, s_2^-)$. Here

$$s_1^+ = \left( (\bar{\theta} + \alpha_{\lambda+2\mu} G)\Lambda_1^{-1}, \alpha_{\lambda+2\mu}(\bar{\theta} + \alpha_{\lambda+2\mu} G)\Lambda_1^{-2} \right)$$

are the eigenvectors of the matrix $M_1(y, \zeta)$ on the sphere $\zeta \in S^2$ which corresponds to the eigenvalue $\Gamma_{\lambda+2\mu}^{\pm}$ and the vectors $s_2^\pm$ are given by the formula

$$S_2^\pm = E_\pm s_1^\pm, \quad E_\pm = \frac{1}{2\pi i} \int_{C_\pm} (z - M_1(y, \zeta))^{-1} \mathrm{d}z,$$

where $C^\pm$ are small circles centered at $\Gamma_{\lambda+2\mu}^{\pm}(\gamma)$ and $s^\pm$ solves the equation $M_1(\gamma)s^\pm - \Gamma_{\lambda+2\mu}^{\pm}(\gamma)s^\pm = s_1^\pm(\gamma)$. Since $\zeta^* \in \mathcal{M}$ and $\zeta_0^* = 0$, we have $\xi_0^* \neq 0$. Therefore the circles $C^\pm$ may be taken such that the disks bounded by these circles do not intersect, provided that $\delta_1, \delta$ are taken sufficiently small. Note that the vectors $s_j^\pm \in C^2(B_2 \times \mathcal{O}_{\delta_1})$ are homogeneous functions of the order zero in $(s, \xi_0, \xi_1)$. Now using a standard argument (see [36], p. 241), we can estimate the last two components of $W$ as follows

$$\|(W_3, W_4)\|_{L^2(R^1; (H^2(\Sigma))^2)} \leq C_6 \left( \|\mathcal{P}_\sigma \mathcal{F}_{\sigma}^{-1} U\|_{L^2(R^1; (H^1(\Sigma))^2)} + \|\mathcal{F}_{\sigma}^{-1} U\|_{L^2(R^1; (H^1(\Sigma))^2)} \right),$$

where the constant $C_6$ is independent of $N$.

Now we need to estimate the first two components of the vector function $W$ on $\Sigma$. Thanks to the zero boundary conditions for $U_3$ and $U_4$, we have

$$S_{11}(y_0, y_1, 0, D')(W_1, W_2) = -S_{12}(y_0, y_1, 0, D')(W_3, W_4) + T_{-1}(y_0, y_1, 0, D')\mathcal{F}_{\sigma}^{-1} U,$$

where we set

$$S(y, \zeta) = \begin{pmatrix} S_{11}(y, \zeta) & S_{12}(y, \zeta) \\ S_{21}(y, \zeta) & S_{22}(y, \zeta) \end{pmatrix}, \quad T_{-1} : (H^1(\Sigma))^2 \rightarrow (H^2(\Sigma))^2.$$ 

The principal symbol of the pseudo-differential operator $S_{11}$ is the $2 \times 2$ matrix such that the first column equals the last two coordinates of the vector $s_1^+$ and the second column equals the last two coordinates of the vector $s_2^+$. At the point $\gamma$, these vectors are given by the formulæ

$$\tilde{\eta} = (\xi_1^* + i|s^*|\varphi_{y_1}(y^*), \text{isign}(\xi_1^*)(\xi_1^* + i|s^*|\varphi_{y_1}(y^*)))$$

$$s_1^+(\gamma) = \left( \tilde{\eta}, \frac{\text{isign}(\xi_1^*)(\xi_1^* + i|s^*|\varphi_{y_1}(y^*))}{\sqrt{(|\xi_1^*|)^2 + (s^*)^2}} \tilde{\eta} \right),$$

$$s_2^+(\gamma) = \left( \tilde{\eta}, \frac{\text{isign}(\xi_2^*)(\xi_2^* + i|s^*|\varphi_{y_1}(y^*))}{\sqrt{(|\xi_2^*|)^2 + (s^*)^2}} \tilde{\eta} \right).$$
\[ \zeta = \frac{-1}{\sqrt{(\xi_1^*)^2 + (s^*)^2}} \frac{\lambda + 3\mu}{2(\lambda + \mu)} (y^*)(isgn(\xi_1^*), 1), \]

\[ s_2^+(\gamma) = \left( \zeta, \frac{1}{\sqrt{(\xi_1^*)^2 + (s^*)^2}} \{isgn(\xi_1^*)(\xi_1^* + i|s^*|\varphi_{\zeta_1}(y^*))\zeta + \bar{\eta} \} \right). \]

Therefore \( \det S_{11}(\gamma) \neq 0 \). From (8.15), (8.16) and Gårding's inequality, we obtain

\[ \left\| \left( \frac{\partial \mathbf{w}_x}{\partial y_2}, \mathbf{w}_x \right) \right\| \leq C_7(\| P_\sigma F^{-1}_\sigma U \|_{H^1(Q)}^2 + \| F^{-1}_\sigma U \|_{H^2(Q)}^2), \tag{8.18} \]

where the constant \( C_7 \) is independent of \( N \). By (8.9), (5.30) and (5.31), we obtain (5.15). The proof of Lemma 8.1 is finished. \( \square \)

9. PROOFS OF THEOREMS 2.2 AND 2.3

In this section we prove Theorems 2.2 and 2.3. The proof is based on the duality argument and the scenario is described as follows. In view of the fact that observability implies controllability and vice versa, we will introduce an extremal problem, and, using Carleman estimate (2.9), we show that there exists a solution to this problem which solves the control problem for the operator \( P^* \) and minimizes weighted \( L^2(Q) \)-norm. At the next step, we obtain an estimate of this solution in the weighted \( H^1 \)-norm. This estimate implies (2.11) and (2.12).

We introduce the Banach space \( \mathcal{X} = (H^1(Q))^2 \) with the norm

\[ \| \mathbf{w} \|_{\mathcal{X}}^2 = \int_Q (|\nabla \mathbf{w}|^2 + s^2|\mathbf{w}|^2)dx. \]

In order to prove the theorems, we consider the following extremal problem

\[ J(z, v_1, v_2) = \frac{1}{2} \| ze^{-s\phi} \|_{(L^2(Q))^2}^2 + \frac{1}{2} \| v_1 e^{-s\phi} \|_{(L^2(Q_\omega))^2}^2 + \frac{1}{2s^2} \| v_2 e^{-s\phi} \|_{(L^2(Q_\omega))^2}^2 \rightarrow \text{inf}, \tag{9.1} \]

\[ Pz = u e^{2s\phi} + \frac{\partial v_1}{\partial x_0} + v_2 \quad \text{in } Q, \tag{9.2} \]

\[ \text{supp } v_j \subset Q_\omega, \quad j = 1, 2, \quad z|_{(0,T) \times \partial \Omega} = 0, \quad \frac{\partial z}{\partial x_0}(0, \cdot) = \frac{\partial z}{\partial x_0}(T, \cdot) = 0. \tag{9.3} \]

Denote by \((z, v_1, v_2)\) the solution to extremal problem (9.1)–(9.3).

We have

Lemma 9.1. Under the conditions of Theorem 2.2 for all \( u \in (L^2(Q))^2 \), there exists a unique solution \((z, v_1, v_2) \in (H^1(Q))^2 \times (H^1(0,T; L^2(Q)))^2 \times (H^1(Q))^2 \) to problem (9.1)–(9.3). Moreover this solution satisfies the optimality system

\[ Pp + ze^{-2s\phi} = 0 \quad \text{in } Q, \tag{9.4} \]

\[ p|_{(0,T) \times \partial \Omega} = 0, \quad \frac{\partial p}{\partial x_0}(0, \cdot) = \frac{\partial p}{\partial x_0}(T, \cdot) = 0, \tag{9.5} \]

\[ p = \frac{1}{s^2} v_2 e^{-2s\phi} \quad \text{in } Q_\omega, \quad \frac{\partial p}{\partial x_0} = -v_1 e^{-2s\phi} \quad \text{in } Q_\omega, \tag{9.6} \]

\[ Pz = u e^{2s\phi} + \frac{\partial v_1}{\partial x_0} + v_2 \quad \text{in } Q, \quad \text{supp } v_j \subset Q_\omega, \quad j = 1, 2, \tag{9.7} \]

\[ z|_{(0,T) \times \partial \Omega} = 0, \quad \frac{\partial z}{\partial x_0}(0, \cdot) = \frac{\partial z}{\partial x_0}(T, \cdot) = 0, \tag{9.8} \]
and the following estimate holds true:

\[
\|z e^{-s\phi}\|_{L^2}^2 + \|\frac{\partial v_1}{\partial x_0} e^{-s\phi}\|_{L^2(\Omega)}^2 + s^2 \|v_1 e^{-s\phi}\|_{L^2(Q_\omega)}^2 + \|v_2 e^{-s\phi}\|_{L^2(Q_\omega)}^2 \leq C_1 \|u e^{s\phi}\|_{L^2(Q)}^2.
\] (9.9)

**Proof of Lemma 9.1.** The proof is done along the standard argument (e.g., [38]) and for completeness we will give it. For any \( \varepsilon \in (0, 1) \), we consider the following extremal problem

\[
J_\varepsilon(z, v_1, v_2, w) = \frac{1}{2} \int_Q |v|^2 e^{-2s\phi} dx + \frac{1}{2} \int_Q \varepsilon \left( \|v_1\|^2 + \frac{|v_1|^2}{s^2} \right) e^{-2s\phi} dx + \frac{1}{2\varepsilon} \int_Q |w|^2 dx \rightarrow \inf,
\] (9.10)

\[
P_\varepsilon z = \frac{\partial v_1}{\partial x_0} + v_2 + u e^{2s\phi} + w \quad \text{in } Q,
\] (9.11)

\[
z|_{(0,T) \times \partial \Omega} = 0, \quad \frac{\partial n}{\partial x_0}(0, x') = \frac{\partial n}{\partial x_0}(T, x') = 0,
\] (9.12)

where \( m_\varepsilon \in C^2(\overline{\Omega}), m_\varepsilon(x') > 0 \) on \( \overline{\Omega} \).

Denote by \((\hat{z}_\varepsilon, \hat{v}_1, \hat{v}_2, \hat{w}_\varepsilon)\) a solution to extremal problem (9.10)–(9.12).

**Remark.** We understand equation (9.11) and the boundary Conditions (9.12) in the sense of the equality:

\[(z, P_\delta)(L^2(Q)) = -(v_1, \partial x_0, \delta)(L^2(Q))^2 + (v_2 + u e^{2s\phi} + w, \delta)(L^2(Q))^2\]

for any \( \delta \in (H^1(Q))^2 \) satisfying \( P_\delta \in (L^2(Q))^2, \delta|_{(0,T) \times \partial \Omega} = 0, \frac{\partial \delta}{\partial x_0}(0, \cdot) = \frac{\partial \delta}{\partial x_0}(T, \cdot) = 0 \). If \( z, v_1 \) are regular, then \( \frac{\partial n}{\partial x_0}(0, \cdot) - v_1(0, \cdot) = 0 \) and \( \frac{\partial n}{\partial x_0}(T, \cdot) - v_1(T, \cdot) = 0 \).

We have

**Proposition 9.1.** Under conditions of Theorem 2.2 for all \( u \in (L^2(Q))^2 \), there exists a unique solution \((\hat{z}_\varepsilon, \hat{v}_1, \hat{v}_2, \hat{w}_\varepsilon) \in (H^1(\Omega))^2 \times (H^1(0, T; L^2(\Omega)))^2 \times (H^2(Q))^2 \times (H^2(Q))^2 \) to problem (9.10)–(9.12). Moreover, this solution satisfies the optimality system:

\[
p_\varepsilon(x) = \hat{w}_\varepsilon(x) \quad \text{in } Q,
\] (9.13)

\[
P_\varepsilon e^{-2s\phi} \hat{z}_\varepsilon = 0 \quad \text{in } Q,
\] (9.14)

\[
\frac{\partial p_\varepsilon}{\partial x_0}(0, \cdot) = \frac{\partial p_\varepsilon}{\partial x_0}(T, \cdot) = 0,
\]

\[
\frac{\partial p_\varepsilon}{\partial x_0}(0, \cdot) = \frac{\partial p_\varepsilon}{\partial x_0}(T, \cdot) = 0,
\] (9.15)

\[
P_\varepsilon \hat{z}_\varepsilon = \frac{\partial \hat{v}_1, \hat{v}_2}{\partial x_0} + \hat{v}_3 + u e^{2s\phi} + \hat{w}_\varepsilon \quad \text{in } Q,
\] (9.16)

\[
\frac{\partial p_\varepsilon}{\partial x_0} + m_\varepsilon \hat{v}_1 e^{-2s\phi} = 0 \quad \text{in } Q,
\] (9.17)

\[
p_\varepsilon - m_\varepsilon \frac{\hat{v}_2}{s^2} e^{-2s\phi} = 0 \quad \text{in } Q,
\] (9.18)
and the following estimate holds:

\[ \| \tilde{z}_e e^{-s \phi} \|_X^2 + \left\| \frac{\partial \tilde{v}_1}{\partial x_0} e^{-s \phi} \right\|_{L^2(Q_e)}^2 + s^2 \left\| \tilde{v}_1 e^{-s \phi} \right\|_{L^2(Q_e)}^2 + \left\| \tilde{v}_2 e^{-s \phi} \right\|_{L^2(Q_e)}^2 \leq C_2 \| u e^{-2s \phi} \|_{L^2(Q)}^2. \]  

(9.19)

**Proof of Proposition 9.1.** Since the functional \( J_\varepsilon \) is strictly convex and the set of admissible elements is a linear space, problem (9.10)–(9.12) has at most one solution. First let us prove that there exists a solution to (9.10)–(9.12): an element \((\tilde{z}, \tilde{v}_1, \tilde{v}_2, \tilde{w})\) in the space \((L^2(Q))^3\). Obviously \((0, 0, 0, -u e^{-2s \phi})\) is an admissible element and the set of an admissible elements is not empty. Hence there exists a minimizing sequence \(\{ (\tilde{z}_j, \tilde{v}_{1,j}, \tilde{v}_{2,j}, \tilde{w}_j) \}_{j=1}^{\infty} \) such that

\[
(\tilde{z}_j, \tilde{v}_{1,j}, \tilde{v}_{2,j}, \tilde{w}_j) \rightarrow (\tilde{z}, \tilde{v}_1, \tilde{v}_2, \tilde{w}) \text{ weakly in } (L^2(Q))^3.
\]  

(9.20)

Passing to the limit in (9.11), (9.12) and using (9.20), we obtain that \((\tilde{z}_j, \tilde{v}_{1,j}, \tilde{v}_{2,j}, \tilde{w}_j)\) is an admissible element. On the other hand, since the functional \( J_\varepsilon \) is lower semi-continuous with respect to the weak convergence in \((L^2(Q))^3\), this is a solution to problem (9.10)–(9.12).

In order to obtain optimality system (9.13)–(9.18), we introduce the function \( q(\delta_1, \delta_2, \delta_3) = J_\varepsilon(\tilde{z}_e + \delta d_1, \tilde{v}_1 e^{-2s \phi} + \delta_2 d_2, \tilde{v}_2, \tilde{w}) \), where \( d_1 \in (L^2(Q))^2 \) with \( P d_1 \in (L^2(Q))^2 \), \( d_2 \in (H^1(Q))^2 \), \( d_3 \in (L^2(Q))^2 \),

\[ r(\delta_1, \delta_2, \delta_3) = P(\tilde{z}_e + \delta d_1) - \left( \frac{\partial}{\partial x_0}(\tilde{v}_1 e^{-2s \phi} + \tilde{v}_2) + \delta_3 d_3 \right) - u e^{-2s \phi}. \]

Obviously the function \( q \) attains the minimum in \( \mathbb{R}^3 \) at \((0, 0, 0)\) if the variation is admissible. Thus \( \nabla q(0, 0, 0) = 0 \). Moreover the equalities \( \frac{\partial q}{\partial \delta_2}(0, 0, 0) = \frac{\partial q}{\partial \delta_3}(0, 0, 0) = 0 \) imply

\[
- \frac{1}{\varepsilon} \int_Q \tilde{w}_e \frac{\partial d_2}{\partial x_0} dx + \int_Q m_e \tilde{v}_1 e^{-2s \phi} dx = 0, \quad \forall d_2 \in (H^1(Q))^2 \text{ such that } d_2(0, \cdot) = d_2(T, \cdot) = 0, 
\]

\[
- \frac{1}{\varepsilon} \int_Q \tilde{w}_e \frac{\partial d_3}{\partial x_0} dx + \int_Q m_e \tilde{v}_2 e^{-2s \phi} dx = 0, \quad \forall d_3 \in (L^2(Q))^2. 
\]

On the other hand, these equalities are equivalent to

\[
\frac{1}{\varepsilon} \frac{\partial \tilde{w}_e}{\partial x_0} + m_e \tilde{v}_1 e^{-2s \phi} = 0 \text{ in } Q, 
\]

(9.21)

\[
\frac{\tilde{w}_e}{\varepsilon} - m_e \frac{\tilde{v}_2}{s^2} e^{-2s \phi} = 0 \text{ in } Q. 
\]

(9.22)

By the equality \( \frac{\partial q}{\partial \delta_1}(0, 0, 0) = 0 \), we obtain

\[
\left( \frac{\tilde{w}_e}{\varepsilon}, P d_1 \right)_{(L^2(Q))^2} + \int_Q \tilde{z}_e d_1 e^{-2s \phi} dx = 0, \quad \forall d_1 \in X, 
\]

(9.23)

where \( X = \{ d_1 \in L^2(0, T; (H^2(\Omega))^2); P d_1 \in (L^2(Q))^2, d_1|_{(0,T) \times \partial \Omega} = 0, \frac{\partial d_1}{\partial x_0}(0, \cdot) = \frac{\partial d_1}{\partial x_0}(T, \cdot) = 0 \}. \)

Since \( \tilde{v}_1 e^{-2s \phi} \in (L^2(Q))^2 \), we obtain immediately from (9.21) that \( \frac{\partial \tilde{w}_e}{\partial x_0} \in (L^2(Q))^2 \). Since \( d_1(0, \cdot) \) and \( d_1(T, \cdot) \) can be chosen arbitrarily, it follows from (9.23) that

\[
\frac{\partial \tilde{w}_e}{\partial x_0}(0, \cdot) = \frac{\partial \tilde{w}_e}{\partial x_0}(T, \cdot) = 0, \quad \tilde{w}_e|_{(0,T) \times \partial \Omega} = 0. 
\]
Introducing the function $p_e$ by formula (9.13), in terms of (9.21)-(9.23), we immediately obtain equalities (9.17), (9.18) and (9.14). Equation (9.17) implies that $\frac{\partial \phi}{\partial x_0} \in (L^2(Q))^2$. From (9.14), (9.15) we obtain $p_e \in (H^1(Q))^2$.

Next we will show that $p_e \in (H^2(Q))^2$. We extend $p_e$ on the set $[-T, T] \times \Omega$ by the formula: $p_e(x_0, x') = p_e(x_0, x') = p_e(2T - x_0, x')$ for $x \in [-T, 0] \times \Omega$ and $p_e(x_0, x') = p_e(2T - x_0, x')$ for $x_0, x' \in [T, 2T] \times \Omega$. In the same way, we extend $-\partial\phi_* e^{-2\phi}$ on the domain $[-T, 2T] \times \Omega$ and denote the extended function by $\tilde{f}$. Since $\frac{\partial \phi}{\partial x_0}(T, x') < 0$ for all $x' \in \Omega$ and $\frac{\partial \phi}{\partial x_0}(0, x') > 0$ for all $x' \in \Omega$ by (2.10), there exists $\delta > 0$ such that we can continue the function $\phi$ on $[-\delta, T+\delta] \times \Omega$ up to a $C^3$-function such that $\frac{\partial \phi}{\partial x_0}(x) < 0$ for all $x \in [-T, T+\delta] \times \Omega$ and $\frac{\partial \phi}{\partial x_0}(x) > 0$ for all $x \in [-\delta, 0] \times \Omega$. By (9.14), we have

$$Pp_e = \tilde{f} \quad \text{in} \quad \tilde{Q} = [-\delta, T+\delta] \times \Omega. \quad (9.24)$$

Also Condition 2.1 for the function $\phi$ holds true if we replace the domains $Q, Q_\omega$ by $\tilde{Q}, [-\delta, T+\delta] \times \omega$ respectively.

Let $D_h f = \frac{f(x_0, x') - f(x_0, x)}{h}$ and $D_h^2 f = \frac{f(x_0, x') - f(x_0, x)}{2h}$. For the function $D_h D_h^2 p$, we have

$$\frac{\partial}{\partial x_0} D_h D_h^2 p\big|_{x_0 = 0} = \frac{\partial}{\partial x_0} D_h D_h^2 p\big|_{x_0 = T} = 0.$$

Note that $PD_h D_h^2 p = D_h D_h^2 \tilde{f}$. Hence

$$(\tilde{z}_e, D_h D_h^2 \tilde{f}(L^2(Q))^2) = -((\tilde{v}_1, \partial_{x_0} D_h D_h^2 p)(L^2(Q))^2) + (\tilde{v}_2 + u e^{2\phi} + \tilde{w}_e, D_h D_h^2 p)(L^2(Q))^2).$$

Using (9.17), (9.18) and the definition of the function $\tilde{f}$, we have

$$\|D_h \tilde{z}_e\|_{(L^2(Q))^2} + \|D_h \tilde{v}_1,\|_{(L^2(Q))^2} + \|D_h \tilde{v}_2,\|_{(L^2(Q))^2} \leq C_0'\|D_h u\|_{(L^2(Q))^2} + \|\tilde{z}_e, \tilde{v}_1, \tilde{v}_2\|_{(L^2(Q))^2},$$

where the constant $C_0' > 0$ is independent of $h$. Therefore

$$(\partial_{x_0} \tilde{z}_e, \partial_{x_0} \tilde{v}_1, \partial_{x_0} \tilde{v}_2) \in (L^2(Q))^6.$$

Equations (9.11) - (9.18) imply that $\tilde{z}_e \in (H^1(Q))^2$ and $p_e \in (H^2(Q))^2$.

Let $\chi_1 \in C_0^\infty(-\delta, T + \delta)$ be a cut-off function such that $\chi_1\big|_{[-\delta, T+\delta]} = 1$. Then

$$P(\chi_1 p_e) = \chi_1 \tilde{f} - [\chi_1, P] p_e \quad \text{in} \quad \tilde{Q}, \quad (9.25)$$

where $\supp \chi_1, P p_e \subset ((T + \frac{\delta}{2}, T + \delta) \times \Omega) \cup (-\delta, -\frac{\delta}{2}) \times \Omega$. We will apply Carleman estimate (2.9) to equation (9.25).
Thus the verification of (9.27) is complete.

Moreover, by a way similar to Appendix II in [26] (i.e., the final step of the proof of Lem. 2.3 in [26]), we can prove that at the right hand side of (2.9), we can replace the integral over \( Q \) by the following integral

\[
\int_{Q} \left( \left| \frac{\partial^2 u}{\partial x_0^2} \right|^2 + s^2 \left| \frac{\partial u}{\partial x_0} \right|^2 + s^4 |u|^2 \right) e^{2s\phi} \, dx.
\]

Note that thanks to the choice of the extension of the function \( \phi \), we have

\[
\int_{(-\delta,T+\delta)\times \omega} \left( \left| \frac{\partial^2 \phi_c}{\partial x_0^2} \right|^2 + s^2 \left| \frac{\partial \phi_c}{\partial x_0} \right|^2 + s^4 |\phi_c|^2 \right) e^{2s\phi} \, dx \leq C_5 \int_{Q} \left( \left| \frac{\partial^2 \phi_c}{\partial x_0^2} \right|^2 + s^2 \left| \frac{\partial \phi_c}{\partial x_0} \right|^2 + s^4 |\phi_c|^2 \right) e^{2s\phi} \, dx.
\]

In fact, let us denote the left and the right hand sides of (9.27) respectively by \( I_1 \) and \( I_2 \). First we can easily see

\[ I_1 \leq C_5 \int_{(-\delta,T+\delta)\times \omega} \left( \left| \frac{\partial^2 \phi_c}{\partial x_0^2} \right|^2 + s^2 \left| \frac{\partial \phi_c}{\partial x_0} \right|^2 + s^4 |\phi_c|^2 \right) e^{2s\phi} \, dx. \]

On the other hand, since \( \phi_c(x_0, x') = \phi_c(-x_0, x'), \) \(-\delta \leq x' \leq 0\) by the extension, we have

\[
\int_{-\delta}^{0} \int_{\omega} \left( \left| \frac{\partial^2 \phi_c}{\partial x_0^2} \right|^2 + s^2 \left| \frac{\partial \phi_c}{\partial x_0} \right|^2 + s^4 |\phi_c|^2 \right) e^{2s\phi(x_0, x')} \, dx_0 \, dx' = \]

\[
\int_{0}^{\delta} \int_{\omega} \left( \left| \frac{\partial^2 \phi_c}{\partial x_0^2} \right|^2 + s^2 \left| \frac{\partial \phi_c}{\partial x_0} \right|^2 + s^4 |\phi_c|^2 \right) e^{2s\phi(-x_0, x')} \, dx_0 \, dx'.
\]

By (2.10), we have \( \partial_{x_0} \phi(0, x') > 0 \). Therefore, for all sufficiently small \( \delta > 0 \), we obtain \( \partial_{x_0} \phi(x) > 0 \) for all \( x_0 \in [-\delta, \delta] \). This implies \( e^{2s\phi(-x_0, x')} \leq e^{2s\phi(x_0, x')} \) for \( 0 < x_0 < \delta \). Hence

\[
\int_{0}^{\delta} \int_{\omega} \left( \left| \frac{\partial^2 \phi_c}{\partial x_0^2} \right|^2 + s^2 \left| \frac{\partial \phi_c}{\partial x_0} \right|^2 + s^4 |\phi_c|^2 \right) e^{2s\phi(x_0, x')} \, dx_0 \, dx' \leq \int_{0}^{\delta} \int_{\omega} \left( \left| \frac{\partial^2 \phi_c}{\partial x_0^2} \right|^2 + s^2 \left| \frac{\partial \phi_c}{\partial x_0} \right|^2 + s^4 |\phi_c|^2 \right) e^{2s\phi(x_0, x')} \, dx_0 \, dx' \leq I_2.
\]

We can similarly estimate

\[
\int_{T}^{T+\delta} \int_{\omega} \left( \left| \frac{\partial^2 \phi_c}{\partial x_0^2} \right|^2 + s^2 \left| \frac{\partial \phi_c}{\partial x_0} \right|^2 + s^4 |\phi_c|^2 \right) e^{2s\phi} \, dx_0 \, dx'.
\]

Thus the verification of (9.27) is complete.
Using equations (9.17), (9.18), (9.24) and estimate (9.27), by Theorem 2.1, we obtain
\[ \sum_{|α|=2} \| (Δ^p_0 p_ε)e^{εφ} \|^2_{L^2(Q)^2} + s^2 \| p_ε e^{εφ} \|^2_{X} \leq C_6 M(\tilde{z}_ε, \tilde{v}_{1,ε}, \tilde{v}_{2,ε}), \]  
(9.28)
where we set
\[ M(\tilde{z}_ε, \tilde{v}_{1,ε}, \tilde{v}_{2,ε}) = \| \tilde{z}_ε e^{-sφ} \|^2_X + \int_{Q_ω} \left( |\partial \tilde{v}_{1,ε} / \partial x_0|^2 + s^2 |\tilde{v}_{1,ε}|^2 + |\tilde{v}_{2,ε}|^2 \right) e^{-2sφ} \, dx. \]

By (9.14)–(9.18) and integration by parts, we have
\[ \left( \partial \tilde{v}_{1,ε} / \partial x_0 + \tilde{v}_{2,ε} + u e^{2sφ} + \tilde{w}_ε, p_ε \right)_{(L^2(Q))^2} = (P \tilde{z}_ε, p_ε)_{(L^2(Q))^2} = (\tilde{z}_ε, P p_ε)_{(L^2(Q))^2} = -(\tilde{z}_ε, \tilde{z}_ε e^{-2sφ})_{(L^2(Q))^2}. \]

Therefore, taking the scalar product of (9.16) and \( p_ε \) in \( (L^2(Q))^2 \) and using (9.17) and (9.18), we obtain
\[ 2 J_ε(\tilde{z}_ε, \tilde{v}_{1,ε}, \tilde{v}_{2,ε}, \tilde{w}_ε) = -\frac{1}{2} \int_{\Omega} (u e^{2sφ}, p_ε) \, dx. \]

By (9.28), we obtain from this inequality that
\[ s^2 J_ε(\tilde{z}_ε, \tilde{v}_{1,ε}, \tilde{v}_{2,ε}, \tilde{w}_ε) \leq C_7 \| u e^{sφ} \|_{(L^2(Q))^2}^2 M(\tilde{z}_ε, \tilde{v}_{1,ε}, \tilde{v}_{2,ε})^{1/2}. \]
(9.29)

Next we differentiate equations (9.14) and (9.16) with respect to the variable \( x_0 \):
\[ P \frac{∂ p_ε}{∂ x_0} = \frac{∂}{∂ x_0} \tilde{f} \quad \text{in } Q, \]
(9.30)
\[ \frac{∂ \tilde{z}_ε}{∂ x_0} = \frac{∂^2 \tilde{v}_{1,ε}}{∂ x_0^2} + \frac{∂ \tilde{v}_{2,ε}}{∂ x_0^2} + \frac{∂ (u e^{2sφ})}{∂ x_0} + \frac{∂ \tilde{w}_ε}{∂ x_0} \quad \text{in } Q. \]
(9.31)

Taking the scalar product of (9.31) and \( \frac{∂ p_ε}{∂ x_0} \) in \( (L^2(Q))^2 \) and integrating by parts, in terms of (9.14)–(9.18), we similarly obtain
\[ 2 J_ε \left( \frac{∂ \tilde{z}_ε}{∂ x_0}, \frac{∂ \tilde{v}_{1,ε}}{∂ x_0}, \frac{∂ \tilde{v}_{2,ε}}{∂ x_0}, \frac{∂ \tilde{w}_ε}{∂ x_0} \right) = \int_{Q} \left\{ \left( u e^{2sφ}, \frac{∂ p_ε}{∂ x_0} \right) + 2 s \frac{∂ \tilde{z}_ε}{∂ x_0} \left( \frac{∂ \tilde{z}_ε}{∂ x_0}, \tilde{z}_ε \right) e^{-2sφ} + 2 s \frac{∂ φ}{∂ x_0} m_ε \left( \frac{∂ \tilde{v}_{1,ε}}{∂ x_0}, \tilde{v}_{1,ε} \right) e^{-2sφ} + \frac{2 m_ε}{s} \frac{∂ φ}{∂ x_0} \left( \frac{∂ \tilde{v}_{2,ε}}{∂ x_0}, \tilde{v}_{2,ε} \right) e^{-2sφ} \right\} \, dx. \]

This equality and (9.28), (9.29) imply
\[ J_ε \left( \frac{∂ \tilde{z}_ε}{∂ x_0}, \frac{∂ \tilde{v}_{1,ε}}{∂ x_0}, \frac{∂ \tilde{v}_{2,ε}}{∂ x_0}, \frac{∂ \tilde{w}_ε}{∂ x_0} \right) \leq C_8 \| u e^{sφ} \|_{(L^2(Q))^2} M(\tilde{z}_ε, \tilde{v}_{1,ε}, \tilde{v}_{2,ε})^{1/2}. \]
(9.32)
Let $\tilde{L}$ denote the part of first order of $L_{\lambda, \mu}$, that is, $(\tilde{L}\mathbf{v})(x') = \text{div} \mathbf{v}(x')\nabla_x \lambda(x') + (\nabla_x \mathbf{v} + (\nabla_x \mathbf{v})^T) \nabla_x \mu(x')$. Taking the scalar product of (9.16) with $\hat{z}_u e^{-2s\phi}$ in $(L^2(Q))^2$, we obtain
\begin{align}
\int_Q \left( \|\nabla_x \hat{z}_u\|^2 + (\lambda + \mu)(\text{div} \hat{z}_u)^2 \right) e^{-2s\phi} \, dx - \int_Q (\tilde{L}\hat{z}_u, \hat{z}_u e^{-2s\phi}) dx
\end{align}
\begin{align}
= \int_Q \left( \left| \frac{\partial \hat{z}_u}{\partial x_0} \right|^2 - 2s \frac{\partial \phi}{\partial x_0}(\hat{z}_u) \right) e^{-2s\phi} dx
\end{align}
\begin{align}
+ \int_Q \left( 2\mu s \sum_{k=1}^n (\partial_{x_k} \hat{z}_u, (\partial_{x_k} \phi) \hat{z}_u) + 2(\lambda + \mu) s (\text{div} \hat{z}_u)(\nabla_x \phi, \hat{z}_u) \right) e^{-2s\phi} dx
\end{align}
\begin{align}
- \int_Q \frac{2}{\sqrt{2}} (\hat{z}_u, \frac{\partial \hat{z}_u}{\partial x_0} \hat{z}_u)(\partial_{x_k} \mu) e^{-2s\phi} dx - \int_Q (\text{div} \hat{z}_u)(\nabla_x \lambda + \mu, \hat{z}_u) e^{-2s\phi} dx
\end{align}
\begin{align}
+ \int_Q \left( u e^{2s\phi} + \tilde{w}_u, \hat{z}_u \right) e^{-2s\phi} dx + \int_Q \left( \frac{\partial \tilde{v}_1}{\partial x_0} + \tilde{v}_2, \hat{z}_u e^{-2s\phi} \right) dx.
\end{align}

We note that $|\partial_{x_k} z_k| \leq \frac{4}{2^2} \left| \frac{\partial_{x_k} z_k}{2} \right| + \frac{1}{2^2} |z_k|^2$ for any $\delta > 0$. Therefore if we take sufficiently small $\delta > 0$ and sufficiently large $s > 0$, then by (9.28), (9.29) and (9.32), we obtain (9.19). The proof of Proposition 9.1 is complete. □

Now we finish the proof of Lemma 9.1. Obviously $\tilde{w}_e \to 0$ in $(L^2(Q))^2$ and $\tilde{v}_1, e_\xi, \tilde{v}_2 e_\xi \to 0$ in $(L^2(Q \setminus \Omega))^2$ as $\varepsilon \to +0$. In terms of (9.19), from the sequence $\{ (\hat{z}_u, \hat{v}_1, \hat{v}_2, p) \}$, one can extract a subsequence $\{ (\hat{z}_{e_k}, \hat{v}_{1,e_k}, \hat{v}_{2,e_k}, p_{e_k}) \}$ such that
\begin{align}
(\hat{z}_{e_k}, \hat{v}_{1,e_k}, \hat{v}_{2,e_k}, p_{e_k}) \to (\hat{z}, \hat{v}_1, \hat{v}_2, p) \text{ weakly in } X \times (L^2(0; T; L^2(\Omega))^2 \times (L^2(Q))^4)
\end{align}
Thanks to (9.33), we can pass to the limit in (9.14)-(9.18), so that the element $\{ \hat{z}, \hat{v}_1, \hat{v}_2, p \}$ satisfies the equations
\begin{align}
P p + e^{-2s\phi} \hat{z} &= 0 \text{ in } Q,
\end{align}
\begin{align}
p |_{\partial(0,T) \times \partial \Omega} = \hat{z} |_{\partial(0,T) \times \partial \Omega} = 0,
\end{align}
\begin{align}
\frac{\partial p}{\partial x_0}(0, \cdot) = \frac{\partial p}{\partial x_0}(T, \cdot) = \frac{\partial \hat{z}}{\partial x_0}(0, \cdot) = \frac{\partial \hat{z}}{\partial x_0}(T, \cdot) = 0,
\end{align}
\begin{align}
P \hat{z} = \frac{\partial \hat{v}_1}{\partial x_0} + \hat{v}_2 + u e^{2s\phi} \text{ in } Q,
\end{align}
\begin{align}
\frac{\partial \hat{v}_2}{\partial x_0} + \hat{v}_1 e^{-2s\phi} &= 0 \text{ in } Q,
\end{align}
\begin{align}
p - \hat{v}_2 e^{-2s\phi} = 0 \text{ in } Q, \text{ supp } \hat{v}_j \subset \overline{Q}_j, \ j = 1, 2.
\end{align}

Estimate (9.9) follows from (9.19). Finally we note that $J(\hat{z}, \hat{v}_1, \hat{v}_2) \leq J(\mathbf{z}, \mathbf{v}_1, \mathbf{v}_2)$ for all $\varepsilon \in (0, 1)$. Hence $J(\hat{z}, \hat{v}_1, \hat{v}_2) \leq J(\mathbf{z}, \mathbf{v}_1, \mathbf{v}_2)$, the element $\{ \hat{z}, \hat{v}_1, \hat{v}_2 \}$ is a solution to extremal problem (9.1)–(9.3). Since a solution to this problem is unique, we have $\{ \hat{z}, \hat{v}_1, \hat{v}_2 \} = \{ \mathbf{z}, \mathbf{v}_1, \mathbf{v}_2 \}$. The proof of Lemma 9.1 is complete. □

Proof of Theorem 2.2. Taking the scalar product of (2.1) with $\mathbf{z}$ in $(L^2(Q))^2$ and integrating by parts, in terms of (2.1), (2.2), (9.7) and (9.8), we obtain the equality
\begin{align}
\|u e^{2s\phi}\|_{L^2(Q))^2}^2 = \int_Q (f, \mathbf{z}) dx - \int_{Q_{\varepsilon}} \left( \frac{\partial \mathbf{v}_1}{\partial x_0} + \mathbf{v}_2 \right) dx.
\end{align}
Applying (9.9) to this equality and using again an inequality $|ab| \leq \frac{\delta}{2} |a|^2 + \frac{1}{2} |b|^2$ for any $\delta > 0$, we obtain
\[
\int_Q s^2|u|^2 e^{2s\phi} \, dx \leq C_0 \left( \|fe^{s\phi}\|_{L^2(Q)}^2 + \int_{Q_{\omega}} (|\nabla u|^2 + s^2|u|^2)e^{2s\phi} \, dx \right), \quad \forall s \geq s_0(\tau). \tag{9.40}
\]

In order to estimate the derivatives of first order for the function $u$, replacing $u$ by $\frac{\partial u}{\partial x_0}$, we consider extremal problem (9.1)–(9.3). Let $(\tilde{z}, \tilde{v}_1, \tilde{v}_2)$ be the corresponding solution. Then Lemma 9.1 yields
\[
\|\tilde{z}e^{-s\phi}\|^2_{L^2} + \left\|\frac{\partial \nabla_1}{\partial x_0} e^{-s\phi}\right\|^2_{(L^2(Q_{\omega}))^2} \leq C_{10} \left\|\frac{\partial u}{\partial x_0} e^{s\phi}\right\|^2_{(L^2(Q))^2}. \tag{9.41}
\]

Since the Lamé coefficients are independent of $x_0$, we have
\[
P \frac{\partial u}{\partial x_0} = \frac{\partial f}{\partial x_0} \quad \text{in } Q, \quad \frac{\partial u}{\partial x_0}(0,T\times \partial Q) = 0, \quad \frac{\partial u}{\partial x_0}(T,\cdot) = \frac{\partial u}{\partial x_0}(0,\cdot) = 0. \tag{9.42}
\]

Taking the scalar product of (9.42) with $\tilde{z}$ in $(L^2(Q))^2$ and integrating by parts, we obtain the equality
\[
\left\|\frac{\partial u}{\partial x_0} e^{s\phi}\right\|^2_{(L^2(Q))^2} = \int_Q \left( \frac{\partial f}{\partial x_0} \right) \, dx - \int_{Q_{\omega}} \left( \frac{\partial u}{\partial x_0} \frac{\partial \nabla_1}{\partial x_0} + \tilde{v}_2 \right) \, dx.
\]

Applying the inequality $2|ab| \leq \delta|a|^2 + \frac{1}{2} |b|^2$ to the second term at the right hand side of this equality, by means of (9.41), we obtain
\[
\int_Q \left( s^2|u|^2 + 2s^2|u|^2 e^{2s\phi} \, dx \leq C_{11} \left\{ \|fe^{s\phi}\|_{L^2(Q)}^2 + \int_{Q_{\omega}} (|\nabla u|^2 + s^2|u|^2)e^{2s\phi} \, dx \right\}, \quad \forall s \geq s_0(\tau). \tag{9.43}
\]

Finally, taking the scalar product of (2.1) with $ue^{s\phi}$ in $(L^2(Q))^2$, we obtain
\[
\int_Q \left( \mu \nabla \cdot u^2 + (\lambda + \mu) (\text{div } u)^2 \right) e^{2s\phi} \, dx = \int_Q \left( \frac{\partial u}{\partial x_0} \right)^2 + 2s \frac{\partial \nabla_1}{\partial x_0} \cdot \left( \frac{\partial u}{\partial x_0} \right) \right) e^{2s\phi} \, dx
\]
\[
- \int_Q \left( 2s \sum_{k=1}^2 (\frac{\partial u}{\partial x_k}) (\text{div } u)(\nabla \phi, \phi, u) \right) e^{2s\phi} \, dx - \int Q \left( (\text{div } u)(\nabla \cdot (\lambda + \mu), u) e^{2s\phi} \, dx + \int Q \left( (f(\text{div } u) e^{2s\phi} \, dx + \int Q (f, u) e^{2s\phi} \, dx.
\]

This equality and (9.43) imply (2.11), the conclusion of Theorem 2.2.

\textbf{Proof of Theorem 2.3.} In order to complete the proof, it is sufficient to estimate $\int_Q (f, z) \, dx$ in (9.39) as follows:
\[
\left| \int_Q (f_{-1}, z) \, dx \right| \leq \|f_{-1} e^\phi\|_{(H^{-1}(Q))^2} \|z e^{-s\phi}\|_{(H^1(\Omega))^2} \leq \|f_{-1} e^\phi\|_{(H^{-1}(Q))^2} \|z e^{-s\phi}\|_{x}
\]

and
\[
\left| \int_Q (\partial_j f_j, z) \, dx \right| = \left| \int_Q (f_j, \partial_j z) \, dx \right| \leq \|f_j e^\phi\|_{(L^2(\Omega))^2} \|\partial_j z e^{-s\phi}\|_{(L^2(\Omega))^2} \leq C_{12} \|f_j e^\phi\|_{(L^2(\Omega))^2} \|\nabla (z e^{-s\phi})\|_{(L^2(\Omega))^2} + s \|z e^{-s\phi}\|_{(L^2(\Omega))^2} \leq C_{13} \|f_j e^\phi\|_{(L^2(\Omega))^2} \|z e^{-s\phi}\|_{x}.
\]
Therefore
\[
\left| \int_{\Omega} \left( \sum_{j=0}^{2} \partial_{x_j} f_j \right) \cdot \mathbf{z} \, dx \right| \leq C_{14} \left( \left\| f_1 e^{s\phi} \right\|_{H^{1-1}(Q)}^2 + \sum_{j=0}^{2} \left\| f_j e^{s\phi} \right\|_{L^2(Q)}^2 \right) \| \mathbf{z} e^{-s\phi} \|_\mathbf{V}.
\]

Then, again by using the inequality \(|ab| \leq \frac{1}{2}|a|^2 + \frac{1}{2\delta}|b|^2\) for \(\delta > 0\), this inequality and estimates (9.9), (9.39) imply (2.12).

**APPENDIX I. PROOF OF PROPOSITION 5.1**

In order to prove the proposition, it is convenient to use the coordinate \(x\) instead of \(y\). Moreover it suffices to prove the estimate for an arbitrary but fixed \(x_0 \in [0,T]\). Therefore we should establish the estimate: there exist \(\tilde{T} > 1\) and \(N_0 > 1\) such that for any \(\tau > \tilde{T}\) and \(N > N_0\), there exists \(s_0(\tau, N)\) such that
\[
\int_{\Omega_{1/N^2}} \left( \sum_{j,k=1}^{2} \partial_{x_j} \partial_{x_k} u \right)^2 + |\nabla_x^2 u|^2 + k \phi^2 |u|^2 \right) e^{2\phi} \, dx' \leq C_0 \left( \left\| \text{rot} u e^{s\phi} \right\|_{H^1(\Omega_{1/N}^2)}^2 + \left\| \text{div} u e^{s\phi} \right\|_{H^1(\Omega_{1/N}^2)}^2 \right),
\]

∀\(u \in (H^1(\Omega_{1/N}^2))^2, \forall s \geq s_0(\tau, N), \text{supp } u \subset B_2 \cap \Omega_{1/N^2},\)

where the constant \(C_0\) is independent of \(N\). Recall that \(\Omega_{1/N^2} = \{ x' \in \Omega; \text{dist} (x', \partial \Omega) \leq \frac{1}{N} \} \).

First we choose \(N_0 > 0\) sufficiently large such that
\[
\nabla_x \psi(x) \neq 0, \quad \forall x' \in \Omega_{1/N^2}, \forall x_0 \in (0,T).
\]
The existence of such \(N_0\) follows from (2.6).

Denote \(\text{rot } u \equiv \frac{\partial u}{\partial x_2} - \frac{\partial u}{\partial x_1} = \mathbf{y}\) and \(\text{div } u \equiv \mathbf{w}\). Let \(\text{rot}^* \mathbf{v} = (\partial \mathbf{v}/\partial x_2, -\partial \mathbf{v}/\partial x_1)\). Using a formula \(\text{rot}^* \text{rot} = -\Delta_x + \nabla_x \text{div}\), we obtain
\[
-\Delta_x u = -\text{rot}^* \mathbf{y} - \nabla_x \mathbf{w} \quad \text{in } \Omega_{1/N^2}, \quad u|_{\partial \Omega_{1/N^2}} = 0.
\]
The function \(\tilde{u} = u e^{s\phi}\) satisfies the equation
\[
L_1 \tilde{u} + L_2 \tilde{u} = \mathbf{q}, \quad \text{in } \Omega_{1/N^2}, \quad \tilde{u}|_{\partial \Omega_{1/N^2}} = 0,
\]
where \(L_1 \tilde{u} = -\Delta_x \tilde{u} - s^2 |\nabla_x \phi|^2 \tilde{u}, \quad L_2 \tilde{u} = 2s \sum_{k=1}^{2} (\partial_{x_k} \tilde{u}) \partial_{x_k} \phi + s(\Delta_x \phi) \tilde{u} \) and \(\mathbf{q} = -\text{rot}^* \mathbf{y} - \nabla_x \mathbf{w} e^{s\phi}\).

Taking the \(L^2\) norms of the right and the left hand sides of equation (2), we obtain
\[
\| L_1 \tilde{u} \|_{(L^2(\Omega_{1/N^2}))^2}^2 + \| L_2 \tilde{u} \|_{(L^2(\Omega_{1/N^2}))^2}^2 + 2(\langle L_1 \tilde{u}, L_2 \tilde{u} \rangle_{(L^2(\Omega_{1/N^2}))^2})^2 = \| \mathbf{q} \|_{(L^2(\Omega_{1/N^2}))^2}^2.
\]

Therefore we can obtain the formula
\[
(\langle L_1 \tilde{u}, L_2 \tilde{u} \rangle_{(L^2(\Omega_{1/N^2}))^2})^2 = \int_{\Omega_{1/N^2}} \left( 2s \sum_{k,j=1}^{2} \partial_{x_k} \tilde{u} \partial_{x_j} \phi \partial_{x_k} \phi + s^2 (\text{div} |\nabla_x \phi|^2 \nabla_x \phi) 
\]
\[
- |\nabla_x \phi|^2 \Delta_x \phi |\tilde{u}|^2 - \frac{s}{2} \sum_{j=1}^{2} \left( \partial_{x_j}^2 \phi \tilde{u} \right) |\tilde{u}|^2 \right) \, dx' - s \int_{\partial \Omega_{1/N^2}} \left| \frac{\partial \tilde{u}}{\partial \mathbf{n}} \right|^2 (\nabla_x \phi, \tilde{u}) \, d\sigma.
\]
By (2.6), the last integral in (3) is nonnegative. Denote \( \psi_1(x) = \psi(x) - \hat{\ell}_1(x) \). Then

\[
\text{div}(|\nabla_x \varphi|^2 \nabla_x \varphi) - |\nabla_x \varphi|^2 \Delta_x \varphi = 2 \sum_{k,j=1}^2 \varphi_{x_k} \varphi_{x_j} \varphi_{x_k x_j}
\]
\[
= 2 \varphi^3 \sum_{k,j=1}^2 \tau^4 (\partial_{x_k} \psi_1 + 2 N \ell_1 \partial_{x_k} \ell_1)^2 (\partial_{x_j} \psi_1 + 2 N \ell_1 \partial_{x_j} \ell_1)^2
\]
\[
+ \tau^3 (\partial_{x_k} \psi_1 + 2 N \ell_1 \partial_{x_k} \ell_1)(\partial_{x_j} \psi_1 + 2 N \ell_1 \partial_{x_j} \ell_1)(\partial_{x_k} \partial_{x_j} \psi_1 + 2 N \partial_{x_k} \ell_1 \partial_{x_j} \ell_1 + 2 N \ell_1 \partial_{x_k} \partial_{x_j} \ell_1).
\]

Since \( (\nabla_x \psi_1, \nabla_x \ell_1) > 0 \) on \( \partial \Omega \), there exists a constant \( C_1 > 0 \) which is independent of \( N, \tau, s \) such that

\[
\text{div}(|\nabla_x \varphi|^2 \nabla_x \varphi) - |\nabla_x \varphi|^2 \Delta_x \varphi \geq 2 \varphi^3 \tau^4 |\nabla_x \psi_1|^4 + C_1 N \tau^3 \varphi^3 + \varphi^2 O(\tau^3).
\]  
(4)

On the other hand, by the definition of \( \tilde{\psi} = \psi - \hat{\ell}_1 + N \ell_1^2 = \psi_1 + N \ell_1^2 \),

\[
\sum_{k,j=1}^2 (\partial_{x_k} \tilde{u})(\partial_{x_k} \tilde{u}) \varphi_{x_k x_k} = \tau^2 (\nabla_x \tilde{u}, \nabla_x \tilde{\psi})^2 \varphi
\]
\[
+ \tau \sum_{k,j=1}^2 (\partial_{x_k} \tilde{u})(\partial_{x_k} \tilde{u}) (\partial_{x_j} \tilde{u})(\partial_{x_j} \partial_{x_k} \psi_1 + 2 N \ell_1(\partial_{x_j} \partial_{x_k} \ell_1)) \varphi + 2 N \tau (\nabla_x \tilde{u}, \nabla_x \ell_1)^2 \varphi.
\]  
(5)

Note that there exists a constant \( C_2 > 0 \), independent of \( N \), such that

\[
\|N \ell_1 \partial_{x_j} \ell_1\|_{C^0(\Omega_1/N^2)} \leq C_2/N.
\]  
(6)

By (3)–(6), we obtain

\[
\|L_1 \tilde{u}\|_{(L^2(\Omega_1/N^2))^2} + \|L_2 \tilde{u}\|_{(L^2(\Omega_1/N^2))^2} + \int_{\Omega_1/N^2} (2 \varphi^3 \tau^4 |\nabla_x \psi_1|^4 + C_1 N \tau^3 \varphi^3) |\tilde{u}|^2 dx'
\]
\[
= -\sigma T \int_{\Omega_1/N^2} \varphi |\nabla_x \tilde{u}|^2 dx' \leq \|q_s\|_{(L^2(\Omega_1/N^2))^2}^2.
\]  
(7)

Multiplying equation (2) by \( sN \varphi \tilde{u} \) and integrating by parts, we obtain

\[
\int_{\Omega_1/N^2} (sN \varphi |\nabla_x \tilde{u}|^2 + s^2 N(\Delta_x \varphi) \varphi |\tilde{u}|^2 - s^3 \varphi^3 |\nabla_x \varphi|^2 |\tilde{u}|^2 - sN \div \varphi |\tilde{u}|^2) dx' = \int_{\Omega_1/N^2} q_s sN \varphi \tilde{u} dx'.
\]  
(8)

Next we note that

\[
\Delta_x \varphi = (|\nabla_x \tilde{u}|^2)^2 + \tau \Delta_x \psi_1 + 2 \tau N |\nabla_x \ell_1|^2 + 2 \tau N \ell_1 \Delta_x \ell_1 \varphi \geq C_4 \tau N \varphi.
\]

This inequality and (8) imply

\[
\int_{\Omega_1/N^2} \left\{ sN \varphi |\nabla_x \tilde{u}|^2 + \frac{1}{2} s^2 N(\Delta_x \varphi) \varphi |\tilde{u}|^2 - s^3 \varphi^3 |\nabla_x \varphi|^2 |\tilde{u}|^2 \right\} dx' \leq C_4 \|q_s\|_{(L^2(\Omega_1/N^2))^2}^2.
\]  
(9)
By (7) and (9), we obtain
\[ \| L_1 \tilde{u} \|_{L^2(\Omega)}^2 + \| L_2 \tilde{u} \|_{L^2(\Omega)}^2 + \int_{\Omega} \left( \frac{1}{2} \varphi \Delta \tilde{u} + C_1 \varphi + C_2 \right) |\tilde{u}|^2 \, dx \]
\[ + sN \int_{\Omega} \varphi |\nabla \tilde{u}|^2 \, dx' \leq C_0 \| \mathbf{q} \|_{L^2(\Omega)}^2. \]
(10)

Let \( \tilde{u} = \tilde{u}_1 + \tilde{u}_2 \) where the functions \( \tilde{u}_j \) are solutions to the boundary value problems
\[ -\Delta x' \tilde{u}_1 = L_1 \tilde{u}_1 \quad \text{in} \quad \Omega, \quad \tilde{u}_1|_{\partial \Omega} = 0, \quad -\Delta x' \tilde{u}_2 = s^2 |\nabla x' \varphi|^2 \tilde{u}_2 \quad \text{in} \quad \Omega, \quad \tilde{u}_2|_{\partial \Omega} = 0. \]

By means of a standard \textit{a priori} estimate for the Laplace operator, we have
\[ \| \tilde{u}_1 \|_{H^2(\Omega)} \leq C_6 \| L_1 \tilde{u} \|_{L^2(\Omega)}^2, \]
(11)
\[ \frac{\sqrt{N}}{\sqrt{s}} \| \tilde{u}_2 \|_{H^2(\Omega)} \leq C_7 \sqrt{N} \| s |\nabla x' \varphi|^2 \tilde{u} \|_{L^2(\Omega)}^2, \]
(12)
where the constants \( C_6 \) and \( C_7 \) are independent of \( N \). Taking \( s_0(\tau, N) \geq N \), we obtain (1) from (9)–(12). The proof of Proposition 5.1 is finished. \( \square \)

**APPENDIX II. PROOF OF ESTIMATE (5.28)**

We prove (5.28) for a more general hyperbolic operator. Denote \( y = (y_0, y') = (y_0, y_1, \ldots, y_n), \xi = (\xi_0, \xi') = (\xi_0, \xi_1, \ldots, \xi_n) \) and \( \mathcal{G}_N = \mathbb{R}^n \times [0, 1/N] \).

Let a function \( w \in H^1(\mathcal{G}_N) \) satisfy the equations:
\[ R(y', D)w = \frac{\partial^2 w}{\partial y_0^2} - \sum_{j,k=1}^n \frac{\partial}{\partial y_j} \left( a_{jk}(y') \frac{\partial w}{\partial y_k} \right) + \sum_{j=0}^n b_j(y') \frac{\partial w}{\partial y_j} + c(y')w = g \quad \text{in} \quad \mathcal{G}_N, \]
(1)
\[ w|_{y_0=\frac{1}{N}} = \frac{\partial w}{\partial y_0}|_{y_0=\frac{1}{N}} = 0, \quad \text{supp} \, w \subset B_0(x^*), \]
(2)
where \( x^* \) is an arbitrary point on \( \partial \mathcal{G}_N \) and \( B_0(x^*) \) is a ball of radius \( \delta \) centered at \( x^* \).

We assume that the coefficients of the linear operator \( R \) satisfy the conditions
\[ a_{jk} \in C^1(\mathcal{G}_N), \quad a_{jk} = a_{kj}, \quad 1 \leq j, k \leq n, \quad b_\ell \in L^\infty(\mathcal{G}_N), \quad 0 \leq \ell \leq n, \quad c \in L^\infty(\mathcal{G}_N) \]
(3)
and the uniform ellipticity: there exists \( \delta > 0 \) such that
\[ a(y', \xi, \xi) \equiv \sum_{j,k=1}^n a_{jk}(y') \xi_j \xi_k \geq \delta |\xi|^2, \quad \forall \xi \in \mathbb{R}^{n+1}, \quad \forall y \in \mathcal{G}_N. \]
(4)

By \( R(y', \xi) \), we denote the principal symbol of the operator \( R \):
\[ R(y', \xi) = -\xi_0^2 + \sum_{j,k=1}^n a_{jk}(y') \xi_j \xi_k. \]
and by $\tilde{R}(y', \xi^1, \xi^2)$ the quadratic form

$$\tilde{R}(y', \xi^1, \xi^2) = \xi_0^2 - \sum_{j,k=1}^n a_{j,k}(y') \xi_j^1 \xi_k^2$$

with $\xi^1 = (\xi_1^1, ..., \xi_n^1)$ and $\xi^2 = (\xi_1^2, ..., \xi_n^2)$. Following [15], we introduce the notations:

$$R^{(j)}(y', \xi) = \frac{\partial R(y', \xi)}{\partial \xi_j}, \quad R^{(j,k)}(y', \xi) = \frac{\partial^2 R(y', \xi)}{\partial \xi_j \partial \xi_k}, \quad R^{(j)}(y', \xi) = \frac{\partial R(y', \xi)}{\partial y_j}.$$  

We assume that there exists a function $\psi_1 \in C^2(\mathcal{G}_N)$ such that

$$\{R, \{R, \psi_1\}\}(y, \xi) > 0$$

if $(y, \xi) \in (\mathcal{G}_N \setminus B_\delta(x^*)) \times (\mathbb{R}^{n+1} \setminus \{0\})$ satisfies

$$R(y', \xi) = \langle \nabla_\xi R(y', \xi), \nabla \psi_1(y) \rangle = 0,$$

and

$$\{R(y', \xi - is\nabla \psi_1(y)), R(y', \xi + is\nabla \psi_1(y))\}/2is > 0$$

if $(y, \xi, s) \in (\mathcal{G}_N \setminus B_\delta(x^*)) \times (\mathbb{R}^{n+1} \setminus \{0\}) \times (\mathbb{R} \setminus \{0\})$ satisfies

$$R(y', \xi + is\nabla \psi_1(y)) = \langle \nabla_\xi R(y', \xi + is\nabla \psi_1(y)), \nabla \psi_1(y) \rangle = 0,$$

$$R(y, \nabla \psi_1) < 0.$$  

Using the function $\psi_1$ and following [15], we construct the function $\phi$ by

$$\phi(y) = e^{\tau \psi_1(y)}, \quad \tau > 1.$$  

It is known (see e.g., Th. 8.6.2, p. 205 [15]) that provided that the parameter $\tau$ is sufficiently large,

$$\{R, \{R, \phi\}\}(y, \xi) > 0$$

if $(y, \xi) \in (\mathcal{G}_N \setminus B_\delta(x^*)) \times (\mathbb{R}^{n+1} \setminus \{0\})$ satisfies

$$R(y', \xi) = 0,$$

and

$$\{R(y', \xi - is\nabla \phi(y)), R(y', \xi + is\nabla \phi(y))\}/2is > 0$$

if $(y, \xi, s) \in (\mathcal{G}_N \setminus B_\delta(x^*)) \times (\mathbb{R}^{n+1} \setminus \{0\}) \times (\mathbb{R} \setminus \{0\})$ satisfies

$$R(y', \xi + is\nabla \phi(x)) = 0.$$  

Now we fix the parameter $\tau$ such that inequalities (8) and (9) hold true. Let $\ell_1 \in C^2(\mathcal{G}_N)$ be a function such that $\ell_1|_{y_n=0} = 0$. Let $\tilde{\psi}(y) = \psi_1(y) + N\ell_1(y)$ and $\varphi = e^{\tilde{\psi}}$. Since $\varphi(y) = \phi(y)e^{N\ell_1(y)}$, using $\ell_1|_{y_n=0} = 0$, we have

$$\varphi \to \phi \quad \text{in} \quad C^1(\mathcal{G}_N) \quad \text{as} \quad N \to +\infty.$$  

LAMÉ SYSTEM
Moreover
\[
\{R(y', \xi - i s \nabla \varphi(y)), R(y', \xi + i s \nabla \varphi(y))\}/2is
- 2N_T \sum_{j,k=1}^{n} (\partial_{y_j} \ell_1(y)) (\partial_{y_k} \ell_1(y)) (R^{(j)}(y', \xi) R^{(k)}(y', \xi) + s^2 R^{(j)}(y', \xi) R^{(k)}(y', \xi))
\rightarrow \{R(y', \xi - i s \nabla \phi(y)), R(y', \xi + i s \nabla \phi(y))\}/2is \text{ in } C(\mathcal{G}_N \times \mathbb{S}^n) \text{ as } N \to +\infty. \tag{11}
\]

Here we set \( \mathbb{S}^n = \{ \xi \in \mathbb{R}^{n+1}; |\xi| = 1 \} \). By (8)–(11), there exists \( N_0 > 0 \) such that for any \( N > N_0 \), the following inequalities hold true:
\[
\{R_s, \{R, \varphi\}\}(y, \xi) > 0 \tag{12}
\]
if \((y, \xi) \in (\mathcal{G}_N \setminus B_3(x^*)) \times (\mathbb{R}^{n+1} \setminus \{0\}) \) satisfies \( R(y, \xi) = 0 \), and
\[
\{R(y', \xi - i s \nabla \varphi(y)), R(y', \xi + i s \nabla \varphi(y))\}/2is > C_1(|\xi|^2 + N s^2) \tag{13}
\]
if \((y, \xi, s) \in (\mathcal{G}_N \setminus B_3(x^*)) \times (\mathbb{R}^{n+1} \setminus \{0\}) \times (\mathbb{R} \setminus \{0\}) \) satisfies \( R(y', \xi + i s \nabla \varphi(y)) = 0 \), where the constant \( C_1 > 0 \) is independent of \( \xi, s, N \).

Denote \( \tilde{w}(y) = w(y)e^{s\varphi} \). By (11), the following equality holds:
\[
e^{s\varphi} R(y', D)(e^{-s\varphi} \tilde{w}) = g e^{s\varphi} \text{ in } \mathcal{G}_N. \tag{14}
\]

The short calculations give the equation
\[
L_{2, \varphi} \tilde{w} + L_{1, \varphi} \tilde{w} = g, \text{ in } \mathcal{G}_N, \tag{15}
\]

where
\[
L_{1, \varphi} \tilde{w} = - \sum_{j=0}^{n} s \varphi_{y_j} R^{(j)}(y', \nabla \tilde{w}), \quad L_{2, \varphi} \tilde{w} = R \tilde{w} + s^2 R(y', \nabla \varphi) \tilde{w},

\]
\[
g_s(y) = g e^{s\varphi} + \tilde{w} R \varphi. \tag{16}
\]

Taking the \( L_2 \)-norms of the both sides of (15), we obtain
\[
\|g_s\|_{L^2(\mathcal{G}_N)}^2 = \|L_{2, \varphi} \tilde{w}\|_{L^2(\mathcal{G}_N)}^2 + \|L_{1, \varphi} \tilde{w}\|_{L^2(\mathcal{G}_N)}^2 + 2(L_{1, \varphi} \tilde{w}, L_{2, \varphi} \tilde{w})_{L^2(\mathcal{G}_N)}. \tag{17}
\]

Denote
\[
G_\varphi(y, s, \tilde{w}) = \{R_s, \{R, \varphi\}\}(y', \nabla \tilde{w}) + s^2 \sum_{j,k=0}^{n} R_{(j)}(y', \nabla \varphi) R^{(j)}(y', \nabla \varphi) \tilde{w}^2
\]
\[+ s^2 \sum_{j,k=0}^{n} \phi_{y_j y_k} \tilde{w} R^{(j)}(y', \nabla \varphi) \tilde{w} R^{(k)}(y', \nabla \varphi) \tilde{w}^2 \tag{18}
\]

and \( G_\varphi(y, s, \tilde{w}) \) is defined similarly.
Lemma 1. Let $\mathbf{n}$ be the unit outward normal vector to $\partial G_N$ and

$$\theta(y) = \sum_{l,m=0}^{n} (\varphi_{y,m} R^{(l,m)}(y', \nabla \tilde{w}) + \varphi_{y} R^{(0,m)}(y', \nabla \tilde{w})).$$

Now we need the following Lemma proved in [18].

**Lemma 1.** Let $w \in H^1(\mathcal{G}_N)$ be a solution to (1) and (2).

$$s \int_{\mathcal{G}_N} (|\nabla \tilde{w}|^2 + s^2 \tilde{w}^2) dy \leq C_2 \int_{\mathcal{G}_N} sG_\phi(y, s, \tilde{w}) dy + C_3 \left( \frac{1}{s} \|L_{2,y} \tilde{w}\|_{L^2(\mathcal{G}_N)}^2 + \frac{1}{s} \|L_{1,y} \tilde{w}\|_{L^2(\mathcal{G}_N)}^2 + s \|\tilde{w}\|_{L^2(\mathcal{G}_N)} \|\partial_{y_n} \tilde{w}\|_{L^2(\partial \mathcal{G}_N)} \right), \quad \forall s \geq s_0(\tau), \quad (20)$$

where the constants $C_2$ and $C_3$ are independent of $s, N$.

We claim:

$$s \int_{\mathcal{G}_N} \left( \sum_{j,k=0}^{n} R^{(k)}(y', \nabla \tilde{w}) \varphi_{y_j} R^{(j)}(y', \nabla \tilde{w}) - \theta \{ R(y', \nabla \tilde{w}) - s^2 R(y', \nabla \varphi) \tilde{w}^2 \} \right) dy \leq \frac{s}{2} \int_{\mathcal{G}_N} \left( \sum_{j,k=0}^{n} R^{(k)}(y', \nabla \tilde{w}) \varphi_{y_j} R^{(j)}(y', \nabla \tilde{w}) dy \right) \left( \int_{\mathcal{G}_N} \theta(R(y', \nabla \tilde{w}) - s^2 R(y', \nabla \varphi) \tilde{w}^2) dy \right) \leq \frac{\varepsilon s}{2} \int_{\mathcal{G}_N} (|\nabla \tilde{w}|^2 + s^2 \tilde{w}^2) dy + C_4 \left( \frac{1}{s \varepsilon} \|L_{1,y} \tilde{w}\|_{L^2(\mathcal{G}_N)}^2 + \frac{1}{s \varepsilon} \|L_{2,y} \tilde{w}\|_{L^2(\mathcal{G}_N)}^2 + s \|\tilde{w}\|_{L^2(\mathcal{G}_N)} \|\partial_{y_n} \tilde{w}\|_{L^2(\partial \mathcal{G}_N)} \right).\quad (21)$$

In fact, by the Cauchy-Bunyakovskii inequality,

$$s \int_{\mathcal{G}_N} \left( \sum_{j,k=0}^{n} R^{(k)}(y', \nabla \tilde{w}) \varphi_{y_j} R^{(j)}(y', \nabla \tilde{w}) dy \right) \left( \int_{\mathcal{G}_N} \theta(R(y', \nabla \tilde{w}) - s^2 R(y', \nabla \varphi) \tilde{w}^2) dy \right) \leq \frac{\varepsilon s}{4} \|\tilde{w}\|_{H^1(\mathcal{G}_N)}^2 + C_5 \frac{s}{\varepsilon} \|L_{1,y} \tilde{w}\|_{L^2(\mathcal{G}_N)}^2. \quad (22)$$
Since the function $\theta$ is continuous, there exists $\theta_\varepsilon \in C^2(\overline{G_N})$ such that $\|\theta - \theta_\varepsilon\|_{C(\overline{G_N})} \leq \frac{\varepsilon}{s}$. Taking the scalar product in $L^2(\overline{G_N})$ of the functions $\theta_\varepsilon \tilde{w}$ and $L_{2,\varphi} \tilde{w}$, we obtain the equality

$$\int_{\overline{G_N}} \theta_\varepsilon(sR(y', \nabla \tilde{w}) - s^3 R(y', \nabla \varphi)\tilde{w}^2)dy = -s \int_{\overline{G_N}} (L_{2,\varphi} \tilde{w})\theta_\varepsilon \tilde{w}dy + s \int_{\overline{G_N}} \sum_{j,k=1}^n \left( \frac{\partial a_{jk}}{\partial y_j} \theta_\varepsilon \tilde{w} - \tilde{R}(y', \nabla \tilde{w}, \nabla \theta_\varepsilon) \tilde{w} \right)dy + \int_{\partial G_N} a(y, \tilde{n}, \nabla \tilde{w})\theta_\varepsilon \tilde{w}d\Sigma.$$

Thus

$$\left| \int_{\overline{G_N}} \theta(sR(y', \nabla \tilde{w}) - s^3 R(y', \nabla \varphi)\tilde{w}^2)dy \right| \leq \int_{\overline{G_N}} \left( \theta - \theta_\varepsilon \right)(sR(y', \nabla \tilde{w}) - s^3 R(y', \nabla \varphi)\tilde{w}^2)dy \right| + \left| \int_{\overline{G_N}} \theta_\varepsilon(sR(x', \nabla \tilde{w}) - s^3 R(x', \nabla \varphi)\tilde{w}^2)dy \right| \leq \frac{\varepsilon s}{4} \int_{\overline{G_N}} (|\nabla \tilde{w}|^2 + s^2 \tilde{w}^2)dy + C_6 \left( \frac{1}{8} \|L_{1, \varphi} \tilde{w}\|^2_{L^2(\overline{G_N})} + \frac{1}{8} \|L_{2, \varphi} \tilde{w}\|^2_{L^2(\overline{G_N})} + s \|\tilde{w}\|_{L^2(\partial G_N)} \|\partial_{\tilde{n}} \tilde{w}\|_{L^2(\partial G_N)} \right).$$

Inequalities (22) and (23) imply (21).

By Lemma 1, we have

$$s \int_{\overline{G_N}} (|\nabla \tilde{w}|^2 + s^2 \tilde{w}^2)dy + \int_{\overline{G_N}} 4N\tau \sum_{j,k=1}^n \partial_{y_j} \ell_1(y') \partial_{y_k} \ell_1(y') \{R^{(j)}(y', \nabla \tilde{w}) R^{(k)}(y', \nabla \tilde{w}) \}
+ s^2 R^{(j)}(y', \nabla \varphi) R^{(k)}(y', \nabla \varphi) \}dy \leq \int_{\overline{G_N}} 2sG_{\varphi}(y, s, \tilde{w})dy + \int_{\overline{G_N}} \left\{ 2sG_{\varphi}(y, s, \tilde{w}) - 2sG_{\varphi}(y, s, \tilde{w}) \right\}dy
+ 4N\tau \sum_{j,k=1}^n \partial_{y_j} \ell_1(y') \partial_{y_k} \ell_1(y') \{R^{(j)}(y', \nabla \tilde{w}) R^{(k)}(y', \nabla \tilde{w}) + s^2 R^{(j)}(y', \nabla \varphi) R^{(k)}(y', \nabla \varphi) \}dy
+ C_8 \left( \frac{1}{8} \|L_{2, \varphi} \tilde{w}\|^2_{L^2(\overline{G_N})} + \frac{1}{8} \|L_{1, \varphi} \tilde{w}\|^2_{L^2(\overline{G_N})} + s \|\tilde{w}\|_{L^2(\partial G_N)} \|\partial_{\tilde{n}} \tilde{w}\|_{L^2(\partial G_N)} \right).$$

Note that there exists a constant $C_9 > 0$, independent of $N$, such that

$$\int_{\overline{G_N}} 4N\tau \sum_{j,k=1}^n \partial_{y_j} \ell_1(y') \partial_{y_k} \ell_1(y') \{R^{(j)}(y', \nabla \tilde{w}) R^{(k)}(y', \nabla \tilde{w}) + s^2 R^{(j)}(y', \nabla \varphi) R^{(k)}(y', \nabla \varphi) \}dy \geq C_9 N \int_{\overline{G_N}} \tilde{w}^2dy$$

for all sufficiently large $N$.

By (11), we have

$$\int_{\overline{G_N}} \left( 2sG_{\varphi}(y, s, \tilde{w}) - 2sG_{\varphi}(y, s, \tilde{w}) \right) - 4N\tau \sum_{j,k=1}^n \partial_{y_j} \ell_1(y') \partial_{y_k} \ell_1(y') \{R^{(j)}(y', \nabla \tilde{w}) R^{(k)}(y', \nabla \tilde{w}) + s^2 R^{(j)}(y', \nabla \varphi) R^{(k)}(y', \nabla \varphi) \}dy \leq C_{10}(N)s \int_{\overline{G_N}} (|\nabla \tilde{w}|^2 + s^2 \tilde{w}^2)dy,$$
where $C_{10}(N) \to 0$ as $N \to +\infty$. By (10), we obtain
\[
\frac{1}{s}||L_2 \phi \tilde{w}||_{L^2(\Gamma_N)}^2 + \frac{1}{s}||L_1 \phi \tilde{w}||_{L^2(\Gamma_N)}^2 - \frac{1}{s}||L_2 \varphi \tilde{w}||_{L^2(\Gamma_N)}^2 - \frac{1}{s}||L_1 \varphi \tilde{w}||_{L^2(\Gamma_N)}^2 \leq C_{11}(N)s \int_{\Gamma_N} (|\nabla \tilde{w}|^2 + s^2 \tilde{w}^2)dy,
\]
where $C_{11}(N) \to 0$ as $N \to +\infty$. Using (25)–(27), from (24) we obtain
\[
\frac{1}{C_7}s \int_{\Gamma_N} (|\nabla \tilde{w}|^2 + s^2 \tilde{w}^2)dy \leq \frac{1}{4}||L_1 \phi \tilde{w}||_{L^2(\Gamma_N)}^2 + \frac{1}{4}||L_2 \varphi \tilde{w}||_{L^2(\Gamma_N)}^2 + \int_{\Gamma_N} 2sG_\phi(y, s, \tilde{w})dy + sC_9||\tilde{w}||_{L^2(\partial \Omega_N)}^2||\partial_{0n} \tilde{w}||_{L^2(\partial \Omega_N)}, \quad \forall s \geq s_0(\tau). \tag{28}
\]
Inequalities (21), (28) imply (5.28). The proof is finished. \qed

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