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Orthogonal, symplectic and unitary polar spaces sub-weakly embedded in projective space

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Abstract. We show that every sub-weak embedding of any non-singular orthogonal or unitary polar space of rank at least 3 in a projective space $\mathbb{P}G(d, \mathbb{K})$, $\mathbb{K}$ a commutative field, is a full embedding in some subspace $\mathbb{P}G(d, \mathbb{F})$, where $\mathbb{F}$ is a subfield of $\mathbb{K}$; the same theorem is proved for every sub-weak embedding of any non-singular symplectic polar space of rank at least 3 in $\mathbb{P}G(d, \mathbb{K})$, where the field $\mathbb{F}'$ over which the symplectic polarity is defined is perfect in the case that the characteristic of $\mathbb{F}'$ is two and the secant lines of the embedded polar space $\Gamma$ contain exactly two points of $\Gamma$. This generalizes a result announced by LEFÈVRE-PERCY [5] more than ten years ago, but never published. We also show that every quadric defined over a subfield $\mathbb{F}$ of $\mathbb{K}$ extends uniquely to a quadric over the groundfield $\mathbb{K}$, except in a few well-known cases.

Key words: polar space, weak embedding, sub-weak embedding, projective space

1. Introduction and statement of the results

In this paper we always assume that $\mathbb{K}$ and $\mathbb{F}$ are commutative fields. Any polar space considered in this paper is assumed to be non-degenerate (which means that no point of the polar space is collinear with all points of the polar space), unless explicitly mentioned otherwise.

A weak embedding of a point-line geometry $\Gamma$ with point set $S$ in a projective space $\mathbb{P}G(d, \mathbb{K})$ is a monomorphism $\theta$ of $\Gamma$ into the geometry of points and lines of $\mathbb{P}G(d, \mathbb{K})$ such that

(WE1) the set $S^\theta$ generates $\mathbb{P}G(d, \mathbb{K})$,
(WE2) for any point $x$ of $\Gamma$, the subspace generated by the set $X = \{y^\theta \mid y \in S$ is collinear with $x\}$ meets $S^\theta$ precisely in $X$,
(WE3) if for two lines $L_1$ and $L_2$ of $\Gamma$ the images $L_1^\theta$ and $L_2^\theta$ meet in some point $x$, then $x$ belongs to $S^\theta$.

In such a case we say that the image $\Gamma^\theta$ of $\Gamma$ is weakly embedded in $\mathbb{P}G(d, \mathbb{K})$.

A full embedding in $\mathbb{P}G(d, \mathbb{K})$ is a weak embedding with the additional property that for every line $L$, all points of $\mathbb{P}G(d, \mathbb{K})$ on the line $L^\theta$ have an inverse image under $\theta$.  

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Weak embeddings were introduced in [3,5]; in these papers she announced the classification of all weakly embedded finite polar spaces (clearly the polar spaces are considered here as point-line geometries) having the additional property that there exists a line of $\mathbf{PG}(d, \mathbb{K})$ which does not belong to $\Gamma^\theta$ and which meets $S^\theta$ in at least three points. Only the case $d = 3$, $|\mathbb{K}| < \infty$ and rank $(\Gamma) = 2$ was published [4]. The question arose again in connection with full embeddings of generalized hexagons (see [7]) and a proof seemed desirable. In the present paper, we will first show that the condition (WE3) is superfluous and then classify all – finite and infinite – weakly embedded non-singular polar spaces of rank at least 3 of orthogonal, symplectic or unitary type, assuming that for the symplectic type the field $\mathbb{F}^\theta$ over which the symplectic polarity is defined is perfect in the case that $\mathbb{F}^\theta$ has characteristic two and no line of $\mathbf{PG}(d, \mathbb{K})$ which does not belong to $\Gamma^\theta$ intersects $S^\theta$ in at least three points. The classification of all generalized quadrangles weakly embedded in finite projective space can be found in [8].

We call a monomorphism $\theta$ from the point-line geometry of a polar space $\Gamma$ with point set $S$ to the point-line geometry of a projective space $\mathbf{PG}(d, \mathbb{K})$ a sub-weak embedding if it satisfies conditions (WE1) and (WE2). Usually, we simply say that $\Gamma$ is weakly or sub-weakly embedded in $\mathbf{PG}(d, \mathbb{K})$ without referring to $\theta$, that is, we identify the points and lines of $\Gamma$ with their images in $\mathbf{PG}(d, \mathbb{K})$. In such a case the set of all points of $\Gamma$ on a line $L$ of $\Gamma$ will be denoted by $L^\ast$.

If the polar space $\Gamma$ arises from a quadric it is called orthogonal, if it arises from a hermitian variety it is called unitary, and if it arises from a symplectic polarity it is called symplectic. In these cases $\Gamma$ is called non-singular either if the hermitian variety is non-singular, or if the symplectic polarity is non-singular, or if the quadric is non-singular (in the sense that the quadric $Q$, as algebraic hypersurface, contains no singular point over the algebraic closure of the ground field over which $Q$ is defined); in the symplectic and hermitian case this is equivalent to assuming that the corresponding matrix is non-singular. In the orthogonal case with characteristic not 2, in the symplectic case and in the hermitian case, $\Gamma$ is non-singular if and only if it is non-degenerate; in the orthogonal case with characteristic 2, non-singular implies non-degenerate, but when not every element of $\mathbb{K}$ is a square, and only then, a non-degenerate quadric may be singular.

Our main results read as follows.

**THEOREM 1** Let $\Gamma$ be a non-singular polar space of rank at least 3 arising from a quadric, a hermitian (unitary) variety or a symplectic polarity, and let $\Gamma$ be sub-weakly embedded in the projective space $\mathbf{PG}(d, \mathbb{K})$, where for $\Gamma$ symplectic the polarity is defined over a perfect field $\mathbb{F}^\theta$ in the case that $\mathbb{F}^\theta$ has characteristic two and the secant lines of $\Gamma$ contain exactly two points of $\Gamma$. Then $\Gamma$ is fully embedded in some subspace $\mathbf{PG}(d, \mathbb{F})$ of $\mathbf{PG}(d, \mathbb{K})$, for some subfield $\mathbb{F}$ of $\mathbb{K}$.

If $\Gamma$ is finite, then it is automatically of one of the three types mentioned. Moreover, it is non-degenerate if and only if it is non-singular. Combining this with [8], we have
COROLLARY 1 (i) Let \( \Gamma \) be a non-degenerate polar space sub-weakly embedded in the finite projective space \( \text{PG}(d, q) \). Then \( \Gamma \) is fully embedded in some subspace \( \text{PG}(d, q') \) of \( \text{PG}(d, q) \), for some subfield \( \text{GF}(q') \) of \( \text{GF}(q) \), unless \( \Gamma \) is the unique generalized quadrangle of order \((2, 2)\) universally embedded in \( \text{PG}(4, q) \) with \( q \) odd.

(ii) Let \( \Gamma \) be a finite non-degenerate polar space of rank at least 3 sub-weakly embedded in the projective space \( \text{PG}(d, \mathbb{K}) \). Then \( \Gamma \) is fully embedded in some subspace \( \text{PG}(d, q) \) of \( \text{PG}(d, \mathbb{K}) \), for some subfield \( \text{GF}(q) \) of \( \mathbb{K} \).

Our second main result might belong to folklore but we give a full proof here.

THEOREM 2 (i) Let \( Q \) be a non-degenerate non-empty quadric of \( \text{PG}(d, \mathbb{F}) \), \( d \geq 2 \), and let \( \mathbb{K} \) be a field containing \( \mathbb{F} \). Then in the corresponding extension \( \text{PG}(d, \mathbb{K}) \) of \( \text{PG}(d, \mathbb{F}) \) there exists a unique quadric containing \( Q \), except if \( d = 2 \) and \( \mathbb{F} \in \{ \text{GF}(2), \text{GF}(3) \} \), or \( d = 3 \), \( \mathbb{F} = \text{GF}(2) \) and \( Q \) is of elliptic type.

(ii) Let \( \Gamma \) be a non-singular symplectic polar space defined by a symplectic polarity in \( \text{PG}(d, \mathbb{F}) \), \( d \geq 3 \), and let \( \mathbb{K} \) be a field extending \( \mathbb{F} \). Then in the corresponding extension \( \text{PG}(d, \mathbb{K}) \) of \( \text{PG}(d, \mathbb{F}) \), there exists a unique symplectic polarity whose corresponding polar space contains \( \Gamma \).

(iii) Let \( H \) be a non-singular non-empty hermitian variety of \( \text{PG}(d, \mathbb{F}) \), \( d \geq 2 \), with associated \( \mathbb{F} \)-involution \( \sigma \), and let \( \mathbb{K} \) be a field containing \( \mathbb{F} \) admitting a \( \mathbb{K} \)-involution \( \tau \) the restriction of which to \( \mathbb{F} \) is exactly \( \sigma \). Then in the corresponding extension \( \text{PG}(d, \mathbb{K}) \) of \( \text{PG}(d, \mathbb{F}) \) there exists a unique hermitian variety with associated field involution \( \tau \) and containing \( H \).

Remark It is now easy to extend Theorem 2 to the singular cases with at least one non-singular point over \( \mathbb{F} \). Again the extension of the polar space \( \Gamma \) is unique, except for \( \Gamma \) orthogonal and \( \mathbb{F} \in \{ \text{GF}(2), \text{GF}(3) \} \).

2. Proof of Theorem 1

In the sequel, we adopt the notation \( x^\perp \) for the set of all points collinear with the point \( x \) in a polar space. After having coordinatized \( \text{PG}(d, \mathbb{K}) \), we denote by \( e_i \), \( 1 \leq i \leq d+1 \), the point with coordinates \((0, \ldots, 0, 1, 0, \ldots, 0)\), where the 1 is in the \( i \)th position. By generalizing this, we denote by \( e_J \) the point with every coordinate equal to 0 except in each position belonging to the set \( J, J \subseteq \{1, 2, \ldots, d+1\} \), where the coordinate equals 1. We also remark that polar spaces are Shult spaces, i.e. for every point \( x \) and every line \( L \), \( x^\perp \) contains either all points of \( L \) or exactly one point of \( L \) (we will call that property the Buekenhout–Shult axiom).

We prove Theorem 1 in a sequence of lemmas.

LEMMA 1 If \( L \) is a line of the sub-weakly embedded polar space \( \Gamma \), then the only points of \( \Gamma \) on \( L \) are the points of \( L^* \).

Proof Let \( x \) be a point of \( \Gamma \) on \( L \) with \( x \notin L^* \). By the Buekenhout–Shult axiom \( L^* \) contains a point \( y \) collinear with \( x \). So the lines \( xy \) and \( L \) of \( \Gamma \) coincide in
LEMMA 2 Every sub-weak embedding of a non-degenerate polar space is also a weak embedding.

Proof Let \( \Gamma \) be a polar space sub-weakly embedded in \( \text{PG}(d, K) \) for some field \( K \). Let \( L_1 \) and \( L_2 \) be two lines of \( \Gamma \) meeting in a point \( x \) of \( \text{PG}(d, K) \) which does not belong to \( S \), the point set of \( \Gamma \). If some point \( y \) of \( \Gamma \) is collinear with all points of \( L_1^* \), then \( y^\perp \) contains a triangle of the plane \( L_1L_2 \) of \( \text{PG}(d, K) \) (\( y^\perp \) contains some point of \( L_2^* \) by the Buekenhout–Shult axiom). Hence (WE2) implies that \( y \) is collinear with all points of \( L_2^* \). If we let \( y \) vary on \( L_1^* \), then we see that all points of \( L_1^* \) are collinear with all points of \( L_2^* \), in other words, \( L_1^* \) and \( L_2^* \) span a 3-dimensional singular subspace \( S \) of \( \Gamma \). Since \( \Gamma \) is non-degenerate, no point of \( S \) is collinear with all other points of \( \Gamma \), hence there exists a point \( z \) of \( \Gamma \) not collinear with all points of \( S \). It is easily seen that \( z^\perp \) meets \( S \) in the point set of a plane \( \pi \) of \( \Gamma \). Since any two lines of \( \Gamma \) in \( \pi \) generate the plane \( L_1L_2 \), the points of \( \pi \) span the plane \( L_1L_2 \) of \( \text{PG}(d, K) \). By (WE2), \( z^\perp \) must contain all points of \( S \) (since they all lie in \( L_1L_2 \)), a contradiction. \( \square \)

Let \( L \) be any line of \( \text{PG}(d, K) \) containing at least two points of \( \Gamma \) which are not collinear in \( \Gamma \). Then we call \( L \) a secant line. By Lemma 1, no secant line contains two collinear points. The following result is due to Lefèvre-Percsy [3].

LEMMA 3 The number of points of \( \Gamma \) on a secant line is a constant.

We put that number equal to \( \delta \) (\( \delta \) is possibly an infinite cardinal) and call it the degree of the embedding.

We now prepare the proof of the case \( \delta = 2 \) by first proving a lemma which certainly belongs to folklore.

A kernel of a non-empty non-singular quadric in a projective space is any point belonging to every tangent hyperplane of the quadric. As the quadric is non-singular a kernel does not belong to the quadric. The subspace of all kernels is sometimes called the radical of the quadric.

LEMMA 4 Every non-empty non-singular quadric has at most one kernel.

Proof Suppose that the non-singular non-empty quadric \( \Gamma \) of \( \text{PG}(d, K) \) has a radical \( V \) of dimension at least one. Extend \( \Gamma \) over the algebraic closure \( \overline{K} \) of \( K \) to the non-singular quadric \( \overline{\Gamma} \). Then \( \overline{\Gamma} \cap \overline{V} \), with \( \overline{V} \) the corresponding extension of \( V \), is a non-empty quadric. Let \( x \) be a point of it. Every line \( xp \) with \( p \in \overline{\Gamma}, p \neq x \), is a tangent line of \( \overline{\Gamma} \) and all these lines generate the whole projective space \( \text{PG}(d, \overline{K}) \). This yields a contradiction as all tangent lines of \( \overline{\Gamma} \) at \( x \) lie in the tangent hyperplane of \( \overline{\Gamma} \) at \( x \). \( \square \)

LEMMA 5 Let \( \Gamma \) be a non-singular polar space of rank at least 3 arising from a quadric, a hermitian (unitary) variety or a symplectic polarity, where for \( \Gamma \) sym-
plectic the polarity is defined over a perfect field $\mathbb{F}$ in the characteristic two case, and let $\Gamma$ be sub-weakly embedded of degree 2 in the projective space $\text{PG}(d, \mathbb{K})$. Then $\Gamma$ is fully embedded in some subspace $\text{PG}(d, \mathbb{F})$ of $\text{PG}(d, \mathbb{K})$, for some subfield $\mathbb{F}$ of $\mathbb{K}$.

Proof We label the steps of the proof for future reference.

(a) Let $\Gamma$ be a non-singular orthogonal polar space sub-weakly embedded in $\text{PG}(d, \mathbb{K})$, $d \geq 3$, and suppose that $\Gamma$ has rank at least 3. We identify the points and lines of $\Gamma$ with the corresponding points and lines of $\text{PG}(d, \mathbb{K})$. Let $\pi$ be any plane of $\Gamma$. Three non-concurrent lines of $\pi$ span a unique plane $\pi'$ of $\text{PG}(d, \mathbb{K})$. Any other line of $\pi$ meets these three lines in at least two points, hence we see that $\pi'$ is uniquely determined by $\pi$; moreover, the points and lines of $\pi$ determine a unique subplane of $\pi'$. Hence $\pi$ is isomorphic to a projective plane over some subfield $\mathbb{F}$ of $\mathbb{K}$. Moreover, since $\Gamma$ is residually connected (as a polar space or a building, see e.g. [1]), $\mathbb{F}$ is independent from $\pi$. Hence, if we coordinatize $\text{PG}(d, \mathbb{K})$, then every re-coordinatization by means of a linear transformation (so without using a field automorphism) which maps the points $e_1, e_2, e_3$ and $e_{\{1,2,3\}}$ onto points of $\pi$, defines a subfield $\mathbb{F}$ of $\mathbb{K}$ which is independent of the choice of $\pi$ and where $\mathbb{F}$ is equal to the set of quotients of possible coordinates (in the new coordinate system) for points of $\pi$. This implies that the set of all points of $\Gamma$ on any line of $\Gamma$ is uniquely determined in $\text{PG}(d, \mathbb{K})$ by any three of its points; indeed, re-coordinatize so that these points become $e_1, e_2$ and $e_{\{1,2\}}$, and then all points of the line are obtained by taking all linear combinations of the vectors $(1, 0, \ldots, 0)$ and $(0, 1, 0, \ldots, 0)$ over $\mathbb{F}$. All this shows that not only the isomorphism type of $\mathbb{F}$ is fixed, but also the subfield $\mathbb{F}$ itself.

(b) Now consider a line $L_1$ of $\Gamma$ and a point $x_1$ of $\Gamma$ on it. Through $x_1$ there is a line $M_1$ of $\Gamma$ with the property that $L_1$ and $M_1$ are not in a common plane of $\Gamma$. Now we take a point $y_1$ of $\Gamma$ not collinear with $x_1$ and we consider the unique line $L_2$ of $\Gamma$ passing through $y_1$ and meeting $M_1$ in a point of $\Gamma$. Now we show that in $\Gamma$ no point on $L_2$ is collinear with all points of $L_1$. The point $x_1$ is not collinear with $y_1$, and as $L_1$ and $M_1$ are not in a common plane of $\Gamma$ the point $M_1 \cap L_2$ is not collinear with all points of $L_1$. As $x_1$ is not collinear with $y_1$, it is not collinear with two distinct points of $L_2$; hence no point of $L_2$ different from $y_1$ and $M_1 \cap L_2$ is collinear with all points of $L_1$. Similarly, in $\Gamma$ no point on $L_1$ is collinear with all points on $L_2$. If $L_1$ and $L_2$ would span a plane $L_1L_2$, then every point of $L_2$ is in the space spanned by $x_1$ for every $x \in L_1^*$, since there is at least one point of $x_1$ on $L_2$. So by (WE2) the point $x \in L_1^*$ is collinear with every point of $L_2^*$, a contradiction. Hence $L_1$ and $L_2$ generate a 3-space $U$ of $\text{PG}(d, \mathbb{K})$. In $\Gamma$ the lines $L_1$, $L_2$ and their points generate a polar space $\Omega$; $\Omega$ corresponds to a hyperbolic quadric $Q_3^+$ (of a 3-space) on the non-singular quadric from which $\Gamma$ arises. The point set of $\Omega$ will also be denoted by $Q_3^+$, and the sets of lines of $\Omega$ corresponding to the reguli of $Q_3^+$ will also be called the reguli of $\Omega$. Since all points of $\Omega$ lie on lines meeting both $L_1$ and $L_2$, we see that $\Omega$ is entirely contained in $U$. Let $M_2 \neq M_1$ belong to the regulus of $\Omega$ defined by $M_1$. Put $x_2 = L_1 \cap M_2$, $x_3 = L_2 \cap M_1$ and $x_4 = L_2 \cap M_2$. 

ORBITAL, SYMPLECTIC AND UNITARY POLAR SPACES 79
Let $x_5$ be one further point of $\Omega$ not on one of the lines $L_1, L_2, M_1, M_2$ and let $L_3$, respectively $M_3$, be the line of $\Omega$ through $x_5$ and belonging to the regulus defined by $L_1$, respectively $M_1$. No four of the points \{x_1, x_2, x_3, x_4, x_5\} are coplanar, so they determine a unique subspace $V$ of $U$ over $F$.

(c) We claim that $\Omega$ is fully embedded in $V$, that is, we claim that all points of $\Omega$ are contained in $V$. Indeed, the points on $L_1$ in $V$ are uniquely determined by the three points $x_1, x_2$ and $M_3 \cap L_1$. But as remarked above, these points are precisely all points of $\Gamma$ on $L_1$. Similarly for $L_2, M_1$ and $M_2$. Let $M_4$ be a line of $\Omega$ meeting $L_1, L_2$ in points of $\Omega$, so of $V$, with $M_1 \neq M_4 \neq M_2$; then $M_4$ is a line of $V$. As $L_3$ is a line of $V$, also $L_3 \cap M_4$ is a point of $V$. It follows that the points of $M_4$ in $V$ are exactly the points of $M_4$ in $\Omega$. Similarly, for any line $L_4$ of $\Omega$ meeting $M_1, M_2$ in points of $\Omega$, the points of $L_4$ in $V$ are exactly the points of $L_4$ in $\Omega$. If $y$ is any point of $\Omega$, then the line of $\Gamma$ through $y$ meeting $L_1, L_2$, respectively $M_1, M_2$, contains at least two points of $V$, and hence the intersection $y$ of these two lines also belongs to $V$. This shows our claim.

(d) Next we prove that no other point of $\Gamma$ belongs to $U$. Indeed, suppose the point $z$ of $\Gamma$ lies in $U$, but is not contained in $\Omega$. Then $z$ does not belong to $V$ since the unique line $M$ in $V$ through $z$ meeting both $L_1$ and $L_2$ contains three points of $\Gamma$, say $z, x_1, x_4$, hence belongs to $\Gamma$, contradicting the fact that $z$ does not belong to $\Omega$. In $\Gamma$ the points of $\Omega$ collinear with $z$ either are all the points of $\Omega$, or are the points of a point set $C$ of $\Omega$ corresponding to a non-singular conic of the hyperbolic quadric $Q_3^+$, or are the points of $\Omega$ on two lines of $\Omega$, say $L_1$ and $M_1$. Noticing that for every point $y$ of $\Omega$, the space generated by $y$ in $\text{PG}(d, \mathbb{K})$ meets $U$ in a plane (by axiom (WE2)), we see that in the first case $z$ must lie in every plane containing two lines of $\Omega$. This yields a contradiction since these planes have no intersection point in $V$, hence neither in $U$. In the second case $z$ must lie in the planes tangent to $Q_3^+$ at points of $C$. These planes meet in at most one point, which lies in $V$, a contradiction. In the third case $z$ must lie in all planes of $V$ containing $L_1$ or $M_1$, hence $z = x_1$, a contradiction. This proves our claim.

(e) An orthogonal subspace of $\Gamma$ containing lines is called $s$-dimensional if the corresponding subquadric on the quadric from which $\Gamma$ arises generates an $(s+1)$-dimensional space. Now suppose that any $(c-1)$-dimensional non-singular orthogonal subspace $\Omega'$ of $\Gamma$ containing lines is fully embedded in a $c$-dimensional projective subspace over $F$ of $\text{PG}(d, \mathbb{K})$, $3 \leq c \leq d - 1$. We show that, if $\Omega$ is a $c$-dimensional non-singular orthogonal subspace of $\Gamma$ containing lines, then $\Omega$ is fully embedded in some $(c+1)$-dimensional projective subspace $\text{PG}(c+1, \mathbb{K})$ of $\text{PG}(c+1, \mathbb{K})$. Since $\Omega$ is non-singular, it contains some $(c-1)$-dimensional non-singular orthogonal subspace $\Omega'$ containing lines. By assumption $\Omega'$ is contained in a $c$-dimensional projective space $V'$ over $F$. Let $U'$ be the extension of $V'$ over $\mathbb{K}$. We first show that $U'$ does not contain any point of $\Omega \setminus \Omega'$. Let the point $x$ of $\Omega \setminus \Omega'$ belong to $U'$. Then $x$ and the point set of $\Omega'$ intersect in a point set $Q''$ which corresponds to a non-singular subquadric of the quadric from which $\Gamma$ arises. By (WE2) $Q''$ is contained in a $(c-1)$-dimensional subspace $V''$ of $V'$. 

Assume that \( Q'' \) does not generate \( V'' \). Then \( \Omega' \) contains a point \( u \) of \( V'' \) not on \( Q'' \). Every line of \( \Omega' \) through \( u \) contains a point of \( x_\perp \), so every line of \( \Omega' \) through \( u \) contains a point of \( Q'' \). Hence \( V'' \) contains all lines of \( \Omega' \) through \( u \). Analogously, \( V'' \) contains all lines of \( \Omega' \) through \( u' \), with \( u' \neq u \) a second point of \( \Omega' \) in \( V'' \setminus Q'' \).

So the tangent hyperplanes of the point set of \( \Omega' \) at \( u \) and \( u' \) coincide with \( V'' \), a contradiction. We conclude that \( Q'' \) generates \( V'' \). The extension of \( V'' \) over \( K \) will be denoted by \( U'' \). If \( x \notin U'' \), then \( x_\perp \cap U' \) spans \( U' \), hence by (WE2) all points of \( \Omega' \) are collinear with \( x \), a contradiction. So \( x \in U'' \). Let \( y \) be a point of \( Q'' \) and let \( V'_y \) be the tangent hyperplane of \( \Omega' \) at \( y \); the extension of \( V'_y \) to \( K \) is denoted by \( U'_y \). If \( x \notin U'_y \), then the space generated by \( x \) and \( U'_y \) is \( U' \), so by (WE2) \( y_\perp \) contains all points of \( \Omega' \), a contradiction. Hence \( x \in U'_y \). Let \( V''_y \) be the tangent hyperplane of \( Q'' \) at \( y \), and let \( U''_y \) be the extension of \( V''_y \) to \( K \); then \( V''_y = V'_y \cap V'' \) and \( U''_y = U'_y \cap U'' \). As \( x \in U''_y \), we have \( x \in U'_y \) for every point \( y \) of \( Q'' \). This implies that \( x \in V'' \) and that \( x \) is the unique kernel of \( Q'' \) in \( V'' \). Since \( Q'' \) has a unique kernel, the dimension \( c - 1 \) of the space generated by \( Q'' \) is even and the matrix defined by \( Q'' \) has rank equal to \( c - 1 \). If \( x \) is also a kernel of \( \Omega' \), then as \( c + 1 \) is even \( \Omega' \) admits at least a line \( L \) of kernels. Over the algebraic closure \( \overline{F} \) of \( F \) the extension \( \overline{L} \) of \( L \) contains a point \( r \) of the extension \( \overline{\Omega} \) of \( \Omega' \). The point \( r \) is singular for \( \overline{\Omega} \), hence \( \Omega' \) is singular, a contradiction. Consequently \( x \) is not a kernel for \( \Omega' \). Hence there is a line \( N \) of \( V' \) containing \( x \) and two distinct points \( y_1, y_2 \) of \( \Omega' \). Since the degree of the weak embedding is equal to 2, \( N \) is a line of \( \Gamma \), so \( y_1 = y_2 \in Q'' \), a contradiction. It follows that \( U' \) does not contain any point of \( \Omega \setminus \Omega' \).

(f) Let \( x_1 \) be any point of \( \Omega \setminus \Omega' \) and let \( L_1 \) be any line of \( \Omega \) through \( x_1 \). Evidently, \( L_1 \) meets \( \Omega' \) in a unique point \( y \). Let \( L_2 \) be any line of \( \Omega' \) such that \( L_1, L_2 \) and their points in \( \Omega' \) generate a polar space in \( \Omega \) with as point set a hyperbolic quadric \( Q = Q^+_3 \). Take any point \( x_2 \neq x_1 \) on \( L_1 \) with \( x_2 \neq y \). The space \( V' \) together with the two points \( x_1, x_2 \) defines a unique \((c + 1)\)-dimensional subspace \( V \) over \( F \), which contains \( x_1, x_2 \) and \( y \) and hence all points of \( \Omega \) on \( L_1 \). Also, \( V \) contains all points of \( \Omega \) on \( L_2 \) and all points of the line of \( \Omega' \) containing \( y \) and concurrent with \( L_2 \). Similarly as in (c), one now shows that \( Q^+_3 \) is completely contained in a 3-dimensional subspace over \( F \) which clearly belongs to \( V \).

(g) We now show that all points of \( \Omega \) belong to \( V \). Let \( z \) be any point of \( \Omega \setminus \Omega' \). First suppose that \( z \) is not collinear with \( y \). Consider a line \( M_1 \) on \( \Omega' \) through \( y \) and such that \( L_1 \) and \( M_1 \) are not contained in a plane of \( \Omega \). Let \( L_3 \) be the unique line of \( \Omega \) through \( z \) meeting \( M_1 \) in a point of \( \Omega \). Then clearly \( L_1 \) and \( L_3 \) define a hyperbolic quadric \( Q' \) over \( F \) on \( \Omega \). We show that the polar subspace of \( \Omega \) with point set \( Q' \) has two different lines \( M_1 \) and \( L'_2 \) in common with \( \Omega' \). If we identify the point set of \( \Omega \) with a quadric in some \( \text{PG}(c + 1, F) \), then the 3-space of \( Q' \) and the hyperplane defined by \( \Omega \) have a plane \( \zeta \) in common, which intersects \( Q' \) in two distinct lines. Hence \( Q' \) has two different lines \( M_1 \) and \( L'_2 \) in common with \( \Omega' \). Interchanging roles of \( L_2 \) and \( L'_2 \), we now see that \( z \) also belongs to the space
V. Now suppose that the point \( z \) of \( \Omega \setminus \Omega', z \neq y \), is collinear with \( y \). Let \( L_3 \) and \( L_4 \), with \( L_3 \neq yz \neq L_4 \), be two distinct lines of \( \Omega \) through \( z \) for which \( yL_3 \) and \( yL_4 \) are not planes of \( \Omega \). By the foregoing all points of \( L_3 \setminus \{z\} \) and \( L_4 \setminus \{z\} \) belong to \( V \). Hence also the intersection of \( L_3 \) and \( L_4 \), that is, \( z \), belongs to \( V \). So we conclude that each of the points of \( \Omega \) belongs to \( V \), and consequently \( \Omega \) is fully embedded in the space \( V \) over \( \mathbb{F} \).

(h) Applying consecutively the previous paragraphs for \( c = 3, 4, \ldots, d - 1 \), we finally obtain that \( \Gamma \) is fully embedded in some \( \text{PG}(d, \mathbb{F}) \).

(i) Now let \( \Gamma \) be a non-singular hermitian polar space sub-weakly embedded in \( \text{PG}(d, \mathbb{K}), d \geq 3 \), and suppose that the degree is 2. On the non-singular hermitian variety \( \mathcal{H} \) from which \( \Gamma \) arises we consider a non-singular hermitian variety \( \mathcal{H}' \), where \( \mathcal{H}' \) generates a 3-dimensional space. The corresponding point set on \( \Gamma \) will be denoted by \( \mathcal{H} \) and the corresponding polar subspace of \( \Gamma \) by \( \Omega \). Let \( L, M \) be two non-intersecting lines of \( \Omega \). In \( \text{PG}(d, \mathbb{K}) \), the lines \( L \) and \( M \) generate a 3-dimensional subspace \( U = \text{PG}(3, \mathbb{K}) \), which contains all points of \( \mathcal{H} \) (\( \Omega \) is generated by \( L, M \) and their points in \( \Omega \)). Now consider two points \( x \) and \( y \) in \( \mathcal{H} \) which are not collinear in \( \mathcal{H} \). Let \( U(x) \) and \( U(y) \) be the set of points of \( \mathcal{H} \) collinear in \( \mathcal{H} \) with \( x \) and \( y \) respectively. Clearly neither \( U(x) \) nor \( U(y) \) can be contained in a line of \( U \). Also, by condition (WE2), neither \( U(x) \) nor \( U(y) \) generates \( U \). Hence \( U(x) \) and \( U(y) \) define unique planes \( U(x) \) and \( U(y) \) respectively. These planes meet in a unique line \( N \) of \( U \). Clearly \( N \) contains all points of \( \mathcal{H} \) collinear in \( \Omega \) with both \( x \) and \( y \). Assume that \( z \) is any point of \( \Gamma \) on \( N \). Further, let \( u, v \in N \cap \mathcal{H}, u \neq v \). Then \( z \) is collinear in \( \Gamma \) with all points of \( u^\perp \cap v^\perp \). Let \( u', v', z' \) be the points of \( \overline{\mathcal{H}} \) which correspond to \( u, v, z \) respectively. As \( z' \) is collinear in \( \overline{\mathcal{H}} \) with all points of \( u'^\perp \cap v'^\perp \), it belongs to \( \overline{\mathcal{H}} \cap u'v' = \mathcal{H}' \cap u'v' \). Hence \( z \) belongs to \( \mathcal{H} \cap uu \). It follows that the set of all points of \( \Gamma \) on \( N \) corresponds to the point set \( \overline{\mathcal{H}} \cap u'v' = \mathcal{H}' \cap u'v' \). As \( N \) meets \( \Gamma \) in more than 2 points, we are in contradiction with \( \delta = 2 \).

(j) Finally let \( \Gamma \) be a non-singular symplectic polar space sub-weakly embedded in \( \text{PG}(d, \mathbb{K}), d \geq 3 \). Let \( \mathbb{F}' \) be the ground field over which the symplectic polarity \( \zeta \) from which \( \Gamma \) arises is defined. If the characteristic of \( \mathbb{F}' \) is not two, then a similar proof as for the hermitian case leads to a contradiction; here the secant line \( N \) will contain \( |\mathbb{F}'| + 1 \) points (note that the secant lines of \( \Gamma \) correspond (bijectively) to the non-isotropic lines of the symplectic polarity \( \zeta \)).

If the characteristic of \( \mathbb{F}' \) is two, then \( \mathbb{F}' \) is perfect, hence \( \Gamma \) is also orthogonal. Now it follows from (a)–(h) that \( \Gamma \) is fully embedded in some \( \text{PG}(d, \mathbb{F}) \). \( \square \)

The next lemma is a result similar to Theorem 1 for projective spaces. A sub-\( n \)-space of a projective space \( \text{PG}(n, \mathbb{K}) \) is any space \( \text{PG}(n, \mathbb{F}) \), \( \mathbb{F} \) a subfield of \( \mathbb{K} \), obtained from \( \text{PG}(n, \mathbb{K}) \) by restricting coordinates to \( \mathbb{F} \) (with respect to some coordinatization). Note that, for many fields \( \mathbb{K} \) and positive integers \( n \), there exist subsets \( S \) of the point set of \( \text{PG}(n, \mathbb{K}) \) such that the linear space induced in \( S \) by the lines of \( \text{PG}(n, \mathbb{K}) \) is the point-line space of a \( \text{PG}(m, \mathbb{F}) \) with \( m > n \). The
following result gives a necessary and sufficient condition for such a structure to be a sub-$n$-space. These conditions are basically (WE1) and some analogue of (WE2).

**Lemma 6** Let $S$ be a generating set of points in the projective space $\mathbb{P}G(n, \mathbb{K})$, $\mathbb{K}$ a skewfield and let $L$ be the collection of all intersections of size $> 1$ of $S$ with lines of $\mathbb{P}G(n, \mathbb{K})$. Suppose $(S, L)$ is the point-line space of some projective space $\mathbb{P}G(m, \mathbb{F})$, for some skewfield $\mathbb{F}$ and some positive integer $m$. Then $\mathbb{F}$ is a subfield of $\mathbb{K}$, $m = n$ and $S$ and $L$ are the point set and line set respectively of some sub-$n$-space $\mathbb{P}G(n, \mathbb{F})$ of $\mathbb{P}G(n, \mathbb{K})$ if and only if there exists a dual basis of hyperplanes in $\mathbb{P}G(m, \mathbb{F})$ such that each element $H$ of that basis is contained in a hyperplane $H'$ of $\mathbb{P}G(n, \mathbb{F})$ with $H' \cap S = H$.

**Proof** It is clear that the given condition is necessary. Now we show that it is also sufficient. If $m + 1$ points of $S$ generate $\mathbb{P}G(m, \mathbb{F})$, then by the condition that lines of $\mathbb{P}G(m, \mathbb{F})$ are line intersections of $\mathbb{P}G(n, \mathbb{K})$ with $S$, these $m + 1$ points must also span $\mathbb{P}G(n, \mathbb{K})$ (otherwise $S$ is contained in some proper subspace of $\mathbb{P}G(n, \mathbb{K})$). Hence $m \geq n$. Now let $\{H_i : i = 0, 1, \ldots, m - 1, m\}$ be a collection of hyperplanes of $\mathbb{P}G(m, \mathbb{F})$ meeting the requirements of the lemma. Put $S_i = H_0 \cap H_1 \cap \ldots \cap H_i$, $i = 0, 1, \ldots, m$. Suppose that $S_i$ generates the same space as $S_{i+1}$ in $\mathbb{P}G(n, \mathbb{K})$ for some $j$, $0 \leq j \leq m - 1$. Let $H_i$ be contained in the hyperplane $H'_i$ (not necessarily unique at this point) of $\mathbb{P}G(n, \mathbb{K})$, $i = 0, 1, \ldots, m$. If $x$ is a point of $S_j$ not lying in $S_{j+1}$ ($x$ exists by the assumptions on $H_i$), then in $\mathbb{P}G(n, \mathbb{K})$ $x$ is not generated by the points of $H_{j+1}$, since $H'_{j+1}$ meets $S$ precisely in $H_{j+1}$. But $S_{j+1} \subseteq H_{j+1}$, hence in $\mathbb{P}G(n, \mathbb{K})$ $x$ is not in the space generated by $S_{j+1}$, a contradiction. So $S_j$ generates a space in $\mathbb{P}G(n, \mathbb{K})$ which is strictly larger than $S_{j+1}$. That means that we have a chain of $m + 1$ subspaces of $\mathbb{P}G(n, \mathbb{K})$ consecutively properly contained in each other and all contained in $H'_0$, hence $n \geq m$. We conclude that $n = m$.

Now if we choose a basis of $\mathbb{P}G(n, \mathbb{F})$ (this is also a basis of $\mathbb{P}G(n, \mathbb{K})$), then it is clear that the corresponding coordinatization of $\mathbb{P}G(n, \mathbb{F})$ is the restriction of the coordinatization of $\mathbb{P}G(n, \mathbb{K})$ to the field $\mathbb{F}$. The result follows. $\square$

**Lemma 7** Let $\Gamma$ be a non-singular polar space of rank at least 3 arising from a quadric, a symplectic polarity or a hermitian variety, and let $\Gamma$ be sub-weakly embedded of degree $\delta > 2$ in the projective space $\mathbb{P}G(d, \mathbb{K})$. Then $\Gamma$ is fully embedded in some subspace $\mathbb{P}G(d, \mathbb{F})$ of $\mathbb{P}G(d, \mathbb{K})$, for some subfield $\mathbb{F}$ of $\mathbb{K}$.

**Proof** Let $\mathbb{F}'$ be the field underlying $\Gamma$.

(1) First, let the characteristic of $\mathbb{F}'$ be odd and let $\Gamma$ be a non-singular symplectic polar space. By (j) in the proof of Lemma 5, secant lines of $\Gamma$ correspond (bijectively) with non-isotropic lines of the symplectic polarity $\zeta$ from which $\Gamma$ arises. Now the space $\Omega$ with point set $S$, the point set of $\Gamma$, and line set $\{L^* : L$ is a line of $\Gamma\} \cup\{S \cap S : S$ is the point set in $\mathbb{P}G(d, \mathbb{K})$ of a secant line of $\Gamma\}$ is a projective space. Every hyperplane $H$ in that projective space $\Omega$ is the set of points of $S$ collinear in $\Gamma$ with some fixed point $x$ of $S$. It is easy to see that, as $S$ is a generating set of $\mathbb{P}G(d, \mathbb{K})$, the hyperplane $H$ of $\Omega$ generates a hyperplane
$H'$ of $\mathbf{PG}(d, K)$. Now by (WE2) the assumptions of Lemma 6 are satisfied and the result follows.

Next, assume that the characteristic of $\mathbb{F}'$ is two and let $\Gamma$ be a non-singular symplectic polar space. Let $\zeta$ be again the symplectic polarity from which $\Gamma$ arises. If $\zeta$ is defined in $\mathbf{PG}(d', \mathbb{F}')$, then we consider a subspace $\mathbf{PG}(3, \mathbb{F}')$ of $\mathbf{PG}(d', \mathbb{F}')$ in which $\zeta$ induces a non-singular symplectic polarity $\eta$. The polar space defined by $\zeta$ is $\Gamma'$, and the polar space defined by $\eta$ is $\Omega'$. With $\Omega'$ corresponds the polar subspace $\Omega$ of $\Gamma$. Let $L, M$ be two non-intersecting lines of $\Omega$ and let $L', M'$ be the corresponding lines of $\Omega'$. Let $x$ be a point of $\Omega$ on $L$ and $y$ a point of $\Omega$ on $M$, where $x$ and $y$ are not collinear in $\Omega$. The points of $\mathbf{PG}(3, \mathbb{F}')$ which correspond to $x, y$ are denoted by $x', y'$ respectively. As $\delta > 2$ the line $xy$ contains a third point $z$ of $\Gamma$. As, by (WE2), $z$ is collinear in $\Gamma$ to all points of $x^\perp \cap y^\perp$, the corresponding point $z'$ of $\mathbf{PG}(d', \mathbb{F}')$ is collinear in $\Gamma'$ to all points of $x'^\perp \cap y'^\perp$. Hence $z'$ belongs to the line $x'y'$, so belongs to $\Omega'$. It follows that $z$ belongs to $\Omega$. As $\Omega'$ is generated by $z', L', M'$ and all points of $L'$ and $M'$, also $\Omega$ is generated by $z, L, M$ and all points of $L$ and $M$. Hence $\Omega$ is contained in a subspace $\mathbf{PG}(3, \mathbb{F})$ of $\mathbf{PG}(d, \mathbb{F})$. Then a similar argument as in (i) of Lemma 5 shows that the secant lines of $\Gamma$ correspond (bijectively) to the non-isotropic lines of $\zeta$. Now, analogously as in the odd characteristic case, the result follows.

(2) Now suppose that $\Gamma$ is of orthogonal type. Let $\Gamma'$ be the image of a natural full embedding of $\Gamma$ in a projective space $\mathbf{PG}(d', \mathbb{F}')$ where the point set of $\Gamma'$ is a non-degenerate quadric $Q'$ of $\mathbf{PG}(d', \mathbb{F}')$. Denote by $x'$ the element of $\Gamma'$ corresponding to any element $x$ of $\Gamma$. Let $M$ be a secant line in $\mathbf{PG}(d, \mathbb{F})$. Let $p_1, p_2, p_3$ be three points of $\Gamma$ on $M$. Consider a point $r$ of $\Gamma$ collinear with both $p_1$ and $p_2$. By (WE2) all points of $\Gamma$ on $M$ are collinear with $r$. If the lines $r_p^1, r_p^2, r_p^3$ lie in a plane of $\mathbf{PG}(d', \mathbb{F}')$, then this must be a plane of $\Gamma'$ and hence $M$ is a line of $\Gamma$, a contradiction. Consequently $r, p_1, p_2, p_3$ generate a 3-dimensional subspace $\mathbf{PG}(3, \mathbb{F}')$ of $\mathbf{PG}(d', \mathbb{F}')$. Let $\mathbf{PG}(4, \mathbb{F}') \supseteq \mathbf{PG}(3, \mathbb{F}')$ intersect $Q'$ in a non-singular quadric $Q'_1$. Suppose the characteristic of $\mathbb{F}'$ is not 2. Then there is a unique second point $s'$ of $Q'_1$ collinear with $p_1', p_2', p_3'$. So $s$ is collinear with $p_1, p_2, p_3$. Since $s$ and $r$ are not collinear in $\Gamma$, $s$ is not in the plane $r p_1 p_2 p_3$ by (WE2). Let $N$ be a line of $\Gamma$ concurrent with $r p_1$ and $s p_2$ in $\Gamma$, but not incident with $r$ or $s$. The line $R$ of $\Gamma$ through $p_3$, meeting $N^*$ lies in the 3-dimensional space $s r p_1 p_2$. By (WE2) $R$ is in the plane $p_3 r s$. Let $w$ be the unique point of $R^*$ collinear with $p_1$; then $w$ is also collinear with $p_2$ (by (WE2)). Clearly $w' \in Q'_1$, a contradiction. Hence the characteristic of $\mathbb{F}'$ is equal to 2.

Let $p_1', p_2', p_3'$ and $r'$ be as above, and let $p_1' p_2' p_3' \cap Q' = C'$; further let $Q'_1$ be as above. Let $s' \neq r'$ be a point of $Q'_1$ collinear with $p_1', p_2'$ ($s'$ exists since $Q'_1$ defines itself a polar space). By (WE2), $s'$ is also collinear with $p_3'$. As in the previous paragraph, we construct the line $R$ and the point $w$. Let $V'$ be a line on $Q'_1$ through $w'$, not containing $p_1', p_2'$. There is a line $L'$ meeting $r' p_1', s' p_2'$ and $V'$, thus implying that $V$ belongs to the space $s r w p_1 p_2 = r s p_1 p_2$. By (WE2), $V$ is contained in the plane $w p_1 p_2$. Let $W$ be a line of $\Gamma$ containing $r$ and meeting $V^*$. Then $W$ is in the
plane \( rp_1p_2 \neq wp_1p_2 \), hence \( V \cap W \) is on \( M \). So \( M \) contains all the points \( x \) such that \( x' \) is on the conic \( C' \). Note that the kernel \( k' \) of \( C' \) coincides with the kernel of \( Q_1' \) (as all tangents \( k'r', k's' \) and \( k'p' \) with \( p' \in C' \) generate the 4-space of \( Q_1' \)). We now show that for any point \( x \) of \( \Gamma \) on \( M \), the point \( x' \) belongs to \( C' \). By (WE2), each point of \( \Gamma \) on \( M \) lies in \((\{p_1, p_2\}^\perp)^\perp \). But \((\{p_1', p_2'\}^\perp)^\perp \) is the intersection of \( Q' \) with either a line (and this happens if and only if \( d' \) is odd) or a plane \( \pi \) (and this happens if and only if \( d' \) is even) containing the kernel \( k' \) of \( Q' \). The first case contradicts \( \delta > 2 \), hence only the latter case occurs. But clearly \( \pi \) must meet \( Q' \) in \( C' \) and our claim follows.

Note that the argument of the previous paragraph also shows that all points of every conic on \( Q' \) lying in a plane which contains the kernel \( k' \) of \( Q' \) correspond to the points of intersection of \( \Gamma \) with some secant line \( M \). Also, every two non-collinear points of \( Q' \) lie in such a unique plane. Projecting \( \Gamma' \) from the kernel \( k' \) onto some hyperplane \( PG(d' - 1, \mathbb{F}') \) not containing \( k' \), we obtain an embedding of \( \Gamma' \) into \( PG(d' - 1, \mathbb{F}') \) such that secant lines of \( \Gamma \) correspond with secant lines of the image \( \Gamma'' \) of \( \Gamma' \) in \( PG(d' - 1, \mathbb{F}') \). Note that if \( \mathbb{F}' \) is perfect, in particular when \( \mathbb{F}' \) is finite, then \( \Gamma'' \) is a non-singular symplectic space and the result follows from the first part of the proof.

(3) Remark that in (1) and (2) the proof does not depend on the rank of \( \Gamma \), as long as it is at least 2.

From now on we use the fact that the rank of the orthogonal polar space \( \Gamma \) is at least 3. By the last part of (2) we may assume that the field \( \mathbb{F}' \) is not perfect. As in paragraph (a) of the proof of Lemma 5, one shows that any set \( L^* \), with \( L \) a line of \( \Gamma \), is a subline of \( L \) over a subfield \( \mathbb{F} \) of \( K \) which is independent of \( L \) (and clearly \( \mathbb{F} \) is isomorphic to \( \mathbb{F}' \)). We now proceed in the same style as in the proof of Lemma 5, adapting the arguments to our present case \( \delta > 2 \).

We denote by \( x'' \) the element of \( \Gamma'' \) in \( PG(d' - 1, \mathbb{F}') \) corresponding to any element \( x \) of \( \Gamma \) in \( PG(d, \mathbb{K}) \). Let \( L_1 \) and \( L_2 \) be two lines of \( \Gamma \) such that in \( PG(d', \mathbb{F}') \) \( L_1' \) and \( L_2' \) span a 3-space which intersects \( Q' \) in a non-singular quadric \( Q^+ \). Let \( Q_1' \) be the intersection of \( Q' \) with the 4-dimensional subspace of \( PG(d', \mathbb{F}') \) generated by \( L_1' \) and \( L_2' \) and the kernel \( k' \) of \( Q_1' \); note that \( Q_1' \) is non-singular. Let \( \Omega \) be the polar subspace of \( \Gamma \) which corresponds with the quadric \( Q^+ \). As in paragraphs (b) and (c) of the proof of Lemma 5, one shows that \( \Omega \) is fully embedded in a unique 3-dimensional subspace \( V \) over \( \mathbb{F} \) of the 3-dimensional subspace \( U \) (over \( \mathbb{K} \)) of \( PG(d, \mathbb{K}) \) generated by \( L_1 \) and \( L_2 \). Let \( V'' \) be the 3-dimensional subspace of \( PG(d' - 1, \mathbb{F}') \) generated by \( L_1'' \) and \( L_2'' \) (where \( L_1'' \) and \( L_2'' \) are the respective projections of \( L_1' \) and \( L_2' \)). Let \( x'' \) be any point of \( \Gamma'' \) in \( V'' \). Then \( x' \in Q_1' \) and since \( Q_1' \) is non-singular, \( x \) is not collinear with all points of \( L_i' \), \( i = 1, 2 \). Suppose \( x' \) does not lie on \( Q^+ \) and let \( y \) be the unique point on \( L_1 \) collinear with \( x \) in \( \Gamma \). Let \( x_1, x_2 \) be two other points of \( \Gamma \) on \( L_1 \). Let \( x \) be the line of \( \Gamma \) containing \( y \) and concurrent with \( L_2 \). The lines \( x'y', L' \) and \( L_i' \) define a cone on \( Q_1' \) and consequently there is a unique conic \( C_i' \) on that cone with kernel \( k' \) and containing \( x' \) and \( x_i' \), \( i = 1, 2 \). These conics correspond with the respective secant lines \( M_1 \) and \( M_2 \) of...
Γ. Hence $M_i$, $i = 1, 2$, contains $x_i$ and another point $y_i$ of $\Gamma$ on $L$. But $x_i, y_i \in V$, hence $M_i$ defines a line of $V$, $i = 1, 2$. Since $x$ is the intersection of $M_1$ and $M_2$, it belongs to $V$. So we obtain a full embedding of the polar subspace of $\Gamma$ determined by $Q_1'$.

Now let $z$ be any other point of $\Gamma$ contained in $U$. If $z$ belongs to $V$ then there is a unique line $M$ in $V$ meeting both $L_1$ and $L_2$ and containing $z$. The extension of $M$ to $\mathbb{K}$ is a secant line of $\Gamma$ and hence it corresponds with a conic on $Q_1'$; hence $z'$ belongs to $Q_1'$, a contradiction.

Suppose now $z \in U \setminus V$. Considering the polar subspace of $\Gamma$ generated by $L_1$, $L_2$ and their points in $\Gamma$, one shows as in paragraph (d) of the proof of Lemma 5 that $z \in V$, a contradiction. Hence the only points $x$ of $\Gamma$ in $U$ satisfy $x' \in Q_1'$.

As in paragraphs (e), (f), (g) and (h) of the proof of Lemma 5 we use an inductive argument. The assumption is that any $(2c - 1)$-dimensional non-singular orthogonal subspace $\Gamma_1$ of $\Gamma$, whose corresponding subspace $V_1'$ in $\text{PG}(d', \mathbb{F}')$ contains $k'$, is fully embedded in a $(2c - 1)$-dimensional projective subspace $V_1$ over $\mathbb{F}$ of $\text{PG}(d, \mathbb{K})$, $2 \leq c < \frac{d}{2}$. We want to show that every $(2c + 1)$-dimensional non-singular orthogonal subspace $\Gamma_2$ of $\Gamma$, whose corresponding subspace of $\text{PG}(d', \mathbb{F}')$ contains $k'$, is fully embedded in a $(2c + 1)$-dimensional projective subspace over $\mathbb{F}$ of $\text{PG}(d, \mathbb{K})$.

Let $\Gamma_2$ be a $(2c + 1)$-dimensional non-singular subspace of $\Gamma$, whose corresponding subspace $V_2'$ of $\text{PG}(d', \mathbb{F}')$ contains $k'$, $2 \leq c < \frac{d}{2}$. Further, let $\Gamma_1$ be a $(2c - 1)$-dimensional non-singular subspace of $\Gamma_2$, whose point set corresponds to the set of all points of $\Gamma_2'$ collinear with two given non-collinear points $u'$ and $v'$ of $\Gamma_2$. Then the subspace $V_1'$ of $\text{PG}(d', \mathbb{F}')$ containing $\Gamma_1'$, also contains the kernel $k'$. Hence $\Gamma_1$ is fully embedded in a $(2c - 1)$-dimensional projective subspace $V_1$ over $\mathbb{F}$ of $\text{PG}(d, \mathbb{K})$.

First, suppose there is a point $x$ of $\Gamma_2 \setminus \Gamma_1$ with the property that the subspace $V_3'$ of $\text{PG}(d', \mathbb{F}')$ generated by $V_1'$ and $x'$ meets the point set of $\Gamma_2'$ in a non-degenerate quadric $Q_3'$, i.e. the singular point of $Q_3'$ lies in a proper extension of $V_3'$ over some extension field $\mathbb{F}_1$ of $\mathbb{F}$, but not in $V_3'$ itself. Let $U_1$ be the extension of $V_1$ over $\mathbb{K}$. We first show that $U_1$ does not contain any point of $\Gamma_3 \setminus \Gamma_1$, where $\Gamma_3$ is the polar subspace of $\Gamma$ which corresponds to $Q_3'$. Let the point $z$ of $\Gamma_3 \setminus \Gamma_1$ belong to $U_1$. Since $\Gamma_3$ is generated by $\Gamma_1$ and $z$, all points of $\Gamma_3$ belong to $U_1$. All points of $\Gamma_1$ are collinear with $u$. Since the point set of $\Gamma_1$ generates $U_1$, by (WE2) all points of $\Gamma_3$ are collinear with $u$. As $\Gamma_3$ is non-degenerate the point $u$ does not belong to $\Gamma_3$, and so the set of all points of $\Gamma_3$ collinear with $u$ is just the point set of $\Gamma_1$. This yields a contradiction. Consequently no point of $\Gamma_3 \setminus \Gamma_1$ is contained in $U_1$. Similarly to parts (f) and (g) of the proof of Lemma 5 we can now show that $\Gamma_3$ is fully embedded in a subspace $\text{PG}(2c, \mathbb{F})$ of $\text{PG}(d, \mathbb{K})$. Let $\text{PG}(2c, \mathbb{K})$ be the extension of $\text{PG}(2c, \mathbb{F})$ over $\mathbb{K}$. Assume, by way of contradiction, that $\text{PG}(2c, \mathbb{K})$ contains a point $r$ of $\Gamma_2 \setminus \Gamma_3$. Since $\Gamma_2$ is generated by $\Gamma_3$ and $r$, all points of $\Gamma_2$ belong to $\text{PG}(2c, \mathbb{K})$. Hence $u$ belongs to $\text{PG}(2c, \mathbb{K})$. By (WE2) the points $u$ and $v$ belong to the $(2c - 1)$-dimensional space $U_1$. Since $\Gamma_2$ is generated by $\Gamma_1$, $u$
and \( v \), the polar space \( \Gamma_2 \) belongs to \( U_1 \). Hence \( \Gamma_3 \) belongs to \( U_1 \), a contradiction. Consequently no point of \( \Gamma_2 \setminus \Gamma_3 \) is contained in \( \text{PG}(2c, \mathbb{K}) \). Similarly to parts (f) and (g) of the proof of Lemma 5 we now show that \( \Gamma_2 \) is fully embedded in a subspace \( \text{PG}(2c + 1, \mathbb{F}) \) of \( \text{PG}(d, \mathbb{K}) \).

Next, suppose that for each point \( x \) of \( \Gamma_2 \setminus \Gamma_1 \) the subspace \( V'_3 \) of \( \text{PG}(d', \mathbb{F}') \) generated by \( V'_1 \) and \( x' \) meets the point set of \( \Gamma'_2 \) in a degenerate quadric \( Q'_3 \), that is, the singular point \( y'_3 \) of \( O'_3 \) belongs to \( V'_3 \). The set of all singular points \( y'_3 \) is a non-singular conic \( C' \) with kernel \( k' \). Let \( L' \) be any line through \( k' \) in the plane \( \pi' \) of \( C' \). Then the \((2c + 1)\)-dimensional space generated by \( V'_1 \) and \( L' \) intersects the point set of \( \Gamma'_2 \) in a degenerate quadric with singular point on \( C' \) and \( L' \). It follows that each line \( L' \) in \( \pi' \) through \( k' \) contains a point of \( C' \). Consequently the field \( \mathbb{F}' \) is perfect, a contradiction.

As in (h) of the proof of Lemma 5, induction now shows that \( d = d' - 1 \) and that \( \Gamma \) is fully embedded in a subspace \( \text{PG}(d, \mathbb{F}) \) of \( \text{PG}(d, \mathbb{K}) \).

(4) Finally suppose that \( \Gamma \) is a non-singular unitary polar space of rank at least 3 arising from some hermitian variety \( \mathcal{H}' = H(d', \mathbb{F}', \sigma) \) in \( \text{PG}(d', \mathbb{F}') \) with \( \sigma \) an involutory field automorphism of \( \mathbb{F}' \). Again we can copy part (a) of the proof of Lemma 5. As in (b) of that proof we can choose two lines \( L_1 \) and \( L_2 \) of \( \Gamma \) generating a 3-space \( U \) of \( \text{PG}(d, \mathbb{K}) \). In \( \Gamma \) the lines \( L_1 \) and \( L_2 \) and their points generate a non-singular polar space \( \Omega \) which corresponds to a hermitian surface \( \mathcal{H}'_3 \) (of a 3-space) on \( \mathcal{H}' \). Now \( L_1 \) and \( L_2 \) (but not all their points) are contained in a polar subspace \( \Omega_0 \) corresponding to a symplectic space \( W(3, \mathbb{F}_\sigma) \) in a 3-dimensional subspace \( \text{PG}(3, \mathbb{F}_\sigma) \) of \( \text{PG}(d, \mathbb{F}) \) over the field \( \mathbb{F}_\sigma \) which consists of all elements of \( \mathbb{F}' \) fixed by \( \sigma \). By part (1) of this proof we know that there exists a subfield \( \mathbb{F}_\sigma \) of \( \mathbb{K} \) isomorphic to \( \mathbb{F}_\sigma \) and a 3-dimensional subspace \( V_\sigma \) of \( \text{PG}(d, \mathbb{K}) \) over \( \mathbb{F}_\sigma \) such that \( \Omega_0 \) is fully embedded in \( V_\sigma \). We also know that for any line \( L \) of \( \Gamma \) the set \( L^* \) is a projective subline of \( L \) in \( \text{PG}(d, \mathbb{K}) \) over some field \( \mathbb{F} \), which is independent of \( L \). Evidently \( \mathbb{F} \) contains \( \mathbb{F}_\sigma \). Let \( V \) be the extension of \( V_\sigma \) over \( \mathbb{F} \). Let \( L \) be a line of \( \Omega_0 \) and let \( x \) be a point on \( L \) belonging to \( \Omega \setminus \Omega_0 \). Then clearly \( x \) lies in \( V \). We will show that every point \( x \) of \( \Omega \) lies on a line of \( \Omega_0 \).

Let \( x \) be an arbitrary point of \( \Omega \setminus \Omega_0 \) and let \( x' \) be the corresponding point of \( \mathcal{H}'_3 \). Since \( \text{PG}(3, \mathbb{F}_\sigma) \) is a Baer subspace of \( \text{PG}(3, \mathbb{F}) \), there is a unique line \( L' \) of \( \text{PG}(3, \mathbb{F}_\sigma) \) containing \( x' \). If \( L' \) were not a line of \( W(3, \mathbb{F}_\sigma) \), then it would meet \( \mathcal{H}'_3 \) in a subline of \( L' \) over \( \mathbb{F}_\sigma \), hence \( x' \) would be a point of \( \text{PG}(3, \mathbb{F}_\sigma) \), a contradiction. So \( L' \) is a line of \( \mathcal{H}'_3 \) (alternatively, this can be easily seen by considering the dual generalized quadrangle). The corresponding line \( L \) of \( \Omega \) is incident with \( x \) and belongs to \( \Omega_0 \). Hence \( \Omega \) is fully embedded in \( V \) and \( U \) is the extension of \( V \) over \( \mathbb{K} \).

Now we show that no other point of \( \Gamma \) belongs to \( U \). Suppose, by way of contradiction, that the point \( z \) of \( \Gamma \) lies in \( U \) but is not contained in \( \Omega \). Let \( z' \) be the corresponding point of \( \mathcal{H}' \). If \( T' \) is the set of all points of \( \mathcal{H}'_3 \) collinear with \( z' \), then either \( \mathcal{H}'_3 = T' \), or \( T' \) is a non-singular hermitian curve, or \( T' \) is a singular hermitian curve. Let \( T \) be the corresponding point set of \( \Omega \). First, let \( \mathcal{H}'_3 = T' \).
Noticing that for every point \( y \) of \( \Omega \), the space generated by \( y^\perp \) in \( \text{PG}(d, \mathbb{K}) \) meets \( U \) in a plane (by axiom (WE2)), we see that \( z \) must lie in every plane containing two intersecting lines of \( \Omega \). Hence the extensions over \( \mathbb{K} \) of all tangent planes of the unitary polar space \( \Omega \) (the point set of \( \Omega \) is a hermitian variety of \( V \)) have a common point, clearly a contradiction. Hence \( \mathcal{H}_3 \neq T' \). Then, by (WE2), \( T \) and \( z \) are contained in a common plane \( \text{PG}(2, \mathbb{K}) \). Assume that \( T' \) is a singular hermitian curve, with singular point \( u' \). Let \( r' \in T' \setminus \{u'\} \). As \( r \) is collinear with \( u \) and \( z \) in \( \Gamma \), by (WE2) it is collinear in \( \Gamma \) with all points of \( T \), clearly a contradiction. Finally, let \( T' \) be a non-singular hermitian curve. Let \( s \) be any point of \( T \), and let \( M_1, M_2 \) be any two distinct lines of \( \Omega \) through \( s \). By (WE2) the lines \( M_1, M_2 \), \( zs \) are contained in a common plane, which is the extension over \( \mathbb{K} \) of the tangent plane of the unitary polar space \( \Omega \) at \( s \). Hence \( z \) belongs to the extensions of all tangent planes of \( \Omega \) at points of \( T \), so \( z \) belongs to \( V \). It follows that all tangent lines of the hermitian curve \( T \) concur at \( z \), a contradiction. We conclude that the only points of \( \Gamma \) in \( U \) are the points of \( \Omega \).

As in paragraphs (e), (f), (g) and (h) of the proof of Lemma 5 (and as in (3) of the present proof) we use an inductive argument. Let \( \Gamma_1 \) be the polar subspace of \( \Gamma \) arising from a non-degenerate hermitian subvariety \( \mathcal{H}'_1 \) of \( \mathcal{H}' \) containing lines, and obtained from \( \mathcal{H}' \) by intersecting it with a \( c \)-dimensional subspace \( W'_1 \) of \( \text{PG}(d', \mathbb{F'}) \), \( 3 \leq c < d' \). Suppose that \( \Gamma_1 \) is fully embedded in a \( c \)-dimensional subspace \( V_1 \) over \( \mathbb{F} \) of \( \text{PG}(d, \mathbb{K}) \). Let \( \Gamma_2 \) be the polar subspace of \( \Gamma \) arising from a non-degenerate hermitian subvariety \( \mathcal{H}'_2 \) of \( \mathcal{H}' \) obtained from \( \mathcal{H}' \) by intersecting it with a \( (c + 1) \)-dimensional subspace \( W'_2 \) of \( \text{PG}(d', \mathbb{F'}) \) containing \( W'_1 \). Then we will show that \( \Gamma_2 \) is fully embedded in some \((c + 1)\)-dimensional subspace \( V_2 \) over \( \mathbb{F} \) of \( \text{PG}(d, \mathbb{K}) \). Let \( x \) be a point of \( \Gamma_2 \setminus \Gamma_1 \). Let \( U_1 \) be the extension of \( V_1 \) over \( \mathbb{K} \). Suppose by way of contradiction that \( x \) belongs to \( U_1 \). The points of \( \Gamma_1 \) collinear with \( x \) in \( \Gamma_2 \) form a point set \( \mathcal{H}_3 \) corresponding to a non-singular hermitian subvariety \( \mathcal{H}'_3 \) of \( \mathcal{H}'_1 \) obtained by intersecting \( \mathcal{H}'_1 \) with a hyperplane of \( W'_1 \). By (WE2), \( x \) must belong to the extension over \( \mathbb{K} \) of every hyperplane of \( V_1 \) tangent to \( \Gamma_1 \) at a point of \( \mathcal{H}_3 \). Also by (WE2), \( x \) and \( \mathcal{H}_3 \) are contained in a common hyperplane \( W_3 \) of \( U_1 \). As the polar space with point set \( \mathcal{H}'_1 \) is generated by \( \mathcal{H}'_3 \) and any point of \( \mathcal{H}'_1 \setminus \mathcal{H}'_3 \), also \( \Gamma_1 \) is generated by \( \mathcal{H}_3 \) and any point of \( \Gamma_1 \) not in \( \mathcal{H}_3 \). Hence \( \mathcal{H}_3 \) generates a hyperplane \( R_3 \) of \( V_1 \). Clearly \( W_3 \) is the extension over \( \mathbb{K} \) of the hyperplane \( R_3 \). It follows that the extensions over \( \mathbb{K} \) of the tangent hyperplanes of \( \Gamma_1 \) at points of \( \mathcal{H}_3 \) intersect in a unique point which belongs to \( V_1 \setminus R_3 \). Hence \( x \not\in W_3 \), a contradiction. Consequently no point of \( \Gamma_2 \setminus \Gamma_1 \) belongs to \( U_1 \). Let \( L \) be any line of \( \Gamma_2 \setminus \Gamma_1 \); then \( L^* \) defines a projective subline over \( \mathbb{F} \) and hence there is a unique \((c + 1)\)-dimensional subspace \( V_2 \) over \( \mathbb{F} \) of \( \text{PG}(d, \mathbb{K}) \) containing \( V_1 \) and all elements of \( L^* \). We now show that all points of \( \Gamma_2 \) are contained in \( V_2 \). Let \( x \) be any point of \( \Gamma_2 \). Clearly we may assume that \( x \) does not belong to \( \Gamma_1 \) nor to \( L^* \).

In the sequel, we again denote the corresponding element in \( \text{PG}(d', \mathbb{F'}) \) of an element \( e \) of \( \Gamma \) by \( e' \).
First suppose that \( x \) is collinear in \( r_2 \) with a point \( y \in L^* \) which does not belong to \( r_1 \). All points of the line \( x'y' \) belong to \( \mathcal{H}'_2 \) and hence there is a unique point \( z' \) of \( x'y' \) in \( \mathcal{H}'_2 \). Let \( w \) be the unique point of \( \Gamma_1 \) on \( L^* \). The line \( wz \) is either a line of \( \Gamma_1 \) or a secant line. In the first case the points of \( r_2 \) in the plane \( xwz \) of \( \mathbf{PG}(d, \mathbb{K}) \) form a projective subplane over \( \mathbb{F} \) sharing all points of at least two lines with \( V_2 \). Hence all points of that subplane belong to \( V_2 \) and so does \( x \). In the second case let \( u \) be any point of \( \Gamma_1 \) on \( wz, \ w \neq u \neq z \) (this is possible by the assumption \( \delta > 2 \)). By Proposition 4 of [5] the line \( xu \) meets \( L \) in a point of \( \Gamma \). Hence both \( xu \) and \( xz \) are lines of \( V_2 \) and the result follows.

Now suppose that \( x \) is not collinear in \( \Gamma_2 \) with an element of \( L^* \) not belonging to \( \Gamma_1 \). By the Buekenhout–Shult axiom \( x \) is collinear in \( \Gamma_2 \) with the unique point \( w \) of \( L^* \) in \( \Gamma_1 \). Let \( y \in L^*, \ y \neq w \). It is easy to see that there is at most one point on the line \( y'w' \) collinear in \( \mathcal{H}'_2 \) to all points of \( \mathcal{H}'_1 \) which are collinear to \( x' \) (since all such points belong to a secant line of \( \mathcal{H}'_2 \)). So there is a point \( y_1 \neq w \) on \( L^* \) and a point \( r \) of \( \Gamma_1 \) collinear with \( y_1 \) in \( \Gamma_2 \), but not collinear with \( x \) in \( \Gamma_2 \). By the Buekenhout–Shult axiom, there exists a unique line \( M \) of \( \Gamma_2 \) incident with \( x \) and containing a point \( s \) of \( \Gamma_2 \) on the line \( ry_1 \). By assumption \( s \neq r \), so \( s \) does not belong to \( \Gamma_1 \). By the previous paragraph, all points of \( \Gamma \) on \( ry_1 \) belong to \( V_2 \). Interchanging the roles of \( ry_1 \) and \( L \), we now see that \( x \) belongs to \( V_2 \). We conclude that \( \Gamma_2 \) is fully embedded in a \((c + 1)\)-dimensional subspace over \( \mathbb{F} \) of \( \mathbf{PG}(d, \mathbb{K}) \). Applying this for \( c = 3, 4, \ldots, d' - 1 \), we finally obtain that \( \Gamma \) is fully embedded in some \( \mathbf{PG}(d', \mathbb{F}) \) from which immediately follows that \( d' = d \).

This completes the proof of the lemma.

The previous lemmas prove Theorem 1.

Remarks 1. When \( \Gamma \) arises from a non-degenerate but singular quadric (and that can only happen if the characteristic of the ground field \( \mathbb{F}' \) is equal to 2), Theorem 1 is not valid. For example consider in \( \mathbf{PG}(7, \mathbb{F}') \) where \( \mathbb{F}' \) is a non-perfect field with characteristic 2, the quadric \( Q \) with equation

\[
X_0^2 + X_1^2 + X_0 X_1 + X_2^2 + aX_3^2 + X_4^2 + X_5^2 + X_4X_5 + X_6X_7 = 0,
\]

where \( a \in \mathbb{F}' \) is a non-square. Let \( \mathbb{K} \) be the algebraic closure of \( \mathbb{F}' \) and let \( \mathbf{PG}(7, \mathbb{K}) \) be the corresponding extension of \( \mathbf{PG}(7, \mathbb{F}') \). The point \( x(0, 0, \sqrt{a}, 1, 0, 0, 0, 0) \) is the unique singular point of \( Q \). If we project \( Q \) from \( x \) onto a hyperplane \( \mathbf{PG}(6, \mathbb{K}) \) of \( \mathbf{PG}(7, \mathbb{K}) \) which does not contain \( x \), then we obtain a weakly embedded polar space which is not fully embedded in any subspace \( \mathbf{PG}(6, \mathbb{F}) \) for any subfield \( \mathbb{F} \) of \( \mathbb{K} \). In a forthcoming paper, we will classify sub-weakly embedded singular polar spaces, degenerate or not, arising from quadrics, symplectic polarities or hermitian varieties.

2. When \( \Gamma \) has \( \delta = 2 \) and arises from a non-singular symplectic polar space of rank at least three over a non-perfect field of characteristic two, then Theorem 1 is not valid. We give an example. Let \( \mathbb{K} \) be a field of characteristic two for which the subfield \( \mathbb{F} \) of squares is not perfect. Then also \( \mathbb{K} \) is not perfect. Now consider in
PG(6, K) the set S of points \((x_0, x_1, \ldots, x_6)\) with \(x_0, x_1, \ldots, x_5 \in \mathbb{F}, x_6 \in K\), and lying on the quadric \(Q\) with equation
\[X_0X_3 + X_1X_4 + X_2X_5 = X_6^2.\]

Then \(S\), provided with lines and planes induced by \(Q\), is a polar space \(\Gamma\) isomorphic to the non-singular symplectic polar space \(W(5, \mathbb{F})\) in PG(5, \(\mathbb{F}\)) by projecting \(S\) from \((0, 0, 0, 0, 0, 1)\) into the subspace \(U\) with equation \(X_6 = 0\) over \(\mathbb{F}\). Clearly \(\Gamma\) is sub-weakly embedded in PG(6, \(\mathbb{K}\)). Let \(e_i, 0 \leq i \leq 5\), be the point of PG(6, \(\mathbb{K}\)) with all coordinates 0 except the \((i + 1)\)th coordinate, which is equal to 1. Let \(e\) be the point all coordinates of which are equal to 1 and let \(e_0\) be the point with coordinates \((1, 1, 0, 0, 0, 0)\). Then it is easy to see that the set \(V\) of points of \(S\) on the lines \(e_ie_{i+1}, i \in \{0, 1, \ldots, 4\}\), on \(e_0e_5\) and on \(ee_0\) generates the subspace PG(6, \(\mathbb{F}\)) of PG(6, \(\mathbb{K}\)) consisting of all points with coordinates in \(\mathbb{F}\). Hence, if \(S\) were fully embedded in a subspace of PG(6, \(\mathbb{K}\)) over a subfield of \(\mathbb{K}\), then this subspace would be PG(6, \(\mathbb{F}\)). As \(S\) contains the point \((0, 0, 1, 0, 0, a^2, a), a \in K\backslash F\), the polar space \(\Gamma\) is not fully embedded in a subspace of PG(6, \(\mathbb{K}\)).

3. Proof of Theorem 2

(i) First suppose that the non-degenerate quadric \(Q\) does not contain lines. Since by assumption the points of \(Q\) span PG(\(d, \mathbb{F}\)), we may assume that \(e_i = (0, \ldots, 0, 1, 0, \ldots, 0)\), where the 1 is in the \(i\)th position, lies on \(Q\) for every \(i\). The plane \(e_ie_je_k\), \(1 \leq i < j < k \leq d + 1\), meets \(Q\) in a non-singular non-empty conic. Assume that the coefficient of \(X_iX_m\) in a fixed equation for \(Q\) over \(\mathbb{F}\) is \(a_{\ell m} = a_{m\ell}\). Let the quadric \(Q'\) of PG(\(d, \mathbb{K}\)), with \(\mathbb{K}\) an extension of \(\mathbb{F}\) and PG(\(d, \mathbb{K}\)) the corresponding extension of PG(\(d, \mathbb{F}\)), contain \(Q\). The coefficient of \(X_iX_m\) in a fixed equation for \(Q'\) over \(\mathbb{K}\) is denoted by \(a'_{\ell m} = a'_{m\ell}\). If \(|\mathbb{F}| \geq 4\), then, either \(e_ie_je_k \cap Q'\) is a non-singular non-empty conic or the plane \(e_ie_je_k\) itself. As a non-singular non-empty conic is uniquely defined by any five of its points, we have \(a'_{\ell m} = c_{\{i,j,k\}} a_{\ell m}\) with \(\ell, m \in \{i, j, k\}\) and \(c_{\{i,j,k\}} \in \mathbb{K}\) (as \(e_ie_je_k \cap Q\) is non-singular we have \(a_{\ell m} \neq 0\)). By fixing \(i\) and \(j\) we see that \(c_{\{i,j,k\}} = c_{\{i,j,k'\}}\), for every \(k, k'\) and now it is easy to see that \(c_{\{i,j,k\}}\) is a constant \(c\); it is clear that \(c \neq 0\), whence the result for \(|\mathbb{F}| \geq 4\).

Suppose now \(|\mathbb{F}| = 3\). As \(Q\) does not contain lines we have \(d \in \{2, 3\}\). For \(d = 2\), there are indeed distinct conics in PG(2, \(\mathbb{K}\)), where \(\mathbb{K}\) is a field of characteristic 3 with \(|\mathbb{K}| > 3\), containing the four points of a conic in a subplane isomorphic with PG(2, 3), and the same remark holds for \(|\mathbb{F}| = 2 \text{ and } d = 2\). If \(d = 3\) and \(|\mathbb{F}| = 3\), then a direct and straightforward computation shows that the ten points of \(Q\) are on a unique quadric in every extension PG(3, \(\mathbb{K}\)). For \(|\mathbb{F}| = 2 \text{ and } d = 3\), the five points of \(Q\) are contained in several non-singular quadrics over every proper extension of \(\mathbb{F}\). This completes the case where \(Q\) does not contain lines.

Now suppose that \(Q\) contains lines. Let \(Q'\) be a quadric in PG(\(d, \mathbb{K}\)) containing \(Q\), with \(\mathbb{K}\) an extension of \(\mathbb{F}\) and PG(\(d, \mathbb{K}\)) the corresponding extension of PG(\(d, \mathbb{F}\)).
Again we can assume that \( e_i \in Q \) for all \( i \). Let \( a_{ij} = a_{ji} \) respectively \( a'_{ij} = a'_{ji} \) be the coefficient of \( x_i x_j \) in the equation of \( Q \) respectively \( Q' \). The tangent hyperplane \( U_i \) of \( Q \) at \( e_i \) is spanned by all lines through \( e_i \) contained in \( Q \). If \( e_i \) is not singular for \( Q' \), then also the tangent hyperplane \( U'_i \) of \( Q' \) at \( e_i \) is spanned by all lines through \( e_i \) contained in \( Q' \); in such a case the hyperplane \( U_i \) is necessarily a subhyperplane of \( U'_i \). The equation of \( U_i \) is \( \sum_j a_{ij} x_j = 0 \) (note that \( a_{ii} = a'_{ii} = 0 \) for all \( i \)). If \( e_i \) is not singular for \( Q' \), then the equation of \( U'_i \) is \( \sum_j a'_{ij} x_j = 0 \); if \( e_i \) is singular for \( Q' \), then \( a'_{ij} = 0 \) for all \( j \). From the foregoing it follows that \( a'_{ij} = c_i a_{ij} \) for all \( j \), with \( c_i \in \mathbb{K} \). Hence if \( a_{ij} = 0 \), then also \( a'_{ij} = 0 \). Now consider \( 1 \leq i < j \leq d + 1 \) and \( 1 \leq k < \ell \leq d + 1 \) with \( \{i, j\} \cap \{k, \ell\} = \emptyset \) and suppose that \( a_{ij} \neq 0 \neq a_{k\ell} \). From the preceding it immediately follows that if \( a_{ik}, a_{i\ell}, a_{jk} \) and \( a_{j\ell} \) are not all zero, then

\[
\frac{a'_{ij}}{a_{ij}} = \frac{a'_{k\ell}}{a_{k\ell}}.
\]

On the other hand, if \( a_{ik} = a_{i\ell} = a_{jk} = a_{j\ell} = 0 \), then the same equality follows from considering the tangent hyperplane of \( Q \) at the point \( e_{ik} = (0, \ldots, 0, 1, 0, \ldots, 0, 1, 0, \ldots, 0) \), with the 1 in the \( i \)th and the \( k \)th position, from considering the tangent hyperplane of \( Q' \) at \( e_{ik} \) if this point is not singular for \( Q' \) (if this point is singular for \( Q' \), then \( a'_{ij} = a'_{k\ell} = 0 \)), and from considering the coefficients of \( x_j \) and \( x_{i} \) in the equations of these hyperplanes. Now it immediately follows that \( Q' \) is uniquely determined by \( Q \).

(ii) The proof is similar to the last part of (i) and in fact it can be simplified a great deal because we can immediately use standard equations.

(iii) First suppose that the non-singular non-empty hermitian variety \( H \) does not contain lines. Since the points of \( H \) span \( \text{PG}(d, \mathbb{F}) \), \( d \geq 2 \), we may assume that \( e_i = (0, \ldots, 0, 1, 0, \ldots, 0) \), where the 1 is in the \( i \)th position, lies on \( H \) for every \( i \). The plane \( e_i e_j e_k \), \( 1 \leq i < j < k \leq d + 1 \), meets \( \tilde{H} \) in a non-singular non-empty hermitian curve \( C \). Assume that the coefficient of \( x_i x_j \) in a fixed equation for \( H \) over \( \mathbb{F} \) is \( a_{\ell m} \). Let \( \mathbb{K} \) be a field containing \( \mathbb{F} \) admitting a \( \mathbb{K} \)-involution \( \tau \) the restriction of which to \( \mathbb{F} \) is \( \sigma \), let \( \text{PG}(d, \mathbb{K}) \) be the corresponding extension of \( \text{PG}(d, \mathbb{F}) \), and let the hermitian variety \( H' \) of \( \text{PG}(d, \mathbb{K}) \) contain \( H \). The coefficient of \( x_i x_j \) in a fixed equation for \( H' \) over \( \mathbb{K} \) is denoted by \( a'_{\ell m} \). The intersection of \( C \) with the line \( e_i e_j \) is determined by the equation \( a_{ij} x_i x_j + a_{ji} x_j x_i = 0 \) (as \( C \) is non-singular we have \( a_{ij} \neq 0 \)). For each point of that intersection also the equation \( a'_{ij} x_i x_j + a'_{ji} x_j x_i = 0 \) is satisfied. Let \((0, \ldots, 0, 1, 0, \ldots, 0, u, 0, \ldots, 0)\) be a point of \( C \cap e_i e_j \) with \( u \neq 0 \). Then \( a_{ij} u + a_{ji} u = a'_{ij} u + a'_{ji} u = 0 \). Hence

\[
\frac{a'_{ij}}{a_{ij}} = \frac{a'_{ji}}{a_{ji}}.
\]


Let us now consider a point \((0, \ldots, 0, 1, 0, \ldots, 0, u, 0, \ldots, 0, v, 0, \ldots, 0)\) of \(C \cap e_i e_j \in k\) with the \(u\) as above and \(v \neq 0\). Then
\[
a_{ik} v^\sigma + a_{ki} v + a_{jk} u v^\sigma + a_{kj} v^2 u^\sigma = a'_{ik} v^\sigma + a'_{ki} v + a'_{jk} u v^\sigma + a'_{kj} v u^\sigma = 0.
\]
As
\[
\frac{a'_{ik}}{a_{ik}} = \frac{a'_{ki}}{a_{ki}} \quad \text{and} \quad \frac{a'_{jk}}{a_{jk}} = \frac{a'_{kj}}{a_{kj}},
\]
we have
\[
a_{ik} v^\sigma + a_{ki} v + a_{jk} u v^\sigma + a_{kj} v^2 u^\sigma = b(a_{ik} v^\sigma + a_{ki} v) + c(a_{jk} u v^\sigma + a_{kj} v u^\sigma)
\]
\[= 0,
\]
with \(b, c \in \mathbb{K}\). Assume, by way of contradiction, that
\[
\begin{align*}
a_{ij} u^\sigma + a_{ji} u &= 0, \\
a_{ik} v^\sigma + a_{ki} v &= 0, \\
a_{jk} u v^\sigma + a_{kj} v u^\sigma &= 0.
\end{align*}
\]
Then it readily follows that \(a_{ij} a_{jk} a_{ki} + a_{ji} a_{ik} a_{kj} = 0\). As \(C\) is non-singular, we have \(a_{ij} a_{jk} a_{ki} + a_{ji} a_{ik} a_{kj} \neq 0\), a contradiction. Hence \(a_{ik} v^\sigma + a_{ki} v\) and \(a_{jk} u v^\sigma + a_{kj} v u^\sigma\) are not both zero, so that \(b = c\). Hence
\[
\frac{a'_{ik}}{a_{ik}} = \frac{a'_{ki}}{a_{ki}} = \frac{a'_{jk}}{a_{jk}} = \frac{a'_{kj}}{a_{kj}}.
\]
Now it readily follows that \(H'\) is uniquely determined by \(H\).

Now suppose that \(H\) contains lines. If the line \(e_i e_j, i \neq j\), does not belong to \(H\), then as in the first part of (iii) we obtain
\[
\frac{a'_{ij}}{a_{ij}} = \frac{a'_{ji}}{a_{ji}}.
\]
If the line \(e_i e_j, i \neq j\), belongs to \(H\), then \(a_{ij} = a_{ji} = a'_{ij} = a'_{ji} = 0\). Now we proceed as in the second part of the proof of (i).

Remark In the finite case, any \(\mathbf{GF}(q^2)\) contains a unique involution. But in the infinite case, examples arise where distinct choices for \(\tau\) can be made. For instance, one can extend the unique involution \(x \mapsto x^q\) of \(\mathbf{GF}(q^2)\), \(q\) odd, to the involutions \(\sum a_i t^i \mapsto \sum a_i^q t^i\) and \(\sum a_i t^i \mapsto \sum a_i t (-t)^i\) of \(\mathbf{GF}(q^2)(t)\).

References