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ON A FOURTH ORDER EQUATION IN 3-D

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Abstract. In this article we study the positivity of the 4-th order Paneitz operator for closed 3-manifolds. We prove that the connected sum of two such 3-manifold retains the same positivity property. We also solve the analogue of the Yamabe equation for such a manifold.

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1. Introduction

In the analytic study of conformal structures in dimensions greater than two, it is fruitful to consider the family of Q-curvature equations as natural generalization of the Yamabe equation. Since the work of Paneitz [9] there has been a number of such equations introduced by Branson [2] and Fefferman and Graham [7].

In a series of papers [3, 4] it is demonstrated that solutions of these equations lead to significant results for nonlinear analysis as well as for conformal geometry in dimension four. A number of authors have investigated these equations in dimensions higher than four, for example Djadli *et al.* [6], Hebey and Robert [8] and Ahmedou *et al.* [1]. In this paper, we call attention to the validity of the equation in dimension three and begin a preliminary investigation of the fourth order Paneitz equation in the most favorable situation. Let us recall the Paneitz operator

$$P = (-\Delta)^2 + \delta\left(\frac{5}{4}Rg - 4Ric\right)d - \frac{1}{2}Q, \tag{1.1}$$

where

$$Q = -2|Ric|^2 + \frac{23}{32}R^2 - \frac{1}{4}\Delta R. \tag{1.2}$$

Under a conformal change of metrics $\bar{g} = u^{-4}g$ with $u > 0$, the Paneitz operator enjoys the following conformal covariance property:

$$\bar{P}w = u^7\, P(uw). \tag{1.3}$$

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In fact the Q-curvatures are related by the nonlinear equation

$$Pu = -\frac{1}{2}Q_{\bar{g}}u^{-7}. \tag{1.4}$$

In this article, we investigate the conformal classes for which the Paneitz operator is positive. In a previous article [10], we have shown that positivity is preserved under the connected sum operation in dimensions greater than four. While we do not know if the same property continues to hold in dimension four, we will show here that it remains true in dimension three.

Theorem 1.1. *Let (M, g) and (M', g') be closed 3-manifolds on which the Paneitz operator is positive, then there exists a conformal structure on the connected sum $M\#M'$ for which the Paneitz operator is positive.*

The basic reason underlying this result is that on the cylinder, the energy integrand

$$e(u) = \left[(\Delta u)^2 + \left(\frac{5}{4}Rg - 4Rc\right)(\nabla u, \nabla u) - \frac{1}{2}Qu^2\right]\mathrm{d}V \tag{1.5}$$

for the Paneitz operator becomes positive after integrating over the 2-spheres.

The proof of this result is essentially the same as that for the higher dimensional situation given in [10]. We take this opportunity to supply a detailed proof of Lemma 3.1 which works in all dimensions and hence clarify a point in the corresponding argument in [10]. As a corollary, we find a large number of conformal classes with positive Paneitz operators.

Corollary 1.2. *There exist conformal structures on the connected sum of a finite number of copies of $S^1 \times S^2$ for which the Paneitz operator is positive.*

On the other hand, the Paneitz operator on the standard 3-sphere is not positive as its lowest eigenvalue is negative. Since the 3-sphere is conformally the same as a long cylinder capped off at the ends by spherical caps, it is somewhat surprising that the operator should pick up a negative eigenvalue. To understand this situation, it is helpful to remark that contrary to the case of the Yamabe equation, the energy integrand (1.5) is not pointwise conformally covariant, it is only so after integration by parts. The boundary term thus becomes important in any gluing construction. Since the class of conformal structures with positive conformal Laplacian and positive Paneitz operators do not contain the most singular case of the standard 3-sphere, it is possible to solve the equation to prescribe the Q-curvature in this case.

Theorem 1.3. *If (M, g) is a three dimensional closed manifold such that the Paneitz operator P is positive, then the equation (1.4) has a positive solution with $Q_{\bar{g}}$ being a negative constant.*

Our approach to this problem is variational. We consider the functional $Q[u] = (\int_M u^{-6}\mathrm{d}v)^{1/3}\int_M Pu\cdot u\mathrm{d}v$. Due to the presence of the negative power nonlinearity, it is not possible to localize the analysis. The analysis of $Q[u]$ presents a number of features that are distinct from that of the Yamabe quotient. The presence of the negative exponent term means that the analysis is centered on preventing the conformal factor from touching zero. Fortunately in the case under consideration, there is compactness in the minimizing sequence of the energy functional. Thus as a corollary of the proof we obtain an Sobolev type inequality of the form:

$$0 < Q_p[M] = \inf\left(\int_M u^{-p}\mathrm{d}v\right)^{2/p}\int_M Pu\cdot u\mathrm{d}v,\ 1 < p < \infty. \tag{1.6}$$

The key fact we need (Lem. 4.3) is motivated by the classification of entire solutions on R^3 of the equation $\Delta^2 u = -u^{-7}$ which appears in [5]. In the case of the standard 3-sphere, it is possible to determine the best constant in (1.6) for the standard 3-sphere with a negative Q_6. However there are other issues to be resolved in the more general situation. We hope to return to this question on a later occasion.

Finally we outline the paper. In Section 2 we set the notations as well as some elementary examples of Paneitz operators. In Section 3 we prove Theorem 1.1. In Section 4 we prove Theorem 1.3.

2. Notations and examples

In this section we compute a few examples of the Paneitz operator.

$$P = (-\Delta)^2 + \delta\left(\frac{5}{4}Rg - 4Ric\right)d - \frac{1}{2}Q, \tag{2.1}$$

where

$$Q = -2|Ric|^2 + \frac{23}{32}R^2 - \frac{1}{4}\Delta R. \tag{2.2}$$

For the cylinder, which is conformally equivalent to the punctured 3-space $\mathcal{R}^3 - \{0\}$, we write $x = |x| \cdot \frac{x}{|x|} = \mathrm{e}^t \cdot \sigma$ with $|x| = \mathrm{e}^t$ and $\sigma = \frac{x}{|x|} \in S^2$. The metric is given by $g = \frac{|\mathrm{d}x|^2}{|x|^2} = \mathrm{d}t^2 + \mathrm{d}\sigma^2$ where $\mathrm{d}\sigma^2$ the canonical metric on the 2-sphere. We have $R = 2$, $|Ric|^2 = 2$, $Q = -\frac{9}{8}$. With the notation ∇^T, Δ^T denoting gradient and Laplacian in the directions tangential to the 2-sphere, we find

$$P = (\partial_t^2 + \Delta^T)^2 - (5/2)\Delta + 4\Delta^T + \frac{9}{16}\,\cdot \tag{2.3}$$

For functions u with compact support on the cylinder, we find upon integrating by parts,

$$\int Pu \cdot u\mathrm{d}v = \int [|u''|^2 + (5/2)|u'|^2 + |\Delta^T u|^2]\mathrm{d}v + \int \left[2|\nabla^T u'|^2 - (3/2)|\nabla^T u|^2 + \frac{9}{16}u^2\right]\mathrm{d}v \tag{2.4}$$

which is positive after integrating in the 2-sphere direction.

For the standard 3-sphere, we have $Rc = 2g$, so that $R = 6$, and $Q = \frac{15}{8}$.

$$P = (-\Delta)^2 + (1/2)\Delta - \frac{15}{16}\,\cdot \tag{2.5}$$

To determine the base eigenvalue of the operator P, we rewrite the energy integral, using the Bochner formula,

$$\int_{S^3} Pu \cdot u\,\mathrm{d}v = \int_{S^3} \left\{|\nabla^2 u|^2 + (3/2)|\nabla u|^2 - \frac{15}{16}u^2\right\}\mathrm{d}v. \tag{2.6}$$

It follows that the base eigenvalue of the operator P is $-\frac{15}{16}$.

3. Connected sum

In this section we show that the positivity of the Paneitz operator is preserved in taking a connected sum provided the gluing cylinder is sufficiently long and thin. Let (M, g) and (M', g') be compact 3-manifolds whose fourth order Paneitz operators are positive. Due to the conformal covariance property of the Paneitz operator, the operator will remain positive when g is replaced by a conformal metric. From continuity consideration, we may assume that M and M' contain sufficiently small balls B and B' of radius ϵ in which the metrics g and g' are conformally flat. By rescaling of the coordinates, we may assume these are balls of unit radius. In forming a connected sum, it is advantageous to use the cylindrical coordinates for B: $x = |x| \cdot \frac{x}{|x|}$, where we set $|x| = \mathrm{e}^t$ and $\sigma = \frac{x}{|x|}$. Likewise, for B' we have $x' = \mathrm{e}^{t'} \cdot \sigma'$. We identify (t, σ) with $(-L - t', \sigma')$ for $-L \le t \le 0$, and call L the length parameter. We then glue the metrics together over the common region $-L \le t \le 0$. We may also suppose, using the conformal covariance property of the Paneitz operator that the original metrics g and g' agree with the cylinder metric $\mathrm{d}t^2 + \mathrm{d}\sigma^2$ over the common region $-L \le t \le 0$.

To prove that the conformal structure on the connected sum has a positive Paneitz operator, we proceed by contradiction. Assume to the contrary, that there is a sequence of length parameters L tending to infinity, for which the Paneitz operator P_L on the connected sum has a nonpositive eigenvalue. We will show that there is a subsequence of eigenfunctions u_L which converges on either M or M' to a function v for which $\int Pv \cdot v\,\mathrm{d}V \leq 0$. This will be a contradiction to our assumption.

To make use of the assumption that (M, g) and (M', g') have positive Paneitz operators, we will cap off the cylinder at various values of t parameter and extend the eigenfunctions into the caps.

Lemma 3.1. *Let v be a smooth function in a neighborhood of the unit sphere S^2 in $\mathcal{R}^3$, let V be the biharmonic function defined on the unit ball B having the same boundary value and normal derivative as v on S^2. Then there is a constant C such that*

$$\int_B |\Delta V|^2 + |\nabla V|^2 + V^2 \mathrm{d}x \leq C\left\{\int_{S^2} |\Delta^T v|^2 + |\nabla v|^2 + v^2 \mathrm{d}\sigma\right\}\cdot \tag{3.1}$$

Proof. We begin with the representation formula for V:

$$V(x) = \int_{S^2} (\partial_n \Delta)_y G(x,y) v(y) \mathrm{d}\sigma(y) - \int_{S^2} \Delta_y G(x,y) \partial_n v(y) \mathrm{d}\sigma(y), \tag{3.2}$$

where

$$G(x,y) = -\frac{1}{4\pi}\left\{|x-y| - |y||x-\bar{y}| + \frac{1}{2}\frac{(1-|x|^2)(1-|y|^2)}{|y||x-\bar{y}|}\right\} \tag{3.3}$$

and $\bar{y}$ is the inversion of y in the unit sphere $S^2 = \{y||y| = 1\}$. Denote by I the second integral in (3.2) and by II the first integral in (3.2), so that $V(x) = I + II$. A direct calculation yields for the most singular term for $|y| = 1$:

$$I = -\frac{1}{4\pi}\int_{S^2}\left\{\frac{(1-|x|^2)^2}{|x-y|^3}\right\}\partial_n v(y) \mathrm{d}\sigma(y) \tag{3.4}$$

and

$$II = \frac{1}{4\pi}\int_{S^2} (1-|x|^2)\left\{\frac{6(|x|^2 - y\cdot x)^2}{|x-y|^5} + \frac{2 + 3y\cdot x - 5|x|^2}{|x-y|^3}\right\} v(y) \mathrm{d}\sigma(y). \tag{3.5}$$

It will suffice to prove

$$\int_B |\Delta V|^2 + V^2 \mathrm{d}x \leq C\left\{\int_{S^2} |\nabla^2 v|^2 + |\nabla v|^2 + v^2 \mathrm{d}\sigma\right\}\cdot \tag{3.6}$$

Let us consider the new function

$$w(v) = \frac{1}{4\pi}(1-|x|^2)\int_{S^2} \frac{v}{|x-y|^3} \mathrm{d}\sigma(y),$$

for any given continuous function v on S^2.

It is clear from Poisson integral formula, $w(v)$ is a harmonic function in B^3. We recall that for harmonic functions h:

$$\int_B h^2 \mathrm{d}x \leq \int_{S^2} h^2 \mathrm{d}\sigma.$$

Apply this to w, keep in mind that $w = v$ on the unit sphere S^2 to get

$$\int_B w^2 \mathrm{d}x \leq \int_{S^2} v^2 \mathrm{d}\sigma. \tag{3.7}$$

To bound the first integral, we observe that $1 - |x|^2 \leq 1$, thus

$$|I| \leq C\frac{1-|x|^2}{4\pi}\int_{S^2} \frac{|\nabla v(y)|}{|x-y|^3}\mathrm{d}\sigma(y) \leq Cw(|\nabla v|). \tag{3.8}$$

Similarly,

$$|\text{the second integral of} \quad II| \leq C\frac{1-|x|^2}{4\pi}\int_{S^2} \frac{|v|}{|x-y|^3}\mathrm{d}\sigma(y) \leq Cw(|v|). \tag{3.9}$$

and now since $||x|^2 - x\cdot y| = |x\cdot(x-y)| \leq |x||x-y|$;

$$|\text{the first integral of} \quad II| \leq C\frac{1-|x|^2}{4\pi}\int_{S^2} \frac{|v|}{|x-y|^3}\mathrm{d}\sigma(y) \leq Cw(|v|).$$

Then we square V and integrate over B and apply (3.7) to find,

$$\int_B V^2 \mathrm{d}x \leq C\int_{S^2} |\nabla v|^2 + v^2 \mathrm{d}\sigma.$$

To bound the integral $\int_B |\Delta V|^2 \mathrm{d}x$, we apply the inequality (3.7) to ΔV to obtain

$$\int_B |\Delta V|^2 \mathrm{d}x \leq \int_{S^2} |\Delta V|^2 \mathrm{d}\sigma.$$

The desired bound then follows from the following explicit formula [10] for Δu in terms of the intrinsic Laplacian Δ^T operating on the boundary data $\phi = \partial_n u$ and $\psi = u$ that can be derived *via* a calculation utilizing the spherical harmonics.

$$\Delta u = 2\left(\sqrt{-\Delta^T + \frac{1}{4}} + 1\right)\phi + \left\{2\Delta^T - \left(\sqrt{-\Delta^T + \frac{1}{4}} - \frac{1}{2}\right)\right\}\psi. \tag{3.10}$$

This finishes the proof of Lemma 3.1. □

To apply Lemma 3.1, we split the manifold M into $M_0 = M - B$ and B, and likewise M' into $M_0' = M' - B'$ and B' so that $M\#N = M_0 \cup M_0' \cup [0,L]\times S^2$. When we cap off $M_0 \cup [0,t]\times S^2$ by attaching B to the boundary $\{t\}\times S^2$ we view the resulting manifold as conformally the same as M, and we extend the boundary data of the eigenfunction u_L to a biharmonic function on B. (More precisely, in switching over to a conformal metric, we need to use the conformal covariance property of the operators to transform the relevant boundary data.) We denote the resulting function on M by v_L. Coming from the other end, we view M' as $M_0' \cup [t,L]\times S^2$, and extend the boundary data of the function u_L to M' and denote the resulting function on M' by v_L'.

To account for the energy of the several functions we have

$$E(u) = \int Pu\,u\mathrm{d}V = \int e(u)\mathrm{d}V$$

where

$$e(u) = |\Delta u|^2 + \frac{5}{4}R|\nabla u|^2 - 4Rc(\nabla u, \nabla u) - \frac{1}{2}Qu^2. \tag{3.11}$$

The positivity assumption on M and Lemma 3.1 imply

$$\int_{M_0} e(u)\mathrm{d}V + \int_0^t \int_{S^2} e(u(s,\sigma))\mathrm{d}\sigma\mathrm{d}s + C_n \int_{S^2} e(u(t,\sigma))\mathrm{d}\sigma > 0. \tag{3.12}$$

Similarly,

$$\int_{M_0'} e(u)\mathrm{d}V + \int_t^L \int_{S^2} e(u(s,\sigma))\mathrm{d}\sigma\mathrm{d}s + C_n \int_{S^2} e(u(t,\sigma))\mathrm{d}\sigma > 0. \tag{3.13}$$

The nonpositivity assumption on $M\#N$ implies that

$$\int_{M_0} e(u)\mathrm{d}V + \int_{M_0'} e(u)\mathrm{d}V + \int_0^L \int_{S^2} e(u(s,\sigma))\mathrm{d}\sigma\mathrm{d}s \le 0. \tag{3.14}$$

Since on the tube $[0,L]\times S^2$ the energy integral $\int_{S^2} e(u(s,\sigma))\mathrm{d}\sigma$ is positive for each $0 < s < L$, we observe that at least one of the numbers: $\int_{M_0} e(u)\mathrm{d}v$ and $\int_{M_0'} e(u)\mathrm{d}v$ is negative. Hence there are numbers $0 < t_0 \le t_1 < L$ so that

$$\int_{M_0} e(u)\mathrm{d}V + \int_0^{t_1} \int_{S^2} e(u(s,\sigma))\mathrm{d}\sigma\mathrm{d}s = 0 \tag{3.15}$$

and

$$\int_{M_0'} e(u)\mathrm{d}V + \int_{t_0}^L \int_{S^2} e(u(s,\sigma))\mathrm{d}\sigma\mathrm{d}s = 0. \tag{3.16}$$

In case the lowest eigenvalue of the Paneitz on $M\#M'$ is zero, and u the corresponding eigenfunction, we have $t_0 = t_1$. Since

$$-\int_{t_0}^{t_1} \int_{S^2} e(u(s,\sigma))\mathrm{d}\sigma\mathrm{d}s = \int_{M_0} e(u)\mathrm{d}V + \int_{M_0'} e(u)\mathrm{d}V + \int_0^L \int_{S^2} e(u(s,\sigma))\mathrm{d}\sigma\mathrm{d}s.$$

We can rewrite (3.12) and (3.13) by using (3.15) and (3.16) as

$$C_n \int_{S^2} e(u(t,\sigma))\mathrm{d}\sigma > \int_t^{t_1} \int_{S^2} e(u(s,\sigma))\mathrm{d}\sigma\mathrm{d}s \quad \text{for } 0 \le t \le t_1, \tag{3.17}$$

and

$$C_n \int_{S^2} e(u(t,\sigma))\mathrm{d}\sigma > \int_{t_0}^t \int_{S^2} e(u(s,\sigma))\mathrm{d}\sigma\mathrm{d}s \quad \text{for } t_0 \le t \le L. \tag{3.18}$$

Consequently, we find

$$2C_n \int_{S^2} e(u(t,\sigma))\mathrm{d}\sigma \geq \int_{t_0}^{t_1} \int_{S^2} e(u(s,\sigma))\mathrm{d}\sigma\mathrm{d}s \quad \text{for } t_0 \leq t \leq t_1.$$

Upon integration we find

$$2C_n \geq (t_1 - t_0).$$

Hence the middle interval is bounded independent of L. So as L tends to infinity we have at least one of t_1 and $L - t_0$ tends to infinity. Further, on one of these long tubes, equations (3.17) or (3.18) implies exponential decay of the energy on the t slices. A subsequence of the eigenfunctions u_L then converges uniformly on compact subsets of $M_0 \cup [0,\infty) \times S^2$ or $M_0' \cup [0,\infty) \times S^2$ to an eigenfunction u_∞ which has exponential decay in the tube. The conformal transform of this function to $\tilde{u}_\infty$ on M or M' is then a function with $W^{2,2}$ norm bounded but satisfies the condition

$$\int e(\tilde{u}_\infty)\mathrm{d}V \leq 0.$$

This is a contradiction to our assumption. Thus we have proved the Theorem 1.1.

4. Proof of Theorem 1.3

We will consider the variational problem to minimize the functional

$$F_6(u) = \left(\int_M u^{-6}\mathrm{d}v\right)^{1/3} \int_M Pu \cdot u\,\mathrm{d}v. \tag{4.1}$$

The critical points of the functional F_6 satisfy the equation (1.4) with $Q_{\bar{g}}$ given by a constant. The functional F_6 is invariant under scaling, and more generally, invariant under the action induced by conformal transformations T of the manifold M. To break this natural symmetry, we will consider the more general functional

$$F_p(u) = \left(\int_M |u|^{-p}\mathrm{d}v\right)^{2/p} \int_M Pu \cdot u\,\mathrm{d}v \tag{4.2}$$

for every $p \geq 6$ over the space $H^{2,2}(M)$.

For simplicity, we denote

$$H_p = \left\{u \in H^{2,2}(M) : \int_M |u|^{-p}\mathrm{d}v = 1\right\} \tag{4.3}$$

and assume that $\int_M \mathrm{d}v = 1$.

Lemma 4.1. *There exists a positive smooth function which minimizes the functional $F_p(u)$ over H_p if $p > 6$.*

Proof. Since P is positive, there exists a positive constant $\lambda > 0$ such that for every $u \in H_p$,

$$\int_M Pu \cdot u\,\mathrm{d}v \geq \lambda \int_M u^2\mathrm{d}v. \tag{4.4}$$

Thus $F_p(u)$ has a lower bound over H_p (nonnegative). Let $\{u_k\}$ be a minimizing sequence of $F_p(u)$ over H_p. First of all, since $u_k \in H_p$, $\int_M |u_k|^{-p}\mathrm{d}v = 1$. Thus for every k,

$$\int_M u_k^2 \mathrm{d}v \le \frac{1}{\lambda}\int_M Pu_k \cdot u_k \,\mathrm{d}v = \frac{1}{\lambda}F_p(u_k) \le C, \tag{4.5}$$

for some constant C since $\{u_k\}$ is a minimizing sequence.

From $F_p(u_k) \le C$ and the definition of the Paneitz operator, we have the following estimates:

$$\begin{aligned}\int_M (\Delta u_k)^2 \mathrm{d}v &= F_p(u_k) - \frac{5}{4}\int_M R|\nabla u_k|^2\mathrm{d}v + 4\int_M Ric(\nabla u_k, \nabla u_k)\mathrm{d}v + \frac{1}{2}\int_M Qu_k^2\mathrm{d}v \\ &\le \left[\frac{5}{4}\max|R| + 4\max|Ric|\right]\int_M |\nabla u_k|^2 + \frac{1}{2}\max|Q|\int_M u_k^2\mathrm{d}v + C \\ &\le \frac{1}{2}\int_M (\Delta u_k)^2\mathrm{d}v + C_1\int_M u_k^2\mathrm{d}v + C.\end{aligned}$$

Clearly from this and (4.5), we conclude that $\{u_k\}$ is a bounded sequence in $H^{2,2}(M)$. Thus it is standard that there exists a subsequence, still denoted by $\{u_k\}$, such that u_k weakly converges to some function $u_p \in H^{2,2}(M)$. As a consequence, u_k converges to u_p strongly in $H^{1,2}(M)$ and almost everywhere on M.

Now let $\alpha = \frac{p}{3} - 1 > 1$. It follows from $\int_M (\Delta u_k)^2\mathrm{d}v \le C$, $\int_M |\nabla u_k|^2\mathrm{d}v \le C$ and Sobolev's inequality that

$$\int_M |\nabla u_k|^6 \mathrm{d}v \le C. \tag{4.6}$$

Hence by Hölder inequality, we have

$$\int_M \left|\nabla\left(\frac{1}{|u_k|^\alpha}\right)\right|^2 \mathrm{d}v = \alpha^2\int_M \frac{|\nabla|u_k||^2}{|u_k|^{2(\alpha+1)}} \le \left(\int_M |\nabla u_k|^6\mathrm{d}v\right)^{1/3}\cdot\left(\int_M |u_k|^{-3(\alpha+1)}\mathrm{d}v\right)^{2/3}. \tag{4.7}$$

Now the choice of α implies $3(\alpha+1) = p$. Thus we conclude from the fact that $\int_M |u_k|^{-p}\mathrm{d}v = 1$ that

$$\int_M \left|\nabla\frac{1}{|u_k|^\alpha}\right|^2 \mathrm{d}v \le C \tag{4.8}$$

with C depending only on the upper bound of F_p and the geometry of g.

Also it follows from Hölder inequality that

$$\int_M (|u_k|^{-\alpha})^2 \mathrm{d}v \le C \tag{4.9}$$

for some constant C, again depending only on the same data as above.

The estimates (4.8) and (4.9) show that $\{|u_k|^{-\alpha}\}$ is a bounded sequence in $H^{1,2}(M)$. Thus there is a subsequence, again denoted by $\{u_k\}$ such that $|u_k|^{-\alpha}$ weakly converges to $\bar{u}_p^{-\alpha}$ in $H^{1,2}(M)$. However this implies the convergence holds almost everywhere on M. As we have shown that u_k also converges to u_p almost everywhere, $|u_p| = \bar{u}_p$ almost everywhere on M.

Now let $s = \frac{3p}{p-3}$. The fact that $p > 6$ implies $s < 6$. Thus by the Rellich–Kondrachov compactness theorem, we get

$$1 = \lim_{k\to\infty}\int_M (|u_k|^{-\alpha})^s \mathrm{d}v = \int_M (|u_p|^{-\alpha})^s \mathrm{d}v. \tag{4.10}$$

Since α and s are chosen so that $\alpha \cdot s = p$, we conclude that $u_p \in H_p$. Thus u_p is a critical point of F_p over H_p. Hence it weakly satisfies the equation

$$Pu_p = \lambda_p |u_p|^{-p} u_p^{-1}, \tag{4.11}$$

where λ_p is the minimum value of F_p over H_p. Since u_p is not identically zero and P is positive, λ_p is positive.

Since

$$||u_p||_{H^{2,2}} \leq \liminf_{k\to\infty} ||u_k||_{H^{2,2}},$$

$||u_p||_{H^{2,2}} \leq C$.

We will use the equation to improve the regularity of the solution u_p. We begin with the $|\nabla u_p|$ is in $L^6(M)$ by Sobolev's embedding. Thus by the estimate (4.7), we know that if $\alpha = \alpha_0 = (p-3)/3 > 1$, then $|\nabla |u_p|^{-\alpha}|$ is in $L^2(M)$. It follows that $|u_p|^{-\alpha}$ is in $L^6(M)$. Notice that $6\alpha = 2(p-3) > p$ if $p > 6$. Then in (4.7), we choose $\alpha = \alpha_1 = 2\alpha_0 - 1$, then by Sobolev's Embedding, we conclude that $|u_p|^{-\alpha_1}$ is in $L^6(M)$. Repeating this process, define

$$\alpha_{n+1} = 2\alpha_n - 1 = 2^n(\alpha_0 - 1) + 1,$$

to find that for any $n \geq 0$, $|u_p|^{-\alpha_n}$ is in $L^6(M)$. It is clear that α_n goes to infinity as n goes to infinity. Hence we can choose n such that $6\alpha_n \geq 2(p+1)$.

What we have shown in previous paragraph is that the right hand side of equation (4.11) is in $L^2(M)$. Then it follows that $(|\Delta^2 u|)^2$ is integrable. Thus Δu_p is a continuous function on M and bounded on M in absolute value. It follows from

$$\Delta\left(\frac{1}{u_p}\right) = -\frac{\Delta u_p}{u_p^2} + 2\frac{|\nabla u_p|^2}{u_p^3}, \tag{4.12}$$

that $\Delta(u_p^{-1})$ is in $L^2(M)$. Hence u_p^{-1} is a continuous function on M. Thus u_p can never take zero value. Without loss of generality, we can assume that $u_p > 0$, since otherwise we can take $-u_p$ as our solution. It will follow easily that u_p is a C^∞ smooth function. Therefore we have completed the proof of Lemma 4.1.

Lemma 4.2. *Let* $\lambda_p = \min_{u\in H_p} F_p(u)$ *for* $p \geq 6$. *Then*

$$\lim_{p\to 6} \lambda_p = \lambda_6. \tag{4.13}$$

Proof. Since

$$\lambda_6 = \inf_{u\in H_6} F_6(u),$$

let $\{u_i\}$ be a sequence in H_6 such that

$$F_6(u_i) \to \lambda_6$$

as $i \to \infty$. For each fixed i, we have

$$\lambda_p \leq F_p(u_i) \to F_6(u_i),$$

as $p \to 6$. It follows that

$$\overline{\lim}_{p\to 6} \lambda_p \leq \lambda_6.$$

On the other hand, by Hölder inequality, we have

$$F_6(u_p) = F_p(u_p) \cdot \frac{||u_p^{-1}||_6^2}{||u_p^{-1}||_p^2} \leq F_p(u_p) \left(\int_M \mathrm{d}v\right)^{2(1-\frac{6}{p})}.$$

Thus it follows that

$$\lambda_6 \leq \lambda_p \left(\int_M \mathrm{d}v\right)^{2(1-\frac{6}{p})}.$$

Hence, we get

$$\lambda_6 \leq \underline{\lim}_{p\to 6}\lambda_p \left(\int_M \mathrm{d}v\right)^{2(1-\frac{6}{p})} = \underline{\lim}_{p\to 6}\lambda_p.$$

Therefore

$$\lim_{p\to 6}\lambda_p = \lambda_6.$$

This finishes the proof of Lemma 4.2.

□

Lemma 4.3. *There exists no C^4 function v satisfying*

(1) $\int_{\mathcal{R}^3} v^{-6}\mathrm{d}v < \infty$;
(2) $v \geq 1$ *and* $v(0) = 1$;
(3) $\int_{\mathcal{R}^3}(\Delta v)^2\mathrm{d}v < \infty$;
(4) $\Delta^2 v = v^{-7}$.

Proof. It follows from conditions (1) and (2) that for any $q \geq 6$,

$$\int_{\mathcal{R}^3} v^{-q}\mathrm{d}v < \infty.$$

Define

$$w(x) = \frac{1}{4\pi}\int_{\mathcal{R}^3}\frac{v^{-7}(y)}{|x-y|}\mathrm{d}y.$$

Then for any $q > 1$, $w(x)$ is in $L^q(\mathcal{R}^3)$ and

$$\Delta w + v^{-7} = 0. \tag{4.14}$$

For any $r > 0$ and $x \in \mathcal{R}^3$, integrate condition (4) over the ball $B_r(x)$ to get

$$\int_{B_r(x)} v^{-7}\mathrm{d}x = \int_{B_r(x)}\Delta^2 v\mathrm{d}x = r^2\frac{\partial}{\partial r}\left[r^{-2}\int_{\partial B_r(x)}\Delta v\mathrm{d}\sigma\right]. \tag{4.15}$$

Multiply r^{-2} on both sides of (4.15) and integral from 0 to r to have

$$\int_0^r t^{-2}\int_{B_t(x)} v^{-7}\mathrm{d}y\mathrm{d}t = r^{-2}\int_{\partial B_r(x)}\Delta v\mathrm{d}\sigma - \omega_3(\Delta v)(x). \tag{4.16}$$

Now multiply r^2 on both sides of (4.16)and integrate the resulting equation from 0 to r to get

$$\begin{aligned}\int_0^r s^2\left[\int_0^s t^{-2}\int_{B_t(x)} v^{-7}\mathrm{d}y\mathrm{d}t\right]\mathrm{d}s &= \int_{B_r(x)}\Delta v\mathrm{d}x - \omega_3(\Delta v)(x)r^3/3\\ &= r^2\frac{\partial}{\partial r}\left[r^{-2}\int_{\partial B_r(x)} v\mathrm{d}\sigma\right] - \omega_3(\Delta v)(x)r^3/3.\end{aligned} \tag{4.17}$$

Finally multiply both sides of (4.17) by r^{-2} and integrate the resulting equation from 0 to R to get

$$\begin{aligned} g(R) &:= \int_0^R r^{-2}\left\{\int_0^r s^2\left[\int_0^s t^{-2}\int_{B_t(x)} v^{-7}\mathrm{d}y\mathrm{d}t\right]\mathrm{d}s\right\}\mathrm{d}r \\ &= R^{-2}\int_{\partial B_R(x)} v\mathrm{d}\sigma - \omega_3 v(x) - \omega_3(R^2/6)(\Delta v)(x). \end{aligned} \tag{4.18}$$

This is known as the mean value property for biharmonic function. By using L'Hospital's rule, we can see that

$$\lim_{R\to\infty}\frac{g(R)}{R^2}$$

exists and is less than a constant independent of x. Hence we can conclude that

$$\Delta v(x) + C \ge 0 \tag{4.19}$$

for all $x \in \mathcal{R}^3$.

Notice that condition (4) and equation (4.14) imply that

$$\Delta(\Delta v + w) = 0. \tag{4.20}$$

Clearly w is nonnegative and by equation (4.19), we see that $w + \Delta v$ is bounded from below, hence Liouville's theorem implies that $w + \Delta v$ is a constant:

$$\Delta v = C - w. \tag{4.21}$$

As pointed out earlier, w is in $L^q(\mathcal{R}^3)$ for every $q > 1$. Thus as $|x| \to \infty$, $w(x) \to 0$. This, together with the condition (3), we can see that the constant C must be zero.

It follows from the definition of w that Δv is nonpositive. Notice that

$$\Delta\left(\frac{1}{v}\right) = -\frac{\Delta v}{v^2} + 2\frac{|\nabla v|^2}{v^3} \ge 0.$$

It follows that $\frac{1}{v}$ must be a constant. Clearly this, together with condition (1), implies that v must be everywhere infinity, which is a contradiction to condition (2).

We have completed the proof of Lemma 4.3. □

Lemma 4.4. *Let $\{u_p\}$ be the positive smooth function found in Lemma 4.1 for every $p > 6$. Then there is a positive constant c_0, independent of p such that*

$$u_p \ge c_0 > 0.$$

Proof. Assume such a c_0 does not exist. Thus there exists a subsequence $p_k \to 6, u_k = u_{p_k}, z_k \in M$ such that $u_k(z_k) = \min u_k := m_k \to 0$. Since M is compact, there exists a point z_0 such that $z_k \to z_0$ as $k \to \infty$. Take a normal coordinate at z_0. In this coordinate,

$$g_{ij}(x) = \delta_{ij} + O(|x|^2),\ \ \mathrm{det}g_{ij}(x) = 1 + O(|x|^2). \tag{4.22}$$

Now let x_k be the coordinate of z_k. Then as $k \to \infty, x_k \to 0$. Notice that u_k satisfies

$$\Delta^2 u_k - \delta\left[\frac{5}{4}Rg - 4Ric\right]\mathrm{d}u_k - \frac{1}{2}Qu_k = \lambda_k u_k^{-(p+1)}, \tag{4.23}$$

where

$$\Delta = \frac{1}{\sqrt{\det g}}\partial_i\left(\sqrt{\det g}g^{ij}\partial_j\right)$$

and $\lambda_k = \lambda_{p_k}$.

Suppose this equation holds on the ball $|x| < 1$. Define

$$v_k(x) = m_k^{-1}u_k(\delta_k x + x_k), \tag{4.24}$$

where $\delta_k = m_k^{(p_k+2)/4} \to 0$ as $k \to \infty$. Thus v_k is well defined on the ball $B_{\rho_k}(0)$ where $\rho_k = (1-|x_k|)/\delta_k \to \infty$ as $k \to \infty$. Denote $\sqrt{(\det g)(\delta_k x + x_k)}$ by b_k and $g^{ij}(\delta_k x + x_k)$ by a_k^{ij}. In terms of v_k, they satisfy the equation:

$$\begin{aligned}
\frac{1}{b_k}\frac{\partial}{\partial x_i}\left\{b_k a_k^{ij}\frac{\partial}{\partial x_j}\left[\frac{1}{b_k}\frac{\partial}{\partial x_l}\left(b_k a_k^{lm}\frac{\partial v_k}{\partial x_m}\right)\right]\right\} &= m_k^{-1}\delta_k^4(\Delta^2 u_k)(\delta_k x + x_k)\\
&= m_k^{-1}\delta_k^4\left[\delta\left(\frac{5}{4}Rg - 4Ric\right)\mathrm{d}u_k + \frac{1}{2}Qu_k + \lambda_k u_k^{-(p+1)}\right]\\
&= -\delta_k^2\frac{1}{b_k}\frac{\partial}{\partial x_i}\left[(b_k a_k^{ij})\left(\frac{5}{4}R(\delta_k x + x_k)g_{lj}(\delta_k x + x_k)\right.\right.\\
&\quad \left.\left. -\ 4R_{lj}(\delta_k x + x_k)\right)a_k^{lm}\frac{\partial}{\partial x_m}(v_k)\right]\\
&\quad + \frac{1}{2}Q(\delta_k x + x_k)\delta_k^4 v_k + \lambda_k v_k^{-(p+1)}.
\end{aligned} \tag{4.25}$$

On the other hand, the left hand side of the equation (4.25) can be written as

$$\frac{1}{b_k}\frac{\partial}{\partial x_\alpha}\left[b_k^2 a_k^{\alpha\beta}\frac{\partial}{\partial x_\beta}\left(\frac{1}{b_k}\right)a_k^{\gamma\eta}\frac{\partial}{\partial x_\eta}v_k + a_k^{\alpha\beta}\frac{\partial}{\partial x_\beta}\frac{\partial}{\partial x_\eta}(a_k^{\gamma\eta}b_k)\frac{\partial}{\partial x_\eta}v_k + b_k a_k^{\alpha\beta}a_k^{\gamma\eta}\frac{\partial}{\partial x_\beta}\frac{\partial}{\partial x_\gamma}\frac{\partial}{\partial x_\eta}v_k\right]. \tag{4.26}$$

Now first we notice that

$$a_k^{ij} = g^{ij}(\delta_k x + x_k) \to \delta^{ij}; \tag{4.27}$$

$$b_k = \sqrt{\det g(\delta_k x + x_k)} \to 1; \tag{4.28}$$

$$\frac{1}{2}Q(\delta_k x + x_k)\delta_k^4 \to 0, \tag{4.29}$$

as $k \to \infty$ where the convergence is C^1 uniform on any bounded set in $\mathcal{R}^3$.

Also notice that in (4.26), $b_k^2 a_k^{\alpha\beta}a_k^{\gamma\eta}\partial_\beta(b_k^{-1})$ and $a_k^{\alpha\beta}\partial_\beta\partial_\eta(a_k^{\gamma\eta}b_k)$ converge to 0 in C^1 norm uniformly.

Now since $v_k(x) \geq v_k(0) = 1$, by L^q and Schauder estimates for v_k from equation (4.25), we will be able to conclude that for any $R > 0$, there exist $C(R) > 0$ and $k(R) > 0$ such that

$$||v_k||_{C^{4,\alpha}(\bar{B}_k)} \leq C(R),\ \text{for any } k \geq k(R). \tag{4.30}$$

Take a sequence $R_n \to \infty$. By the diagonal procedure, we can get a subsequence such that $v_n \to v \in C^4(\mathcal{R}^3)$. This convergence is C^4 convergent on every $\bar{B}_{R_n}$. Then from equation (4.25), we get that v satisfies the equation

$$\Delta^2 v = \lambda_6 v^{-7}. \tag{4.31}$$

Without loss of generality, we can assume $\lambda_6 = 1$ since it is positive and multiplying v by suitable constant will reduce the general case to this case.

Now by changing variables, we have

$$\int_{|x|<(1/2)\delta_k^{-1}} v_k^{-p_k} b_k \mathrm{d}x = \int_{B_{1/2}(x_k)} u_k^{-p_k} \sqrt{\det g} \mathrm{d}x \cdot m_k^{(p_k-6)/4} \leq ||u_k^{-1}||_{p_k}^{p_k} m_k^{(p_k-6)/4} = m_k^{(p_k-6)/4}. \tag{4.32}$$

Since $m_k^{(p_k-6)/4} < 1$ when k large and $v_k^{-p_k} b_k$ uniformly converges to v^{-6} on any bounded set. Thus by Fatou's lemma, we conclude that

$$\int_{\mathcal{R}^3} v^{-6} \mathrm{d}x \leq 1. \tag{4.33}$$

Now since P is positive, it follows from equation for u_k that

$$\int_M u_k^2 \mathrm{d}v \leq \frac{\lambda_{p_k}}{\lambda} \int_M u_k^{-p_k} \mathrm{d}v. \tag{4.34}$$

It implies that

$$\int_M (\Delta u_k)^2 \mathrm{d}v \leq C, \tag{4.35}$$

for some constant independent of k.

Thus, similar to the argument for (4.33), it is not hard to see that

$$\int_{\mathcal{R}^3} (\Delta v)^2 \mathrm{d}x < \infty. \tag{4.36}$$

Since every $v_k \geq 1$, thus $v \geq 1$. Now Lemma 4.3 implies that such v does not exist. Thus we get a contradiction which allows us to conclude the proof of Lemma 4.4.

Proof of Theorem 1.3. By Lemma 4.4, all solutions we have found in Lemma 4.1 are uniformly bounded from below as well as from above since Δu_p are uniformally bounded in L^2. Thus the argument we have used in Lemma 4.1 will show that u_p is uniformly bounded in $C^{k,\alpha}$ for any positive integer and any real number $0 < \alpha < 1$. Therefore there exists a subsequence $\{u_{p_k}\}$ such that $u_{p_k} \to u \in C^\infty(M)$ as $k \to \infty$ and $u > 0$ will satisfy

$$Pu = \lambda_6 u^{-7}. \tag{4.37}$$

This finishes the proof of Theorem 1.3. □

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