AN EXAMPLE IN THE GRADIENT THEORY OF PHASE TRANSITIONS

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Abstract. We prove by giving an example that when \( n \geq 3 \) the asymptotic behavior of functionals
\[
\int_{\Omega} \left( \varepsilon |\nabla^2 u|^2 + \frac{(1 - |\nabla u|^2)^2}{\varepsilon} \right) \nabla \Omega = \mathbb{R}^n
\]
as \( \varepsilon \downarrow 0 \), where \( u \) maps \( \Omega \) into \( \mathbb{R} \). This problem was raised by Aviles and Giga in [2] in connection with the mathematical theory of liquid crystals and more recently by Gioia and Ortiz in [9] for modeling the behavior of thin film blisters. Recently many authors have studied the planar case giving strong evidences that, as conjectured by Aviles and Giga in [2], the sequence \( \{ F_{\varepsilon}\} \) \( \Gamma \)-converge (in the strong topology of \( W^{1,3} \); see [1] for a discussion of such a choice and a rigorous setting) to the functional

\[
F^{\Omega}_\infty(u) := \begin{cases} 
\frac{1}{3} \int_{J_{\nabla u} + \infty} |\nabla u^+ - \nabla u^-|^3 \, dH^{n-1} & \text{if } |\nabla u| = 1, u \in W^{1,\infty} \\
\text{otherwise.} & 
\end{cases}
\]

Here \( J_{\nabla u} \) denotes the set of points where \( \nabla u \) has a jump and \( |\nabla u^+ - \nabla u^-| \) is the amount of this jump. Of course the first line of the previous definition makes sense only for particular choices of \( u \), such as piecewise \( C^1 \). For a rigorous setting the reader should think about a suitable function space \( S \) which contains piecewise \( C^1 \) functions and on which we can give a precise meaning to the above integral (for example a natural choice would be \( \{ u | \nabla u \in BV \} \); however this space turns out not to be the natural one: we refer again to [1] for a discussion of this topic).

Partial results in proving Aviles and Giga’s conjecture (i.e. compactness of minimizers of \( F^{\Omega}_\varepsilon \), estimates from below on \( F^{\Omega}_\varepsilon(u) \) and a suitable weak formulation for the problem of minimizing \( F \) subject to some boundary conditions) can be found in [1,3,5–8].

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1. INTRODUCTION

This paper is devoted to the study of the asymptotic behavior of functionals

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Keywords and phrases: Phase transitions, \( \Gamma \)-convergence, asymptotic analysis, singular perturbation, Ginzburg–Landau.

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The proof can be easily generalized to every \( n \) given by sequence \((\cdots)\) converging to \( E \), which is a dimensional problem in the calculus of variations which can be explicitly solved. This analysis leads to the ansatz has been proved by Jin and Kohn in [8] for \( n = 2 \). It reduces the problem of finding \( E \) to a one-dimensional problem in the calculus of variations which can be explicitly solved. This analysis leads to the result \( E = F_\Omega^2(w) \), which means that at \( w \) the \( \Gamma \)-limit of \( F_\Omega^2 \) exists and coincides with \( F_\Omega^2(w) \). With a standard cut and paste argument (see [4]) it can be proved that the same happens for every \( w \) which is piecewise affine. In the next section we will prove the following theorem:

**Theorem 1.2.** Let \( u \) be the function \( u(x_1, x_2, x_3) = |x_3| \) and \( C \) the cylinder \( \{|x_1|^2 + |x_2|^2 < 1\} \). Then there exists \( (u_k) \) such that:

1. every \( u_k \) is piecewise affine (being the union of a finite number of affine pieces) and satisfies the eikonal equation;
2. \( \lim_k F_\infty^C(u_k) < F_\infty^C(u) \);
3. \( u_k \rightharpoonup u \) strongly in \( W^{1,p} \) for every \( p < \infty \).

The proof can be easily generalized to every \( n \geq 3 \). As an easy corollary we get that the one-dimensional ansatz fails for \( n \geq 3 \). Moreover this failure means that \( F \) cannot be the \( \Gamma \)-limit of \( F_\Omega^2 \) for \( n \geq 3 \).

**Corollary 1.3.** The one-dimensional ansatz is not true for \( n \geq 3 \).

**Proof.** As already observed, being every \( u_k \) piecewise affine, there is a family of functions \( u_{k,\varepsilon} \) such that \( u_{k,\varepsilon} \) converge to \( u_k \) in \( W^{1,p} \) (for every \( p < \infty \)) and \( \lim\inf F_\infty^C(u_{k,\varepsilon}) = F_\infty^C(u_k) \). A standard diagonal argument gives a sequence \( (u_{k,\varepsilon(k)}) \) strongly converging to \( u \) in \( W^{1,p} \) such that \( \lim\inf F_\infty^C(u_{k,\varepsilon(k)}) < F_\infty^C(u) \).

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2. THE EXAMPLE

In this section we prove Theorem 1.2. First of all we recall the following fact:

(Curl) If \( v : \mathbb{R}^n \to \mathbb{R}^n \) is a piecewise constant vector field, then \( v \) is a gradient if and only if for every hyperplane of discontinuity \( \pi \) the right trace and the left trace of \( v \) have same component parallel to \( \pi \).

The building block of the construction of Theorem 1.2 is the following vector field, depending on a parameter \( \phi \in (0, \pi/2) \). First of all we fix in \( \mathbb{R}^3 \) a system of cylindrical coordinates \((r, \theta, z)\) and then we call \( A \) the cone given by \( \{ z > 0, r < 1, (1-r) > z\tan \phi \} \) and \( A' \) the reflection of \( A \) with respect to the plane \( \{ z = 0 \} \). Hence we put

\[
\begin{align*}
v(r, \theta, z) &= (0, 0, 1) & \text{if } z > 0 \text{ and } (r, \theta, z) \notin A \\
v(r, \theta, z) &= (\sin(2\phi), \theta + \pi, \cos(2\phi)) & \text{if } z > 0 \text{ and } z \in A \\
v(r, \theta, z) &= (0, 0, -1) & \text{if } z < 0 \text{ and } (r, \theta, z) \notin A' \\
v(r, \theta, z) &= (\sin(2\phi), \theta + \pi, -\cos(2\phi)) & \text{if } z < 0 \text{ and } z \in A'.
\end{align*}
\]
It is easy to see that \( v \) maps every plane \( \{ \theta = \alpha \} \cup \{ \theta = \alpha + \pi \} \) into itself. Moreover the restrictions of \( v \) to these planes all look like as in the following picture.

![Planar section of the field \( v \)](image)

Lemma 2.1. The vector field \( v \) is the gradient of a function \( w \). Moreover there is a sequence of piecewise affine functions \( w_k \) such that:

(a) \( w_k \to w \) strongly in \( W^{1,p} \) for every \( p \);
(b) \( F^{
abla}_{\Omega}(w_k) \to F^{
abla}_{\Omega}(w) \) for every open set \( \Omega \subset \mathbb{R}^3 \).

Proof. We consider the restriction of \( v \) to the plane \( P := \{ \theta = 0 \} \cup \{ \theta = \pi \} \). As already noticed \( v \) maps this plane into itself. Moreover its restriction to it satisfies condition (Curl), hence on \( P \) \( v \) is the gradient of a scalar function \( w \). Moreover we can find such a \( w \) so that it is identically zero on the line \( \{ z = 0 \} \cap P \). Hence \( w \) is symmetric with respect to the \( z \) axis and so we can extend \( w \) to the whole three-dimensional space so to build a cylindrically symmetric function. It is easy to check that the gradient of such a function is equal to \( v \).

We call this function \( w \) as well and we will prove that it satisfies conditions (a) and (b) written above.

(a) Our goal is approximating \( v \) with piecewise constant gradient fields. First of all we do it in the upper half-space \( \{ z > 0 \} \). For every \( n \) we take a regular \( n \)-agon \( B_n \) which is inscribed to the circle of radius 1 and lies on the plane \( \{ z = 0 \} \). The vertices of this \( n \)-agon are given by \( V_i := (1, 2i\pi/n, 0) \).

Hence we construct the pyramid \( A^n \) with vertex \( V := (0, 0, \cot \phi) \) and base \( B_n \). In the pyramid we identify \( n \) different regions \( A^n_1, \ldots, A^n_n \), where every \( A^n_i \) is given by the tetrahedron with vertices \( (0,0,0), V, V_i, V_{i+1} \). After this we put \( v_n \) equal to \( (0,0,1) \) outside \( A^n \) and in every \( A^n_i \) we put

\[
v_n(r, \theta, z) \equiv (\sin 2\phi, \pi + (2i + 1)\pi/n, \cos 2\phi).
\]

It is easy to see that \( v_n \) satisfies condition (Curl), hence it is the gradient of some function \( w_n \). Moreover we can choose \( w_n \) in such a way that it is identically zero on \( \{ z = 0 \} \). Then we extend \( w_n \) to the lower half space \( \{ z < 0 \} \) just by imposing \( w_n(r, \theta, -z) = w_n(r, \theta, z) \). It is not difficult to see that \( \nabla w_n \) converges strongly to \( \nabla w \) in \( L^p_{\text{loc}} \) for every \( p \).

(b) Now we check that the previous construction satisfies also the second condition of the lemma. We fix an open set \( \Omega \subset \mathbb{R}^3 \) and we observe that both \( w_k \) and \( w \) satisfy the eikonal equation in \( \Omega \). Moreover we call \( L^n \) the triangle with vertices \( V, V_i, V_{i+1} \) and \( L^n \) the union of \( L^n_i \) (so \( L^n \) is the “lateral surface” of the pyramid \( A^n \)). Finally we denote by \( L \) the lateral surface of the cone \( A \), i.e. the set \( \{(1-r) = z \tan \phi \} \).

(i) The amount of jump of \( v_n \) (i.e. \( |v^n_+ - v^n_-| \)) on \( L^n \) is constant and equal to the value of \( |v^+ - v^-| \) on \( L \). Moreover the area of \( L^n \) is converging to the area of \( L \). The same happens on the symmetric sets in the lower half-space \( \{ z < 0 \} \).
(ii) Let us call $B$ the base of the cone. The right and left traces of $v_n$ coincide with those of $v$ on $B_n \cup \{z = 0\} \setminus B$. Moreover the area of $B \setminus B_n$ is converging to zero.

(iii) The vector fields $v_n$ are discontinuous also on the triangles $T_i^n$ joining $V_i$, $(0,0,0)$ and $V_i$ (and on the symmetric triangles lying on $\{z < 0\}$). The amount of jump of $v_n$ on each of these triangles is given by

$$|v_n^+ − v_n^-| = 2 \sin(\pi/n).$$

Moreover the area of everyone is given by $(\cot \phi)/2$. Hence

$$\int_{\cup_i T_i^n} |v_n^+ − v_n^-|^3 d\mathcal{H}^2 = 4n \cot \phi \sin^3 \pi/n.$$

The right hand side goes to zero as $n \to \infty$ and this completes the proof.

Proof of Theorem 1.2. First of all we pass from the cartesian coordinates of the statement to the cylindrical coordinates $(r, \theta, z)$ given by $x_3 = z$, $x_1 = r \cos \theta$, $x_2 = r \sin \theta$ (and sometimes we will denote the elements of $\mathbb{R}^3$ with $(y, z)$, where $y \in \mathbb{R}^2$ and $z \in \mathbb{R}$).

We take $w$ as in the previous lemma. First of all we compute $F_C^c(w)$ where $C$ is the cylinder $\{r < 1\}$. As in the previous proof we call $L$ the lateral surface of the cone, that is the set $\{r − 1 = z \tan \phi\}$. The value of $|\nabla w^+ − \nabla w^-|$ on the surface $L$ is given by $2 \sin \phi$ and the area of $L$ is given by $\pi/\sin \phi$: the same happens for the symmetric of $L$ lying on the half-space $\{z < 0\}$. On the base of the cylinder we have $|\nabla w^+ − \nabla w^-| = 2 |\cos 2\phi|$. Hence

$$a(\phi) := F_C^c(u) − F_C^c(w) = \frac{\pi}{3}(8 − 8 \cos^3 2\phi − 16 \sin^2 \phi)$$

and it can be easily checked that for $\phi$ close enough to zero, $a(\phi)$ is positive.

Therefore let us fix an $\alpha$ for which $a(\alpha) > 0$ and let us agree that $w$ is constructed as in the previous lemma by choosing $\phi = \alpha$. Given $\rho > 0$ and $x \in \mathbb{R}^2$ we define $w_{x,\rho}$ in the cylinder $C_{x,\rho} := \{(y, z) : |y − x| \leq \rho\} \subset \mathbb{R}^3$ as $w_{x,\rho}(y, z) = \rho w((y − x)/\rho, z/\rho)$. It is easy to see that

$$F_{C_{x,\rho}}^c(u) − F_{C_{x,\rho}}^c(w_{x,\rho}) = a(\alpha) \rho^2. \tag{2}$$

Let us fix $\varepsilon$ and take $\rho$ such that $\rho \cot \alpha < \varepsilon$. Thanks to Besicovitch Covering lemma we can cover $\mathcal{H}^2$ almost all $D := \{z = 0, r ≤ 1\}$ with a disjoint countable family of closed discs $D_i$ such that every $D_i$ has radius $r_i < \rho$, center $x_i$ and is contained in $D$. We construct $u_\varepsilon$ by putting $u_\varepsilon \equiv w_{x_i,\rho_i}$ in the cylinder $C_{x_i,\rho_i}$.

Since $\nabla u_\varepsilon$ coincides with $\nabla u$ in $\{z ≥ \varepsilon\}$ and satisfies the eikonal equation, it is easy to see that $u_\varepsilon \to u$ locally in the strong topology of $W^{1,p}$. Moreover equation (2) implies that

$$F_C^c(u) − F_C^c(u_\varepsilon) = \sum_i a(\alpha) r_i^2 = a(\alpha).$$

At this point, using the previous lemma we can approximate the function $u_\varepsilon$ in the cylinders $C_{x_i,\rho_i}$ with piecewise affine functions in such a way that their traces coincide with the trace of $u_\varepsilon$ on the boundary of $C_{x_i,\rho_i}$. Using standard diagonal arguments for every $\varepsilon$ we can find a sequence of piecewise affine functions $u_k^\varepsilon$ which converge in $W^{1,p}$ to $u_\varepsilon$ and such that $F_C^c(u_k^\varepsilon) \to F_C^c(u_\varepsilon)$. Moreover, again using diagonal arguments, we can construct the sequence $u_k^\varepsilon$ so that each one is a finite union of affine pieces.

Finally, one last diagonal argument, gives a sequence $u_k$ such that:

(a) $u_k$ is a finite union of affine pieces;

(b) $\lim_k F_C^c(u_k) < F_C^c(u)$;

(c) $u_k \to u$ strongly in $W^{1,p}$ for every $p < \infty$. 

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REFERENCES


