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Special values of twisted symmetric square L-functions and the trace formula*

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Abstract. We use the trace formula to compute explicitly the trace, over a Hecke eigenbasis, of the algebraic part of the special values. The case of twisting holomorphic level one modular forms by a quadratic character modulo q is considered. The result involves both class numbers of binary quadratic forms with discriminant depending on q, and also the number of points on certain elliptic curves reduced modulo q.

0. Introduction

We let $K = \mathbb{Q}(\sqrt{q})$, with ring of integers $\mathcal{O}$ for an odd prime $q \equiv 1 \bmod 4$. We use χ to denote $(q/*)$, the quadratic character modulo q. The integral kernel for the base change lifting $f \to \tilde{f}$ from $\mathrm{SL}(2,\mathbb{Z})$ cusp forms to those of $\mathrm{SL}(2,\mathcal{O})$ is denoted $\Omega(\tau, z, z')$ (see [12]).

Integrating Ω against a Hilbert modular form F in the (z, z') variable gives a linear map from $S_k(\mathrm{SL}(2,\mathcal{O}))$ to $S_k(\mathrm{SL}(2,\mathbb{Z}))$. Its easy to see this is the adjoint of the lift Ω, so we denote this linear map Ω^*. What can be said about this map? From a version of the Strong Multiplicity One theorem due to Ramakrishnan, we get that the lift is 1-1, and from this follows

EASY LEMMA. *Every Hecke eigenform f is also an eigenform of the map $\Omega^*\Omega$, with eigenvalue equal to $\langle \tilde{f}, \tilde{f}\rangle / \langle f, f\rangle$. If F is a Hilbert modular eigenform which is not a lift, then $\Omega^* F = 0$.*

As we will show below

$$\frac{\langle \tilde{f}, \tilde{f}\rangle}{\langle f, f\rangle} = \beta(f)\langle f, f\rangle, \qquad \beta(f) \in \mathbb{Q}(f).$$

(Zagier did the analogous result for forms of nebentypus in [12], Corollary 1 to Theorem 5 by a different method.) Thus the map B, defined on eigenforms by

$$Bf = \langle f, f\rangle^{-1}\Omega^*\Omega f = \beta(f) f, \tag{0.1}$$

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is in the Hecke algebra. In this paper we will use the trace formula to get an explicit computation of the trace of B, and more generally the trace of $T(n)B$ for any Hecke operator $T(n)$. The trace is a sum over l of class numbers $H(q(l^2 - 4q^2n))$ and $H(q(l^2 - 4n))$ of orders in complex quadratic fields. These class numbers are weighted, respectively, by the character $\chi(l)$ and by the number of points

$$a(l,n) = \sharp E - q - 1, \qquad E/\mathbb{F}_q\colon y^2 = x^3 + lx^2 + nx. \tag{0.2}$$

0.1. Why is this interesting?

The method of computing the trace of the period is one Zagier suggested in Section 5 of [12]. He did the analogous trace formula for forms of level D, quadratic character in Section 6 of that paper; including the technically much more difficult case of weight 2 forms. (This is implicit in formulae (91) and (98). All that is missing is to subtract off an appropriate multiple of an Eisenstein series to get a cusp form.) His interest was in connection with intersection numbers of cycles on the Hilbert modular surface. Gordon [3] has considered higher weight analogs of intersection numbers; presumably $\beta(f)$ has an interpretation in terms of these intersection numbers.

The value at $s = k$ (in the standard normalization) of the twisted symmetric square L-function $D(s, f, \chi)$ is $\beta(f)\langle f, f\rangle$. This is also the residue of the corresponding Asai L-function $L(s, \tilde{f}, \rho \otimes \tilde{\chi})$. By the work of Harder, Langlands and Rapoport, the Asai L-functions occur as factors of the Hasse–Weil zeta function of the Hilbert modular surface $\mathcal{H}^2/\mathrm{SL}(2, \mathcal{O})$ (for weight 2 forms). There are lots of conjectures on the arithmetic significance of the special values of such zeta functions, in the context of higher K-theory and regulators for algebraic varieties. See [8] and [9] for an exposition.

Notation. The prime $q \equiv 1 \bmod 4$ is fixed throughout, as is a positive integer n indexing a Fourier coefficient. The weight k of $\mathrm{SL}(2,\mathbb{Z})$ cusp forms is fixed throughout, but k is also used sometimes as a subscript in sums. $\delta(*)$ is 0 if the argument is not an integer, and $e(*)$ denotes $\exp(2\pi i *)$. Δ denotes a typical non-primitive discriminant, either $l^2 - 4n$ or $l^2 - 4q^2n$, written as Df^2 with D primitive. Subscripts q on L-functions indicate local factors at q omitted, while superscripts L^q are those local factors.

1. Special values of L-series

It will be very useful to view the constant $\beta(f)$ of (0.1) in terms of L-series. Write

$$f(\tau) = \sum_n a(n) n^{(k-1)/2} \exp(2\pi i n \tau).$$

and let $f_\chi(\tau)$ be the quadratic twist (with level q^2 and trivial character.) Note we have a non-classical normalization of the Fourier coefficients. Letting $\alpha + \alpha' = a(p)$

and $\alpha\alpha' = 1$, we write $L(s, f\times f)$ and $L(s, f\times f_\chi)$ as degree four Euler products; a computation then gives the splitting formula $L(s, f\times f)L(s, f\times f_\chi) = L(s, \tilde{f}\times\tilde{f})$. The poles of the Eisenstein series produce poles in the Rankin convolutions

$$\frac{\Gamma(k)\mathrm{res}_{s=1}L(s, f\times f)}{(2\pi)^2} = (4\pi)^{k-1}\langle f, f\rangle,$$

$$\frac{q^2\Gamma(k)^2\mathrm{res}_{s=1}L(s, \tilde{f}\times\tilde{f})}{(2\pi)^4} = (4\pi)^{2k-2}\langle\tilde{f}, \tilde{f}\rangle\mathrm{res}_{s=1}\zeta_K(s),$$

so $L(s, f\times f_\chi)$ is entire at $s = 1$ and

$$\frac{q^2\Gamma(k)L(1, f\times f_\chi)}{(2\pi)^2} = (4\pi)^{k-1}\frac{\langle\tilde{f}, \tilde{f}\rangle}{\langle f, f\rangle}\mathrm{res}_{s=1}\zeta_K(s).$$

We can also consider the twisted symmetric square L function

$$D(s, f, \chi) = \prod_p (1 - \chi(p)\alpha^2 p^{-s})^{-1}(1 - \chi(p)p^{-s})^{-1}(1 - \chi(p)\alpha'^2 p^{-s})^{-1}.$$

Thus $L(s, \chi)D(s, f, \chi) = L(s, f\times f_\chi)$, and by the above

$$D(1, f, \chi) = C_k\frac{\langle\tilde{f}, \tilde{f}\rangle}{\langle f, f\rangle},$$

where $C_k = (2\pi)^2(4\pi)^{k-1}(q^2\Gamma(k))^{-1}$. We can build a cusp form $\Phi_s(\tau)$ depending also on s which satisfies

$$\langle\Phi_s, f_i\rangle = C_k^{-1}D(s, f_i, \chi),$$

for each eigenform f_i. One the one hand, the eigenforms give an orthogonal basis, so

$$\Phi_s(\tau) = C_k^{-1}\sum_i \frac{D(s, f_i, \chi)}{\langle f_i, f_i\rangle}f_i(\tau).$$

This implies that the Fourier expansion of $\Phi_s(\tau)$ looks like

$$\Phi_s(\tau) = C_k^{-1}\sum_{n=1}^{\infty}\left\{\sum_i \frac{a_i(n)n^{(k-1)/2}D(s, f_i, \chi)}{\langle f_i, f_i\rangle}\right\}\exp(2\pi i n\tau).$$

Plugging in $s = 1$ and using the above gives

$$\Phi_1(\tau) = \sum_{n=1}^{\infty}\left\{\sum_i \frac{a_i(n)n^{(k-1)/2}\langle\tilde{f}_i, \tilde{f}_i\rangle}{\langle f_i, f_i\rangle^2}\right\}\exp(2\pi i n\tau).$$

On the other hand we can define the usual the Poincare series $G_r(\tau)$ which satisfies

$$\langle G_r, f\rangle = \frac{\Gamma(k-1)}{(4\pi)^{k-1}}a(r)r^{(1-k)/2},$$

and use the identity (formula (0.2) of [10])

$$D(s,f,\chi)=\zeta(2s)_q\sum_{m=1}^{\infty}\chi(m)a(m^2)m^{-s}.$$

This gives

$$\Phi_s(\tau)=\frac{(k-1)q^2}{(2\pi)^2}\zeta(2s)_q\sum_{m=1}^{\infty}\chi(m)m^{-s+k-1}G_{m^2}(\tau). \tag{1.1}$$

The series converges absolutely and uniformly on compact subsets of $\{\mathrm{re}(s)>1\}\times\mathcal{H}$. Writing out the Fourier expansion of the Poincare series $G_{m^2}(\tau)$, interchanging the order of summation, and continuing to $s=1$ will give an explicit computation of the Fourier series coefficients

$$\Phi_s(\tau)=\sum_{n=1}^{\infty}b(n,s)\exp(2\pi in\tau), \tag{1.2}$$

and $b(n,1)$ gives an explicit computation of the (algebraic) expression

$$\sum_i\frac{a_i(n)n^{(k-1)/2}\langle\tilde{f}_i,\tilde{f}_i\rangle}{\langle f_i,f_i\rangle^2}=\sum_i a_i(n)n^{(k-1)/2}\beta(f_i)=\mathrm{trace}(T(n)B).$$

2. Poisson summation

In computing the Fourier expansion of $\Phi_s(\tau)$, we will suppose first $1<\mathrm{re}(s)<k-1$ and then find analytic continuation to include $\mathrm{re}(s)=1$. It is well known the Fourier expansion of the Poincare series $G_r(\tau)$ is

$$\sum_{n=1}^{\infty}\delta_{r,n}+2\pi(-1)^{k/2}\left(\frac{n}{r}\right)^{(k-1)/2}\left\{\sum_{c=1}^{\infty}\frac{1}{c}K_c(r,n)J_{k-1}\left(\frac{4\pi\sqrt{rn}}{c}\right)\right\}\mathrm{e}(n\tau).$$

Plugging this into (1.1) with $r=m^2$ we get

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\chi(m)m^{-s+k-1}\left(\delta_{m^2,n}+2\pi(-1)^{k/2}\left(\frac{n}{m^2}\right)^{(k-1)/2}\right.$$
$$\left.\times\left\{\sum_{c=1}^{\infty}\frac{1}{c}K_c(m^2,n)J_{k-1}\left(\frac{4m\pi\sqrt{n}}{c}\right)\right\}\right)\mathrm{e}(n\tau).$$

Letting

$$S=2\sum_{c=1}^{\infty}\sum_{m=1}^{\infty}\frac{\chi(m)}{c}m^{-s}K_c(m^2,n)J_{k-1}\left(\frac{4m\pi\sqrt{n}}{c}\right),$$

we see the nth Fourier coefficient $b(n,s)$ of $\Phi_s(\tau)$ in (1.2) satisfies

$$\frac{(2\pi)^2}{(k-1)q^2\zeta(2s)_q} b(n,s) = \pi(-1)^{k/2} n^{(k-1)/2} S + \delta(\sqrt{n})\chi(\sqrt{n}) n^{(-s+k-1)/2}.$$

PROPOSITION. *The 'main term', S is equal*

$$\sum_{l\in\mathbb{Z}} \sum_{(c,q)=1} c^{-1}(cq)^{-s} I(l, nq^2, s) \left\{ \sum_{r(cq)} \chi(r) K_c(r^2, n) e\left(\frac{lr}{cq}\right) \right\}$$

$$+ \sum_{a=1}^{\infty} (cq^a)^{-1-s} I(l,n,s) \left\{ \sum_{r(cq^a)} \chi(r) K_{cq^a}(r^2,n) e\left(\frac{lr}{cq^a}\right) \right\}. \tag{2.1}$$

where the special function $I(l,n,s)$ is defined by (2.2), *and $K_c(r^2,n)$ is a Kloosterman sum.*

Proof. Put

$$A(x) = |x|^{-s} J_{k-1}\left(\frac{4\pi\sqrt{n}|x|}{c}\right), \qquad A(0) = 0,$$

then S equals

$$\sum_{c=1}^{\infty} \sum_{m\in\mathbb{Z}} \frac{\chi(m)}{c} K_c(m^2,n) A(m).$$

Before applying Poisson summation we need to relate the modulus q of χ to the modulus c of the Kloosterman sum K_c. We first break the sum on c into a double sum of c prime to q and a sum on prime powers q^a

$$\sum_{(c,q)=1} \sum_{a=0}^{\infty} \sum_{m\in\mathbb{Z}} (cq^a)^{-1} \chi(m) K_{cq^a}(m^2,n) A(m).$$

Now $\chi(m) K_{cq^a}(m^2,n)$ depends only on m modulo $cq^{a+\varepsilon}$ with

$$\varepsilon = \begin{cases} 1 & \text{if } a = 0, \\ 0 & \text{if } a \geqslant 1. \end{cases}$$

We can then replace the sum on m with a double sum over r modulo $cq^{a+\varepsilon}$ and m in $\mathbb{Z}$, $m \equiv r$. We then consider

$$\sum_{m\equiv r} A(m) = (cq^{a+\varepsilon})^{-s} \sum_{j\in\mathbb{Z}} \left| j + \frac{r}{cq^{a+\varepsilon}} \right|^{-s} J_{k-1}\left(4\pi\sqrt{n}q^{\varepsilon} \left| j + \frac{r}{cq^{a+\varepsilon}} \right|\right),$$

and let

$$B(x) = \sum_{j\in\mathbb{Z}} |j+x|^{-s} J_{k-1}(4\pi\sqrt{n}q^{\varepsilon}|j+x|).$$

Write the Fourier expansion of $B(x)$ as $\Sigma_l c_l e(lx)$. By Poisson summation we have

$$c_l = \int_{-\infty}^{\infty} |x|^{-s} J_{k-1}(4\pi\sqrt{n}q^{\varepsilon}|x|)e(-lx)\,dx = (\text{definition})\ I(l, nq^{2\varepsilon}, s). \qquad (2.2)$$

Thus

$$\begin{aligned}\sum_{m\equiv r} A(m) &= (cq^{a+\varepsilon})^{-s} B\left(\frac{r}{cq^{a+\varepsilon}}\right) \\ &= (cq^{a+\varepsilon})^{-s} \sum_{l\in\mathbb{Z}} I(l, nq^{2\varepsilon}, s) e\left(\frac{lr}{cq^{a+\varepsilon}}\right).\end{aligned}$$

which proves the proposition. □

3. Character sums

We will work towards getting the character sums in braces in (2.1) into a closed form, to realize the nth Fourier coefficient $b(n, s)$ as an infinite sum over l of Dirichlet series times the special functions $I(l, n, s)$. Consider first when $a \geqslant 1$

$$\sum_{r(cq^a)} \chi(r) \sum_{(x,cq^a)=1} \mathrm{e}\left(\frac{r^2x^{-1} + nx}{cq^a}\right) \mathrm{e}\left(\frac{lr}{cq^a}\right). \qquad (3.1)$$

After interchanging the sums and changing variables $r \to rx$, one sees the character sum depends on the behavior of the two counting functions (choosing either + or − and fixed t)

$$\sharp\left\{r \bmod cq^a \mid \chi(r) = \pm 1 \quad \text{and} \quad r^2 + lr + n \equiv t \bmod cq^a\right\},$$

and particularly their difference. Each of these counting functions can be written as a product of a term depending only on c and a term depending only on a. In particular the former can be written

$$N(t, x^2 + lx + n, c) = \sharp\left\{r \bmod c \,|\, r^2 + lr + n \equiv t \bmod c\right\},$$

while for the latter, the relevant term is the difference

$$\begin{aligned}R(t, q^a) &= \sharp\left\{r \bmod q^a | \, \chi(r) = +1 \quad \text{and} \quad r^2 + lr + n \equiv t \bmod q^a\right\} \\ &\quad -\sharp\left\{r \bmod q^a | \, \chi(r) = -1 \quad \text{and} \quad r^2 + lr + n \equiv t \bmod q^a\right\},\end{aligned}$$

and in this case the dependence on the quadratic is suppressed in the notation. By direct computation one sees that the character sum (3.1) simplifies to

$$\begin{aligned}&cq^{a-1/2} \sum_{d|c} \chi\left(\frac{c}{d}\right) \mu\left(\frac{c}{d}\right) \chi(d) N(0, x^2 + lx + n, d) \\ &\quad \times \sum_{j=1}^{q} \chi(j) R(jq^{a-1}, q^a).\end{aligned}$$

The character sum $\Sigma_{j=1}^{q}\chi(j)R(j,q)$ is equal $\sharp E-q-1$, where $\sharp E$ is the number of projective points over $\mathbb{Z}/q\mathbb{Z}$ of the curve E: $y^2 = x^3+lx^2+nx$ with discriminant $(4n)^2(l^2-4n)$. We denote this character sum $a(l,n)$. In particular if the curve is singular, the character sum is $-\chi(l/2)$. For the general a, it is convenient to introduce the generating function $F_{l,n}(T)$ defined by

$$\sum_{a=1}^{\infty}\sum_{j=1}^{q}\chi(j)R(jq^{a-1},q^a)T^a =$$

$$a(l,n)\times\begin{cases} T & \text{if } q\nmid\Delta \\ T-\sum_{\beta=1}^{\alpha-1}(q-1)q^{\beta}T^{2\beta+1}+q^{\alpha}T^{2\alpha+1} & \text{if } q\nmid D, \alpha>0 \\ T-\sum_{\beta=1}^{\alpha}(q-1)q^{\beta}T^{2\beta+1}+\chi\left(\frac{D}{q}\right)q^{\alpha+1}T^{2\alpha+2} & \text{if } q\mid D \\ T-\sum_{\beta=1}^{\infty}(q-1)q^{\beta}T^{2\beta+1} & \text{if } \Delta=0. \end{cases}$$

Here the discriminant of the quadratic l^2-4n is written $\Delta = Df^2$ with D a fundamental discriminant. In the first three cases $q^{\alpha}\parallel f$, while in the last three cases the factor $a(l,n)$ reduces to $-\chi(l/2)$. By summing the geometric series one can write this in the alternative form (with $\chi=\chi(D/q)$ when $q\mid D$)

$$F_{l,n}(T)=\frac{a(l,n)T}{1-qT^2}\times\begin{cases} 1-qT^2, \\ 1-q^2T^2+q^{\alpha+1}T^{2\alpha}(1-T^2), \\ 1-q^2T^2-\chi q^{\alpha+1}T^{2\alpha+1}(1+\chi T)(1-\chi qT), \\ 1-q^2T^2. \end{cases} \tag{3.2}$$

For the other sum in braces in (2.1), coming from '$a=0$', similar methods show that

$$\sum_{r(cq)}\chi(r)K_c(r^2,n)e\left(\frac{lr}{cq}\right)$$
$$=\chi(l)cq^{1/2}\sum_{d\mid c}\chi\left(\frac{c}{d}\right)\mu\left(\frac{c}{d}\right)\chi(d)N(0,qx^2+lx+qn,d). \tag{3.3}$$

4. Zeta functions

We now sum the Dirichlet series

$$\sum_{(c,q)=1}c^{-s}\sum_{d\mid c}\chi\left(\frac{c}{d}\right)\mu\left(\frac{c}{d}\right)\chi(d)N(0,q^{\varepsilon}x^2+lx+q^{\varepsilon}n,d), \tag{4.1}$$

where ε is 0 or 1. This is a convolution of two Dirichlet series, so we have

$$L(s,\chi)^{-1}\sum_{c=1}^{\infty}\chi(c)N(0,q^{\varepsilon}x^2+lx+q^{\varepsilon}n,c)c^{-s}.$$

We will write $N_{c,\Delta}$ instead of $N(0,q^{\varepsilon}x^2+lx+q^{\varepsilon}n,c)$, where $\Delta=l^2-4q^{2\varepsilon}n$ is the discriminant. Elementary considerations tell us that in this case $N_{c,\Delta}$ is multiplicative. And for $k>0$, $N_{p^k,\Delta}=N_{p,\Delta}=1+(\Delta/p)$ whenever the discriminant Δ is prime to p. More generally write $\Delta=Df^2$ with D the discriminant of $\mathbb{Q}(\sqrt{\Delta})$. One can compute

$$L(s,\chi)^{-1}\sum_{c=1}^{\infty}\chi(c)N_{c,\Delta}c^{-s}$$
$$=\begin{cases}\dfrac{L(s,(D/*)\chi)}{\zeta(2s)_q}\displaystyle\sum_{\substack{c|f\\(c,q)=1}}\mu(c)\left(\frac{D}{c}\right)\chi(c)c^{-s}\sigma_{1-2s}(fc^{-1})\\[2ex] \dfrac{\zeta(2s-1)_q}{\zeta(2s)_q},\end{cases}$$

if $\Delta\neq 0$ (resp. $\Delta=0$).

The term $\zeta(2s)_q^{-1}$ in the lemma will be canceled out by the corresponding $\zeta(2s)_q$ in (1.1). What remains will simplify if for each $\Delta=Df^2$, we write $q\Delta=D'f'^2$. The term on the right side above is very nearly the one associated to the non-fundamental discriminant $q\Delta$

$$L(s,q\Delta)$$
$$=\text{(definition) } L\left(s,\left(\frac{D'}{*}\right)\right)\sum_{c|f'}\mu(c)\left(\frac{D'}{c}\right)c^{-s}\sigma_{1-2s}(f'c^{-1}),\qquad(4.2)$$

the only local factor missing is the one at q, $L^q(s,q\Delta)$. If $(q,\Delta)=1$, this local factor is 1. We can always assume this is the case for Δ of the form l^2-4q^2n, since otherwise $q\mid\Delta$ implies $q\mid l$. The relevant character sum (3.3) is then zero; these terms will disappear from the trace. In the other case $\Delta=l^2-4n$; the missing Euler factor will be obtained from the sum on a in (2.1). We introduce a fudge factor $\Gamma(s,\Delta)$ (which turns out to be related to the q-adic Γ function) so that

$$F_{l,n}(q^{-s})=a(l,n)q^{-s}\Gamma(s,\Delta)L^q(s,q\Delta).\qquad(4.3)$$

Then (3.2) implies

(i) If $(q,\Delta)=1$, $\Gamma(s,\Delta)=1$ for all s.
(ii) If $q|D$ and $\chi(Dq^{-1})=1$, then $\Gamma(s,\Delta)$ vanishes at $s=1$.
(iii) If $\Delta=0$, $\Gamma(s,\Delta)=1-q^{2-2s}$ vanishes at $s=1$.

5. Special functions

After a change of variables the special function

$$I(l,n,s) = 2^s \pi^{s-1} \int_0^\infty x^{-s} J_{k-1}(2\sqrt{n}x) \cos(|l|x)\, \mathrm{d}x,$$

is found in the tables to have an analytic continuation to $\frac{1}{2} < \mathrm{re}(s) < k$. For example, if n is a square, Δ will be 0 when $l^2 = 4n$ or $l^2 = 4q^2 n$, and we see from [1], vol. 2 (19.2.24) on p. 342 that $I(2\sqrt{n}, n, s)$ is a quotient of Gamma functions, and has a simple zero at $s = 1$.

The general case of this special function is found in [4], (6.561.14) on p. 684 and (6.699.2) on p. 747, in terms of hypergeometric functions. In particular if $l^2 > 4n$, [2] vol. 2, 2.8 (6) shows that $I(l,n,s)$ has a simple zero at $s = 1$. If $4n > l^2$ we can get the value at $s = 1$ from [7], (1.13.12), p. 67

$$I(l,n,1) = (-1)^{(k-2)/2} \frac{(4n - l^2)^{1/2}}{(k-1)\sqrt{n}} C^1_{k-2}\left(\frac{|l|}{(\sqrt{4n})}\right). \tag{5.1}$$

Here C^1_{k-2} is a Gegenbauer polynomial.

6. Evaluate at $s = 1$

We return to consideration of the Fourier expansion. Combine the proposition in Section 2 with formula (4.1) and (4.3) to see that the Fourier coefficient $b(n,s)$ is written in closed form as

THEOREM.

$$\begin{aligned}
&\frac{(-1)^{k/2}(2\pi)^2 b(n,s)}{(k-1)q^{3/2}} \\
&\quad = \pi \sum_{l\in\mathbb{Z}} \chi(l) q^{(1-s)} n^{(k-1)/2} I(l, q^2 n, s) L(s, q(l^2 - 4q^2 n)) \\
&\qquad + \pi \sum_{l\in\mathbb{Z}} a(l,n) q^{-s} n^{(k-1)/2} I(l,n,s) L(s, q(l^2 - 4n)) \Gamma(s, l^2 - 4n) \\
&\qquad + \delta(\sqrt{n})(-1)^{k/2} \sqrt{q}\, \zeta(2s)_q \chi(\sqrt{n}) n^{(-s+k-1)/2}.
\end{aligned}$$

For the reader inclined to skim, we recall that $L(s, q(l^2 - 4n))$ is defined by formula (4.2) to be the Dirichlet series associated to a (non-fundamental) discriminant, and $a(l,n)$ is defined by (0.2). The special functions $I(l,n,s)$ are defined by (2.2), and $\Gamma(s, l^2 - 4n)$ by (4.3).

The infinite sum converges uniformly and absolutely in the strip $\frac{1}{2} < \mathrm{re}(s) < k - 1$, which by Fubini justifies changing the order of summation in Section 2.

This is as far as we will go with the variable s present; the expression will simplify when we evaluate at $s = 1$. In terms of the Eichler–Selberg trace formula, this is the 'Selberg principle' that the orbital integrals coming from hyperbolic conjugacy classes should not contribute to the trace.

PROPOSITION. *The terms in the theorem with* $\Delta \geqslant 0$ *all vanish at* $s = 1$.

Proof. The discriminant Δ can only be zero if n is a square. We observed in Section 5 that $I(2\sqrt{n}, n, s)$ has a simple zero at $s = 1$, while $L(s, 0)$ has a simple pole at $s = 1$. However if $l^2 - 4q^2 n$ is 0, then l is 0 modulo q and so $\chi(l) = 0$. And (4.3 (iii)) shows that $\Gamma(s, 0)$ has a simple zero at $s = 1$. This completes the $\Delta = 0$ case. We know from Section 5 that $I(l, n, 1) = 0$ when $\Delta = l^2 - 4n > 0$. These terms contribute nothing unless $L(s, q\Delta)$ has a pole at $s = 1$. This happens exactly when $\Delta = qf^2$ for some $f \neq 0$, then $L(s, q^2 f^2)$ is $\zeta(s)$ times a Dirichlet polynomial. If $l^2 - 4q^2 n = qf^2$, then $\chi(l) = 0$; these terms drop out. On the other hand $\Delta = l^2 - 4n = qf^2$ terms are in one to one correspondence with divisors of $n = \nu\nu'$, $\nu \in \mathcal{O}, \nu \notin \mathbb{Z}$. (There will be infinitely many such, if any.) But the simple zero of $I(l, n, s)$ will cancel the pole of $\zeta(s)$, and (4.3 (ii)) shows that $\Gamma(s, qf^2)$ has a simple zero at $s = 1$, since Dq^{-1} is a square in this case. □

There are finitely many l such that $\Delta < 0$. We use (5.1) in this case to evaluate the special function $I(l, n, s)$ at $s = 1$ in terms of Gegenbauer polynomials (see also [2], vol. 1, Sect. 3.15.1)

$$n^{(k-2)/2} C^1_{k-2}\left(\frac{|l|}{\sqrt{4n}}\right) = \frac{\rho^{k-1} - \bar{\rho}^{k-1}}{\rho - \bar{\rho}}, \qquad \rho + \bar{\rho} = |l|, \qquad \rho\bar{\rho} = n,$$

which is $P_{k,1}(l, n)$ in the notation of Zagier ([12] formula (18).) The value of $L(1, q\Delta) = \pi H(q\Delta)/\sqrt{|q\Delta|}$ is classical, $H(q\Delta)$ being the number of equivalence classes of binary quadratic forms ϕ of discriminant $q\Delta$ weighted by $1/\mathrm{Aut}(\phi)$. Equivalently, $H(q\Delta)$ is the sum of class numbers of orders of discriminant $q\Delta/f^2$ in $\mathbb{Q}(\sqrt{q\Delta})$.

THEOREM. *For the map* B *defined in* (0.1), *the trace of the Hecke operator* $T(n)B$ *is* $b(n, 1)$ *where*

$$\begin{aligned} b(n,1) = &-\frac{1}{4}\sum_{l^2<4q^2n} \chi(l) q^{2-k} P_{k,1}(l, q^2 n) H(q(l^2 - 4q^2 n)) \\ &-\frac{1}{4}\sum_{l^2<4n} a(l,n) P_{k,1}(l,n) H(q(l^2-4n))\Gamma(1, l^2-4n) \\ &+\delta(\sqrt{n})\chi(\sqrt{n}) n^{(k-2)/2}\frac{k-1}{24}(q^2-1). \end{aligned}$$

COROLLARY. *Let* $\beta(f) = \langle \tilde{f}, \tilde{f}\rangle / \langle f, f\rangle^2$, *then* $\beta(f) \in \mathbb{Q}(f)$.

Proof. This follows from the fact that the Fourier coefficients $b(n, 1)$ of $\Phi_1(\tau)$ are rational in the theorem above, and Lemma 4, p. 792 of [11]. □

7. Examples

To convince ourselves the computations above are correct, we did some examples using *Mathematica*. In the case of weight $k = 10$, there are no cusp forms so $\Phi_s(\tau) = 0$ for all s. With $q = 5$ we verified $b(n, 1) = 0$ for $1 \leqslant n \leqslant 20$. In the case of weight $k = 12$, the space of cusp forms is spanned by the discriminant cusp form, so the Fourier coefficient $b(n, 1) = \tau(n)b(1, 1)$, with $\tau(n)$ Ramanujan's tau function, and again the above relation holds for $1 \leqslant n \leqslant 20$. The coefficient $b(1, 1) = \beta$ was then computed (still weight $k = 12$) for some small primes q

$$\begin{array}{ll} q = 5 & \beta = 2^{12} \cdot 3^6 \cdot 7 \,/\, 5^8, \\ q = 13 & \beta = 2^{12} \cdot 3^9 \cdot 5^3 \cdot 7 \cdot 563 \,/\, 13^{10}, \\ q = 17 & \beta = 2^{19} \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 2389 \,/\, 17^{10}, \\ q = 29 & \beta = 2^{12} \cdot 3^{10} \cdot 5^4 \cdot 7^2 \cdot 13 \cdot 17 \cdot 317 \,/\, 29^{10}, \\ q = 37 & \beta = 2^{12} \cdot 3^{10} \cdot 5^6 \cdot 7 \cdot 89 \cdot 3889 \,/\, 37^{10}, \\ q = 41 & \beta = 2^{23} \cdot 3^6 \cdot 5^4 \cdot 7^2 \cdot 117413 \,/\, 41^{10}. \end{array}$$

The formula for $b(n, s)$ also reduces to a finite sum for $s = 3, 5, \ldots, k-1$, with a different Gegenbauer polynomial and Cohen's function instead of the Hurwitz–Kronecker class number. The computations are analogous to those in [12]. In this case the Dirichlet series $D(s, f, \chi)$ converges absolutely, and we computed the first 100 terms of the series with $s = 3, 5, 7$

$$\frac{D(3, f, \chi)}{C_k\langle f, f\rangle} \approx 22.5795 \qquad b(1,3) = \frac{2^{12} \cdot 3^2 \cdot 7 \cdot 2851}{5^{12} \cdot 13}\pi^4 \approx 22.5794729896,$$

$$\frac{D(5, f, \chi)}{C_k\langle f, f\rangle} \approx 20.30838094 \qquad b(1,5) = \frac{2^{15} \cdot 3 \cdot 1511599}{5^{17} \cdot 7 \cdot 13}\pi^8 \approx 20.3083809367,$$

$$\frac{D(7, f, \chi)}{C_k\langle f, f\rangle} \approx 19.903417716 \qquad b(1,7) = \frac{2^{13} \cdot 3^2 \cdot 521 \cdot 295387}{5^{22} \cdot 13 \cdot 17}\pi^{12} \approx 19.9034177155.$$

Here again $q = 5$ and $k = 12$. The value $1.03536205679 \times 10^{-6}$ for the square of the norm of the discriminant function was taken from [12].

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