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On Siegel modular forms

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1. Introduction and statement of result

Let F be a holomorphic cusp form of integral weight k on the Siegel modular group $\Gamma = \mathrm{Sp}_g(\mathbb{Z})$ of genus g and denote by $a(T)$ (T a positive definite symmetric even integral (g, g) -matrix) its Fourier coefficients.

If $g = 1$ and $k \geq 2$, then by Deligne's theorem (previously the Ramanujan-Petersson conjecture) one has

$$a(T) \ll_{\varepsilon, F} T^{k/2-1/2+\varepsilon}, \quad (\varepsilon > 0),$$

and since by [10]

$$\limsup_{T \rightarrow \infty} |a(T)|/T^{k/2-1/2} = \infty,$$

this bound is best possible.

For arbitrary $g \geq 2$ our knowledge of how to obtain good bounds for the coefficients $a(T)$ in terms of $\det(T)$ is still extremely limited. For $g \geq 2$ and $k > g + 1$ Böcherer and the author in [4] proved that

$$a(T) \ll_{\varepsilon, F} (\det(T))^{k/2-\delta_g+\varepsilon}, \quad (\varepsilon > 0), \quad (1)$$

where

$$\delta_g := \frac{1}{2g} + \left(1 - \frac{1}{g}\right) \frac{1}{4(g-1) + 4[(g-1)/2] + 2/(g+2)}.$$

The bound (1) for arbitrary g seems to be the best one known so far. Note, however, that for $g \rightarrow \infty$ it is still of the same order of magnitude as Hecke's bound $a(T) \ll_F (\det(T))^{k/2}$.

In the present paper we shall prove

THEOREM. *Suppose that $4/g$. Then there exists $\kappa = \kappa(g) \in \mathbb{N}$ with the following property: for each $N \in \mathbb{N}$ there is an integer $k \in \{N, N+1, \dots, N+\kappa-1\}$ and a non-zero cusp form F of weight k on Γ_g whose Fourier coefficients $a(T)$ satisfy*

$$a(T) \ll_{\varepsilon, F} (\det(T))^{k/2-1/2+\varepsilon}, \quad (\varepsilon > 0). \quad (2)$$

The proof of the Theorem will be given in Section 2. The functions F will be constructed as theta series attached to a positive definite quadratic form of rank $2g$ with certain harmonic forms. For some general comments we refer the reader to Section 3.

NOTATION. If A and B are real resp. complex matrices of appropriate sizes we put $A[B] := B'AB$ resp. $A\{B\} := \bar{B}'AB$; here B' is the transpose of B .

If S is a real symmetric matrix we write $S \geq 0$ resp. $S > 0$ if S is positive semi-definite resp. positive definite. If S is real of size m and $S > 0$, we denote by $S^{1/2}$ the unique real symmetric positive definite matrix of size m satisfying $(S^{1/2})^2 = S$.

2. Proof of theorem

For $\nu, m \in \mathbb{N}$ denote by $H_\nu(m, g)$ the \mathbb{C} -linear space of harmonic forms $P: \mathbb{C}^{(m,g)} \rightarrow \mathbb{C}$ of degree ν , i.e. of polynomial functions $P(X) (X = (x_{ij})_{\substack{1 \leq i \leq m \\ 1 \leq j \leq g}})$ which satisfy $P(XU) = (\det(U))^\nu P(X)$ for all $U \in \text{Gl}_g(\mathbb{C})$ and which are annihilated by the Laplace operator $\sum_{i,j} (\partial^2 / \partial x_{ij}^2)$ [6, 8]. For $m \geq 2g$ the space $H_\nu(m, g)$ is generated by the forms $(\det(L'X))^\nu$ where L is a complex (m, g) -matrix with $L'L = 0$ [8].

Let S be a fixed positive definite symmetric even integral matrix of size m with determinant 1 (such a matrix exists if and only if $8|m$). Then the generalized theta series

$$\vartheta_{S,P}(Z) = \sum_{G \in \mathbb{Z}^{(m,g)}} P(S^{1/2}G) e^{\pi i \cdot \text{tr}(S[G]Z)},$$

($Z \in \mathbf{H}_g =$ Siegel upper half-space of degree g)

is a cusp form of weight $m/2 + \nu$ on Γ_g [5, Kap. II, Sect. 3].

We now specialize to the case $m = 2g$ (supposing $4|g$). Take $L = \begin{pmatrix} E \\ iE \end{pmatrix}$ where E is the unit matrix of size g and define

$$P_\nu(X) := (\det(L'X))^\nu, \quad (\nu \in \mathbb{N}).$$

We write

$$a_\nu(T) = \sum_{\substack{G \in \mathbb{Z}^{(2g,g)} \\ S[G]=T}} (\det((E \ iE)S^{1/2}G))^\nu, \quad (T > 0) \tag{3}$$

for the Fourier coefficients of ϑ_{S,P_ν} .

Put

$$H := T + iR$$

where

$$R := J[S^{1/2}G], \quad J := \begin{pmatrix} 0 & -E \\ E & 0 \end{pmatrix}$$

and observe that R is skew-symmetric and H is hermitian.

From

$$\begin{pmatrix} E \\ iE \end{pmatrix} (E - iE) = E + iJ, \quad T = S[G]$$

we see that

$$H = G'S^{1/2} \begin{pmatrix} E \\ iE \end{pmatrix} \cdot \left(G'S^{1/2} \begin{pmatrix} E \\ iE \end{pmatrix} \right)';$$

in particular

$$\det(H) = |\det((E \ iE)S^{1/2}G)|^2. \tag{4}$$

Choose a unitary matrix $U \in \text{GL}_g(\mathbb{C})$ such that $iR\{T^{-1/2}U\} = D$ is a real diagonal matrix with diagonal entries the eigenvalues of the hermitian matrix $iR[T^{-1/2}]$. As $R[T^{-1/2}]$ is real, these eigenvalues occur in pairs $\pm\alpha_\nu$ ($\nu = 1, \dots, g/2$).

From

$$H\{T^{-1/2}U\} = E + D$$

we find that

$$\det(H\{T^{-1/2}U\}) = \prod_{\nu=1}^{g/2} (1 - \alpha_\nu^2) \leq 1,$$

hence by (4) we obtain

$$\begin{aligned} |\det((E \ iE)S^{1/2}G)|^2 &= \det(H\{T^{-1/2}U\}\{U^{-1}T^{1/2}\}) \\ &\leq \det(E\{U^{-1}T^{1/2}\}) \\ &= \det(T). \end{aligned}$$

From (3) we now infer that

$$|a_\nu(T)| \leq (\det(T))^{\nu/2} \cdot r_S(T), \tag{5}$$

where

$$r_S(T) := \#\{G \in \mathbb{Z}^{(2g,g)} \mid S[G] = T\}$$

is the number of representations of T by S .

Denote by

$$r_S^*(T) := \#\{G \in \mathbb{Z}^{(2g,g)} \mid G \text{ primitive, } S[G] = T\},$$

the number of primitive representations of T by S . By elementary divisor theory we have

$$r_S(T) = \sum_{\substack{D \in \text{GL}_g(\mathbb{Z}) \setminus \mathbb{Z}_*^{(g,g)} \\ T[D^{-1}] > 0 \text{ even integral}}} r_S^*(T[D^{-1}])$$

where $\mathbb{Z}_*^{(g,g)}$ denotes the set of integral (g, g) -matrices of rank g .

Let S_1, \dots, S_h ($h = h(2g)$) be a set of representatives of classes of (the genus of) unimodular positive definite symmetric even integral $(2g, 2g)$ -matrices and $\varepsilon(S_\mu)$ ($\mu = 1, \dots, h$) be the number of units of S_μ . Then according to the ‘primitive’ version of Siegel’s main theorem on quadratic forms [11] one has

$$\begin{aligned} & \left(\sum_{\mu=1}^h \frac{1}{\varepsilon(S_\mu)} \right)^{-1} \sum_{\mu=1}^h \frac{1}{\varepsilon(S_\mu)} r_{S_\mu}^*(T) \\ &= c_g \cdot (\det(T))^{g-(g+1)/2} \cdot \prod_p \alpha_{p,T}^*, \end{aligned}$$

where c_g is a constant depending only on g , and where the $\alpha_{p,T}^*$ are certain local densities which satisfy

$$\prod_{p \nmid \det(T)} \alpha_{p,T}^* \ll 1$$

and

$$\alpha_{p,T}^* \leq 2, \quad (p \mid \det(T))$$

(cf. [1; Sect. 2, esp. (2.6), (2.7b), (2.7d); Sect. 3] in combination with [2; 1.1ff.], and [7; Sect. 6.8, Thm. 6.8.1, iii]); note that in the product in formulas (2.7b) and (2.7d) in [1] the index j should start with 0 (rather than 1) and that the algebraic calculations given in [1; Sect. 3] remain valid also without the assumption on the weight imposed there).

We therefore conclude that

$$r_S^*(T) \ll_\varepsilon (\det(T))^{(g-1)/2+\varepsilon}, \quad (\varepsilon > 0),$$

hence that

$$r_S(T) \ll_\varepsilon (\det(T))^{(g-1)/2+\varepsilon} \sum_{\substack{D \in \text{GL}_g(\mathbb{Z}) \setminus \mathbb{Z}_*^{(g,g)} \\ T[D^{-1}] > 0 \text{ even integral}}} \frac{1}{|\det(D)|^{g-1+2\varepsilon}}, \quad (\varepsilon > 0). \tag{6}$$

The condition $T[D^{-1}]$ integral implies that $(\det(D))^2 \mid \det(T)$. Hence the sum on the right of (6) is majorized by

$$\sum_{d^2 \mid \det(T)} \alpha_g(d) / d^{g-1+2\varepsilon},$$

where for any $n \in \mathbb{N}$ we have put

$$\alpha_n(d) := \#\{D \in \text{GL}_n(\mathbb{Z}) \setminus \mathbb{Z}_*^{(n,n)} \mid |\det(D)| = d\}, \quad (d \in \mathbb{N}).$$

As is well-known and easy to see one has

$$\sum_{d \geq 1} \alpha_n(d) / d^s = \zeta(s)\zeta(s-1) \dots \zeta(s-n+1), \quad (\text{Re}(s) > n).$$

From the latter equality one easily checks by induction on n that

$$\alpha_n(d) \ll_\varepsilon d^{n-1+\varepsilon}.$$

Thus

$$\sum_{d^2 \mid \det(T)} \alpha_n(d) / d^{g-1+2\varepsilon} \ll_\varepsilon (\det(T))^\varepsilon,$$

and by (6) it follows that

$$r_S(T) \ll_\varepsilon (\det(T))^{(g-1)/2+\varepsilon}$$

for any $\varepsilon > 0$.

Together with (5) this implies

$$a_\nu(T) \ll_\varepsilon (\det(T))^{k/2-1/2+\varepsilon}, \quad (\varepsilon > 0),$$

where $k = g + \nu$ is the weight of ϑ_{S,P_ν} .

To complete the proof we proceed as follows. Suppose that ϑ_{S,P_ν} is identically zero for all $\nu \geq 1$, so

$$\sum_{\substack{G \in \mathbb{Z}^{(2g,g)} \\ S[G]=T}} (\det((E \ iE)S^{1/2}G))^\nu = 0 \tag{7}$$

for all $T \geq 0$ and all $\nu \geq 1$. Identity (7) implies that

$$\det((E \ iE)S^{1/2}G) = 0 \tag{8}$$

for all $G \in \mathbb{Z}^{(2g,g)}$; in fact, this follows from the well-known more general result that if $\sum_{n=1}^{\infty} c_n$ is an absolutely convergent series of complex numbers such that $\sum_{n=1}^{\infty} c_n^\nu = 0$ for all $\nu \in \mathbb{N}$, then $c_n = 0$ for all n .

By (4) and the definition of H , (8) is equivalent to

$$\det((S + iJ[S^{1/2}])[G]) = 0. \tag{9}$$

for all $G \in \mathbb{Z}^{(2g,g)}$. Since the left-hand side of (9) is a polynomial in the components of G , equality (9) must hold for all $G \in \mathbb{R}^{(2g,g)}$. Replacing G by $S^{-1/2}G$ we find

$$\det((E + iJ)[G]) = 0$$

for all G , a contradiction (take e.g. $G = \begin{pmatrix} G_1 \\ 0 \end{pmatrix}$ with G_1 invertible). Therefore there exists $\nu \in \mathbb{N}$ with $\vartheta_{S,P_\nu} \neq 0$ (of course, we could have also used the slightly different reasoning suggested by Maass, cf. [9, p. 154f.]).

Repeating the above argument with ν replaced by $N\nu$ where N is an arbitrary positive integer, we deduce inductively that there are infinitely many ν with $\vartheta_{S,P_\nu} \neq 0$.

To obtain the slightly stronger assertion of the Theorem, we follow Maass [9, loc. cit.]. Assume that $a_{\nu_0}(T_0) \neq 0$, say and denote by b_1, \dots, b_κ the distinct non-zero numbers of the form $\det((E \ iE)S^{1/2}G)$ as G runs over all $G \in \mathbb{Z}^{(2g,g)}$ with $S[G] = T_0$. Then there exist $n_1, \dots, n_\kappa \in \mathbb{N}$ such that

$$a_\nu(T_0) = \sum_{j=1}^{\kappa} n_j b_j^\nu$$

for all $\nu \geq 1$. Supposing that

$$a_N(T_0) = a_{N+1}(T_0) = \dots = a_{N+\kappa-1}(T_0) = 0,$$

we obtain $n_1 = n_2 = \dots = n_\kappa = 0$ (Vandermonde determinant), a contradiction.

3. Comments

We conclude the paper with a few general comments.

(i) Certainly the estimate (2) can be proved for the Fourier coefficients of theta series with more general harmonics than the special forms P_ν considered in Section 2, and eventually it would be true for all $P \in H_\nu(2g, g)$. However, we have not checked this, mainly for the following reason: for $\nu \rightarrow \infty$ the dimension of $H_\nu(2g, g)$ grows like $\nu^{g(g+1)/2-g}$ ([6], cf. also [3, formula XI.1]), while the

dimension of the space of cusp forms of weight $g + \nu$ on Γ_g grows like $(g + \nu)^{g(g+1)/2}$; hence there is no hope to eventually proving (2) for all cusp forms on Γ_g of weight $k \gg g$ by the method of this paper.

(ii) If in (2) one drops the condition that S is unimodular (and hence also the condition that $4|g$), one obtains cusp forms on subgroups of Γ_g of finite index with a multiplier system. The same method as before can be applied to estimate their Fourier coefficients. For example, take the simplest case $g = 1$ and let $S = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$. Then

$$\vartheta_{S, P_\nu}(z) = \sum_{x_1, x_2 \in \mathbb{Z}} (x_1 + ix_2)^\nu e^{2\pi i(x_1^2 + x_2^2)z}, \quad (z \in \mathbf{H} := \mathbf{H}_1; \nu \in \mathbb{N})$$

is a cusp form of weight $1 + \nu$ on $\Gamma_0(4) = \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1 \mid 4|c \}$ with character $(\frac{-4}{\cdot})$ (Legendre symbol). If $4|\nu$ it is not identically zero (the coefficient of $e^{2\pi iz}$ is equal to 4). Since (as is of course well-known) $r_S(T) \ll T^\epsilon (\epsilon > 0)$, we obtain Deligne's bound

$$a(T) \ll_\epsilon T^{\nu/2 + \epsilon}, \quad (\epsilon > 0)$$

for the Fourier coefficients $a(T)$ of ϑ_{S, P_ν} .

(iii) One should observe that in general (i.e. for $m \neq 2g$) the coefficients of theta functions with harmonic forms in $H_\nu(m, g)$ cannot be estimated directly in a good way. In fact, for $m < 2g$ one has

$$\begin{aligned} H_\nu(m, g) &= \{0\} & \text{if } m < g, \quad \text{all } \nu \geq 1, \\ H_\nu(m, g) &= \{0\} & \text{if } g \leq m < 2g \quad \text{and } \nu \neq 1 \end{aligned}$$

(cf. [3, p.13]). On the other hand, for $m > 2g$ an estimate with the same method as in Section 2 leads to a bound which is even worse than the usual Hecke bound.

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