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# A generic approach to the structure of certain normal ideals

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**Abstract.** By considering a Rees ring as a projective scheme, the geometric technique of generic Bertini sections is used to examine conditions under which Rees rings satisfy Serre conditions. Liaison emerges naturally in the case of an almost complete intersection ideal and specialisation of liaison is then used to generalise aspects of work of Huneke, Ulrich and Vasconcelos on certain normal ideals.

**Key words:** Rees ring, generic Bertini section, Serre conditions, liaison, specialisation, normal ideal

## Introduction

The aim of this paper is to generalise certain aspects of work by Huneke, Ulrich and Vasconcelos [HUV] which concern normal almost complete intersection ideals. At first sight, their results deal with affine spectra. The point of view taken here is that their work essentially deals with projective schemes; this allows us to use the natural geometrical approach of reduction by generic Bertini sections to obtain a generalisation of their results on the generic level. One can then use the work of Huneke and Ulrich [HU] on the specialisation of linkage (liaison) to descend to the original base ring. Here the approach to the Primbasissatz given by Herrmann, Moonen and Villamayor [HMV], and in particular their discussion of generic sets of 'Primbasen', is very helpful.

In this paper, all rings are commutative and Noetherian and possess an identity element. The reader is referred to [Ei, HIO, Ma, Va] for general background.

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## 1. Basic results

Let  $A = \bigoplus_{n \geq 0} A_n$  be a graded ring and let  $\mathfrak{p}$  be a relevant homogeneous prime ideal, with, as usual,  $A_{(\mathfrak{p})}$  denoting the degree zero part of the homogeneous localisation at  $\mathfrak{p}$ . Then the natural map  $A_{(\mathfrak{p})} \rightarrow A_{\mathfrak{p}}$  is faithfully flat with regular fibres, so that  $A_{(\mathfrak{p})}$  satisfies any one of the Serre conditions  $(R_s)$ ,  $(S_r)$  if and only if  $A_{\mathfrak{p}}$  satisfies the same condition (cf. [HIO, Sect. 12], [Ma, Sect. 23]). Recall also from [CN, 1.3] that if any one of  $(R_s)$ ,  $(S_r)$  is satisfied at every homogeneous prime of  $A$ , then that condition is satisfied on all of  $\text{Spec } A$ .

Let  $R$  be a ring and let  $I$  be an ideal of  $R$ . We shall apply the observations above to the Rees ring  $S := R[It]$ . In connection with the Serre conditions, note that often  $S$  is Cohen–Macaulay, in which case of course the condition  $(S_r)$  holds for all  $r \geq 1$ . Before giving a pertinent case where this happens, we recall some terminology. According to Huneke [H],  $I$  is an *almost complete intersection* if  $I = I' + xR$  where  $I'$  is generated by a regular sequence and  $I' : x = I' : x^2$ ; i.e., in the terminology of [HMV, Sect. 4]  $I$  is a *generalised almost complete intersection*. According to [HMV, Sect. 4]  $I$  is called an almost complete intersection if the minimal number of generators  $\mu(I)$  of  $I$  equals  $ht(I) + 1$  and if  $I$  is a local complete intersection (in the sense that, for all primes  $P$  minimal over  $I$ ,  $I_P$  is a complete intersection in  $R_P$ ); note that, in the latter case,  $I$  is often called a *generic complete intersection*. Fortunately, if  $R$  is Cohen–Macaulay with all localisations  $R_m$ ,  $m \in \text{Max Spec } R$ , having infinite residue fields and if  $\mu(I) = ht(I) + 1$ , the notions of generalised almost complete intersection and of almost complete intersection (in the sense of [HMV]) coincide (cf. [HMV, 4.8]). For simplicity, from now on we assume that all residue fields are infinite and insist, for an almost complete intersection ideal  $I$ , that  $\mu(I) = ht(I) + 1$ .

We can now describe an important situation where  $S$  is Cohen–Macaulay (see [HUV, pp. 25–27], say):

(1.1) *Let  $R$  be a Cohen–Macaulay local ring and let  $I$  be an almost complete intersection ideal of  $R$  such that  $R/I$  is again Cohen–Macaulay. Then  $S := R[It]$  is also Cohen–Macaulay.*

Thus, in the situation of (1.1) and in view of the preceding remarks, the question of the normality of  $S$  devolves onto the question of the Serre condition  $(R_1)$  being satisfied at each homogeneous prime of  $S$ . We next remark that irrelevant prime ideals of  $S$  typically are easy to deal with:

(1.2) *Let  $R$  be a quasi-unmixed local ring and let  $I$  be an ideal of  $R$  with  $ht(I) \geq s$ , where  $s$  is a positive integer. Suppose that  $R$  satisfies the Serre condition  $(R_{s-1})$ . Then  $S := R[It]$  satisfies the Serre condition  $(R_s)$  at every irrelevant homogeneous prime  $\mathfrak{p}$ .*

*Proof.* Let  $\mathfrak{p}$  be such an ideal with  $ht(\mathfrak{p}) \leq s$ . Set  $P = \mathfrak{p} \cap R$ . Then  $S/\mathfrak{p} \approx R/P$ , so  $\dim_S \mathfrak{p} = \dim_R P$ .

Now

$$\begin{aligned} ht(P) &= \dim R - \dim P \\ &= \dim S - 1 - \dim \mathfrak{p} \\ &= ht \mathfrak{p} - 1 \leq s - 1 \end{aligned}$$

(see [Ma, Sect. 31], [HIO, (4.5), (9.7)]).

Hence  $I \not\subseteq P$ , so that  $S_{\mathfrak{p}}$ , which is a localisation of  $R_P[It] = R_P[t]$ , is indeed regular. □

The upshot is the following:

(1.3) *Let  $R$  be a reduced Cohen–Macaulay local ring and let  $I$  be an almost complete intersection ideal of  $R$  such that  $R/I$  is Cohen–Macaulay. Then  $S := R[It]$  is normal if and only if  $\text{Proj } S$  satisfies the Serre condition  $(R_1)$ .*

This sets [HUV, 1.7] (and other aspects of [HUV] as well) in the context of projective schemes and suggests that geometrical techniques might prove useful in examining more general versions of this work. Below we employ one such technique, that of generic Bertini sections.

## 2. Reduction via generic Bertini sections

Let  $I$  be an ideal in the Noetherian local ring  $(R, \mathfrak{m})$  and suppose that  $I = (x_1, \dots, x_d)$  with  $d := \mu(I) > 1$ . Suppose further that  $\text{grade } I > 0$ . Let  $T$  denote the set of indeterminates  $\{T_1, \dots, T_d\}$  and let  $R[T] := R[T_1, \dots, T_d]$  and  $-[T] := - \otimes_R R[T]$ . Set

$$S = R[It] = R[x_1t, \dots, x_dt],$$

so that  $S[T] = R[T][I[T] \cdot t] = R[T][x_1t, \dots, x_dt]$ .

Let  $\mathcal{P} = \{\mathfrak{p}_1, \dots, \mathfrak{p}_r\}$  be a given finite set of relevant homogeneous primes in  $S[T]$  (with  $R[T]$  sitting in degree zero), with each  $\mathfrak{p}_i$  extended from  $S$ ; typically,  $\mathcal{P}$  is the set of relevant associated primes of  $I[T] \cdot S[T] = I \cdot S[T]$ . Then we have:

(2.1) *Claim.* There exists  $zt \in [S[T]]_1$ , with  $z = x'_1T_1 + \dots + x'_dT_d$ , such that  $zt \notin \mathfrak{p}_i, i = 1, \dots, r$ , where  $x'_j = u_jx_j$  for some unit  $u_j \in R$  ( $j = 1, \dots, d$ ). In particular,  $z \in I[T]$  and  $I = (x'_1, \dots, x'_d)$ .

*Proof.* Take an infinite subset  $W$  of  $R \setminus \mathfrak{m}$  with the property that for all distinct  $a, b \in W, a - b \in R \setminus \mathfrak{m}$ . Consider elements  $s_j := (x_1T_1 + a_jx_2T_2 + \dots + a_j^{d-1}x_dT_d)t$ , where  $a_1, \dots, a_d$  are distinct elements of  $W$ . If  $\{s_1, \dots, s_d\} \subseteq \mathfrak{p}_i$  for some  $i$ , then since  $\det(a_j^{k-1}) = \prod_{j>\ell} (a_j - a_\ell)$  is a unit in  $R$ ,  $\{x_1T_1t, \dots, x_dT_dt\} \subseteq \mathfrak{p}_i$ . Since  $\mathfrak{p}_i$  is extended from  $S$ , by assumption, we deduce that  $\{x_1t, \dots, x_dt\} \subseteq \mathfrak{p}_i$ . This contradicts the hypothesis that  $\mathfrak{p}_i$  is relevant.

Since  $W$  is infinite, the claim now follows. □

*Remark 1.* This argument is due to Itoh [It, p. 107].

*Remark 2.* As regards the claim, note that we can subsequently absorb the unit  $u_j$  into the  $T_j$  and use  $x_1, \dots, x_d$  in place of  $x'_1, \dots, x'_d$ , respectively.

Following the notation of (2.1), first consider  $\text{Proj } S[T]/(zt)$ ; as in Remark 2 above, we have  $z = x_1T_1 + \dots + x_dT_d$  (after absorbing units). Then on a typical affine coordinate patch, say  $x_1t \neq 0$ , the affine coordinate ring is

$$R[T][x_2/x_1, \dots, x_d/x_1]/(T_1 + x_2T_2/x_1 + \dots + x_dT_d/x_1)$$

which is isomorphic as an  $R[x_2/x_1, \dots, x_d/x_1]$ -algebra to

$$R[x_2/x_1, \dots, x_d/x_1][T_2, \dots, T_d].$$

Hence if  $\mathbf{P}$  is a local property which is stable under polynomial extension (for example, any one of the Serre conditions) and if  $\mathbf{P}$  holds on  $\text{Proj } S$ , then  $\mathbf{P}$  continues to hold on  $\text{Proj } S[T]/(zt)$ .

Secondly, consider what is happening at base-ring level. Note that  $z$  is a nonzero divisor (cf. [Ho]). Now take the finite set  $\mathcal{P}$  of primes to be (or, more generally, to contain) the relevant associated primes of  $I[T] \cdot S[T] = I \cdot S[T]$ . Then  $z$  is a superficial element of  $I[T]$  of order 1 [Na, p. 72] and the usual Artin–Rees argument [Na, (3.12)] shows that

$$I[T]^n : z = I[T]^{n-1}, \quad n \gg 0. \tag{*}$$

(Note that the associated primes of  $I \cdot S[T]$  are extended from those of  $IS$  and  $[S[T]]_1$  is extended from  $[S]_1$ . Hence the usual theory of superficial elements of order 1 holds in  $S[T]$  even though the base ring  $R[T]$  is no longer (semi-)local (cf. [ZS, pp. 286–7].) Let  $\overline{R}[T]$  (respectively,  $\overline{I}[T]$ ) denote  $R[T]/(z)$  (respectively,  $I[T]/(z)$ ). It follows from (\*) that the natural map

$$S[T]/(zt) \rightarrow \overline{R}[T][\overline{I}[T] \cdot t] =: \overline{S}[T]$$

induces an isomorphism

$$\text{Proj } S[T]/(zt) \approx \text{Proj } \overline{S}[T].$$

As regards a repetition of this procedure (which we term *reduction by a generic Bertini section*) the remarks about (2.1) and about superficial elements show that the procedure continues to work at each step as long as the ideal in play has positive grade. The fact that the base ring ceases to be local causes no problem; we can continue to use the infinite set  $W \subseteq R \setminus \mathfrak{m}$  as before. Note also that we can continue to use the images of  $x_1, \dots, x_d$  as generators for the successive versions of the ideal  $I$ , since the fact that  $d = \mu(I)$  was not used in any way in our description of the reduction process.

We finish this section with a result which is needed because of our interest in ideals which, in particular, are generic complete intersections.

(2.2) *Let  $P$  be a prime ideal minimal over  $I$ . Then  $z/1$  is part of a minimal system of generators of  $I[T]_{P[T]}$ .*

*Proof.* Suppose that  $z/1 \in PI[T]_{P[T]}$ . Then there exists  $g \in R[T] \setminus P[T]$  such that  $g \cdot \sum_{i=1}^d x_i T_i \in PI[T]$ ; we may assume that no coefficient of  $g$  lies in  $P$ . Choose a term ordering [Ei, Va] on the monomials in  $T$  such that (say)  $T_1 = \text{Max}\{T_1, \dots, T_d\}$ . Equating coefficients of maximum monomials, we deduce that  $x_1/1 \in PI_P$ . Repeating the argument in turn for  $T_2, \dots, T_d$ , we deduce that  $I_P = PI_P$  and therefore  $I_P = 0$ . This contradicts the fact that  $\text{grade } I > 0$ .  $\square$

### 3. At the generic level

As we shall see in this section, the reduction process of Section 2 yields generic information about the projective scheme structure of suitable Rees rings under Serre conditions. In the next section we shall consider the effect of specialisation.

From now on let  $(R, \mathfrak{m})$  be a Cohen–Macaulay ring (within infinite residue field) and let  $I$  be an almost complete intersection ideal of  $R$  with  $\mu(I) =: d$ ; typically,  $d \geq 2$ . Let  $\mathbf{T} := \{T_{ij} \mid 1 \leq i, j \leq d\}$  be a set of indeterminates (to be appropriately adjusted at the relevant stage by absorption of units) and let  $z_j = \sum_{i=1}^d x_i T_{ij}$ ,  $1 \leq j \leq d$ . Set  $D = \det(T_{ij})$  and  $B = R[\mathbf{T}]_D$ . It is easy to see that  $z_1, \dots, z_{d-1}$  is a regular sequence in  $B$  (see [Ho]) and it follows from (2.2) that  $z_1, \dots, z_{d-1}$  generate  $J := IB$  generically. Moreover  $d = \mu(J)$ , as can be seen by passing to the further localisation  $R[\mathbf{T}]_{\mathfrak{m}[\mathbf{T}]}$ . Clearly  $ht(I) = ht(J)$ . Hence  $J$  is also an almost complete intersection ideal.

We can now give our main result on the generic level. It can be considered as a generic version of a generalisation of the results from [HUV] discussed previously. Note that the linkage to an ideal with factor ring possessing the appropriate Serre condition emerges in a natural geometrical way from our approach.

**THEOREM 1.** *Let  $R$  be a Cohen–Macaulay local ring with infinite residue field and let  $I$  be an almost complete intersection ideal in  $R$  with  $\mu(I) = d \geq 2$ . Then in the above notation,  $\text{Proj } R[It]$  satisfies the Serre condition  $(R_s)$  if and only if  $B/(z_1, \dots, z_{d-1})B : z_d$  satisfies  $(R_s)$ .*

*Remark 3.* An analogous statement holds for the Serre condition  $(S_r)$ .

*Proof.*  $(\rightarrow)$  Set  $B' = B/(z_1, \dots, z_{d-1})$  and  $J' = J/(z_1, \dots, z_{d-1})$ . Since  $J'$  is an almost complete intersection ideal, it is a generalised almost complete intersection ideal in  $B'$  (see Section 1), so that  $0 : z_d B' = 0 : z_d^n B'$  for all  $n \geq 1$ . Moreover, it follows from Section 2 that  $\text{Proj } B'[J't]$  satisfies  $(R_s)$ , since  $(R_s)$  is a local condition and so is preserved after localising by the determinant  $D$ .

But

$$\begin{aligned} \text{Proj } B'[J't] &= \text{Proj } B'[z_d t] \\ &= \text{Spec } B'/(0 : z_d^n B'), \quad n \gg 0, \\ &= \text{Spec } B'/(0 : z_d B'), \end{aligned}$$

and the result follows.

$(\leftarrow)$  Consider (the fibres of) the natural map  $\text{Proj } B'[J't] \rightarrow \text{Proj } R[IT]$  over the patch  $x_1 t \neq 0$ , say (where  $I = (x_1, \dots, x_d)$ ). By Section 2, we can approach this map by first considering (the fibres of) the map

$$\text{Proj } R[\mathbf{T}][x_1 t, \dots, x_d t]/(z_1 t, \dots, z_{d-1} t) \rightarrow \text{Proj } R[x_1 t, \dots, x_d t]. \quad (*')$$

Over the patch  $x_1 t \neq 0$  (say), the image of  $D$  in the corresponding affine coordinate ring of the first scheme in  $(*)'$  is  $\det(T_{ij})_{1 \leq i, j \leq d}$ , with

$$T_{1j} = -x_2 T_{2j} / x_1 + \cdots + -x_d T_{dj} / x_1, \quad 1 \leq j \leq d - 1.$$

In turn, this image of  $D$  equals

$$(T_{1d} + x_2 T_{2d} / x_1 + \cdots + x_d T_{dd} / x_1) \cdot \det(T_{ij})_{\substack{2 \leq i \leq d, \\ 1 \leq j \leq d-1}}.$$

Hence the image of  $D$  in the affine coordinate ring cannot lie in the extension of any  $P \in \text{Spec } R[x_2/x_1, \dots, x_d/x_1]$ , since the image of  $D$  involves a unique monomial in  $T_{1d}$  with coefficient 1. It follows almost immediately that the map  $(*)'$  is faithfully flat (with regular fibres), and the full result now follows.  $\square$

*Remark 4.* Of course if  $I = aR$  is a principal ideal, then  $\text{Proj } R[at] = \text{Spec } R/0 : a^n, n \gg 0$ .

### 4. Specialisation

We now wish to apply a basic result of Huneke and Ulrich [HU, 2.13] on the specialisation of linkage. Hence from now on we suppose that  $(R, \mathfrak{m}, k)$  is in fact Gorenstein and that  $I$  is a Cohen–Macaulay almost complete intersection ideal of  $R$ , with  $\mu(I) = d$  as before.

We will also need the helpful form of the Primbasissatz given in [HMV, 4.2], namely that there exists a generic set  $\text{Pr}(I) \subseteq I^{[d]}$  (of ‘Primbasen’) such that for each  $(a_1, \dots, a_d) \in \text{Pr}(I)$  the following hold:

- (i)  $\{a_1, \dots, a_d\}$  is a minimal set of generators for  $I$ ;
- (ii) for all primes  $P$  minimal over  $I$ ,  $(a_1, \dots, a_{d-1})_P = I_P$

hence each such  $a_1, \dots, a_{d-1}$  is a regular sequence. Here  $I^{[d]} := I \times \cdots \times I$  ( $d$  times) is equipped with the topology induced by the projection from  $I^{[d]}$  onto  $(I/\mathfrak{m}I)^{[d]}$ , the latter being a finite dimension  $k$ -vector space with the Zariski topology. Note that a subset  $U$  of  $I^{[d]}$  is called *generic* if  $U$  contains a non-empty open subset of  $I^{[d]}$ ; in particular, a finite intersection of generic sets is again generic.

We recall briefly some basic aspects of linkage (‘liaison’) in this context (see [HU], [Va]). Let  $(a_1, \dots, a_d) \in \text{Pr}(I)$  and let  $I' = (a_1, \dots, a_{d-1})R : I$ . Then  $I'$  is said to be *linked* to  $I$  and  $I'$  is a Cohen–Macaulay ideal with  $I = (a_1, \dots, a_{d-1})R : I'$ ; moreover, the primary components of  $(a_1, \dots, a_{d-1})$  are a disjoint union of those of  $I$  and  $I'$  ( $I$  and  $I'$  are then said to be *geometrically linked*) and  $I \cap I' = (a_1, \dots, a_{d-1})$ .

In [HU, 2.3], Huneke and Ulrich have defined the notion of a *generic link*  $L(\underline{x})$ : if  $Q = R[T_{ij}]_{\substack{1 \leq i \leq d, \\ 1 \leq j \leq d-1}}$  and  $\alpha_j = x_1 T_{1j} + \cdots + x_d T_{dj}$ ,  $1 \leq j \leq d - 1$ , we set  $L(\underline{x}) = (\alpha_1, \dots, \alpha_{d-1})Q : I$ . Then the link  $(z_1, \dots, z_{d-1})B : z_d$  in Theorem 1

equals  $L(\underline{x})[T_{d1}, \dots, T_{dd}]_D$ . Clearly the contents of [HU, 2.5] and its proof extend to show that these links in  $B$  are geometric links (as are the corresponding links in  $Q$ ). In particular, the dimensions of the linked ideals are the same.

We now show that Theorem 1, together with specialisation, yields the major part of our generalisation of [HUV, 1.7 (a)→(c)] (cf. [*loc. cit.*, 3.5 and 3.6 (b)] also).

**THEOREM 2.** *Let  $(R, \mathfrak{m})$  be a Gorenstein local ring with infinite residue field and let  $I$  be a Cohen–Macaulay almost complete intersection ideal in  $R$  of dimension  $s$ . If  $\text{Proj } R[It]$  satisfies  $(R_s)$  then  $I$  is geometrically linked to a regular ideal and  $\text{Spec } R \setminus V(I)$  satisfies  $(R_s)$ . Conversely if  $I$  is geometrically linked to a regular ideal and if  $\text{Spec } R \setminus V(I)$  satisfies  $(R_s)$ , then  $\text{Proj } R[It]$  satisfies  $(R_s)$ .*

*Proof.* (→) Consider first the case where  $I$  is principal. Then, by the remark after Theorem 1 and by the above, there exists  $a$  in  $I (= aR)$  such that  $0 : aR = 0 : a^n R$ ,  $n \geq 1$ , and  $\text{Proj } R[It] = \text{Spec } R/0 : aR$ , where  $0 : aR$  has dimension  $s$ ; the result follows.

Now suppose that  $\mu(I) =: d > 1$ . Then, by Theorem 1,  $B/(z_1, \dots, z_{d-1})B : I$  satisfies  $(R_s)$ . Set  $C = R[\mathbf{T}]$ . We claim that  $(z_1, \dots, z_{d-1})C : I \subseteq \mathfrak{m}C$ . For if not, then some  $f$  (say) lies in  $((z_1, \dots, z_{d-1})C : I) \setminus \mathfrak{m}C$ . This gives rise to the generic set  $U$  consisting of all  $(a_1, \dots, a_d)$  in  $I^{[d]}$  such that  $a_j = x_1 b_{1j} + \dots + x_d b_{dj}$ ,  $1 \leq j \leq d$ , with  $f(b_{ij})$  a unit in  $R$ . Picking  $(a_1, \dots, a_d)$  in the generic set given by the intersection of  $U$  with  $\text{Pr}(I)$ , we have  $f(b_{ij})$  lying in  $(a_1, \dots, a_{d-1})R : I$ , by a slight extension of [HU, 2.13] to our context, and this is a contradiction.

Moreover,  $\dim((C/(z_1, \dots, z_{d-1})C : I)_{\mathfrak{m}C}) = s$ . Hence, for some  $g$  in  $C \setminus \mathfrak{m}C$  and  $f_1, \dots, f_s$  in  $\mathfrak{m}C$ ,

$$g \cdot \mathfrak{m}C \subseteq (f_1, \dots, f_s)C + (z_1, \dots, z_{d-1})C : I.$$

(Note that the determinant  $D$  also lies in  $C \setminus \mathfrak{m}C$ .)

Now pick  $(a_1, \dots, a_d)$  in the generic set  $\text{Pr}(I)$  such that, if  $a_j = x_1 b_{1j} + \dots + x_d b_{dj}$ ,  $1 \leq j \leq d$ , with  $b_{ij} \in R$ , then both  $\det(b_{ij})$  and  $g(b_{ij})$  are units in  $R$ . By [HU, 2.13] once again specialisation yields that

$$\mathfrak{m} \subseteq (f_1(b_{ij}), \dots, f_s(b_{ij}))R + (a_1, \dots, a_{d-1})R : I,$$

and the first part of the result follows.

Finally consider  $\mathfrak{p} \in \text{Spec } R \setminus V(I)$  with  $ht \mathfrak{p} \leq s$ . Let  $P = \mathfrak{p}[t] \cap R[It]$ . Then  $P \in \text{Proj } R[It]$ ,  $P \cap R = \mathfrak{p}$  and  $ht P = ht \mathfrak{p} \leq s$  (as can be seen by localising at  $R \setminus \mathfrak{p}$ ): in fact

$$R[It]_P = R_{\mathfrak{p}}[t]_{\mathfrak{p}[t]}.$$

Since  $\text{Proj } R[It]$  satisfies  $(R_s)$ ,  $R[It]_P$  is regular. Hence  $R_{\mathfrak{p}}$  is regular, and the full result follows.

(←) We prove a slightly stronger more flexible result, viz. we change the hypotheses on  $I$  to the following:

let  $I$  be a Cohen–Macaulay generic complete intersection ideal such that  $\mu(I) \leq ht(I) + 1$ , with  $I$  either a regular ideal or geometrically linked to a regular ideal.

Assume this to be the case from now on. (By [HUV, pp. 25–27],  $R[It]$  is Cohen Macaulay.) If  $\mu(I) = ht(I)$  then  $I$  is a complete intersection. On the other hand, if  $\mu(I) = ht(I) + 1$  then  $I$  is an almost complete intersection and so  $I$  is not a regular ideal (cf. [Ku]); hence, in this case,  $I$  is geometrically linked to a regular ideal. Analyzing this last situation, let  $\mu(I) = d$  (say) and let  $x_1, \dots, x_{d-1}$  be a regular sequence in  $I$  such that  $I' := (x_1, \dots, x_{d-1}) : I$  is regular, with  $I \sim I'$  a geometrical link; in particular, note that  $I'$  is a prime ideal minimal over  $(x_1, \dots, x_{d-1})$  and that  $I' \not\subseteq I$ . By a result of Kunz [Ku] (cf. [Va, 4.1.4]), there exists  $x_d$  such that  $I = (x_1, \dots, x_d)$ , so  $x_d \notin I'$ . Since  $I'$  is prime we deduce that

$$I' = (x_1, \dots, x_{d-1}) : x_d = (x_1, \dots, x_{d-1}) : x_d^2$$

and  $I$  is an almost complete intersection

We proceed by induction on  $s$ . If  $s = 0$ , then  $I$  cannot be geometrically linked to a regular ideal  $I'$  (say), since then  $\dim I' = 0$  and so  $I' = \mathfrak{m}$ , a contradiction. It follows that  $I = \mathfrak{m}$  and that  $R$  is regular (since  $I$  is a complete intersection). Hence  $\text{Proj } R[It]$  satisfies  $(R_0)$  in this case.

If  $I$  is a complete intersection so that  $I$  is here a regular ideal, then  $R$  itself is regular. In this case it is classical (cf. [HIO, 14.8]) that  $\text{Proj } R[It]$  is also regular and so satisfies  $(R_s)$  for all values of  $s$ . Hence we suppose from now on that  $I$  is an almost complete intersection.

Suppose that  $s > 0$  and that the result holds for smaller values of  $s$ . Let  $\mathfrak{p}$  be a relevant homogeneous prime in  $R[It]$  with  $ht(\mathfrak{p}) \leq s$  and let  $P = \mathfrak{p} \cap R$ . Now the hypotheses and the conclusion involve only local conditions, so that if  $P \neq \mathfrak{m}$  then by induction  $R[It]_{\mathfrak{p}}$  is regular, as required (note that if  $P \not\subseteq I$ , then  $R[It]_{\mathfrak{p}}$  is a localisation of  $R_P[t]$ , where  $ht(P) \leq s$ ). So suppose that  $P = \mathfrak{m}$ .

Now  $R[It] = S(I)$ , the symmetric algebra on  $I$ , and it follows almost immediately that  $\mathfrak{m}R[It]$  is a prime ideal of height  $s$ ; hence  $\mathfrak{p} = \mathfrak{m}R[It]$  (cf. [HUV]). An easy adaptation of the proof of [HUV, 3.2] shows that  $R[It]_{\mathfrak{p}}$  is regular, and the result follows.  $\square$

*Remark 5.* The latter half of Theorem 2 and its proof are heavily influenced by [HUV, 3.2 and 3.6 (b)].

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