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Uniqueness of linear periods

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1. Introduction

We let k be a local non-Archimedean field of characteristic zero with a finite residual field. We denote by G_n the group $\mathrm{GL}(n)$ regarded as an algebraic group over k. We let $p \geqslant 1$, $q \geqslant 1$ be two integers with $p+q=n$ and denote by $H = H_{p,n}$ the subgroup of G_n of matrices of the form:

$$h = \begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix} \quad \text{with} \quad g_1 \in G_p\,,\, g_2 \in G_q. \tag{1}$$

Suppose that π is an admissible irreducible representation of G_n on a complex vector space V. We let $\mathrm{Hom}_H(\pi, 1)$ be the space of H invariant linear forms on V, i.e. linear forms T on V such that $T(\pi(h)v) = T(v)$ for all $v \in V$ and $h \in H$. Our main result is the following one:

THEOREM 1.1. *For any irreducible admissible representation* π

$$\dim \mathrm{Hom}_H(\pi, 1) \leqslant 1.$$

Furthermore, if $\dim\mathrm{Hom}_H(\pi, 1) = 1$ *then* π *is equivalent to the contragredient representation* $\tilde{\pi}$.

If $\dim\mathrm{Hom}_H(\pi, 1) \neq 0$, we say that π is H distinguished. The importance of this statement comes from the following result. We consider the special case where $p = q$ (and n is even). We let k be a number field. Suppose that π is an automorphic cuspidal representation of $G_n(\mathbb{A})$ with trivial central character. For a form ϕ in the space of π we consider the 'period integral'

$$P(\phi) = \int_{Z(\mathbb{A})H(k)\backslash H(\mathbb{A})} \phi(h)\,\mathrm{d}h,$$

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where Z is the center of G_n. Then the integral $P(\phi)$ is non-zero for some $\phi \in \pi$ if and only if the (partial) exterior square L-function attached to π has a pole at $s = 1$ and the standard L-function $L(s, \pi)$ does not vanish at $s = \frac{1}{2}$ (see [FJ] and [BF]). If this is the case, then the integral defines on the space of π an $H(\mathbb{A})$ invariant linear form. The local components π_v of π are thus H_v-distinguished. The above integral is then given by an Eulerian product in the following sense. There exists an embedding r of $\bigotimes \pi_v$ into the space of cusp forms of $G_m(\mathbb{A})$. If

$$\phi = r\,(\otimes_v \phi_v)$$

then

$$P(\phi) = L(1/2, \pi) \prod_v T_v(\phi_v),$$

where T_v is a certain canonical element of the space

$$\mathrm{Hom}_{H_v}(\pi_v, 1)$$

which is one dimensional if v is finite. In the above formula, at almost all finite places v, the representation π_v is spherical, the vector ϕ_v is invariant under the standard maximal compact subgroup and $T_v(\phi_v) = 1$. This is proved in [FJ] without using the previous theorem. However, it is clear that the theorem could be used also to establish (in part) this assertion and will be used in any application of the period integral to the study of the L-function at $\frac{1}{2}$.

At this point it is natural to go back to a local situation and ask for an explicit construction of a linear form invariant under H. We discuss only the most interesting case where $p = q$. (For some partial results on the general case see [FJ]). To that end, we introduce the parabolic subgroup $P_p = HU_p$ of type (p, p). Its unipotent radical U_p is the subgroup of matrices of the form:

$$u = \begin{pmatrix} I_p & Z \\ 0 & I_p \end{pmatrix}. \tag{2}$$

Let ψ be a non-trivial character of k. We define a character Ψ of U_p by:

$$\Psi(u) = \psi(\mathrm{Tr}(Z)).$$

Then the stabilizer of Ψ in H is the subgroup H_0 of matrices of the form:

$$h_0 = \begin{pmatrix} g & 0 \\ 0 & g \end{pmatrix}. \tag{3}$$

Suppose that π is an admissible irreducible representation of $\mathrm{GL}(2n)$ on a complex vector space V. Then a Shalika functional on V is a linear form l such that

$$l(\pi(uh_0)v) = \Psi(u)l(v),$$

for $u \in U_p$ and $h_0 \in H_0$. Assuming the existence of a Shalika functional $l \neq 0$, we construct an H invariant linear form as follows. We consider the integral

$$H(v,s) = \int_{G_n} l\left(\pi\begin{pmatrix} g & 0 \\ 0 & I \end{pmatrix} v\right) \mid \det g \mid^{s-1/2} \mathrm{d}g \tag{4}$$

In order to show this integral converges for $\Re s$ sufficiently large, we first establish an asymptotic expansion for the functions $l(\pi(a)v)$ where a is diagonal. Then as in [FJ], it follows that this integral is an arbitrary holomorphic multiple of $L(s,\pi)$. We then set

$$I_l(v) = \left.\frac{H(v,s)}{L(s,\pi)}\right|_{s=1/2}$$

and I_l has the required invariance property. The uniqueness of the linear map I_l implies then the uniqueness of l. Also, it follows from the above results that an irreducible representation which has a Shalika model is self-contragredient. This result has been used by Cogdell and Piatetski-Shapiro in their study of the exterior square L-function. At any rate, the above local results supplement the global results of [JS]: there it was proved that an automorphic cuspidal representation π whose exterior square L-function has a pole has a global Shalika model. The local components of the representation π have thus a local Shalika model.

At this point, we formulate a question: let $p = q$ and suppose that the vector space $\mathrm{Hom}_{H_0}(\pi, 1)$ is not zero; we ask whether the representation π is self contragredient.

In order to prove the above theorem we let σ be the involution (antiautomorphism of order 2) defined by $\sigma(g) = g^{-1}$ and we prove that any distribution T on G_m which is bi-H-invariant is fixed by σ (see Theorem 4.1 below). This will imply the theorem as in $[GK]$. Indeed, since the automorphism $g \mapsto {}^t g^{-1}$ takes H to itself and π to $\tilde{\pi}$, the spaces $\mathrm{Hom}_H(\pi, 1)$ and $\mathrm{Hom}_H(\tilde{\pi}, 1)$ have the same dimension. Let $\lambda \in \mathrm{Hom}_H(\tilde{\pi}, 1)$ and $\tilde{\lambda} \in \mathrm{Hom}_H(\pi, 1)$ be non-zero. For every smooth function of compact support f on the group G, there is a smooth vector $\pi(f)\lambda$ in the space of π such that for any smooth vector $\tilde{v}$ in $\tilde{\pi}$ we have:

$$(\pi(f)\lambda, \tilde{v}) = \left(\lambda, \tilde{\pi}(\check{f})\tilde{v}\right).$$

Applying the result to the distribution $f \mapsto (\pi(f)\lambda, \tilde{\lambda})$, we conclude that

$$\left(\pi(f_1)\lambda, \tilde{\pi}(f_2)\tilde{\lambda}\right) = \left(\pi(f_2)\lambda, \tilde{\pi}(f_1)\tilde{\lambda}\right)$$

for any two functions f_1, f_2 smooth of compact support. Since π is irreducible, this implies that if $\pi(f)\lambda = 0$ then $\tilde{\pi}(f)\tilde{\lambda} = 0$. Thus there is a linear operator S from the space of π to the space of $\tilde{\pi}$ such that $S(\pi(f)\lambda) = \tilde{\pi}(f)\tilde{\lambda}$. It is a non-trivial intertwining operator. This already establishes the fact that π is self-contragredient.

Moreover S is unique within a scalar factor. This proves our contention on the dimension of $\mathrm{Hom}_H(\pi, 1)$.

In the case at hand, we do not have the property that $\sigma(g) \in HgH$ for all $g \in G$. Thus we cannot apply directly the method of [GK] to prove the above result. The lack of stability of the double cosets under σ leads us to consider in great detail the structure of the space of double cosets of H. In fact the method of proof given in our case is an adaptation of ideas presented by J. Bernstein (see [Be] and [GPSR]). Bernstein proved that if $G = \mathrm{GL}(n) \times \mathrm{GL}(n-1)$ and $H = \mathrm{GL}(n-1)$ is viewed as the diagonal subgroup of the product, then $\dim \mathrm{Hom}(\pi, 1) \leqslant 1$. We remark that if $p \neq 1, q \neq 1$ we do not expect this to be true for $G = \mathrm{GL}(p+q) \times H$, where $H = \mathrm{GL}(p) \times \mathrm{GL}(q)$ is viewed as the diagonal subgroup, because H does not have an open orbit in the flag variety of G.

To study the double cosets of H, we consider the element

$$\varepsilon = \varepsilon_{p,n} = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}. \tag{5}$$

Then we form the symmetric space

$$Y = \{g \in G_n \,|\, g\varepsilon \text{ is conjugate to } \varepsilon\}. \tag{6}$$

We also introduce the moment map $\rho \colon G \to Y$ given by

$$\rho(g) = g\varepsilon g^{-1}\varepsilon. \tag{7}$$

It satisfies the property that

$$\rho(gxh) = g\rho(x)\varepsilon g^{-1}\varepsilon \tag{8}$$

for all g and x in G, and h in H. In particular, if g is in H then:

$$\rho(gx) = g\rho(x)g^{-1}. \tag{9}$$

Passing to the quotient, ρ defines an isomorphism $G/H \to Y$. We can classify the double cosets of H via the map ρ. In particular, we show that for any $g \in Y$ the semi-simple part g_s and the unipotent part g_u of its Jordan decomposition $g = g_s g_u$ both belong to Y. Suppose that g is a semi-simple element $g \in Y$ and $\rho(x) = g$. Then we show that the double coset HxH is invariant under σ (see Proposition 4.2). Thus 'generically' the double cosets of H are stable under σ. Now let G^g be the centralizer of g in G. For $\xi \in G^g$ we have

$$\sigma(H\xi xH) = H\xi^\sharp xH,$$

where $\xi \mapsto \xi^\sharp$ is a certain involution (antiautomorphism of order 2) of G^g which leaves $H^g = H \cap G^g$ invariant. In fact, in order for $\sharp$ to have order 2, it is necessary to choose x suitably. Let U_x be the open set of ξ such that the map

$\Phi(h_1, \xi, h_2) = h_1 \xi x h_2$ is submersive at $(1, \xi, 1)$. The image of $H \times U_x \times H$ under Φ is an open set Ω_x. The most technical part of this work is to establish the properties of these objects (see Subsection 5.2): the set U_x is invariant under $\sharp$; it is also the set of non-zeroes of a regular function f_x on G^g which is bi-invariant under $H^g = H \cap G^g$ and invariant under $\sharp$; finally, the set Ω_x contains any element y such that the semi-simple part of $\rho(y)$ is g. Now to prove the theorem it suffices to show an H invariant distribution T which is also σ skew invariant vanishes. Suppose that g is semi-simple not central. Then the restriction of the distribution T to Ω_x has a pullback μ_T to U_x which is H^g invariant and $\sharp$ skew invariant. If ψ is a smooth function of compact support on $k^\times$, then $(\psi \circ f_x)\mu_T$ extends to a distribution on G^g with the same properties of invariance under H^g and $\sharp$. In turn, the triple $(G^g, H^g, \sharp)$ decomposes into a product of triples (G_i, H_i, σ_i); here σ_i is an involution of G_i which leaves H_i invariant. For each triple, the assertion corresponding to the theorem is known, either for trivial reasons or inductively because the triple has the form $(\mathrm{GL}(n'), H_{p',n'}, g \mapsto g^{-1})$ with $n' < n$. Thus $(\psi \circ f_x)\mu_T = 0$ and $\mu_T = 0$. It follows that the restriction of T to Ω_x is trivial. This amounts to saying that the support of such a distribution is contained in the complement of the union of the sets Ω_x, that is, the set of x such that the semi-simple part of $\rho(x)$ is ± 1. In other words, the support of T is contained in the union

$$H N_Y H \cup H N_Y w H, \tag{10}$$

where we have set

$$w = \begin{pmatrix} 0 & I_p \\ I_p & 0 \end{pmatrix},$$

and N_Y denotes the set of unipotent elements of Y. Every coset in the first set is invariant under σ. Thus we can reduce ourselves to the case where the distribution has support in the second set. Of course, we have then to assume $p = q$. At this point we introduce the infinitesimal symmetric space, that is, the set L of matrices X such that $\varepsilon X \varepsilon = -X$. Clearly L is invariant under conjugation by H and w. Using the exponential map (or rather the Cayley map) we see that the distribution T gives rise to a distribution T' on L which is invariant under conjugation by H and skew invariant under conjugation by w. Our task is then to show that such a distribution vanishes (Theorem 2.1). Using the same kind of reduction as in the group case, we can show that such a distribution has support in the set $\mathfrak{n}_L$ of nilpotent elements of L. The Fourier transform of T' has the same property. This implies that the distribution T' is invariant under an appropriate oscillator representation of $\mathrm{SL}(2, k)$. In particular, it has a certain property of homogeneity under the dilations $X \mapsto tX$. Now there are only finitely many orbits of H in $\mathfrak{n}_L$. If T' is not zero, one orbit must carry an invariant distribution with the same property of homogeneity. We check this is not the case and so prove the theorem (see Proposition 3.1).

We did not mention a minor complication. In the group case, in order to carry the induction, we have to consider also the involution $x \mapsto wx^{-1}w$. Equivalently, we have to show that any distribution on G invariant under H is also invariant under conjugation by w (see Subsection 5.3).

It will be clear to specialists that we have imitated some reduction techniques that Harish Chandra used in his study of invariant distributions. See [RR] where similar reduction techniques are used in a broader context to study spherical characters.

The paper is organized as follows. In Section 2, we discuss the space of orbits of H on the infinitesimal symmetric space and reduce the problem on the infinitesimal symmetric space to the study of distributions on the cone of nilpotent elements. This study is carried out in Section 3. In Section 4 we discuss the structure of the set of double cosets. In Section 5, we reduce the problem on the symmetric space to the problem on the infinitesimal symmetric space. Finally in Section 6, we discuss the Shalika models.

For a first reading, the reader should read Section 2 and Subsection 3.1, and take for granted the crucial Lemma 3.1, the proof of which is given in Subsection 3.2. Then it would be enough to glance at the results in Section 4 and read Subsection 5.1. The results of Subsection 5.2 can be taken for granted at first. Section 5.3 is similar to Section 5.1. and so can be skipped. Finally, the above introduction gives a sufficient idea of the contents of Section 6.

2. The infinitesimal symmetric space

We let k be a field of characteristic zero and V be a vector space of dimension m over k with a $\mathbb{Z}/2\mathbb{Z}$ grading; thus V is written as the direct sum of its homogeneous components:

$$V = V_0 \oplus V_1. \tag{11}$$

We set $r_i = \dim(V_i)$. We let ε be the element of $\mathrm{GL}(V)$ such that $\varepsilon(v) = (-1)^{\mathrm{degree}(v)}v$. Let L be the subspace of elements of $\mathrm{End}_k(V)$ which are homogeneous of degree 1. Thus

$$L = \mathrm{Hom}(V_1, V_0) \oplus \mathrm{Hom}(V_0, V_1). \tag{12}$$

We write an element X of L as a pair of operators

$$X = (X_1, X_0)$$

with $X_0 \in \mathrm{Hom}(V_0, V_1)$ and $X_1 \in \mathrm{Hom}(V_1, V_0)$. We set

$$q(X) = X_1 X_0. \tag{13}$$

We let H be the subgroup of $g \in \mathrm{GL}(V)$ which are homogeneous of degree 0. Hence

$$H = \mathrm{GL}(V_0) \times \mathrm{GL}(V_1).$$

We write an element g of H as a pair

$$g = (g_0, g_1)$$

with $g_i \in \mathrm{GL}(V_i)$. The group $G = \mathrm{GL}(V)$ operates on $\mathfrak{g} = \mathrm{End}(V)$ by the adjoint representation $\mathrm{Ad}g(X) = gXg^{-1}$. In particular, L is stable under H. Explicitly, if $l = (X_1, X_0)$ and $g = (g_0, g_1)$ then

$$glg^{-1} = (g_0 X_1 g_1^{-1}, g_1 X_0 g_0^{-1}).$$

Let τ be an element of G such that $\tau^2 = 1$. We have then $r_0 = r_1$. We then define an involution σ of L by $\sigma(l) = \tau l \tau$. Our goal in this section is to prove the following theorem:

THEOREM 2.1. *Suppose that V is a $\mathbb{Z}/2\mathbb{Z}$ graded vector space of dimension $m = 2r$ whose homogeneous subspaces have the same dimension. Let τ be an involution of V homogeneous of degree* 1. *Let σ be the corresponding involution of L, the space of linear maps from V to itself which are homogeneous of degree* 1. *Let also H be the group of invertible automorphisms of V (of degree* 0*). Then any distribution T on L which is H-invariant is invariant under σ.*

We first recall some standard facts on the orbits of H on L. The assumption $r_0 = r_1$ is not needed there. For $X \in \mathrm{End}_k(V)$, let

$$X = X_s + X_n$$

be the Jordan decomposition of X as a sum of a semi-simple element X_s and a nilpotent element X_n which commute with one another. Since L is the -1 eigenspace of $\mathrm{Ad}\varepsilon$, we see that if X is in L then X_s and X_n also belong to L. Now suppose that k is algebraically closed and $X \in L$ is semi-simple. If v is an eigenvector of X belonging to the eigenvalue λ let v_0, v_1 be its components. Then $Xv_0 = \lambda v_1$ and $Xv_1 = \lambda v_0$. In particular, $v_0 - v_1$ is an eigenvector for X belonging to the eigenvalue $-\lambda$. One deduces from this observation that one can choose an homogeneous basis with respect to which the operators X_i have diagonal matrices with the same non-zero diagonal entries. It follows that if X is semi-simple then the matrices X_0, X_1 and $q(X) = X_1 X_0$ are semi-simple and have the same rank. This last assertion remains true even if the field is not algebraically closed.

Choosing again a basis of each space V_i, it will be convenient to view the elements X of L as matrices

$$X = (X_1, X_0) = \begin{pmatrix} 0 & X_1 \\ X_0 & 0 \end{pmatrix}.$$

Let R be any integer with $R \leqslant r_i, i = 0, 1$. For any matrix A of size $R \times R$ we let $J(A)$ be the element of L such that

$$X_0 = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}, \quad X_1 = \begin{pmatrix} I_R & 0 \\ 0 & 0 \end{pmatrix}.$$

Thus $J(A)$ is represented by the matrix

$$J(A) = \begin{pmatrix} 0 & 0 & I_R & 0 \\ 0 & 0 & 0 & 0 \\ A & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{14}$$

PROPOSITION 2.1. *Each semi-simple element X of L is H conjugate to an element of the form $J(A)$ where $0 \leqslant R \leqslant r_i$ and A is an invertible semi-simple $R \times R$ matrix.*

Proof. Let $X = (X_1, X_0)$ be a semi-simple element of L. Let R be the rank of the matrix X_1. Then X_0 and $q(X) = X_1 X_0$ have also rank R. There is g_0 and g_1 in $\mathrm{GL}(r_0, k)$ and $\mathrm{GL}(r_1, k)$ such that

$$g_0 X_1 g_1^{-1} = \begin{pmatrix} I_R & 0 \\ 0 & 0 \end{pmatrix}.$$

At the cost of replacing X by gXg^{-1} with $g = (g_0, g_1)$ we may as well assume

$$X_1 = \begin{pmatrix} I_R & 0 \\ 0 & 0 \end{pmatrix}.$$

Let us write then

$$X_0 = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where A is an $R \times R$ matrix. Then

$$q(X) = \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix}.$$

Since $q(X)$ is semi-simple and of rank R, the matrix A is semi-simple and invertible. We can replace X by gXg^{-1} where

$$g = \left(\begin{pmatrix} I_R & \beta \\ 0 & I \end{pmatrix}, \begin{pmatrix} I_R & 0 \\ \gamma' & I \end{pmatrix} \right)$$

without changing X_1. Then we can compute

$$\begin{pmatrix} I & 0 \\ \gamma' & I \end{pmatrix} \cdot \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A & B \\ \gamma' A + C & \gamma' B + D \end{pmatrix}$$

and

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot \begin{pmatrix} I & \beta \\ 0 & I \end{pmatrix} = \begin{pmatrix} A & A\beta + B \\ C & C\beta + D \end{pmatrix} .$$

Since A is invertible there is β and γ' such that $\gamma' A + C = 0$ and $A\beta + B = 0$. Thus there is a $g = (g_0, g_1)$ of the above form such that

$$g_1 X_0 g_0^{-1} = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} .$$

Since this matrix has rank R the matrix D must have rank zero, i.e., $D = 0$. This proves our contention. □

Recall $G = \mathrm{GL}(V)$. Let $\mathfrak{g} = M(m \times m, k)$ be the Lie algebra of G. For $Y \in \mathfrak{g}$ we will denote by G^Y the centralizer of Y in G and by $\mathfrak{g}^Y$ the centralizer of Y in $\mathfrak{g}$.

Let $X = (X_1, X_0)$ be an element of L. We will denote by H^X its centralizer $H \cap G^X$ in H and by L^X its centralizer $\mathfrak{g}^X \cap L$ in L.

LEMMA 2.1. *Suppose that*

$$X = J(A)$$

where A is an invertible matrix of size $R \times R$. Then H^X is the group of all pairs of the form:

$$\left(\begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix}, \begin{pmatrix} \alpha & 0 \\ 0 & \delta' \end{pmatrix} \right), \tag{15}$$

where $\alpha \in \mathrm{GL}(R)$ commutes with A, $\delta \in \mathrm{GL}(r_0 - R)$ and $\delta' \in \mathrm{GL}(r_1 - R)$. Similarly, L^X consists of all pairs of the form:

$$\left(\begin{pmatrix} X & 0 \\ 0 & \xi \end{pmatrix}, \begin{pmatrix} XA & 0 \\ 0 & \xi' \end{pmatrix} \right), \tag{16}$$

where $X \in M(R \times R, k)$ commutes with A and ξ, ξ' are arbitrary.

Proof. The first statement is immediate. We prove the second statement. If $(Z, W) \in L$ commutes with X then we write

$$Z = \begin{pmatrix} Z_1 & Z_2 \\ Z_3 & Z_4 \end{pmatrix}, \quad W = \begin{pmatrix} W_1 & W_2 \\ W_3 & W_4 \end{pmatrix},$$

where Z_1 and W_1 are $R \times R$ matrices. We find at once that $Z_1 A = AZ_1 = W_1$, $W_2 = 0$, $W_3 = 0$ and $AZ_2 = 0$ and $Z_3 A = 0$. Since A is invertible we conclude that $Z_3 = 0$ and $Z_2 = 0$ and our conclusion follows. □

Suppose that $X = J(A)$ as above. Then its centralizer H^X in H is isomorphic to the product

$$\mathrm{GL}(R)^A \times \mathrm{GL}(r_0 - R) \times \mathrm{GL}(r_1 - R).$$

On the other hand, the centralizer L^X in L is isomorphic to the product

$$M(R \times R, k)^A \times M(r_0 - R \times r_1 - R, k) \times M(r_1 - R \times r_0 - R, k).$$

The space L^X is invariant under the action of H^X. With the above identifications the element $g = (\alpha, \delta, \delta')$ operates on (X, ξ, ξ') by

$$(X, \xi, \xi') \mapsto (\alpha X \alpha^{-1}, \delta\xi(\delta')^{-1}, \delta'\xi'\delta^{-1}).$$

Thus we have proved the following proposition:

PROPOSITION 2.2. *Let X be a non-zero semi-simple element of L and R the rank of $q(X)$. Then there is an isomorphism of (H^X, L^X) with*

$$(\mathrm{GL}(R, k)^A, M(R \times R, k)^A) \times (H', L'),$$

where A is semi-simple in $\mathrm{GL}(R, k)$. *Here V' is a graded vector space whose graded subspaces have dimension $(r_0 - R, r_1 - R)$, L' is the space of operators of degree 1 on V', H' the group of automorphisms (of degree 0) of V'. The isomorphism is compatible with the respective adjoint actions.* □

For $X, Y \in \mathfrak{g}$, we set $\beta(XY) = \mathrm{Tr}(XY)$. We let $\mathfrak{g}_Y$ be the orthogonal complement of $\mathfrak{g}^Y$ for β. If Y is semi-simple then

$$\mathfrak{g} = \mathfrak{g}^Y \oplus \mathfrak{g}_Y. \tag{17}$$

For $\xi \in \mathfrak{g}^Y$, both summands are stable under $\mathrm{ad}(Y + \xi)$. In particular:

LEMMA 2.2. *If ξ is nilpotent and commutes with Y then* $\mathrm{ad}(Y + \xi)$ *defines a bijection of $\mathfrak{g}_Y$ on itself.*

Proof. This well known result follows from the Jordan normal form for $Y + \xi$. □

Let Y be a semi-simple element of L. Our goal is to construct an open subset Ω_Y of L which is a union of orbits under H of elements belonging to L^Y. Furthermore, the set Ω_Y to be constructed contains any element of the form $Y + \xi$ where ξ is nilpotent in L^Y. Since $L^Y = L$ only if $Y = 0$, it follows that the complement of the union of the sets Ω_Y with $Y \neq 0$ and semi-simple is the set $\mathfrak{n}_L$ of nilpotent elements of L. We will eventually show that a distribution on L which is σ skew invariant and H invariant has a trivial restriction to any of the open sets Ω_Y, that is, is supported in the set $\mathfrak{n}_L$.

We let $\mathfrak{h}$ be the space of linear homogeneous operators of degree 0 on V. We denote by $\mathfrak{h}^Y$ the centralizer of Y in $\mathfrak{h}$, by L^Y the centralizer of Y in L. We set $\mathfrak{h}_Y = \mathfrak{h} \cap \mathfrak{g}_Y$ and similarly $L_Y = L \cap \mathfrak{g}_Y$. Since $\mathfrak{h}$ is the $+1$ eigenspace for $\mathrm{Ad}\varepsilon$ and L the -1 eigenspace, we have

$$\mathfrak{g} = \mathfrak{h} \oplus L,$$

$$\mathfrak{g}^Y = \mathfrak{h}^Y \oplus L^Y, \quad \mathfrak{g}_Y = \mathfrak{h}_Y \oplus L_Y.$$

and the orthogonal decompositions:

$$\mathfrak{h} = \mathfrak{h}^Y \oplus \mathfrak{h}_Y, \quad L = L^Y \oplus L_Y.$$

LEMMA 2.3. *Suppose $Y \in L$ is semi-simple. Then $\mathfrak{h}_Y$ and L_Y have the same dimension. The restriction of β to each space is non-degenerate.*

Proof. Define

$$\langle A, B\rangle_Y = \mathrm{Tr}(Y[A, B]) = \mathrm{Tr}([Y, A]B).$$

Then $\langle A, B\rangle_Y = 0$ for all B if and only if A commutes with Y. Thus the radical of this skew linear form is the centralizer $\mathfrak{g}^Y$ of Y. We have in fact $\mathfrak{g}^Y = \mathfrak{h}^Y \oplus L^Y$, and $\mathfrak{h}$ and L are maximal isotropic subspaces for the form $\langle .,. \rangle_Y$. The conclusion follows. □

We denote by U_Y the set of $\xi \in L^Y$ such that

$$\mathrm{ad}(Y + \xi)\colon \mathfrak{h}_Y \to L_Y$$

is surjective. Since the transpose of this linear map (with respect to β) is the linear map

$$\mathrm{ad}(Y + \xi)\colon L_Y \to \mathfrak{h}_Y$$

and conversely, the previous lemma implies that U_Y is also the set of $\xi \in L^Y$ such that

$$\mathrm{ad}(Y + \xi)^2\colon L_Y \to L_Y$$

is bijective. In particular:

LEMMA 2.4. *For ξ in L^Y set:*

$$f_Y(\xi) = \det([\mathrm{ad}(Y+\xi)^2]|_{L_Y}). \tag{18}$$

The set U_Y is the set of ξ in L^Y such that $f_Y(\xi) \neq 0$. It contains all nilpotent elements of L^Y. The polynomial f_Y is invariant under $\mathrm{Ad}(H^Y)$. *In particular, the set U_Y is a non-empty open set invariant under* $\mathrm{Ad}\, H^Y$.

Proof. The first assertion is clear. The second assertion follows from Lemma 2.2. The third assertion follows from the following formula, where h is in H^Y:

$$\mathrm{ad}(Y + \mathrm{Ad}(h)(\xi)) = \mathrm{Ad}(h) \circ \mathrm{ad}(Y+\xi) \circ \mathrm{Ad}(h)^{-1}.$$

The last assertion is then an easy consequence. □

Consider the map

$$\phi\colon H \times L^Y \to L$$

given by:

$$\phi\colon (g,\xi) \mapsto \mathrm{Ad}g(Y+\xi) = g(Y+\xi)g^{-1}. \tag{19}$$

The map ϕ is clearly submersive on the product $G \times U_Y$. Thus the image Ω_Y of $G \times U_Y$ is open and contains any element of the form $Y+\xi$ with ξ nilpotent in L^Y. We will use these objects to study H invariant distributions on Ω_Y. In a precise way, there is a surjective map $\alpha \mapsto f_\alpha$ from $C_c^\infty(H \times L^Y)$ to $C_c^\infty(\Omega_Y)$ such that for any $F \in C^\infty(\Omega_Y)$

$$\int_{H\times U_Y} \alpha(g,\xi) F(\mathrm{Ad}(g)(Y+\xi))\, \mathrm{d}g\, \mathrm{d}\xi = \int_L f_\alpha(T)F(T)\, \mathrm{d}T, \tag{20}$$

where $\mathrm{d}g, \mathrm{d}\xi, \mathrm{d}T$ are appropriate Haar measure on H, L^Y, L respectively. It follows that for every $\mathrm{Ad}(H)$ invariant distribution T on Ω_Y there is a unique distribution μ_T on U_Y invariant under H^Y such that

$$T(f_{\alpha_1\otimes\alpha_2}) = \mu_T(\alpha_2) \int \alpha_1(g)\, \mathrm{d}g. \tag{21}$$

From now on, we assume $r_0 = r_1$. We consider an element τ of G of degree 1 such that $\tau^2 = 1$. Thus

$$\tau = (\tau_1, \tau_0)$$

with $\tau_0 = \tau_1^{-1}$. We then define an involution σ of L by $\sigma(l) = \tau l \tau$ and an involution $\tilde{\sigma}$ of H by $\tilde{\sigma}(g) = \tau g \tau$. If we use τ to identify V_0 to V_1 (or use an homogeneous basis invariant under τ to identify operators with matrices) then

$$\sigma(X_1, X_0) = (X_0, X_1). \tag{22}$$

and

$$\tilde{\sigma}(g_0, g_1) = (g_1, g_0). \tag{23}$$

We have the following compatibility between the two involutions

$$\sigma(\mathrm{Ad}(g)X) = \mathrm{Ad}(\tilde{\sigma}(g))\sigma(X). \tag{24}$$

We also note the following result:

LEMMA 2.5. *If Y is semi-simple in L, then there is $z \in H$ such that*

$$\mathrm{Ad}(z)(\sigma(Y)) = Y.$$

Then H^Y is invariant under the map

$$\tilde{\sigma}_Y : g \mapsto \mathrm{Ad}(z)(\tilde{\sigma}(g)).$$

Moreover $\tilde{\sigma}_Y$ is an involution. The space L^Y and the open set U_Y are both invariant under the map:

$$\sigma_Y : \xi \mapsto \mathrm{Ad}(z)(\sigma(\xi)) = \mathrm{Ad}(z\tau)(\xi)$$

and σ_Y is an involution. The involutions are compatible in the sense that

$$\sigma_Y(\mathrm{Ad}(h)\xi) = \mathrm{Ad}(\tilde{\sigma}_Y(h))\sigma_Y(\xi).$$

Finally,

$$\sigma(\phi(g, \xi)) = \phi(\tilde{\sigma}(g)z^{-1}, \sigma_Y(\xi)).$$

In particular, the open set Ω_Y is invariant under σ.

Proof. We may assume

$$Y = J(A) = \left[\begin{pmatrix} I_R & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \right],$$

where A is an invertible semi-simple matrix of rank $R \leqslant r$. We define an element z of H by

$$z = \left[\begin{pmatrix} A^{-1} & 0 \\ 0 & I_{r-R} \end{pmatrix}, I_r \right].$$

Then

$$\mathrm{Ad}(z)\,(\sigma(Y)) = z\sigma(Y)z^{-1} = Y$$

which proves our first assertion. The properties of $\tilde{\sigma}_Y$ follow from the explicit description of H^Y given above (In fact $\tilde{\sigma}_Y = \tilde{\sigma}$ on H^Y).

To continue, we recall that an element ξ of L^Y has the form

$$\left[\begin{pmatrix} X & 0 \\ 0 & Z_1 \end{pmatrix}, \begin{pmatrix} XA & 0 \\ 0 & Z_2 \end{pmatrix}\right],$$

where A commutes with X. It follows that

$$\mathrm{Ad}(z)\sigma(\xi) = \left[\begin{pmatrix} X & 0 \\ 0 & Z_2 \end{pmatrix}, \begin{pmatrix} XA & 0 \\ 0 & Z_1 \end{pmatrix}\right],$$

is again in L^Y. Thus σ_Y is indeed an involution of L^Y. To continue, we compute

$$\begin{aligned} \sigma(\phi(g,\xi)) &= \mathrm{Ad}\tilde{\sigma}(g)\,[\sigma(Y)+\sigma(\xi)] = \mathrm{Ad}\tilde{\sigma}(g)\,[\mathrm{Ad}(z^{-1})(Y)+\sigma(\xi)] \\ &= \mathrm{Ad}(\tilde{\sigma}(g)z^{-1})\,[Y+\mathrm{Ad}(z)(\sigma(\xi))] \\ &= \phi\left(\tilde{\sigma}(g)z^{-1}, \mathrm{Ad}(z)\sigma(\xi)\right). \end{aligned}$$

The compatibility of σ_Y and $\tilde{\sigma}_Y$ follows from the compatibility of σ and $\tilde{\sigma}$.

It remains to see that if ξ is in U_Y then $\mathrm{Ad}(z)\sigma(\xi) \in U_Y$. By assumption, $\mathrm{ad}(Y+\xi)$ is injective on L_Y and we have to see that $\mathrm{ad}(Y+\mathrm{Ad}(z)\sigma(\xi))$ is also injective on L_Y. By the very choice of z we have

$$Y+\mathrm{Ad}(z)\sigma(\xi) = \mathrm{Ad}(z)\sigma(Y+\xi) = \mathrm{Ad}(z\tau)(Y+\xi).$$

Hence

$$\mathrm{ad}(Y+\mathrm{Ad}(z)\sigma(\xi)) = \mathrm{Ad}(z\tau)\mathrm{ad}(Y+\xi)\mathrm{Ad}(z\tau)^{-1}.$$

Since L^Y is invariant under $\mathrm{Ad}(z\tau)$ the same is true of L_Y and our conclusion follows. □

If we use the notations of the Proposition 2.1, we see that $\sigma_Y = 1 \times \sigma'$ and $\tilde{\sigma}_Y = 1 \times \tilde{\sigma}'_Y$ where $\sigma'(X') = \tau' X' \tau'$, $\tilde{\sigma}'(g') = \tau' g' \tau'$; here τ' is an element of order 2 in $\mathrm{GL}(V')$, homogeneous of degree 1.

Now we apply the above considerations to the map $\alpha_1 \otimes \alpha_2 \mapsto f_{\alpha_1 \otimes \alpha_2}$ previously defined. If f is a function on Ω_Y (or L) we denote by f^σ the function defined by $f^\sigma(X) = f(\sigma(X))$. If μ is any distribution on Ω_Y we denote by μ^σ the distribution defined by $\mu^\sigma(f) = \mu(f^\sigma)$. We deduce that

$$f^\sigma_{\alpha_1 \otimes \alpha_2} = f_{\alpha_1^\sharp \otimes \alpha_2^{\sigma_Y}},$$

where we have set

$$\alpha_1^\sharp(g) = \alpha_1(\tilde{\sigma}(gz)).$$

In particular, if T is a distribution on Ω_Y invariant under Ad(H) we have

$$T^\sigma(f_{\alpha_1\otimes\alpha_2}) = \mu_T(\alpha_2^{\sigma_Y}) \int \alpha_1^\sharp(h)\,\mathrm{d}h = \mu_T(\alpha_2^{\sigma_Y}) \int \alpha_1(h)\,\mathrm{d}h.$$

We say that a distribution μ is skew invariant under σ if $\mu^\sigma = -\mu$. We have proved the following lemma:

LEMMA 2.6. *Suppose that a distribution T on Ω_Y is* AdH *invariant. Then if T is skew invariant under σ, the distribution μ_T is skew invariant under σ_Y.*

We are now ready to begin the proof of Theorem 2.1.

INDUCTION STEP: Because of the compatibility of the involutions σ and $\tilde{\sigma}$, any H invariant distribution can be written as the sum of a σ invariant and a σ skew invariant distributions, each of which is H invariant. It will suffice to show that the skew invariant component is 0, that is, that a distribution T which is σ skew invariant and H invariant is 0.

Thus let T be such a distribution. Assume that the theorem is true for a graded vector space of dimension $< m$. We will show that the support of T is contained in the set $\mathfrak{n}_L$ of nilpotent elements of L. In view of our previous results, it suffices to show that for any semi-simple element $Y \neq 0$ of L the restriction of T to the open set Ω_Y is 0. In turn, it will suffice to show that the distribution μ_T determined by T is zero. Recall that μ_T is a distribution on the open set U_Y of L^Y invariant under the action of H^Y and skew invariant under the involution σ_Y introduced above. Recall also that U_Y is the set of non-zeroes of the polynomial f_Y on L_Y. This polynomial in invariant under Ad(H^Y). Furthermore $f_Y(\xi) \neq 0$ if and only if $f_Y(\sigma_Y(\xi)) \neq 0$. Thus U_Y is also the set of non-zeroes of the polynomial

$$g_Y(\xi) = f_Y(\xi) f(\sigma_Y(\xi)).$$

The compatibility of σ_Y and $\tilde{\sigma}_Y$ show that the second factor is also invariant under Ad(H^Y). Thus the polynomial g_Y is invariant under Ad H^Y and σ_Y. If ψ is a smooth function of compact support on $F^\times$, the product $(\psi \circ g_Y)\mu_T$ extends to a distribution on the whole vector space L^Y which is Ad H^Y invariant and skew invariant under σ_Y. We will show in the next paragraph that such a distribution vanishes. This will imply that the distribution μ_T vanishes and will give us our conclusion.

Thus we consider now a distribution T on L^Y which is invariant under H^Y and skew invariant under σ_Y. Recall that L^Y decompose into the direct product of $M(R \times R, k)^A$ and the space L' of homogeneous operators of degree 1 on a $\mathbb{Z}/2\mathbb{Z}$ graded vector space V' of dimension $m - 2R$. The group H^Y decomposes into the product of GL$(R)^A$ and the group H' of homogeneous isomorphisms of V'. Finally the involution σ_Y is the product of the identity on $M(R \times R, k)^A$ and an involutive automorphism σ' of V' of degree 1, compatible with an involutive automorphism $\tilde{\sigma}'$ of H'. We have to show that any distribution μ on the product

$M(R \times R, k)^A \times L'$ which is invariant under $\mathrm{GL}(R)^A \times H'$ and skew invariant under $1 \times \sigma'$ is zero. This is clear if $R = m/2$ because the involution is then the identity. If $R < m/2$ then for any function ψ of compact support on $M(R \times R, k)^A$ the distribution $f \mapsto \mu(\psi \otimes f)$ on L' is H' invariant and skew σ' invariant. By the induction hypothesis, it vanishes. Hence μ vanishes as well and we are done.

Coming back to the proof of our theorem, we have established (under the induction hypothesis) that the support of T is contained in the set of nilpotent elements. We will finish the proof of the theorem in the next section.

3. The nilpotent variety

3.1. HOMOGENEITY

We keep to the notations of the previous section. In particular, we assume $\dim V_0 = \dim V_1$. Suppose that T is a distribution on L which is H-invariant and σ skew invariant. By the results of the previous section and the induction hypothesis of the Theorem, the support of T is contained in the set $\mathfrak{n}_L$. Our task is to show that T is actually zero. To that end we introduce the restriction β_L of β to L. Thus $\beta_L(X, Y) = \mathrm{Tr}(XY)$. The bilinear form β_L is invariant under $\mathrm{Ad}\, H$ and σ, since these operators are actually conjugation by an element of $\mathrm{GL}(V)$. We define the Fourier transform $\hat{f}$ of a function $f \in \mathcal{S}(L)$ by:

$$\hat{f}(X) = \int_L f(Y)\psi(\beta_L(X, Y))\, \mathrm{d}Y, \tag{25}$$

where ψ is a non-trivial additive character of k and $\mathrm{d}Y$ a self-dual Haar measure on L. The Fourier transform of a distribution μ is then defined by $\hat{\mu}(f) = \mu(\hat{f})$. Clearly, the Fourier transform $\hat{T}$ of T is also invariant under H and σ skew invariant. Thus its support is also contained in $\mathfrak{n}_L$. Our assertion and the theorem will be proved if we establish the following proposition:

PROPOSITION 3.1. *Let T be any $\mathrm{Ad}(H)$ invariant distribution such that T and $\hat{T}$ have support in the nilpotent set $\mathfrak{n}_L$. Then $T = 0$.*

The remainder of this section is devoted to the proof of the proposition. We first recall results of [KP] on the structure of the set $\mathfrak{n}_L$. For this discussion, we need not have $\dim V_0 = \dim V_1$. Suppose Z is in $\mathfrak{n}_L$. Then we may regard V as a $k[X]$-module, the action of a polynomial $p(X)$ on a vector v being $p(Z)v$. We can write V as a direct sum of indecomposable (cyclic) $k[X]$-modules. The main result is that one can choose the generators of these submodules to be homogeneous. Another result is that H has only finitely many orbits in $\mathfrak{n}_L$.

Now assume $\dim V_0 = \dim V_1$. The representation Ad of H on L gives us an imbedding of H into the orthogonal group $O(\beta_L)$ of the form β_L. Since $(O(\beta_L), \mathrm{SL}(2, k))$ is a dual reductive pair, we have a corresponding oscillator representation ω of $\mathrm{SL}(2, k)$ on $\mathcal{S}(L)$ which is defined as follows:

$$\omega\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}\phi(X) = \langle a|d_L\rangle \mid a \mid^{(\dim V)^2/4} \phi(aX) \tag{26}$$

$$\omega\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\phi(X) = \psi(t\beta_L(X,X))\phi(X), \tag{27}$$

$$\omega\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\phi(X) = \gamma\hat{\phi}(X). \tag{28}$$

Here $d_L \in k^\times/k^{\times 2}$ is the discriminant of the form β_L; we have denoted by $\langle .|. \rangle$ the canonical pairing on $k^\times/k^{\times 2} \times k^\times/k^{\times 2}$; finally γ is a suitable root of unity.

Consider then the distribution T. It has support in $\mathfrak{n}_L$. However we have $\beta_L(X,X) = \mathrm{Tr}(q(X))$. If X is nilpotent then $X^{2r} = 0$. This implies that $q(X)^r = 0$. Thus $q(X)$ is nilpotent and its trace is zero. As a result, any nilpotent element is isotropic for β_L. It follows from the above formula that T is invariant under

$$\omega\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \quad t \in k.$$

Since the distribution

$$\omega\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} T$$

is a scalar multiple of the Fourier transform of T it has the same property. Equivalently, T is invariant under the operators

$$\omega\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}, \quad t \in k,$$

and thus is fixed under the representation ω. In particular, it has the following property of homogeneity

$$\int f(aX)\,\mathrm{d}T(X) = \langle a|d_L\rangle \mid a \mid^{-(\dim V)^2/4} \int f(X)\,\mathrm{d}T(X). \tag{29}$$

We are led to consider similarly the properties of homogeneity of the invariant measures carried by the nilpotent orbits of H in L.

Let $Z \in \mathfrak{n}_L$. We write

$$V = W_1 \oplus W_2 \oplus \cdots \oplus W_K,$$

where each W_i is an indecomposable (graded) $k[X]$-submodule. We let z_i be an homogeneous generator of W_i. Thus $Z^{\dim W_i} z_i = 0$ and

$$\{z_i, Zz_i, \dots, Z^{\dim(W_i)-1} z_i\}$$

is a linear basis of W_i. To continue our study of the homogeneity we introduce an element $D_t \in H$ such that $D_t Z D_t^{-1} = tZ$. Indeed, we define an operator D_t^i on W_i by demanding that $D_t^i(Z^k(z_i)) = t^k Z^k(z_i)$. Then we can choose for D_t the direct sum of the D_t^i. Let $\mathfrak{h}$ be the Lie algebra of H. This is the space of linear operators of degree 0 on V. Consider the centralizer $\mathfrak{h}^Z$ of Z in $\mathfrak{h}$. Since D_t transforms Z into a scalar multiple, it follows that $\mathfrak{h}^Z$ is invariant under $\mathrm{Ad}\, D_t$. We want to compute the determinant of $\mathrm{Ad}\, D_t$ on $\mathfrak{h}^Z$:

LEMMA 3.1. *There is an integer m_Z such that*

$$\det \mathrm{Ad}(D_t)\Big|_{\mathfrak{h}^Z} = t^{m_Z}.$$

Furthermore

$$m_Z < \frac{(\dim V)^2}{4}.$$

Let us show how this lemma will imply our assertion and the theorem. We can write

$$\mathfrak{n}_L = \cup X_j,$$

where X_j, $0 \leqslant j \leqslant R$, is an increasing sequence of closed H invariant subspaces of $\mathfrak{n}_L$, with $X_0 = \emptyset$ and $X_R = \mathfrak{n}_L$ and, in addition, the difference $X_{j+1} - X_j$ is a single orbit of H. We have just verified that for any nilpotent element Z and for any $t \neq 0$, Z and tZ belong to the same orbit of H. Thus the sets X_j are invariant under dilations. We prove by descending induction on j that T vanishes on the complement O_j of X_j. For $j = R$ this is the assumption on the support of T. Assume the restriction of T to O_{j+1} is zero. Consider the restriction of T to O_j. Suppose it is non-zero. Its support is contained in the set $X_{j+1} - X_j$ which is a closed orbit in O_j. The orbit may or may not carry an H invariant measure. If not, there is nothing to prove. Thus we may assume that the orbit carries an invariant measure and then T is a multiple of this measure. Let Z be any point in the orbit. Thus there is an invariant measure on the quotient H/H^Z such that for any $f \in C_c^\infty(O_k)$

$$T(f) = \int_{H/H^Z} f(gZg^{-1})\, dg.$$

For $t \in k^\times$ let $f_t(X) = f(Xt)$. Then f_t is in $C_c^\infty(O_j)$ and

$$T(f_t) = \int f(gtZg^{-1})\,\mathrm{d}g.$$

Introduce as before the element D_t; then $D_tZD_t^{-1} = tZ$ and

$$T(f_t) = \int f\left(\mathrm{Ad}(gD_t)(Z)\right)\,\mathrm{d}g = \left|\det(\mathrm{Ad}\,D_t|_{\mathfrak{h}^Z})\right|^{-1}\int f(gZg^{-1})\,\mathrm{d}g.$$

Thus we see that

$$T(f_t) = \mid t\mid^{-m_Z} T(f).$$

However, we have

$$T(f_t) = \mid t\mid^{-\dim V^2/4}\,\langle t|d_L\rangle T(f).$$

Since $m_Z < \frac{\dim V^2}{4}$ we conclude that $T(f) = 0$. Thus the restriction of T to O_j vanishes. Inductively, the restriction of T to O_0 vanishes and we are done.

3.2. Proof of Lemma 3.1

It remains to prove Lemma 3.1. Let Z be an element of $\mathfrak{n}_L$. As a first step, we determine the centralizer $\mathfrak{g}^Z$ of Z in $\mathfrak{g} = \mathrm{End}(V)$. Suppose that $B = (b_{ij}(X))$ is a matrix in $M(K \times K, k[X])$. We want to associate to B a linear operator η_B on V such that for any i and any m:

$$\eta_B(Z^m z_i) = \sum_j b_{ij}(Z)Z^m z_j.$$

Since $Z^{\dim W_i} z_i = 0$, in order for this expression to make sense, we need to have

$$b_{ij}(X)X^{\dim W_i} \equiv 0 \pmod{X^{\dim W_j}}. \tag{30}$$

Assuming the matrix B satisfies condition (30), there is indeed a unique linear operator η_B on V with the above property. The operator η_B commutes with Z. Every element of $\mathfrak{g}^Z$ is of the form η_B for a suitable B. The map η reverses the order of multiplication:

$$\eta_C\eta_B = \eta_{BC}.$$

For $p \in k[X]$ and $t \in k^\times$ we define a polynomial $\rho(t)(p)$ by $\rho(t)p(X) = p(tX)$. For a matrix of polynomials $B = (b_{ij})$ we set $\rho(t)B = (\rho(t)b_{ij})$. Then

$$\eta_{\rho(t)B} = D_t\eta_B D_t^{-1}. \tag{31}$$

Finally $\eta_B = 0$ if and only if

$$b_{ij} \equiv 0 \pmod{X^{\dim W_j}}. \tag{32}$$

Thus we can consider instead the space $\mathcal{Z}$ of matrices $B = (b_{ij})$ of truncated polynomials:

$$b_{ij} \in k[X]/X^{\dim W_j}, \tag{33}$$
$$b_{ij} \equiv 0 \pmod{X^{\dim W_j - \dim W_i}}. \tag{34}$$

Of course if $\dim W_i \geqslant \dim W_j$ the second condition is empty. The map $B \mapsto \eta_B$ is then a bijection from $\mathcal{Z}$ onto $\mathfrak{g}^Z$.

Our next task is to determine the structure of $\mathfrak{h}^Z$. To that end, we define an element ε of H by $\varepsilon(v) = (-1)^{\text{degree}(v)} v$ if v is an homogeneous vector. Then $X \in \mathfrak{g}$ is in $\mathfrak{h}$ if and only if $\operatorname{Ad}\varepsilon(X) = X$. Since each space W_i is a graded subspace, it is invariant under ε. More precisely, define $\omega_i = (-1)^{\text{degree}(z_i)}$. Then

$$\varepsilon(Z^k z_i) = (-1)^k \omega_i Z^k z_i. \tag{35}$$

The operator η_B is in $\mathfrak{h}_Z$ if and only if

$$\varepsilon\eta_B = \eta_B\varepsilon. \tag{36}$$

This relation is equivalent to

$$\eta_B\varepsilon(z_i) = \sum_j \varepsilon\left(b_{ij}(Z)z_j\right),$$

or, in view of relation (35),

$$\omega_i b_{ij}(Z)z_j = \omega_j b_{ij}(-Z)z_j.$$

If we write

$$b_{ij}(X) = \sum_{\sup(\dim W_j - \dim W_i, 0) \leqslant l < \dim W_j} \alpha_l^{ij} X^l \tag{37}$$

the above condition reads:

$$\alpha_l^{ij}(\omega_i - (-1)^l \omega_j) = 0,$$

or, more explicitly:

$$\alpha_l^{ij} = 0 \text{ when } \omega_i\omega_j \neq (-1)^l. \tag{38}$$

Thus η_B determines a bijection from the space $\mathcal{Z}_1$ of matrices $B = (b_{ij})$ of truncated polynomials satisfying conditions (37) and (38) onto $\mathfrak{h}^Z$. In view of (31) we have

$$\det\left(\mathrm{Ad}\, D_t|_{\mathfrak{h}^Z}\right) = \det \rho(t)|_{\mathcal{Z}_1}.$$

We view $\mathcal{Z}_1$ as the direct sum of spaces S_{ij} of truncated polynomials satisfying the conditions (37) and (38). Then

$$\det \mathrm{Ad}\, D_t|_{\mathfrak{h}^Z} = \prod_{(i,j)} \det \rho(t)|_{S_{ij}}.$$

The following lemma computes the right hand side. We set $r_i = \dim(W_i)$.

LEMMA 3.2. (i) *Suppose $i = j$ and $r_i = 2p_i$. Then*

$$\det(\rho(t)|_{S_{ii}}) = t^{p_i^2 - p_i}$$

(ii) *Suppose $i = j$ and $r_i = 2p_i + 1$. Then*

$$\det(\rho(t)|_{S_{ii}}) = t^{p_i^2 + p_i}$$

(iii) *Let $i \neq j$. Then*

$$\det(\rho(t)|_{S_{ij}}) \det(\rho(t)|_{S_{ji}})$$

is given by the following formulas:

$$\begin{cases} t^{2p_ip_j - 2\min(p_i,p_j)} & \text{if } r_i = 2p_i, r_j = 2p_j & \text{and } \omega_i\omega_j = 1 \\ t^{2p_ip_j} & \text{if } r_i = 2p_i, r_j = 2p_j & \text{and } \omega_i\omega_j = -1 \\ t^{2p_ip_j} & \text{if } r_i = 2p_i, r_j = 2p_j + 1, r_i < r_j & \\ t^{2p_ip_j} & \text{if } r_i = 2p_i, r_j = 2p_j + 1, r_i > r_j & \text{and } \omega_i\omega_j = 1 \\ t^{2p_ip_j + 2(p_i - p_j) - 1} & \text{if } r_i = 2p_i, r_j = 2p_j + 1, r_i > r_j & \text{and } \omega_i\omega_j = -1 \\ t^{2p_ip_j + 2\sup(p_i,p_j)} & \text{if } r_i = 2p_i + 1, r_j = 2p_j + 1 & \text{and } \omega_i\omega_j = 1 \\ t^{2p_ip_j} & \text{if } r_i = 2p_i + 1, r_j = 2p_j + 1 & \text{and } \omega_i\omega_j = -1 \end{cases}$$

Proof. We consider first the case when $i = j$. If $r_i = 2p_i$ then the truncated polynomials $P \in S_{ii}$ have the form

$$\sum_{0 \leqslant l < 2p_i, l \equiv 0 \bmod 2} \alpha_l X^l.$$

Thus the determinant of $\rho(t)$ on that space is t raised to the power

$$\sum_{0 \leqslant l < 2p_i, l \equiv 0 \bmod 2} l = p_i^2 - p_i. \tag{39}$$

If $r_i = 2p_i + 1$ then the truncated polynomials in S_{ii} have the form

$$\sum_{0\leqslant l<2p_i+1, l\equiv 0 \bmod 2} \alpha_l X^l.$$

Thus the determinant of $\rho(t)$ on that space is t raised to the power

$$\sum_{0\leqslant l<2p_i, l\equiv 0 \bmod 2} l = p_i^2 + p_i. \tag{40}$$

This gives the two first assertions of the lemma.

Now we consider the case where $i \neq j$ and $r_j \geqslant r_i$. This time we have

$$\det(\rho(t)|_{S_{ij}}) \det(\rho(t)|_{S_{ji}}) = t^{m(i,j)}$$

where

$$m(i,j) = \sum_{0\leqslant l<r_i} l + \sum_{0\leqslant l<r_j} l - \sum_{0\leqslant l\leqslant r_j - r_i} l,$$

where the summation on l is restricted by the condition that

$$l \equiv \begin{cases} 0 & \text{if } \omega_i\omega_j = 1 \\ 1 & \text{if } \omega_i\omega_j = -1 \end{cases}$$

We also note in addition to the identities (39) and (40):

$$\sum_{0\leqslant l<2p-1, l\equiv 0 \bmod 2} l = p^2. \tag{41}$$

The third assertion of the lemma follows then from a lengthy but elementary computation. □

At this point we have proved the first assertion of Lemma 3.1. The integer $m = m_Z$ is the sum of the exponents occurring in the previous lemma. It remains to establish the upper bound for the integer m in term of the dimension d of V. We have

$$d = \sum_{1\leqslant i\leqslant k} r_i. \tag{42}$$

In general, $\dim(W_i \cap V_0) = \dim(W_i \cap V_1) \pm 1$. Thus if r_i is even, we write $r_i = 2p_i$ and then $\dim(W_i \cap V_0) = \dim(W_i \cap V_1) = p_i$. Suppose that r_i is odd. Then we write $r_i = 2p_i + 1$. If $\omega_i = 1$, then $\dim(W_i \cap V_0) = p_i$ and $\dim(W_i \cap V_1) = p_i + 1$. If $\omega_i = -1$ then $\dim(W_i \cap V_0) = p_i + 1$ and $\dim(W_i \cap V_1) = p_i + 1$. We let X be the number of indices i such that r_i is odd and $\omega_i = 1$. Since V_0 and V_1 have the same dimension, there must be the same number of indices i such that r_i is odd and $\omega_i = -1$. Then

$$\dim V = 2\left(X + \sum_i p_i\right).$$

Thus

$$\frac{\dim V^2}{4} = X^2 + 2X\left(\sum_i p_i\right) + \sum_i p_i^2 + 2\sum_{i<j} p_i p_j. \tag{43}$$

Note that the integer m is determined by the data

$$(r_1, \omega_1, r_2, \omega_2, \ldots, r_K, \omega_K),$$

without reference to the spaces at hand. The proof of the lemma is by induction on the number of indices i so that r_i is even. First assume the number is zero, that is, all the integers r_i are odd. Then $K = 2X$. We order the r_i so that $\omega_i = 1$ for $1 \leqslant i \leqslant X$ and $\omega_i = -1$ for $X + 1 \leqslant i \leqslant 2X$. We further assume that p_i is a decreasing function of i for $1 \leqslant i \leqslant X$ and for $X + 1 \leqslant i \leqslant 2X$. The previous lemma gives then

$$\begin{aligned} m = & \sum_{1\leqslant i\leqslant 2X} (p_i^2 + p_i) + 2\sum_{i<j} p_i p_j + \\ & + \{2(X-1)p_1 + 2(X-2)p_2 + \cdots + 2p_{X-1}\} \\ & + \{2(X-1)p_{X+1} + 2(X-2)p_{X+2} + \cdots + 2p_{2X-1}\}. \end{aligned}$$

Clearly m is less than

$$\sum_{1\leqslant i\leqslant 2X} p_i^2 + 2\sum_{i<j} p_i p_j + 2X \sum_{1\leqslant i\leqslant 2X} p_i$$

which in turn is strictly less than (43). Thus our assertion is proved in this case.

Now we can arrange the data so that r_K (the last term) is even. If $K = 1$ then $r_1 = 2p$ and $m = p^2 - p$ which is strictly less than $\dim V^2/4 = p^2$. By induction on the number of indices i with r_i even, we may assume that the inequality is proved for the data $(r_1, \omega_1, r_2, \omega_2, \ldots, r_{K-1}, \omega_{K-1})$. The induction hypothesis shows that the contribution of the indices (i, j) with $1 \leqslant i \leqslant j \leqslant K - 1$ is strictly less than $d'^2/4$ where $d' = \sum_{1\leqslant i\leqslant K-1} r_i$. Thus we must show that the sums of the contributions of the pairs (i, K) with $i \leqslant K$ is less than or equal to

$$d^2/4 - d'^2/4 = p_K^2 + 2\sum_{i<K} p_i p_K + 2X p_K. \tag{44}$$

The previous lemma shows that the contribution of the pair (K, K) is $p_K^2 - p_K < p_K^2$. Consider now the contribution of a pair (i, K) with $i < K$. It is always less than or equal to $2p_ip_K$ except when $r_i = 2p_i + 1$ with $r_K > r_i$ and $\omega_i\omega_j = -1$; the contribution is then $2p_ip_K + 2(p_K - p_i) - 1 < 2p_ip_K + 2p_K$. There are at most X such terms. Thus the contribution of the pairs (i, K) with $i \leqslant K$ is at most equal to the right hand side of (44). This proves our contention and concludes the proof of the lemma and the theorem.

4. The symmetric space

4.1. Orbits in the symmetric space

We consider here the variety

$$Z = \{s \in M(n \times n, k) | s^2 = 1\}. \tag{45}$$

Under the adjoint action of $G = \mathrm{GL}(n)$, this variety decomposes into a finite number of orbits:

$$Z = \cup Z_{p,n},$$

where

$$Z_{p,n} = \cup_{g \in G} \mathrm{Ad}(g)(\varepsilon_{p,n}) \tag{46}$$

with

$$\varepsilon_{p,n} = \begin{pmatrix} I_p & 0 \\ 0 & -I_{n-p} \end{pmatrix}. \tag{47}$$

Note that $Z_{n,n}$ and $Z_{0,n}$ are reduced to a single point. Each $Z_{p,n}$ admits the structure of a symmetric space. Indeed, let $\theta_{p,n}$ be the involution of G defined by

$$\theta_{p,n}(g) = \varepsilon_{p,n} g \varepsilon_{p,n} \tag{48}$$

and let $H_{p,n}$ be the centralizer of $\varepsilon_{p,n}$. Then the space

$$P_{p,n} = \{g \in \mathrm{GL}(n) | \theta_{p,n}(g) = g^{-1}\} = \{g \in G_n \,|(g\varepsilon_{p,n})(g\varepsilon_{p,n}) = I_n\}$$

contains the set

$$Y_{p,n} = Z_{p,n}\varepsilon_{p,n} = \{g \in G \mid g\varepsilon_{p,n} \text{ is conjugate to } \varepsilon_{p,n}\}.$$

The group G operates on $P_{p,n}$ via the twisted action:

$$x \overset{g}{\mapsto} gx\theta_{p,n}(g^{-1}).$$

The group $H_{p,n}$ operates by conjugation; it is the stabilizer of e and $Y_{p,n}$ is the orbit of e. In particular, we have a surjective polarization map

$$\rho_{p,n} : G \to Y_{p,n}, \; \rho_{p,n}(g) = g\varepsilon_{p,n}g^{-1}\varepsilon_{p,n}.$$

It verifies:

$$\rho_{p,n}(xgh) = x\rho_{p,n}(g)\theta_{p,n}(x^{-1}).$$

In particular $Y_{p,n}$ is isomorphic to $G/H_{p,n}$ as a G-space. We can regard k^n as a $\mathbb{Z}/2\mathbb{Z}$ graded vector space where the homogeneous vectors are the eigenvectors of $\varepsilon_{p,n}$ and have eigenvalue $(-1)^{\text{degree}(v)}$.

In what follows, we often drop the second index n or even both indices from the notations. For instance we write ε_p or even ε for $\varepsilon_{p,n}$, if this does not create confusion. When $n = 2p$ we also introduce

$$w_p = \begin{pmatrix} 0 & I_p \\ I_p & 0 \end{pmatrix}. \tag{49}$$

Recall our goal is to prove the following theorem:

THEOREM 4.1. *Suppose T is a distribution on* $\mathrm{GL}(n,k)$ *which is $H_{p,n}$ bi-invariant. Then T is invariant under $g \mapsto g^{-1}$. If $n = 2p$ it is also invariant under conjugation by w_p.*

We remark that once the first assertion is proved, then the second assertion amounts to saying that the distribution T is invariant under $g \mapsto w_p g^{-1} w_p$.

The proof of the theorem is by induction on n: we assume $n \geqslant 1$ and the theorem true for all groups $\mathrm{GL}(n')$ with $n' < n$. We first study the orbits of H in P_p.

Let N_n be the set of unipotent elements in $\mathrm{GL}(n)$. We first investigate the structure of the intersection $N_n \cap P_p$. We recall that the exponential map defines an isomorphism of $\mathfrak{n}_n$, the set of nilpotent elements in $M(n \times n, k)$, onto N_n. In particular, if $u = \exp(X)$ lies in $N_n \cap P_p$ then the equation $\varepsilon_p u \varepsilon_p = u^{-1}$ implies $\varepsilon_p X \varepsilon_p = -X$. The operator ε_p defines a $\mathbb{Z}/2\mathbb{Z}$ grading of $V = k^n$: an eigenvector v of ε_p has eigenvalue $(-1)^{\text{degree}(v)}$. The above relation means that X is in L (defined in Section 2). In particular $\frac{1}{2}X$ is in L and $v = \exp(\frac{1}{2}X)$ is in P_p. We have then

$$\rho_p(v) = \exp(\tfrac{1}{2}X)\varepsilon_p \exp(-\tfrac{1}{2}X)\varepsilon_p = v^2 = \exp(X) = u.$$

Thus u is actually in Y_p. We set $N_Y = Y_p \cap N_n$. Recall $\mathfrak{n}_L = L \cap \mathfrak{n}_n$. Thus

$$N_n \cap P_{p,n} = N_Y = \exp(\mathfrak{n}_L).$$

A consequence is the following lemma which describes the Jordan decomposition of an element of Y_p:

LEMMA 4.1. *Let $x \in P_p$ and $x = x_s x_u = x_u x_s$ its Jordan decomposition, where x_s is semi-simple and x_u unipotent. Then x_s and x_u are in P_p. If x is in Y_k then x_s and x_u are in Y_p. More precisely, there is $Y \in \mathfrak{n}_L$ and $g_1 \in G$ such that*

$\rho(\exp(Y)) = x_u$, $\rho(g_1) = x_s$; the elements $\exp(Y)$ and x commute to one another and $\rho(\exp(Y)g_1) = x$.

Proof. Assume x is in P_p, that is, $\varepsilon_p x \varepsilon_p = x^{-1}$. The uniqueness of the Jordan decomposition shows that x_s and x_u satisfy the same condition and are thus in P_p. In fact x_u is in Y_p by the arguments above. More precisely, write $x_u = \exp(X)$. Then $x_s X x_s^{-1} = X$. It follows that $v = \exp(X/2)$ commutes with x_s and also with x.

Assume now that x is in Y_p. Thus $x = \rho_k(g)$ for some $g \in G$. Suppose that $\xi \in G$ commutes with x. Then $\varepsilon\xi\varepsilon$ commutes with $\varepsilon x \varepsilon = x^{-1}$ and thus commutes with x as well. As a result:

$$\begin{aligned}\rho_p(\xi g) &= \xi g \varepsilon g^{-1} \xi^{-1} \varepsilon = \xi x \varepsilon \xi^{-1} \varepsilon \\ &= \xi \varepsilon \xi^{-1} \varepsilon x = \rho_p(\xi)\rho_p(g).\end{aligned}$$

We can apply this identity to the element v above. We find

$$\rho_p(v^{-1}g) = \rho_p(v^{-1})x = v^{-2}x_u x_s = x_s.$$

Thus x_s is in Y_p as claimed. If we set $Y = X/2$ and $g_1 = v^{-1}g$ we obtain the last assertion of the lemma. □

Our next task will be to analyze the elements of P_p which are semi-simple.

LEMMA 4.2. *Let*

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in P_p$$

with A, B, C, D are matrices of size $p \times p$, $p \times (n-p)$, $(n-p) \times p$, $(n-p) \times (n-p)$ respectively. Suppose g is semi-simple. Then the matrices A, D, BC, CB and

$$\begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}$$

are (square) semi-simple matrices.

Proof. We may assume the ground field k is algebraically closed, since the condition of being semi-simple does not depend on the ground field. Let $\{T_i\}$ be a basis of eigenvectors for g. We write $T_i = v_i + w_i$ where v_i (resp. w_i) lies in the $+1$ (resp. -1) eigenspace of ε_p. In other words, v_i has degree 0 and w_i has degree 1. We have then:

$$Av_i + Bw_i = \lambda_i v_i, \quad Cv_i + Dw_i = \lambda_i w_i.$$

Since $\varepsilon g \varepsilon = g^{-1}$ we have $g(\varepsilon T_i) = \lambda_i^{-1}(\varepsilon T_i)$. Thus εT_i is an eigenvector of g with the eigenvalue λ_i^{-1}. This gives the relations

$$Av_i - Bw_i = \lambda_i^{-1} v_i, \quad Cv_i - Dw_i = -\lambda_i^{-1} w_i.$$

Combining the above relations we obtain

$$Av_i = \frac{\lambda_i + \lambda_i^{-1}}{2} v_i, \quad Dw_i = \frac{\lambda_i + \lambda_i^{-1}}{2} w_i.$$

Since the vectors v_i (resp. w_i) span V_0 (resp. V_1), this implies that A and D are semi-simple. We have also

$$Bw_i = \frac{\lambda_i - \lambda_i^{-1}}{2} v_i, \quad Cv_i = \frac{\lambda_i + \lambda_i^{-1}}{2} w_i.$$

This implies similarly that CB andBC are semi-simple. In turn (see Section 2) this implies the last assertion of the lemma. □

Now we want to obtain a canonical form for a semi-simple element of P_p. We record the algebraic equations defining P_p: if

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where A is a $p \times p$ matrix, then g is in P_p if and only if

$$A^2 = I_p + BC, \quad D^2 = I_{n-p} + CB; \tag{50}$$

$$AB = BD, \quad DC = CA. \tag{51}$$

Since the elements of H commute to ε, the group $H \simeq \mathrm{GL}(p) \times \mathrm{GL}(n-p)$ operates on P_p via conjugation:

$$\begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix}^{-1} = \begin{pmatrix} g_1 A g_1^{-1} & g_1 B g_2^{-1} \\ g_2 C g_1^{-1} & g_2 D g_2^{-1} \end{pmatrix}.$$

Thus, at the cost of replacing g by a conjugate under H, we may assume

$$B = \begin{pmatrix} I_\nu & 0 \\ 0 & 0 \end{pmatrix},$$

where ν is the rank of B. If g is semi-simple, then C has the same rank as B and the products CB and BC are semi-simple. Arguing as in the infinitesimal case, we see that g is H conjugate to an element of the form

$$\begin{pmatrix} A & \begin{pmatrix} I_\nu & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} C_\nu & 0 \\ 0 & 0 \end{pmatrix} & D \end{pmatrix},$$

where C_ν is a $\nu \times \nu$ invertible semi-simple matrix. Let us write further

$$A = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}, \quad D = \begin{pmatrix} D_1 & D_2 \\ D_3 & D_4 \end{pmatrix},$$

where A_1 and D_1 are $\nu \times \nu$ matrices. From the algebraic equations which define P_p we get

$$A_2 = 0, \quad A_3 = 0, \quad D_2 = 0, \quad D_3 = 0, \quad A_1 = D_1,$$

and

$$C_\nu = A_1^2 - I_\nu, \quad A_4^2 = I_{p-\nu}, \quad D_4^2 = I_{n-p-\nu}.$$

Thus so far we have shown that a semi-simple element g of P_k is conjugate to an element of the form:

$$\begin{pmatrix} A_1 & 0 & I_\nu & 0 \\ 0 & A_4 & 0 & 0 \\ A_1^2 - I_\nu & 0 & A_1 & 0 \\ 0 & 0 & 0 & D_4 \end{pmatrix}, \tag{52}$$

where A_1 is semi-simple of size $\nu \times \nu$, A_4 and D_4 are elements of order 2; moreover $A_1^2 - I_\nu$ is invertible, that is, ± 1 is not an eigenvalue of A_1. Note that the extreme cases $\nu = p$ and $\nu = 0$ may occur.

We first study the case where $\nu = p$.

LEMMA 4.3. *Let $A \in M(r \times r, k)$ be a matrix so that ± 1 is not an eigenvalue of A. Then the matrix*

$$t(A) = \begin{pmatrix} A & I_r \\ A^2 - I_r & A \end{pmatrix} \tag{53}$$

is invertible. It can be expressed in the form $t(A) = \rho_{r,2r}(g)$ for some $g \in \mathrm{GL}(2r)$. In particular, it is in $Y_{r,2r}$. Moreover, there is $h \in H_{r,2r}$ such that

$$\rho_{r,2r}(g) = h\rho_{r,2r}(g^{-1})h^{-1}.$$

The matrix $t(A)$ does not have the eigenvalue ± 1. It is semi-simple if and only if A is semi-simple.

Proof. One checks at once that $t(A)\varepsilon t(A)\varepsilon = I$ so that $t(A)$ is invertible and in $P_{r,2r}$. Since A does not have the eigenvalues ± 1, we can write

$$A = (I_r + U)(I_r - U)^{-1},$$

where U is a square matrix without the eigenvalues 0 or 1. Then we can check at once that $t(A) = \rho_r(x)$ where $x = x(U)$ is defined by

$$x(U) = \begin{pmatrix} I_r & \frac{1}{2}(I_r - U) \\ 2U(I_r - U)^{-1} & I_r \end{pmatrix}. \tag{54}$$

In addition x has the form:

$$x = \begin{pmatrix} I_r & X \\ Y & I_r \end{pmatrix},$$

where $I_r - XY$ and $I_r - YX$ are invertible (in fact both equal to $I_r - U$). Then

$$x^{-1} = \varepsilon x \varepsilon \begin{pmatrix} (I - XY)^{-1} & 0 \\ 0 & (I - YX)^{-1} \end{pmatrix}.$$

Since the matrix on the right is in H, we get

$$\rho_r(x^{-1}) = \varepsilon \rho_r(x) \varepsilon.$$

This establishes the second assertion.

If $t(A)$ is semi-simple we have seen that A is semi-simple. To prove the converse, we may assume k is algebraically closed. As in the previous proposition, if λ is an eigenvalue of $t(A)$ then $(\lambda + \lambda^{-1})/2$ is an eigenvalue of A. Thus $\lambda \neq \pm 1$. Moreover, if v is an eigenvector for the matrix A belonging to the eigenvalue μ then for $\lambda = \mu \pm \sqrt{\mu^2 - 1}$ the column vector

$$T_{v,\pm} = \begin{bmatrix} v \\ (\lambda - \mu) v \end{bmatrix}$$

is an eigenvector for $t(A)$ belonging to the eigenvalue λ. If A is semi-simple we can choose the vectors v among a basis of eigenvectors for A; then the vectors $T_{v,\pm}$ form a basis of k^{2r}. Thus $t(A)$ is semi-simple. □

REMARK. Similarly, it is easily proved that every element $g \in Y_{r,2r}$ of the form

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where A and D are $r \times r$ matrices not having ± 1 as eigenvalues is $H_{r,2r}$ conjugate to a matrix of the form

$$\begin{pmatrix} A' & I_r \\ A'^2 - I_r & A' \end{pmatrix},$$

where A' does not have the eigenvalue ± 1. In fact, this establishes a bijection between $H_{r,2r}$ conjugacy classes of elements $g \in Y_{r,2r}$ satisfying the above conditions and conjugacy classes of $\mathrm{GL}(r)$ in $M(r \times r, k)$ of elements A' which do not have eigenvalues ± 1. As in the proof of the previous lemma, the Cayley transform gives a bijection of the latter set with the set of conjugacy classes of $\mathrm{GL}(r)$ in $M(r \times r, k)$ of elements U which do not have eigenvalues 0 and 1.

Now we go back to the general situation of a semi-simple element of $Y_{p,n}$. Recall that $g \in \mathrm{GL}(n)$ is in $Y_{p,n}$ if and only if $g\varepsilon_p$ is conjugate to ε_p.

PROPOSITION 4.1. *Each semi-simple element $g \in Y_p$ is H conjugate to an element of the form*

$$\begin{pmatrix} A & 0 & I_\nu & 0 \\ 0 & \eta_1 & 0 & 0 \\ A^2 - I_\nu & 0 & A & 0 \\ 0 & 0 & 0 & \eta_2 \end{pmatrix}, \tag{55}$$

where A is a semi-simple element of $M(\nu \times \nu, k)$ without the eigenvalues ± 1 and η_1, η_2 are matrices of the form

$$\eta_1 = \begin{pmatrix} I_\alpha & 0 \\ 0 & -I_\beta \end{pmatrix}, \quad \eta_2 = \begin{pmatrix} I_\gamma & 0 \\ 0 & -I_\delta \end{pmatrix}, \tag{56}$$

with $\alpha + \beta = p - \nu$, $\gamma + \delta = n - p - \nu$ and $\beta = \delta$. The set of H conjugacy classes of semi-simple elements of Y_p is in bijective correspondence with the set of all triples $(\nu, \{A\}, \beta)$, where $0 \leqslant \nu \leqslant p$ is an integer, $\{A\}$ a semi-simple conjugacy class in $M(\nu \times \nu)$ without the eigenvalues ± 1 and β is an integer with $0 \leqslant \beta \leqslant p - \nu$.

Proof. We may assume that g has the canonical form (52). We can view $V = k^n$ as the direct sum of two graded subspaces V' and V''; correspondingly, $g = g' \oplus g''$. With respect to suitable homogeneous bases of V' and V'', the operator g' has the matrix $t(A)$ and the operator g'' has the matrix

$$\begin{pmatrix} A_4 & 0 \\ 0 & D_4 \end{pmatrix}.$$

With obvious notations, the group H contains $H' \times H''$ where $H' \simeq \mathrm{GL}(\nu) \times \mathrm{GL}(\nu)$ and $H'' \simeq \mathrm{GL}(p-\nu) \times \mathrm{GL}(n-p-\nu)$. Since $A_4^2 = I_{p-\nu}$, the matrix A_4 is conjugate under $\mathrm{GL}(p-\nu)$ to a matrix η_1 of the above form. Likewise D_4 is conjugate under $\mathrm{GL}(n-\nu-p)$ to a matrix η_2 of the above form. Thus g'' is indeed conjugate under H'' to an element with a matrix of the form:

$$\begin{pmatrix} \eta_1 & 0 \\ 0 & \eta_2 \end{pmatrix}$$

with η_i of the above form. However, we have still to show that $\beta = \delta$. The previous lemma shows that the product

$$\begin{pmatrix} A & I_\nu \\ A^2 - I_\nu & A \end{pmatrix} \begin{pmatrix} I_\nu & 0 \\ 0 & -I_\nu \end{pmatrix}$$

is $\mathrm{GL}(2\nu)$ conjugate to $\varepsilon_{\nu,2\nu}$. Since $g\varepsilon_{p,n}$ is $\mathrm{GL}(n)$ conjugate to $\varepsilon_{p,n}$, this implies that the product

$$\begin{pmatrix} \eta_1 & 0 \\ 0 & \eta_2 \end{pmatrix} \begin{pmatrix} I_{p-\nu} & 0 \\ 0 & -I_{n-p-\nu} \end{pmatrix}$$

is $\mathrm{GL}(n-2\nu)$ conjugate to $\varepsilon_{p-\nu,n-p-\nu}$. Comparing the eigenvalues of the products we get our result. This gives the first assertion of the proposition.

To prove the second assertion of the proposition we need to show that any matrix of the specified form is actually in Y_k. This amounts to showing that the matrix

$$\eta = \begin{pmatrix} \eta_1 & 0 \\ 0 & \eta_2 \end{pmatrix} \tag{57}$$

is in $Y_{p-\nu,n-2\nu}$. This is easily checked: indeed, if

$$\zeta = \begin{pmatrix} I_\alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & I_\delta \\ 0 & 0 & I_\gamma & 0 \\ 0 & I_\beta & 0 & 0 \end{pmatrix} \tag{58}$$

then (recall $\beta = \delta$)

$$\rho_{p-\nu,n-2\nu}(\zeta) = \eta\,, \ \zeta = \zeta^{-1}. \quad \square \tag{59}$$

REMARK. Suppose that g is the matrix of the proposition. Let us write again $V = V' \oplus V''$ and $g = g' \oplus g''$. Since g' and g'' do not have a common eigenvalue, the centralizer of g in $\mathrm{GL}(V)$ consists of all matrices of the form $\xi' \oplus \xi''$ with $\xi' \in \mathrm{GL}(V')^{g'}$ and $\xi'' \in \mathrm{GL}(V'')^{g''}$.

Our main result is now:

PROPOSITION 4.2. *If $g \in Y_{p,n}$ is semi-simple and $\rho(x) = g$ then*

$$H_{p,n} x H_{p,n} = H_{p,n} x^{-1} H_{p,n}.$$

Proof. We may assume that g has the form (55). Equivalently, we may assume that $V = V' \oplus V''$ where V' and V'' are graded subspaces, and $g = g' \oplus g''$ where g'

has matrix $t(A)$ and g'' has matrix η, with respect to a suitable homogeneous basis. Similarly, $\varepsilon = \varepsilon \oplus \varepsilon''$ where ε' and ε'' are homogeneous of degree 0. We have then, with obvious notations, $\rho'(x(U)) = g'$ and $\rho''(\zeta) = g''$. Thus if $x = x(U) \oplus \zeta$ then $\rho(x) = g$. Since $H'x(U)H' = H'x(U)^{-1}H'$ and $\zeta = \zeta^{-1}$ we obtain our assertion. □

4.2. Induced symmetric spaces

In this subsection, we discuss the symmetric spaces of lower rank which will be used to carry out the induction step needed in the proof of Theorem 4.1. The following simple lemma will be very useful:

LEMMA 4.4. *Suppose x is in* GL(n). *Then*

$$L \cap (xLx^{-1}) = L^{\rho(x)}, \mathfrak{h} \cap (x\mathfrak{h}x^{-1}) = \mathfrak{h}^{\rho(x)};$$

$$H \cap (xHx^{-1}) = H^{\rho(x)}.$$

Proof. Suppose l is in $L \cap xLx^{-1}$. Then

$$\rho(x)l\rho(x)^{-1} = -x\varepsilon x^{-1}lx\varepsilon x^{-1} = xx^{-1}lxx^{-1}$$

since $x^{-1}lx$ is in L. Thus we find that l commutes with $\rho(x)$. Conversely, suppose l is in $L^{\rho(x)}$. Then

$$\varepsilon x^{-1}lx\varepsilon = x^{-1}\varepsilon l\varepsilon x = -x^{-1}lx$$

so that $x^{-1}lx$ is in L and $l \in L \cap xLx^{-1}$. The other assertions are proved in a similar way. □

Recall the form $\beta(X, Y) = \mathrm{Tr}(XY)$ on $\mathfrak{g} = M(n \times n, k)$. We have an orthogonal decomposition: $\mathfrak{g} = \mathfrak{h} \oplus L$. Suppose $x \in \mathrm{GL}(V)$. Since the orthogonal complement of $\mathfrak{h} + x\mathfrak{h}x^{-1}$ is $L \cap (xLx^{-1}) = L^{\rho(x)}$, we have

$$\mathfrak{g} = \left(\mathfrak{h} + x\mathfrak{h}x^{-1}\right) \oplus L^{\rho(x)}.$$

Let x be in G. Suppose that $g = \rho(x)$ is semi-simple. We consider the map

$$\Phi\colon H \times G^{\rho(x)} \times H \to \mathrm{GL}(n)\,,\ (h, \xi, h') \mapsto h\xi xh'. \tag{60}$$

We denote by U_x the set of $\xi \in G^{\rho(x)}$ such that Φ is submersive at $(1, \xi, 1)$ (and thus at any point (h, ξ, h')). In fact U_x is the set of $\xi \in G^{\rho(x)}$ such that

$$\mathfrak{g} = \mathfrak{h} + \mathrm{Ad}(\xi x)(\mathfrak{h}) + L^{\rho(x)}.$$

The previous formula shows that $1 \in U_x$. We will establish additional properties of the set U_x in Subsection 5.2, in particular, the fact that it is open and invariant under left and right multiplication by $H^{\rho(x)}$.

Suppose $\rho(x)$ is semi-simple. We have $HxH = Hx^{-1}H$ so that we can write $g_1 x g_2 = x^{-1}$ with $g_i \in H$. In particular, $\rho(x^{-1}) = g_1 \rho(x) g_1^{-1}$. Now $x^{-1}\rho(x)x = \varepsilon x^{-1}\varepsilon x$. Taking in account these relations and the fact that ε commutes with g_1, g_2, we find

$$g_1(\varepsilon x \varepsilon x^{-1})g_1^{-1} = \varepsilon g_1 x g_2 \varepsilon g_2^{-1} x^{-1} g_1^{-1} = \varepsilon x^{-1} \varepsilon x.$$

This identity can be written in the form:

$$x^{-1}\rho(x)x = g_1 \varepsilon \rho(x) \varepsilon g_1^{-1}$$

or

$$x^{-1}\rho(x)x = g_1 \rho(x)^{-1} g_1^{-1}. \tag{61}$$

It follows that if ξ belongs to $G^{\rho(x)}$ then $x^{-1}\xi x$ commutes with $x^{-1}\rho(x)x = g_1\rho(x)^{-1}g_1^{-1}$. It therefore commutes with $g_1\rho(x)g_1^{-1}$ as well. Equivalently, the element

$$\xi^\sharp = g_1^{-1}x^{-1}\xi^{-1}x g_1 \tag{62}$$

is in $G^{\rho(x)}$. Thus $\xi \mapsto \xi^\sharp$ is an antiautomorphism of $G^{\rho(x)}$. We have

$$(\xi^\sharp)^\sharp = g_1^{-1}x^{-1}g_1^{-1}x^{-1}\xi x g_1 x g_1.$$

Now $\rho(x^{-1}) = g_1\rho(x)g_1^{-1}i$ and similarly $g_2^{-1}\rho(x^{-1})g_2 = \rho(x)$. It follows that $g_2^{-1}g_1$ commutes with $\rho(x)$. Also $xg_1xg_1 = g_2^{-1}g_1$ and

$$(\xi^\sharp)^\sharp = (h)^{-1}\xi h, \quad h = g_2^{-1}g_1 \in H^{\rho(x)}. \tag{63}$$

Furthermore, if ξ is in $H^{\rho(x)} = H \cap xHx^{-1}$, then $\xi^\sharp$ is also in $H^{\rho(x)}$, since $xg_1 = g_2^{-1}x^{-1}$.

We have

$$(HG^{\rho(x)}xH)^{-1} = Hx^{-1}G^{\rho(x)}xx^{-1}H.$$

Since $x^{-1}\rho(x)x = g_1\rho(x)^{-1}g_1^{-1}$ this is also

$$Hg_1G^{\rho(x)}g_1^{-1}x^{-1}H.$$

Since $xg_2 = g_1^{-1}x^{-1}$ this is also equal to $HG^{\rho(x)}xH$. Thus this set is invariant under $g \mapsto g^{-1}$. More precisely, for $h_i \in H$, $i = 1, 2$:

$$(h_1\xi x h_2)^{-1} = h_2^{-1}x^{-1}\xi^{-1}h_1^{-1} = h_2^{-1}g_1\xi^\sharp g_1^{-1}x^{-1}h_1^{-1}$$

or

$$(h_1\xi x h_2)^{-1} = h_2^{-1} g_1 \xi^\sharp x g_2 h_1^{-1}. \tag{64}$$

We will see that the open set U_x is invariant under $\xi \mapsto \xi^\sharp$. It will follow that the image Ω_x of $H \times U_x \times H$ under Φ is also invariant under $g \mapsto g^{-1}$. We will show that any distribution T on Ω_x which is H bi-invariant and skew invariant under $g \mapsto g^{-1}$ gives rise to a distribution μ_T on the open set U_x which is bi-invariant under $H^{\rho(x)}$ and skew invariant under $\xi \mapsto \xi^\sharp$. Note that in general $\sharp$ needs not be an involution. However, if μ is an $H^{\rho(x)}$ biinvariant distribution on $G^{\rho(x)}$ or on Ω_x then $(\mu^\sharp)^\sharp = \mu$.

Thus we must now study the triple $(G^{\rho(x)}, H^{\rho(x)}, \sharp)$. First we study $\mathfrak{g}^{\rho(x)}$. Since $\varepsilon\rho(x)\varepsilon = \rho(x)^{-1}$ we have

$$\mathfrak{g}^{\rho(x)} = \mathfrak{h}^{\rho(x)} \oplus L^{\rho(x)}.$$

Recall that

$$\mathfrak{h}^{\rho(x)} = \mathfrak{h} \cap (x\mathfrak{h}x^{-1}), \quad L^{\rho(x)} = L \cap (xLx^{-1}).$$

We use this observation and the explicit form of the representatives of H orbits given above.

First suppose that $x \in \mathrm{GL}(2\nu)$ has the form

$$x = x(U) = \begin{pmatrix} I_\nu & \frac{1}{2}(I_\nu - U) \\ 2U(I_\nu - U)^{-1} & I_\nu \end{pmatrix}$$

so that in the symmetric space $Y_{\nu,2\nu}$ we have $\rho_{\nu,2\nu}(x) = t(A)$ where $A = (I_\nu + U)(I_\nu - U)^{-1}$. Given Z_1, Z_2 there are Z_1' and Z_2' such that

$$x^{-1}\begin{pmatrix} 0 & Z_1 \\ Z_2 & 0 \end{pmatrix} x = \begin{pmatrix} 0 & Z_1' \\ Z_2' & 0 \end{pmatrix}$$

if and only if $Z_1' = Z_1, Z_2' = Z_2$ and

$$Z_2 = 4U(1-U)^{-1}Z_1(1-U)^{-1} = 4(1-U)^{-1}Z_1(1-U)^{-1}U.$$

The last relation implies that $(1-U)^{-1}Z_1(1-U)^{-1}$ commutes with U. In turn this implies that Z_1 commutes with U. Thus we see that $L^{\rho(x)} = L \cap xLx^{-1}$ is equal to the set of matrices of the form

$$\begin{pmatrix} 0 & Z_1 \\ 4Z_1U(1-U)^{-2} & 0 \end{pmatrix}, \tag{65}$$

where $Z_1 \in M(\nu \times \nu, k)^U$. Similarly, $\mathfrak{h} \cap (x\mathfrak{h}x^{-1})$ is the set of matrices of the form

$$\begin{pmatrix} Z_1 & 0 \\ 0 & Z_1 \end{pmatrix}, \tag{66}$$

where $Z_1 \in M(\nu \times \nu, k)^U$. It follows that for $x = x(U)$

$$M(2\nu \times 2\nu, k)^{\rho_{\nu,2\nu}(x)} = \tag{67}$$

$$\left\{ \begin{pmatrix} Z_1 & Z_2 \\ 4Z_2U(1-U)^{-2} & Z_1 \end{pmatrix} \middle| Z_1, Z_2 \in M(\nu \times \nu, k)^U \right\}.$$

The group $\mathrm{GL}(2\nu, k)^{\rho(x)}$ is just the set of invertible matrices of the above form. The space $\mathfrak{h}^{\rho(x)}$ is just the space of matrices of the above form with $Z_2 = 0$ and the group $H^{\rho(x)}$ the group of matrices of the above form with $Z_2 = 0$ and Z_1 invertible.

Now we determine the effect of the map $\xi \mapsto \xi^\sharp = g_1^{-1}x^{-1}\xi^{-1}xg_1$, where $g_1 \in H$ conjugates $\rho(x)$ to $\rho(x^{-1})$. Here we can take $g_1 = \varepsilon_{\nu,2\nu}$. We have:

$$x = \begin{pmatrix} I_\nu & X \\ Y & I_\nu \end{pmatrix},$$

where $X = (I_\nu - U)/2$ and $Y = 2U(I_\nu - U)^{-1}$. The matrices X and Y are in the bicommutant of U and verify $X\delta = \delta X = Y$ for $\delta = 4U(1-U)^{-2}$. Thus x is actually in the center of the algebra $M(2\nu \times 2\nu)^{\rho(x)}$. Thus $\xi^\sharp = \varepsilon_{\nu,2\nu}\xi^{-1}\varepsilon_{\nu,2\nu}$ in this case. Explicitly, if ξ^{-1} is written in the above form, then

$$\xi^\sharp = \begin{pmatrix} Z_1 & -Z_2 \\ -4Z_2U(1-U)^{-2} & Z_1 \end{pmatrix}.$$

In particular, $(\xi^\sharp)^\sharp = \xi$.

Since the element U is semi-simple in $M(\nu \times \nu, k)$, there exists a k linear isomorphism

$$k^\nu \simeq K_1^{l_1} \oplus \cdots \oplus K_r^{l_r},$$

where K_i/k are field extension and the operator U becomes under this identification

$$U \simeq \zeta_1 I_{l_1} \oplus \cdots \oplus \zeta_r I_{l_r}$$

with $\zeta_i \in K_i$.

If we identify $k^\nu \oplus k^\nu$ with

$$(K_1^{l_1} \oplus K_1^{l_1}) \oplus \cdots \oplus (K_r^{l_r} \oplus K_r^{l_r})$$

then the associative algebra $M(2\nu \times 2\nu, k)^{\rho(x)}$ can be identified with a direct sum of associative algebras:

$$\bigoplus \left\{ \left(\begin{matrix} X & Y \\ Y\delta_i & X \end{matrix} \right) \middle| X, Y \in M(l_i \times l_i, K_i) \right\},$$

where $\delta_i = 4\zeta_i(1-\zeta_i)^{-2}$. The group $G^{\rho(x)}$ in this case is then identified to the product of the multiplicative groups of the algebras. The group $H^{\rho(x)}$ is identified with the product of the groups

$$\left\{ \left(\begin{matrix} X & 0 \\ 0 & X \end{matrix} \right) \middle| X \in \mathrm{GL}(l_i, K_i) \right\}.$$

Each factor of $G^{\rho(x)}$ is invariant under the map (induced by) $\xi \mapsto \xi^\sharp$. The corresponding map changes an element to its inverse and then changes the matrix Y to $-Y$.

Now $\delta_i \neq 0$ and thus δ_i is either represented by a square or not from the multiplicative group $K_i^\times$. In particular, if δ_i is *not* a square, then δ_i determines a unique quadratic extension $L_i = K_i(\sqrt{\delta_i})$ of K_i. Then the algebra

$$\left\{ \left(\begin{matrix} X & Y \\ Y\delta_i & X \end{matrix} \right) \middle| X, Y \in M(l_i \times l_i, K_i) \right\}$$

is isomorphic to $M(l_i \times l_i, L_i)$ via the map

$$X + \sqrt{\delta_i} Y \mapsto \left(\begin{matrix} X & Y \\ Y\delta_i & X \end{matrix} \right).$$

The multiplicative group is then $\mathrm{GL}(l_i, L_i)$, the factor of $H^{\rho(x)}$ is $\mathrm{GL}(l_i, K_i)$ and the map induced by $\sharp$ is $\xi \mapsto \overline{\xi}^{-1}$, where $\overline{z}$ indicates the Galois conjugate of an element $z \in L_i$. If δ_i is a square, then the algebra

$$\left\{ \left(\begin{matrix} X & Y \\ Y\delta_i & X \end{matrix} \right) \middle| X, Y \in M(l_i \times l_i, K_i) \right\}$$

is isomorphic to the direct sum $M(l_i \times l_i, K_i) \oplus M(l_i \times l_i, K_i)$ via the map

$$(X + vY, X - vY) \mapsto \left(\begin{matrix} X & Y \\ Y\delta & X \end{matrix} \right),$$

where $\delta_i = v^2$. The multiplicative group is then $\mathrm{GL}(l_i, K_i) \times \mathrm{GL}(l_i, K_i)$ and the factor of $H^{\rho(x)}$ is the diagonal group $\mathrm{GL}(l_i, K)^{\Delta}$. The map induced by $\sharp$ is $(z_1, z_2) \mapsto (z_1^{-1}, z_2^{-1})$.

On the other hand, suppose $x = \zeta$ where

$$\zeta = \begin{pmatrix} I_\alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & I_\delta \\ 0 & 0 & I_\gamma & 0 \\ 0 & I_\beta & 0 & 0 \end{pmatrix} \tag{68}$$

with $\beta = \delta$, $\alpha + \beta = p$, $\gamma + \delta = n - p$. Then

$$\rho(\zeta) = \begin{pmatrix} I_\alpha & 0 & 0 & 0 \\ 0 & -I_\beta & 0 & 0 \\ 0 & 0 & I_\gamma & 0 \\ 0 & 0 & 0 & -I_\beta \end{pmatrix}$$

and the centralizer of $\rho(\zeta)$, that is, $M(n \times n, k)^{\rho(\zeta)}$, is the algebra of matrices of the form:

$$\begin{pmatrix} A & 0 & B & 0 \\ 0 & A' & 0 & B' \\ C & 0 & D & 0 \\ 0 & C' & 0 & D' \end{pmatrix} \tag{69}$$

with

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in M(n - 2\beta \times n - 2\beta), \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix} \in M(2\beta \times 2\beta).$$

We have $\zeta = \zeta^{-1}$ in this case, so that $g_1 = g_2 = 1$ and $\xi^\sharp = \zeta\xi^{-1}\zeta$. Thus if ξ^{-1} is a matrix of the above form we have

$$\xi^\sharp = \begin{pmatrix} A & 0 & B & 0 \\ 0 & D' & 0 & B' \\ C & 0 & D & 0 \\ 0 & C' & 0 & A' \end{pmatrix}. \tag{70}$$

Thus we see that in this case, the triple $(G^{\rho(x)}, H^{\rho(x)}, \xi \mapsto \xi^\sharp)$ decomposes into a product of two triples:

$$(\mathrm{GL}(\alpha + \gamma), \mathrm{GL}(\alpha) \times \mathrm{GL}(\gamma), x \mapsto x^{-1})$$

and

$$(\mathrm{GL}(2\beta), \mathrm{GL}(\beta) \times \mathrm{GL}(\beta), x \mapsto w_\beta x^{-1} w_\beta).$$

In particular, $(\xi^\sharp)^\sharp = \xi$.

Thus we have proved the following result:

PROPOSITION 4.3. *Let g be a semi-simple element of $Y_{p,n}$. Then one can choose x such that $\rho(x) = g$ and $g_1 \in H$ with $g_1 g g_1^{-1} = g^{-1}$ in such a way that the corresponding antiautomorphism $\sharp$ has order* 2.

Proof. Indeed, using the decomposition of $V = V' \oplus V''$ corresponding to $g = g' \oplus g''$ where g' does not have the eigenvalue ± 1 and g'' has only the eigenvalues ± 1, we see that for a suitable choice of x and g_1, the original triple $(G^{\rho(x)}, H^{\rho(x)}, x \mapsto x^\sharp)$ is isomorphic to a product of triples of the following types

$$\begin{array}{llll} \text{(i)} & \mathrm{GL}(l, K(\sqrt{\delta})), & \mathrm{GL}(l, K), & x \mapsto \overline{x}^{-1} \\ \text{(ii)} & \mathrm{GL}(l, K) \times \mathrm{GL}(l, K) & \mathrm{GL}(l, K)^{\Delta}, & (x_1, x_2) \mapsto (x_2^{-1}, x_1^{-1}) \\ \text{(iii)} & \mathrm{GL}(t_1 + t_2, k) & \mathrm{GL}(t_1, k) \times \mathrm{GL}(t_2, k), & x \mapsto x^{-1} \\ \text{(iv)} & \mathrm{GL}(2t, k) & \mathrm{GL}(t, k) \times \mathrm{GL}(t, k), & x \mapsto w_t x^{-1} w_t \end{array} \tag{71}$$

This proves our assertion. □

In addition, we claim that for every one of the above triples (G', H', σ) we know that every H' bi-invariant distribution is also invariant under the involution σ. For case (i), this is a result of [yF]; in this case, every double coset is actually invariant under σ. For case (ii), we may identify G'/H' to $\mathrm{GL}(l, K)$ via the map $(z_1, z_2) \mapsto z_1 z_2^{-1}$. Then if T is H' biinvariant on G' there is a conjugacy invariant distribution μ on $\mathrm{GL}(l, K)$ such that

$$\int f(g_1, g_2)\, \mathrm{d}T(g_1, g_2) = \iint f(gh, h)\, \mathrm{d}\mu(g)\, \mathrm{d}h$$

where $\mathrm{d}h$ is a Haar Measure on $\mathrm{GL}(l, K)$. We have

$$\begin{aligned} \int f(g_2^{-1}, g_1^{-1})\, \mathrm{d}T(g_1, g_2) &= \iint f(h^{-1}, h^{-1}g^{-1})\, \mathrm{d}\mu(g)\, \mathrm{d}h \\ &= \iint f(h^{-1}g, h^{-1})\, \mathrm{d}\mu(g)\, \mathrm{d}h = \iint f(hg, h)\, \mathrm{d}\mu(g)\, \mathrm{d}h \\ &= \iint f(gh, h)\, \mathrm{d}\mu(g)\, \mathrm{d}h = \int f(g_1, g_2)\, \mathrm{d}T(g_1, g_2). \end{aligned}$$

So our assertion is trivial in this case. Finally (iii) and (iv) are just the two cases of the induction hypothesis, provided the centralizer of $\rho(x)$ is not the whole group, i.e. $\rho(x) \neq \pm 1$.

5. Reduction to the infinitesimal symmetric space

5.1. First reduction

We want to prove that a H bi-invariant distribution T is actually invariant under $g \mapsto g^{-1}$. We may as well assume that T is skew invariant under $g \mapsto g^{-1}$ and then show that $T = 0$.

To that end, we consider a semi-simple element $g \in Y$ and an element x such that $\rho(x) = g$. We choose x in such a way that $\sharp$ is an involution. Then we consider the open set U_x and the image Ω_x of $H \times U_x \times H$ under the map Φ. It is an open set. We will show that the restriction of T to Ω_x vanishes. We shall need another property of the set U_x, namely that it is the set of non-zeroes of a regular function $g_x(\xi)$ on $G^{\rho(x)}$. Furthermore, this function is invariant under right and left multiplication by $H^{\rho(x)}$. In particular, if we set $f_x(\xi) = g_x(\xi)g_x(\xi^\sharp)$ then U_x is also the set of non-zeroes of f_x and f_x is invariant under $\sharp$, and under left and right multiplication by $H^{\rho(x)}$.

There exists a surjective map of $C_c^\infty(H \times U_x \times H)$ onto $C_c^\infty(\Omega_x)$ noted $\alpha \mapsto f_\alpha$ such that

$$\int f_{\alpha\otimes\beta\otimes\gamma}(t)F(t)\,dt = \int \alpha(g_1)\beta(\xi)\gamma(g_2)F(g_1^{-1}\xi x g_2)\,\mathrm{d}g_1\,\mathrm{d}\xi\,\mathrm{d}g_2$$

for all $F \in C^\infty(\Omega_x)$. Here $\mathrm{d}g_1 = \mathrm{d}g_2$ is a Haar measure on H and $\mathrm{d}\xi$ a Haar measure on $G^{\rho(x)}$. In passing we note that $G^{\rho(x)}$ is reductive, hence unimodular, because $\rho(x)$ is semi-simple.

Now suppose that T is a $H \times H$ bi-invariant distribution on Ω_x. Then

$$T(f_{\alpha\otimes\beta\otimes\gamma}) = I(\alpha)I(\gamma)\mu_T(\beta),$$

where μ_T is a distribution on $U_{\rho(x)}$ and $I(\alpha) = \int_H \alpha(g)\,\mathrm{d}g$. The distribution μ_T is uniquely determined by T. It has certain properties of invariance. For instance, it is invariant under left multiplication by $H^{\rho(x)}$. It is also invariant under right multiplication by $H \cap G^{\rho(x)} \cap xHx^{-1}$. Since this group is actually equal to $H^{\rho(x)}$, we see that μ_T is actually bi-invariant under $H^{\rho(x)}$. Recall also the identity which defines $\sharp$:

$$(h_1\xi h_2)^{-1} = h_2^{-1}g_1\xi^\sharp x g_2 h_1^{-1}. \tag{72}$$

It follows that the distribution μ_T is skew invariant under $\sharp$. If ψ is a smooth function of compact support on $k^\times$, then $(\psi \circ f_x)\mu_T$ extends to a distribution on $G^{\rho(x)}$ which is $H^{\rho(x)}$ invariant and $\sharp$ skew invariant. Assume that $\rho(x)$ is not central. Then the triple $(G^{\rho(x)}, H^{\rho(x)}, \sharp)$ is a product of triples (G_i, H_i, σ_i) for which the theorem is true: a H_i invariant distribution on G_i which is σ_i skew invariant is 0. It follows that $(\psi \circ f_x)\mu_T = 0$ and then $\mu_T = 0$. Thus the restriction of T to Ω_x is 0. The open set Ω_x contains the element x and $\rho(x)$ is semi-simple. We will show in the next section that the set U_x also contains all the elements of the form $\exp(\frac{1}{2}X)$ where X

is nilpotent in L and commutes to $\rho(x)$. Thus Ω_x contains the product $\exp(\frac{1}{2}X)x$. However $\rho(\exp(\frac{1}{2}X)x) = \rho(\exp(\frac{1}{2}X))\rho(x)$. Conversely, if g' is an element of Y with Jordan decomposition $g' = gg'_u$ then $g' = \rho(\exp(\frac{1}{2}X)x)$ for a suitable X (Lemma 4.1). Thus Ω_x contains all y such that $\rho(y)$ has semi-simple part $\rho(x) = g$ (and in fact all y such that the semi-simple part of $\rho(y)$ is H conjugate to g).

Thus T vanishes on the union of these open sets, that is, T vanishes on the open set of elements y such that the semi-simple part of $\rho(y)$ is not central. In other words, the support of T is contained in the union of the closed sets:

$$\{y \in \mathrm{GL}(n,k) | \rho(y)_s = I\}, \tag{73}$$

$$\{y \in \mathrm{GL}(n,k) | \rho(y)_s = -I\}. \tag{74}$$

Suppose $\rho(x)_s = I$, that is, $\rho(x)$ belongs to the set N_Y of unipotent elements of Y. Then we have $x = \exp(\frac{1}{2}X)$ with $X \in \mathfrak{n}_L$ and $\rho(x) = \exp(X)$. Thus the first set is in fact HN_YH. The same analysis shows that if $g \in N_Y$ then $g^{-1} = \varepsilon g \varepsilon$. Thus $Hg^{-1}H = HgH$. Let Ω be the open set of $x \in G$ such that $\rho(x)_s \neq -I$. We claim the restriction of T to Ω is 0. Let Ω_0 be the complement of (73) in Ω. We can write Ω has a finite union of increasing open sets $\Omega_j, 0 \leqslant j \leqslant J$ starting with Ω_0, such that $\Omega_j - \Omega_{j-1} = Hx_jH$ with $x_j \in \mathfrak{n}_L$. Since the orbit Hx_jH is invariant under $x \mapsto x^{-1}$ so is each open set Ω_j. We prove inductively that T vanishes on Ω_j. We already know that T vanishes on Ω_0. Thus we may assume that $j > 0$ and T vanishes on Ω_{j-1}. Then its restriction T_j to Ω_j may be viewed as a distribution on $X_j = Hx_jH$ invariant under H and skew invariant under $x \mapsto x^{-1}$. Thus T_j is in fact an invariant measure on X_j. The map $x \mapsto x^{-1}$ changes this measure to a positive multiple hence must leave it invariant. On the other hand, T_j is skew invariant under the same map. This implies that $T_j = 0$. Thus $T_j = 0$ for all j and T vanishes on Ω.

We have now proved that the support of T is contained in the set (74). In order for this set to be non empty we need $-I_n$ to be in $Y_{p,n}$. This happens only if n is even and $p = n/2$. Recall the element

$$w = w_p = \begin{pmatrix} 0 & I_p \\ I_p & 0 \end{pmatrix}. \tag{75}$$

We have $\rho(w) = -I_n$. It follows that the set (74) is actually the set

$$HwN_YH = HN_YwH.$$

We introduce the Cayley map λ from

$$W = \{Z \in M(n \times n, k) | \det(Z + I) \,.\, \det(Z - I) \neq 0\} \tag{76}$$

to

$$U = \{Z \in \mathrm{GL}(n,k) | \det(Z + I) \neq 0\} \tag{77}$$

defined by

$$\lambda(Z) = (I - Z)(I + Z)^{-1}. \tag{78}$$

Note that W is invariant under $Z \mapsto -Z$ and

$$\lambda(-Z) = \lambda(Z)^{-1}. \tag{79}$$

The map is a diffeomorphism of the two given sets. In particular, λ carries the set $\mathfrak{n}_n$ of nilpotent elements of $M(n \times n, k)$ onto the set N_G of unipotent elements of G. We set $W_L = L \cap W$ and define a map

$$\phi\colon\ H \times W_L \times H \to \mathrm{GL}(n)$$

given by

$$\phi(g, \xi, g') = g w \lambda(\xi) g'.$$

It is clear that λ is submersive at every point of $H \times W_L \times H$. Let Ω be its image. Thus Ω contains $H w N_Y H$. Moreover:

$$(h_1 w \lambda(\xi) h_2)^{-1} = h_2^{-1} \lambda(\xi)^{-1} w h_1^{-1} = h_2^{-1} w \lambda(-w\xi w) h_1^{-1}.$$

In particular, the open set Ω is invariant under $g \mapsto g^{-1}$ and the restriction of T to Ω is skew invariant under the same map. Finally the restriction of T to Ω has support in the closed set $H w N_Y H$. We want to show that this restriction is 0.

As usual associated to the submersive map ϕ there is a surjective map $\alpha \mapsto f_\alpha$ from $C_C^\infty(H \times W_L \times H)$ to $C_c^\infty(\Omega)$ such that for $T \in C^\infty(\Omega)$,

$$\int \alpha_1(h_1)\alpha_2(\xi)\alpha_3(h_2) T(h_1 w \lambda(\xi) h_2)\, \mathrm{d}h_1\, \mathrm{d}h_2\, \mathrm{d}\xi \\ = \int f_{\alpha_1 \otimes \alpha_2 \otimes \alpha_3}(g) T(g)\, \mathrm{d}g.$$

To the invariant distribution T is then associated a distribution μ_T on W_L such that

$$T(f_{\alpha_1 \otimes \alpha_2 \otimes \alpha_3}) = I(\alpha_1)\mu_T(\alpha_2) I(\alpha_3).$$

As before $I(\alpha_i) = \int \alpha_i(h) dh$ is a Haar measure on H. The distribution is invariant under conjugation by H. It is also skew invariant under $\xi \mapsto -w\xi w$. However we have $\varepsilon(\xi)\varepsilon = -\xi$ for $\xi \in L$ and $\varepsilon \in H$. Thus in fact μ_T is skew invariant under $\xi \mapsto w\xi w$. As usual, if ψ is in $C_c^\infty(F^\times)$ the product

$$\psi(\det(I + \xi) \,.\, \det(I - \xi))\, \mathrm{d}\mu_T(\xi)$$

extends to a distribution on L which is invariant under $\mathrm{Ad}\, H$ and skew invariant under $\xi \mapsto w\xi w$. By the result on the infinitesimal symmetric space, it follows that this distribution vanishes. Hence $\mu_T = 0$ and T vanishes on Ω. This concludes the proof of the induction step for H bi-invariant distributions skew invariant under $g \mapsto g^{-1}$.

5.2. THE OPEN SET U_x

We let $x \in G$ be an element such that $\rho(x)$ is semi-simple. Recall the map

$$\Phi\colon\ H \times G^{\rho(x)} \times H \to \mathrm{GL}(n)$$

$$\Phi\colon\ (h, \xi, h') \mapsto h\xi x h'.$$

Recall U_x is the set of ξ such that Φ is submersive at $(1, \xi, 1)$, or, what amounts to the same:

$$\mathfrak{h} + \mathrm{Ad}(\xi x)(\mathfrak{h}) + L^{\rho(x)} = \mathfrak{g},$$

a condition which is also equivalent to:

$$\mathfrak{h} + \mathrm{Ad}(\xi x)(\mathfrak{h}) + \mathfrak{g}^{\rho(x)} = \mathfrak{g}.$$

Recall also the decomposition of $\mathfrak{g}$ into the $+1$ and -1 eigenspace for $\mathrm{Ad}\,\varepsilon$: $\mathfrak{g} = \mathfrak{h} \oplus L$. We call p_L the projection on the second factor. Since $\varepsilon\rho(x)\varepsilon = \rho(x)^{-1}$, we have also

$$\mathfrak{g}^{\rho(x)} = \mathfrak{h}^{\rho(x)} \oplus L^{\rho(x)}.$$

We see that ξ is in U_x if and only if

$$p_L\left(\mathrm{Ad}(\xi x)(\mathfrak{h})\right) + L^{\rho(x)} = L.$$

Recall we let $\mathfrak{g}_{\rho(x)}$ denote the orthogonal complement of $\mathfrak{g}^{\rho(x)}$. We also set $\mathfrak{h} \cap (\mathfrak{g}_{\rho(x)}) = \mathfrak{h}_{\rho(x)}$ and $L \cap (\mathfrak{g}_{\rho(x)}) = L_{\rho(x)}$. Since $\rho(x)$ is semi-simple, we have the orthogonal decompositions:

$$\mathfrak{g} = \mathfrak{g}^{\rho(x)} + \mathfrak{g}_{\rho(x)};$$

$$\mathfrak{h} = \mathfrak{h}^{\rho(x)} + \mathfrak{h}_{\rho(x)}, \quad L = L^{\rho(x)} + L_{\rho(x)}.$$

Since $\rho(x^{-1}) = x^{-1}\varepsilon\rho(x)\varepsilon x$, the element $\rho(x^{-1})$ is also semi-simple so that we have similar decompositions for x^{-1}. In particular:

$$\mathfrak{h} = \mathfrak{h}^{\rho(x^{-1})} + \mathfrak{h}_{\rho(x^{-1})}.$$

Recall $\mathfrak{h}^{\rho(x^{-1})} = \mathfrak{h} \cap x^{-1}\mathfrak{h}x$. Then, for $\xi \in G^{\rho(x)}$,

$$\mathrm{Ad}(\xi x)(\mathfrak{h} \cap x^{-1}\mathfrak{h}x) = \mathrm{Ad}\,\xi(\mathfrak{h} \cap x\mathfrak{h}x^{-1}) = \mathrm{Ad}\,\xi(\mathfrak{h}^{\rho(x)}) \subset \mathfrak{g}^{\rho(x)} \subset \mathfrak{h} + L^{\rho(x)}.$$

It follows that ξ is in U_x if and only if

$$p_L\left(\mathrm{Ad}(\xi x)\mathfrak{h}_{\rho(x^{-1})}\right) + L^{\rho(x)} = L.$$

But we claim that the first term in this sum of spaces is actually contained in $L_{\rho(x)}$. Indeed, we note that for $W \in L^{\rho(x)}$ and $T \in \mathfrak{h}_{\rho(x^{-1})}$ we have

$$\beta\left(W, p_L(\mathrm{Ad}(\xi x)(T))\right) = \beta\left(\mathrm{Ad}(x^{-1}\xi^{-1})W, T\right).$$

However, $\mathrm{Ad}(\xi^{-1})W$ is still in $\mathfrak{g}^{\rho(x)}$. On the other hand:

$$\mathrm{Ad}\,x^{-1}(\mathfrak{g}^{\rho(x)}) = \mathfrak{h} \cap x^{-1}\mathfrak{h}x \oplus L \cap x^{-1}Lx = \mathfrak{g}^{\rho(x^{-1})}.$$

Thus $\mathrm{Ad}(x^{-1}\xi^{-1})W$ is in $\mathfrak{g}^{\rho(x^{-1})}$ and, in particular, orthogonal to T. Our assertion follows.

Thus there exists a linear map $\phi_{\xi x}$

$$\phi_{\xi x}\colon\ \mathfrak{h}_{\rho(x^{-1})} \to L_{\rho(x)} \tag{80}$$

such that

$$\phi_{\xi x}(T) = p_L(\mathrm{Ad}(\xi x)T), \tag{81}$$

and ξ is in U_x if and only if the map (80) is surjective.

Next we assert that the spaces in (80) have the same dimension. To that end, we let $\xi = 1$ in the above discussion. We have already observed that $\mathrm{Ad}(x)$ carries $\mathfrak{h}^{\rho(x^{-1})} = \mathfrak{h} \cap x^{-1}\mathfrak{h}x$ to $\mathfrak{h} \cap x\mathfrak{h}x^{-1} = \mathfrak{h}^{\rho(x)}$ which is orthogonal to L. It follows that the map $T \mapsto p_L(\mathrm{Ad}(x)(T))$ from $\mathfrak{h}$ to L has kernel $\mathfrak{h} \cap x^{-1}\mathfrak{h}x = \mathfrak{h}^{\rho(x^{-1})}$. Hence ϕ_x is injective. Now let us find the perpendicular complement of the range of ϕ_x. So suppose W is orthogonal to $p_L(\mathrm{Ad}(x)T)$ for all $T \in \mathfrak{h}_{\rho(x^{-1})}$. Then $\mathrm{Ad}\,x^{-1}(p_L W)$ is orthogonal to $\mathfrak{h}_{\rho(x^{-1})}$ thus is in $L + \mathfrak{h}^{\rho(x^{-1})} = L \oplus \mathfrak{h} \cap x^{-1}\mathfrak{h}x$. This implies in turn that $p_L(W) \in \mathrm{Ad}(x)(L) + \mathfrak{h} \cap x\mathfrak{h}x^{-1}$. Thus in fact $p_L(W)$ belongs to $L \cap xLx^{-1} = L^{\rho(x)}$. Hence the perpendicular complement of the range of ϕ_x is $\mathfrak{h} + L^{\rho(x)}$; that is, the range is $L_{\rho(x)}$. Hence ϕ_x is bijective.

We now choose bases in the spaces of (80). Then we can define the determinant of the map $\phi_{\xi x}$ and set

$$S_x(\xi) = \det(\phi_{\xi x}). \tag{82}$$

Thus ξ is in U_x if and only if $S_x(\xi) \neq 0$.

Next we consider the group $H^{\rho(x)}$. We claim that

LEMMA 5.1. *For all ξ in $G^{\rho(x)}$ and $h_i \in H^{\rho(x)}$, $i = 1, 2$:*

$$S_x(h_1 \xi h_2) = S_x(\xi).$$

Proof. We note that

$$p_L \circ \mathrm{Ad}(h_1 \xi h_2 x) = \mathrm{Ad}(h_1) \circ p_L \circ \mathrm{Ad}(\xi x) \circ \mathrm{Ad}(x^{-1} h_2 x).$$

Next we recall that $h_2 \in H^{\rho(x)} = H \cap xHx^{-1}$ implies $x^{-1}h_2 x \in H \cap x^{-1}Hx = H^{\rho(x^{-1})}$. Thus $\mathrm{Ad}(x^{-1}h_2x)$ defines a bijection of $\mathfrak{h}_{\rho(x^{-1})}$ on itself which is an orthogonal transformation for the restriction of β; in particular, it has determinant $\delta_2(h_2) = \pm 1$. On the other hand, $\mathrm{Ad}\, h_1$ leaves $L_{\rho(x)}$ invariant and define a bijection of that space onto itself which is an orthogonal transformation for the restriction of β, hence has determinant $\delta_1(h_1) = \pm 1$. However, we have seen in the previous subsection that $H^{\rho(x)}$ is a productof linear groups (over k or an extension). Thus $\delta_1(h_1) = \delta_2(h_2) = 1$ and we are done. □

Our next lemma is:

LEMMA 5.2. *The open set U_x is invariant under $\xi \mapsto \xi^\sharp$.*

Proof. Recall that we choose $g_1 \in H$ such that $g_1 \rho(x) g_1^{-1} = \rho(x^{-1})$ and then $\xi^\sharp = g_1^{-1} x^{-1} \xi^{-1} x g_1$. Suppose ξ is in U_x, that is,

$$\mathfrak{h} + \mathrm{Ad}(\xi x)(\mathfrak{h}) + \mathfrak{g}^{\rho(x)} = \mathfrak{g}.$$

We have to see that $\xi^\sharp$ verifies the same condition:

$$\mathfrak{h} + \mathrm{Ad}(\xi^\sharp x)(\mathfrak{h}) + \mathfrak{g}^{\rho(x)} = \mathfrak{g}.$$

The left hand side can be written as

$$\mathrm{Ad}(g_1^{-1}x^{-1}\xi^{-1}) \left[\mathrm{Ad}(\xi x)(\mathfrak{h}) + \mathrm{Ad}(x g_1 x)(\mathfrak{h}) + \mathrm{Ad}(\xi x g_1)(\mathfrak{g}^{\rho(x)})\right]. \tag{83}$$

But $\mathrm{Ad}(g_1)$ takes $\rho(x)$ to $\rho(x^{-1})$ hence takes $\mathfrak{g}^{\rho(x)}$ to $\mathfrak{g}^{\rho(x^{-1})}$. In turn, $\mathrm{Ad}\, x$ takes this space to $\mathfrak{g}^{\rho(x)}$. Thus the third term in (83) is $\mathfrak{g}^{\rho(x)}$. For the middle term, we remark that since $g_1 \in H$ commutes to ε we can write, for $T \in \mathfrak{h}$:

$$\mathrm{Ad}(\varepsilon)\, \mathrm{Ad}(x g_1 x)(T) = \mathrm{Ad}[(\varepsilon x \varepsilon) g_1 (\varepsilon x \varepsilon)](T).$$

However, it is easily checked that

$$\begin{aligned}
(\varepsilon x \varepsilon) g_1 (\varepsilon x \varepsilon) x^{-1} g_1^{-1} x^{-1} &= (\varepsilon x \varepsilon) g_1 [\rho(x)]^{-1} g_1^{-1} x^{-1} \\
&= (\varepsilon x \varepsilon) [\rho(x^{-1})]^{-1} x^{-1} \\
&= (\varepsilon x \varepsilon)(\varepsilon x^{-1} \varepsilon) x x^{-1} = e.
\end{aligned}$$

Thus

$$\mathrm{Ad}(\varepsilon)\,\mathrm{Ad}(xg_1x)(T) = \mathrm{Ad}(xg_1x)(T),$$

so that the middle term is contained and in fact equal to $\mathfrak{h}$. Finally we see that (83) can be rewritten in the form:

$$\mathrm{Ad}(g_1^{-1}x^{-1}\xi^{-1})\left[\mathfrak{h} + \mathrm{Ad}(\xi x)(\mathfrak{h}) + \mathfrak{g}^{\rho(x)}\right] = \mathrm{Ad}(g_1^{-1}x^{-1}\xi^{-1})(\mathfrak{g}) = \mathfrak{g}.$$

Thus $\xi^\sharp$ is in U_x as claimed. □

We next give another formula for S_x. It will be convenient to denote by $\mathfrak{q}$ the subspace $\mathfrak{g}_{\rho(x)}$. It is invariant under $\mathrm{Ad}\,\varepsilon$. If $\xi \in G$ commutes to $\rho(x)$ then so does ξ^{-1}; thus $\mathfrak{q}$ is invariant under $\mathrm{Ad}(\rho(\xi))$. Since $\rho(\xi x) = \rho(\xi)\rho(x)$ we see that $\mathfrak{q}$ is invariant under $\mathrm{Ad}\,\rho(\xi x)$.

LEMMA 5.3. *Suppose $\rho(x)$ is semi-simple. Then, there is $c \in k^\times$ such that, for all $\xi \in G^{\rho(x)}$:*

$$S_x(\xi)^2 = c \ \det\left(I - \mathrm{Ad}\,\rho(\xi x)|_{\mathfrak{q}}\right).$$

Proof. We first compare S_{xh_1} and S_x for $h_1 \in H$. We have $\rho((xh_1)^{-1}) = h_1^{-1}\rho(x)h_1$. Thus

$$\mathfrak{g}^{\rho((xh_1)^{-1})} = h_1^{-1}\mathfrak{g}^{\rho(x^{-1})}h_1.$$

Similarly:

$$\mathfrak{h}_{\rho((xh_1)^{-1})} = \mathrm{Ad}(h_1)^{-1}\mathfrak{h}_{\rho(x^{-1})}.$$

On the other hand $\rho(xh_1) = \rho(x)$ so that

$$L_{\rho(xh_1)} = L_{\rho(x)}.$$

Since $p_L \circ \mathrm{Ad}(\xi x h_1) \circ \mathrm{Ad}(h_1)^{-1} = p_L \circ \mathrm{Ad}(\xi x)$ we see that the determinants of ϕ_{xh_1} and ϕ_x are equal (for a suitable choice of the bases), that is, $S_{xh_1}(\xi) = S_x(\xi)$. Thus we have for $h_1, h_2 \in H^{\rho(x)}$ and $h \in H$:

$$S_{xh}(h_1\xi h_2) = S_x(\xi).$$

On the other hand, we have $\rho(h_1\xi h_2 x h) = \rho(h_1\xi h_2)\rho(x) = h_1\rho(\xi)h_1^{-1}\rho(x)$. It follows that:

$$\det((I - \mathrm{Ad}\,\rho(h_1\xi h_2 x h))|_{\mathfrak{q}}) = \det((I - \mathrm{Ad}\,\rho(\xi x))|_{\mathfrak{q}});$$

$$\det(\mathrm{Ad}\,\rho(h_1\xi h_2 xh)|_{\mathfrak{q}}) = \det(\mathrm{Ad}\,\rho(\xi x)|_{\mathfrak{q}}).$$

Thus to prove the identity above we may modify x by multiplication on the right by H and modify ξ by multiplication on the left and on the right by $H^{\rho(x)}$. Furthermore, we may replace k by its algebraic closure. Fix a torus T of G which is ε invariant in the sense that $\varepsilon t\varepsilon = t^{-1}$ for $t \in T$; suppose further that T is maximal among ε invariant tori. Then $\rho(x)$ is H conjugate to an element of T. Thus we may as well assume $\rho(x) \in T$. We can then write $\rho(x) = \beta^2$ with $\beta \in T$. Then $\rho(\beta) = \beta^2 = \rho(x)$. It follows that $x = \beta h$ for some $h \in H$ (polar decomposition). To prove our identity, we may as well assume $x = \beta$. In other words, we may assume that x is also in the torus T.

Now the group $G^{\rho(x)}$ is invariant under conjugation by ε. Clearly T is a maximal invariant torus in $G^{\rho(x)}$. For $\xi \in G^{\rho(x)}$, the element $\rho(\xi) = \xi\varepsilon\xi^{-1}\varepsilon$ is still in the same group. Thus ρ is the polarization map for a symmetric space of $G^{\rho(x)}$. It follows from a Theorem of Richardson that the set of ξ such that $\rho(\xi)$ is semi-simple is dense in $G^{\rho(x)}$. As a result, it suffices to prove our identity for an element ξ such that $\rho(\xi)$ is semi-simple. As before, ξ has a polar decomposition $\xi = \alpha h_2$ with $\varepsilon\alpha\varepsilon = \alpha^{-1}$ and $h_2 \in H^{\rho(x)}$. We may as well assume $\xi = \alpha$, that is, $\varepsilon\xi\varepsilon = \xi^{-1}$, $\rho(\xi) = \xi^2$ and ξ is semi-simple. Then ξ is conjugate to T by an element $h_1 \in H^{\rho(x)}$. Thus we may as well assume ξ is in T. Thus it suffices to prove our identity for x and ξ in T.

At this point, we choose orthonormal bases Y_i and Z_j (for the restriction of β) on the spaces $\mathfrak{h}_{\rho(x^{-1})}$ and $L_{\rho(x)}$. For $X \in \mathfrak{h}$, we have

$$p_L \circ \mathrm{Ad}(\xi x)(X) = \tfrac{1}{2}\left[\mathrm{Ad}(\xi x) - \mathrm{Ad}(\varepsilon)\,\mathrm{Ad}(\xi x)\,\mathrm{Ad}(\varepsilon)\right] X.$$

Thus we can take:

$$S_x(\xi) = \det\{\beta(\tfrac{1}{2}[\mathrm{Ad}(\xi x) - \mathrm{Ad}(\varepsilon)\,\mathrm{Ad}(\xi x)\,\mathrm{Ad}(\varepsilon)]Y_i|Z_j)\}.$$

Since x is in T we have $\rho(x) = x^2 = \rho(x^{-1})^{-1}$. Hence $\mathfrak{g}^{\rho(x)} = \mathfrak{g}^{\rho(x^{-1})}$ in the case at hand. Thus $\mathrm{Ad}(\varepsilon), \mathrm{Ad}(\xi), \mathrm{Ad}(x)$ leave $\mathfrak{g}^{\rho(x)}$ invariant. Thus they leave $\mathfrak{q}$ invariant as well. We can then consider the restriction of the operator

$$\tfrac{1}{2}[\mathrm{Ad}(\xi x) - \mathrm{Ad}\,\varepsilon\,\mathrm{Ad}(\xi x)\,\mathrm{Ad}\,\varepsilon]$$

to $\mathfrak{q}$; it maps this space to itself. We compute its determinant. The vectors Y_j, Z_j form here a basis of $\mathfrak{q}$. Using the fact that $\varepsilon\xi x\varepsilon = (\xi x)^{-1}$, we get

$$\begin{aligned}&\beta(\tfrac{1}{2}[\mathrm{Ad}(\xi x) - \mathrm{Ad}(\varepsilon)\,\mathrm{Ad}(\xi x)\,\mathrm{Ad}(\varepsilon)]Y_i|Z_j)\\ &\quad= \beta(Y_i|\tfrac{1}{2}[\mathrm{Ad}(\varepsilon)\,\mathrm{Ad}(\xi x)\,\mathrm{Ad}(\varepsilon) - \mathrm{Ad}(\xi x)]Y_i|Z_j).\end{aligned}$$

We also have

$$\beta\tfrac{1}{2}[\mathrm{Ad}(\xi x) - \mathrm{Ad}(\varepsilon)\,\mathrm{Ad}(\xi x)\,\mathrm{Ad}(\varepsilon)]Y_i|Y_j) = 0;$$

$$\beta(Z_i|\tfrac{1}{2}[\mathrm{Ad}(\varepsilon)\,\mathrm{Ad}(\xi x)\,\mathrm{Ad}(\varepsilon) - \mathrm{Ad}(\xi x)]Y_i|Z_j) = 0.$$

We easily find then the matrix of our operator has the form

$$\begin{pmatrix} 0 & S \\ -S & 0 \end{pmatrix},$$

where S is the matrix of $\phi_{\xi x}$. It follows that

$$(S_x(\xi))^2 = \det \tfrac{1}{2}[\mathrm{Ad}(\xi x) - \mathrm{Ad}(\varepsilon)\,\mathrm{Ad}(\xi x)\,\mathrm{Ad}(\varepsilon)]|_{\mathfrak{q}}.$$

Since $\varepsilon(\xi x)\varepsilon = (\xi x)^{-1}$ we have $\rho(\xi x) = (\xi x)^2$ and the above operator can be written as the restriction to $\mathfrak{q}$ of

$$-\mathrm{Ad}(\xi x)^{-1}\tfrac{1}{2}(I - \mathrm{Ad}(\xi x)^2) = -\mathrm{Ad}(\xi x)^{-1}\tfrac{1}{2}(I - \mathrm{Ad}\,\rho(\xi x)).$$

Since, ξx is in $T \subseteq G^{\rho(x)} \subseteq G$, we have:

$$\det\ \mathrm{Ad}\,\xi x|_{\mathfrak{q}} = (\det\ \mathrm{Ad}\,\xi x))\,.\,(\det\ \mathrm{Ad}(\xi x)|_{\mathfrak{g}}^{\rho(x)})^{-1} = 1.$$

So we get our formula for $S_x(\xi)^2$. □

The last result we need is the following lemma:

LEMMA 5.4. *Suppose y is an element such that the semi-simple part of $\rho(y)$ is equal to $\rho(x)$. Then y is in U_x.*

Proof. We have seen that there is $v \in L$, nilpotent, such that v commutes with $\rho(x)$, and, setting $\xi = \exp(v/2)$,

$$\rho(\xi) = \exp(v), \quad \rho(\xi x) = \rho(\xi)\rho(x) = \rho(y).$$

Thus $y = \xi x h$ and we have to see that ξ is in U_x. Since $\mathfrak{g}^{\rho(x)}$ is the $+1$ eigenspace for $\mathrm{Ad}(\rho(x))$, it contains any $+1$ eigenvector for the product of $\mathrm{Ad}(\rho(x))$ and the unipotent operator $\mathrm{Ad}(\rho(\xi))$ which commutes with it. This product is $\mathrm{Ad}(\rho(\xi x))$. Thus

$$\det(I - \mathrm{Ad}\,\rho(\xi x))|_{\mathfrak{q}} \neq 0$$

and our conclusion follows.

5.3. Second reduction

Assume n is even. We still have to show that a distribution T on G which is H invariant is invariant under conjugation by $w = w_p$ where $p = n/2$. We may as well assume that T is skew invariant under conjugation by w and show that it is zero.

LEMMA 5.5. *Suppose $\rho(x)$ is semi-simple. Then $\rho(wxw) = w\rho(x)w$ is semi-simple and there is $h \in H$ such that $hxh^{-1} = wxw$. Finally, wh commutes with $\rho(x)$.*

Proof. Indeed, we have $w\varepsilon w = -\varepsilon$. It follows that:

$$\begin{aligned}\rho(wxw) &= (wxw)\varepsilon(wx^{-1}w)\varepsilon = -wx\varepsilon x^{-1}w\varepsilon \\ &= w(x\varepsilon x^{-1}\varepsilon)w = w\rho(x)w.\end{aligned}$$

Thus if $g = \rho(x)$ is semi-simple so is $\rho(wxw)$. To continue we may write $V = V' \oplus V''$ where V' and V'' are homogeneous subspaces and $\dim V_0' = \dim V_1'$, $\dim V_0'' = \dim V_1''$ and $g = g' \oplus g''$, where g' does not have the eigenvalue ± 1 while g'' has only the eigenvalues ± 1. We have then: $\varepsilon = \varepsilon' \oplus \varepsilon''$ and $w = w' \oplus w''$. We may further assume $x = x' \oplus x''$. Thus it suffices to prove our assertion for x' and x''. Equivalently, we may assume that g does not have the eigenvalue ± 1 or, on the contrary, has only the eigenvalues ± 1. In the first case, at the cost of replacing g by a conjugate under H, we may assume that $g = t(A)$ where A is a $p \times p$ matrix without the eigenvalue ± 1. Then we write

$$A = (I + U)(I - U)^{-1}$$

and we can take

$$x = x(U) = \begin{pmatrix} I & X \\ Y & I \end{pmatrix},$$

where $X(I - U)/2$ and $Y = 2U(I - U)^{-1}$. We have then

$$wxw = \begin{pmatrix} I & Y \\ X & I \end{pmatrix}.$$

We find

$$wxw = hxh^{-1}$$

where

$$h = \begin{pmatrix} 4U & 0 \\ 0 & (I - U)^2 \end{pmatrix}.$$

If on the contrary g has only the eigenvalue ± 1 then, at the cost of replacing g by an H conjugate, we may assume that $g = \rho(x)$ where

$$x = \begin{pmatrix} I_\alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & I_\beta \\ 0 & 0 & I_\alpha & 0 \\ 0 & I_\beta & 0 & 0 \end{pmatrix}.$$

Then $wxw = x$.

Finally, we have

$$wh\rho(x)h^{-1}w = w\rho(hxh^{-1})w = \rho(whxh^{-1}w) = \rho(x).$$

The last assertion of the lemma follows. □

At this point, we argue as before. Let g be a semi-simple element of Y. Let x be such that $\rho(x) = g$ is semi-simple. We recall the map $\Phi\colon\ H \times G^{\rho(x)} \times H \to G$ defined by $\Phi(h, \xi, h') = h\xi xh'$. We claim that the image of Φ is invariant under conjugation by w. Indeed, choose $h \in H$ such that $hxh^{-1} = wxw$. We have then $h^{-1}w\rho(x)wh = \rho(x)$. Thus for $\xi \in G^{\rho(x)}$ we get:

$$w(\xi x)w = w\xi wwxw = h\xi^{\flat}h^{-1}hxh^{-1} = h\xi^{\flat}xh^{-1},$$

where we have set

$$\xi^{\flat} = h^{-1}w\xi wh \in G^{\rho(x)}.$$

Thus we get:

$$w(h_1\xi xh_2)w = (wh_1wh)\xi^{\flat}x(h^{-1}wh_2w). \tag{84}$$

This prove our assertion. We show now:

LEMMA 5.6. *The open set U_x is invariant under $\xi \mapsto \xi^{\flat}$.*

Proof. Suppose that ξ is in U_x. Then

$$\mathfrak{h} + \mathrm{Ad}(\xi x)\mathfrak{h} + \mathfrak{h}^{\rho(x)} = \mathfrak{g}$$

and we have to see that $\xi^{\flat}$ has the same property. Indeed:

$$\begin{aligned}\mathfrak{h} + \mathrm{Ad}(\xi^{\flat}x)\mathfrak{h} + \mathfrak{h}^{\rho(x)} &= \mathfrak{h} + \mathrm{Ad}(h^{-1}w\xi whx)\mathfrak{h} + \mathfrak{h}^{\rho(x)}\\ &= \mathfrak{h} + \mathrm{Ad}(h^{-1}w\xi whxh^{-1}h)\mathfrak{h} + \mathfrak{h}^{\rho(x)}\end{aligned}$$

Since h normalizes $\mathfrak{h}$ and $hxh^{-1} = wxw$, we can write this as:

$$= \mathfrak{h} + \mathrm{Ad}(h^{-1}w\xi wwxw)\mathfrak{h} + \mathfrak{h}^{\rho(x)}$$

or, using the fact that w normalizes $\mathfrak{h}$:

$$\begin{aligned}&\mathfrak{h} + \mathrm{Ad}(h^{-1}w\xi x)\mathfrak{h} + \mathfrak{h}^{\rho(x)}\\ &\quad = \mathrm{Ad}(h^{-1}w)[\mathrm{Ad}(wh)\mathfrak{h} + \mathrm{Ad}(\xi x)\mathfrak{h} + \mathrm{Ad}(wh)\mathfrak{h}^{\rho(x)}].\end{aligned}$$

Again wh normalizes $\mathfrak{h}$ and commutes with $\rho(x)$ hence normalizes $\mathfrak{h}^{\rho(x)}$. Thus the above expression is also

$$\mathrm{Ad}(h^{-1}w)\left[\mathfrak{h} + \mathrm{Ad}(\xi x)\mathfrak{h} + \mathfrak{h}^{\rho(x)}\right] = \mathrm{Ad}(h^{-1}w)(\mathfrak{g}) = \mathfrak{g}.$$

The lemma follows. □

If ξ is in $H^{\rho(x)}$ so is $\xi^\flat$ since wh commutes with $\rho(x)$. We have also

$$((\xi)^\flat)^\flat = h_1^{-1}\xi h_1,$$

where $h_1 = whwh$. Clearly h_1 is in H and commutes with $\rho(x)$ since wh does. Now U_x is the set of non-zeroes of g_x, a regular function invariant under $H^{\rho(x)}$ on the left and the right. It is also the set of non-zeroes of $f_x(\xi) = g_x(\xi)g_x(\xi^\flat)$ which is still invariant under $H^{\rho(x)}$ on the left and the right, but is also invariant under $\xi \mapsto \xi^\flat$.

Suppose that g is a semi-simple not central element of Y. We claim we can choose x with $\rho(x) = g$ and h with $wxw = hxh^{-1}$ in such a way that $\flat$ has order 2. As before, we write $V = V' \oplus V''$ and $g = g' \oplus g''$ where g' does not have the eigenvalue ± 1 and g'' has only the eigenvalues ± 1. We have also $w = w' \oplus w''$. We can choose x of the form $x = x' \oplus x''$. Also $G^{\rho(x)} = \mathrm{GL}(V')^{\rho(x')} \oplus \mathrm{GL}(V'')^{\rho(x'')}$ and we can choose h of the form $h = h' \oplus h''$. Then the automorphism $\xi \mapsto \xi^\flat$ is compatible with this decomposition in the sense that if $\xi = \xi' + \xi''$ then $\xi^\flat = (\xi')^\flat + (\xi'')^\flat$.

We may assume $x' = x(U)$ as before. Then $\mathrm{GL}(V')^{\rho(x')}$ is the set of matrices of the form

$$\xi' = \begin{pmatrix} Z_1 & Z_2 \\ Z_2\delta & Z_1 \end{pmatrix},$$

where Z_i commutes with U and $\delta = 4U(I - U^2)^{-1}$. We have then

$$g'^{-1}w'\xi'w'g' = \xi'$$

by a direct computation. Hence $\xi \mapsto \xi^\flat$ induces the identity on $\mathrm{GL}(V')^{\rho(x')}$.

For x'' we may take $g'' = e$ and

$$\rho(x'') = \begin{pmatrix} I_\alpha & 0 & 0 & 0 \\ 0 & -I_\beta & 0 & 0 \\ 0 & 0 & I_\alpha & 0 \\ 0 & 0 & 0 & -I_\beta \end{pmatrix}.$$

Thus $\xi \mapsto \xi^\flat$ induces conjugation by w'' on the second factor. Further the pair (G'', H'') decomposes into a product of pairs

$$(\mathrm{GL}(2\alpha), \quad \mathrm{GL}(\alpha) \times \mathrm{GL}(\alpha)) \times (\mathrm{GL}(2\beta), \quad \mathrm{GL}(\beta) \times \mathrm{GL}(\beta))$$

with $w'' = w_\alpha \oplus w_\beta$. Thus, for this choice of x the automorphism $\flat$ has indeed order 2. Furthermore the triple $(G^{\rho(x)}, H^{\rho(x)}, \xi \mapsto \xi^\flat)$ decomposes into a product of triples of the form (G_i, H_i, σ_i); for each triple, every distribution biinvaraint under H_i is invariant under σ_i either trivially (σ_i is the identity) or by the induction hypothesis.

Now let T be a distribution which is H invariant and skew invariant under w. Just as before, it follows that the restriction of T to U_x vanishes. The support of T is contained in the set of x such that the semi-simple part of $\rho(x)$ is ± 1, or what amounts to the same, the union of the following closed sets:

$$\{x|\rho(x)_s = I\} = HN_YH, \quad \{x|\rho(x)_s = -I\} = HN_YwH.$$

Now we consider the Cayley map λ from

$$W = \{Z|\det(Z+I)\,.\det(Z-I) \neq 0\} \tag{85}$$

to

$$U = \{Z \in \mathrm{GL}(n,k)|\det(Z+I) \neq 0\} \tag{86}$$

given by

$$\lambda(Z) = (I - Z)(I + Z)^{-1}.$$

Let $W_L = W \cap L$. We define a map ϕ: $H \times W_L \times H \to G$ by:

$$\phi(h, \xi, h') = h\lambda(\xi)h'.$$

This map is submersive at any point. Its image Ω is an open set which contains HN_YH. We have also for $h, h' \in H$

$$w(h\lambda(\xi)h')w = whw\lambda(w\xi w)wh'w.$$

Consider the pullback μ_T of the restriction of T to Ω. Since W_L is invariant under w we see that μ_T is invariant under conjugation by H and skew invariant under conjugation by w. Now W_L is the set of non-zeroes of the function $f(Z) = \det(I+Z)\,.\det(I-Z)$ which is invariant under conjugation by H and w. It follows that if μ_T is non-zero, then there is a non-zero distribution on L invariant under H and skew invariant under w. This contradicts the results on the infinitesimal symmetric space. Thus $\mu_T = 0$ and the restriction of T to Ω is zero.

To continue, we consider similarly the map ϕ' form $H \times W_L \times H$ to G defined by:

$$\phi(h, \xi, h') = h\lambda(\xi)wh'.$$

Let Ω' be its image. It an open set containing HN_YwH. We have:

$$w(h\lambda(\xi)wh')w = whw\lambda(w\xi w)wh'w.$$

As before, we conclude that the restriction of T to Ω' is 0. Now Ω, Ω' and the complement of $HN_YH \cup HN_YwH$ form an open cover; the restriction of T to every open set in the cover vanishes. Thus $T = 0$. This concludes the proof of the induction step and the theorem. □

6. Applications to Shalika models

6.1. Uniqueness

We recall the notion of Shalika model for an admissible irreducible representation π of $G = \mathrm{GL}(n,k), n = 2m$. We consider the parabolic subgroup P_m of type (m,m). Its unipotent radical U_m is the group of matrices of the form:

$$\left\{ u = \begin{pmatrix} I & Z \\ 0 & I \end{pmatrix} \middle| Z \in M(m \times m, k) \right\}. \tag{87}$$

The group $H = H_{m,n}$ is a Levi-factor of P. It acts on U_m. Let ψ be a non-trivial additive character of k. Define a character Ψ of U_m by: $\Psi(u) = \psi(\mathrm{Tr}(Z))$. Then the stabilizer of Ψ in H is the group

$$H_0 = \left\{ \begin{pmatrix} g & 0 \\ 0 & g \end{pmatrix} \middle| g \in \mathrm{GL}(m) \right\}. \tag{88}$$

A linear form l on the space V of π is said to be a Shalika functional if

$$l\left(\pi(u)\pi(h)v\right) = \Psi(u)l(v) \tag{89}$$

for $u \in U_m$, $h \in H_0$ and v in V.

We will need the following lemma on Shalika functionals:

LEMMA 6.1. *Suppose that l is a Shalika functional for π. Then there is $s_0 \in \mathbb{R}$ such that for any $v \in V$ the product*

$$l\left[\pi\begin{pmatrix} g & 0 \\ 0 & I \end{pmatrix} v\right] \mid \det g \mid^{s_0} \tag{90}$$

is bounded in absolute value (independently of g). Furthermore, given v, there is a positive Schwartz-Bruhat function $\Phi \geqslant 0$ on $M(m \times m, k)$ such that

$$\left| l\left[\pi\begin{pmatrix} g & 0 \\ 0 & I \end{pmatrix} v\right]\right| \mid \det g \mid^{s_0} \leqslant \Phi(g).$$

For the moment we take the lemma for granted and derive some consequences. Assuming the lemma, we can form the integral

$$I(v,s) = \int_{\mathrm{GL}(n)} l\left[\pi\begin{pmatrix} a & 0 \\ 0 & I \end{pmatrix} v\right] . \mid \det a \mid^{s-1/2} \, \mathrm{d}a$$

The integral converges for $\Re(s)$ sufficiently large. As in [FJ] one can prove that the integral represents a rational function of q^{-s}. More precisely, it has the form

$$L(s,\pi)P(q^{-s}),$$

where P is a polynomial. Moreover, there is a v so that $P = 1$.

REMARK. We note that these assertions are proved in [FJ] under the assumption that the functions $g \mapsto l(\pi(g)v)$ are bounded. The proof is easily modified to apply to the case at hand. Furthermore, in Lemma 6.1, the fact that s_0 is independent of v is not critical.

If we consider then the quotient

$$I_0(v,s) = I(v,s)/L(s,\pi),$$

it is an entire function of s. Moreover:

$$I_0(\pi(h)v,s) = \left|\frac{\det g_2}{\det g_1}\right|^{s-1/2} I_0(v,s) \quad \text{if } h = \begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix}.$$

For $s = \frac{1}{2}$ we obtain a linear form $I_l = I_0(.,\frac{1}{2})$ on the space of π which is invariant under H and non-zero if l is non-zero. In particular:

PROPOSITION 6.1. *Suppose that π has a non-zero Shalika functional. Then $\pi \simeq \tilde{\pi}$. Moreover, the dimension of the space of Shalika functionals is then* 1.

Proof. The first assertion follows from the theorem of the previous section. To prove the second assertion we let l and l' be non-zero Shalika functionals for π. Let I_l and $I_{l'}$ be the corresponding H invariant functionals. We have $I_l(v) = cI_{l'}(v)$ with $c \neq 0$. Consider then the new Shalika functional $l_1 = l - cl'$. From the explicit construction of the linear forms we have $I_{l_1} = I_l - cI_{l'} = 0$. On the other hand if $l_1 \neq 0$ then $I_{l_1} \neq 0$. Thus $l_1 = 0$. □

6.2. AN ASYMPTOTIC EXPANSION

It remains to prove the lemma. The argument that follows is independent of, but closely related to the techniques used by Casselman and Shalika in [CS]. It is likely that their techniques can be used to obtain asymptotic expansions in more general situations. We may assume the conductor of ψ is the ring $\mathcal{O}_k$ of integers. We denote by A the group of diagonal matrices and by P_0 the group of upper triangular matrices. We denote by α_i the simple roots of A with respect to P_0. We consider an element of H of the form:

$$h = h(g) = \begin{pmatrix} g & 0 \\ 0 & I_m \end{pmatrix}. \tag{91}$$

We write $g = k_1 b k_2$ where $k_i \in \mathrm{GL}(m, \mathcal{O}_k)$ and b is a diagonal matrix with $| b_i / b_{i+1} | \leqslant 1$, for $i \leqslant m-1$. In other words, if we set

$$a = a(b) = \begin{pmatrix} b & 0 \\ 0 & I_m \end{pmatrix}, \tag{92}$$

then $| \alpha_i(a) | \leqslant 1$ for $1 \leqslant i \leqslant m-1$. We claim that, given a vector v, there is r such that $l(\pi(h)v) \neq 0$ implies $| b_m | \leqslant q^r$.

Indeed, we note that we have

$$l(\pi(h)v) = l(\pi(a)\pi(k)v),$$

where

$$k = \begin{pmatrix} k_2 & 0 \\ 0 & k_1^{-1} \end{pmatrix}.$$

Since the vectors $\pi(k)v$ belong to a finite set, we may as well assume $g = a$. If

$$u = \begin{pmatrix} I & Z \\ 0 & I \end{pmatrix},$$

then

$$l(\pi(a)\pi(u)v) = \Psi(aua^{-1}) l(\pi(a)v) = \psi\left(\sum_i b_i Z_{i,i}\right) l(\pi(a)v).$$

Thus if v is invariant under the principal congruence subgroup K_r of $K = \mathrm{GL}(n, \mathcal{O})$, we have for $l(\pi(a)v) \neq 0$

$$\psi\left(\sum_i b_i Z_{i,i}\right) = 1 \quad \text{for} \quad | Z_{i,i} | \leqslant q^{-r}.$$

Thus $| b_m | \leqslant q^r$, as claimed.

The next theorem will imply the lemma. It will be convenient to denote by $m(a_1, a_2, \ldots, a_m)$ the matrix $a = a(b)$ where

$$b_i = a_i a_{i+1} \cdots a_m.$$

Thus $\alpha_i(a) = a_i$ for $i \leqslant m$ and $\alpha_i(a) = 1$ for $i > m$. Recall that a finite function on a locally compact abelian group is a continuous function whose translates span a finite dimensional vector space.

THEOREM 6.1. *There is a finite set X of finite functions on $(k^\times)^m$ with the* *following property: for any v, there are Schwartz-Bruhat functions $\phi_\chi, \chi \in X$, on k^m such that, for $a = m(a_1, a_2, \dots, a_m)$ with $| a_i | \leqslant 1$ for $1 \leqslant i \leqslant m - 1$:*

$$l(\pi(a)v) = \sum_{\chi \in X} \chi(a_1, a_2, \dots, a_m)\phi_\chi(a_1, a_2, \dots, a_m).$$

Let us show how this theorem implies Lemma 6.1. Write as above $h = h(g)$ with $g = k_1 b k_2$ and $b_i = a_i a_{i+1} \cdots a_m$ with $| a_i | \leqslant 1$ for $1 \leqslant i \leqslant m - 1$. Then $l(\pi(h)v) = l(\pi(a)\pi(k)v)$ for a suitable k and $a = m(a_1, a_2, \dots, a_m)$. There is r such that $l(\pi(h)v) \neq 0$ implies $| a_m | \leqslant q^r$. Thus, if Φ is the characteristic function of the set of $X \in M(m \times m, k)$ such that $\| X \| \leqslant q^r$, then $l(\pi(h)v) \neq 0$ implies $\Phi(g) \neq 0$. Let $s > 0$. Then $| \det a |^s = | a_1^s a_2^{2s} \cdots a_m^{ms} |$. We can choose s so large that the products

$$| \det a |^s | \chi(a_1, a_2, \dots, a_m) |$$

with $\chi \in X$ are bounded above for $| a_i | \leqslant 1$ for $1 \leqslant i \leqslant m - 1$, $| a_m | \leqslant q^r$. It follows that $| l(\pi(h)v) | | \det g |^s = | l(\pi(a)\pi(k)v) | | \det a |^s$ is bounded above by a constant C. Finally,

$$| l(\pi(h(g))v) | | \det g |^s \leqslant C\Phi(g)$$

and the lemma follows.

Proof. In view of the discussion above, we may in proving the Theorem restrict our attention to the set of $a = m(a_1, a_2, \dots a_m) \in A$ such that $| a_i | \leqslant 1$ for $1 \leqslant i \leqslant m$. Thus in fact, $| \alpha_j(a) | \leqslant 1$ for all j. We first prove a lemma. For $1 \leqslant i \leqslant m$, we let $P_i = M_i U_i$ be the standard parabolic subgroup of type $(i, n - i)$, A_i the center of M_i:

LEMMA 6.2. *Suppose $v = \pi(u)v_0 - v_0$ with $u \in U_i$. Then there is $c > 0$ such that for any $a = m(a_1, a_2, \dots, a_m) \in A$ with $| a_j | \leqslant 1$ for $1 \leqslant j \leqslant m$ and $| \alpha_i(a) | = | a_i | \leqslant c$:*

$$l(\pi(a)v) = 0$$

Proof. Suppose first $i = m$. Then, with the above notations,

$$l(\pi(a)v) = (\Psi(aua^{-1}) - 1)l(\pi(a)v_0) = \left(\prod_i \psi(b_i Z_{i,i}) - 1\right) l(\pi(a)v_0).$$

Since $| b_i | \leqslant | a_m |$ we see this is zero if $| a_m |$ is small enough and we are done in this case. Now suppose $i < m$. We can write $u = u_1 u_2$ with $u_1 \in U_i \cap U_m$ and $u_2 \in U_i \cap M_m$. Explicitly:

$$u_1 = \begin{pmatrix} I_m & Z \\ 0 & I_m \end{pmatrix}, \quad u_2 = \begin{pmatrix} u' & 0 \\ 0 & I_m \end{pmatrix}, \quad u' = \begin{pmatrix} I_i & Z' \\ 0 & I_{m-i} \end{pmatrix}.$$

Then

$$l(\pi(a)v) = \prod_{1\leqslant j\leqslant i} \psi(b_j Z_{j,j}) \,.\, l\left[\pi(a)\pi\begin{pmatrix} I_m & 0 \\ 0 & bu'^{-1}b^{-1}\end{pmatrix} v_0\right] - l(\pi(a)v_0).$$

As before, for $j \leqslant i$, we have $\mid b_j \mid = \mid a_j a_{j+1}\cdots a_i \cdots a_m \mid \leqslant \mid a_i \mid$. Thus $\prod \psi(b_j Z_{j,j}) = 1$ if $\mid a_i \mid$ is small enough. Suppose v_0 is invariant under the principal congruence subgroup K_r. If $a_i = \alpha_i(a)$ has a small enough absolute value then au_2a^{-1} is in K_r. Thus the matrix

$$\begin{pmatrix} I_m & 0 \\ 0 & bu'^{-1}b^{-1}\end{pmatrix}$$

is also in K_r and the above expression is then 0. □

We finish the proof as in [JPS]. Let V be the space of π, $V(U_i)$ the space spanned by the differences $\pi(u)v - v$ with $u \in U_i$ and $v \in V$. The representation $\pi_{U_i} = \pi_i$ of M_i on the quotient $V_i = V/V(U_i)$ is admissible. In particular, the operators $\pi_i(a)$ for $a \in A_i$ span a finite dimensional algebra $\mathcal{A}$ of operators. In fact, $\mathcal{A}$ is already spanned by the operators $\pi_i(a)$ with $\mid \alpha_i(a) \mid \leqslant 1$. There exists a finite set X of finite functionson A_i and for each χ in X an operator A_χ belonging to $\mathcal{A}$ such that

$$\pi_i(a) = \sum_{\chi\in X} \chi(a)A_\chi.$$

Thus A_χ has the form: $A_\chi = \sum \lambda_{j,\chi}\pi_i(a_j)$ where $a_j \in A_i$ verifies $\mid \alpha_i(a_j) \mid \leqslant 1$. We define $B_\chi = \sum \lambda_{j,\chi}\pi(a_j)$. Then we have for any $v \in V$ and $a \in A_i$

$$\pi(a)v \equiv \sum_\chi \chi(a)B_\chi v \pmod{V(U_i)}.$$

To continue, we let S be the product group $\prod_{j\leqslant m} H_j$ where $H_j \simeq k^\times$. Thus $(a_1, a_2, \ldots, a_m) \mapsto m(a_1, a_2, \ldots, a_m)$ gives a mapping $S \to A$ which identifies the factor H_i to the subgroup of A_i of matrices of the form

$$\begin{pmatrix} a_i I_i & 0 \\ 0 & I_{n-i}\end{pmatrix}.$$

Let C be the cone of $m-$tuples in S with $\mid a_i \mid \leqslant 1$ for all i. For $v \in V$ let ϕ_v be the function on C defined by:

$$\phi_v(a_1, a_2, \ldots, a_m) = l\left(\pi(m(a_1, a_2, \ldots, a_m)v)\right).$$

Denote by $\mathcal{V}$ the space spanned by the functions ϕ_v. For $x \in k^\times$ with $\mid x \mid \leqslant 1$ let $\rho_i(x)$ be the operator on the space of functions on C defined by:

$$\rho_i(x)\phi(a_1, \ldots, a_i, \ldots, a_m) = \phi(a_1, \ldots, xa_i, \ldots, a_m).$$

Thus $\mathcal{V}$ is invariant under these operators. Also, for each i, there is a finite set X_i of finite functions on $k^\times$ and operators B_χ such that, for any ϕ, the difference

$$\rho_i(x)\phi - \sum_{\chi \in X_i} \chi(x) B_\chi(\phi)$$

vanishes for $| a_i | \leqslant C_{x,\phi}$. The operators B_χ are themselves linear combinations of operators $\rho_i(x)$ with $|x| \leqslant 1$. Since the vectors $v \in V$ are K finite, we may write any function ϕ as a sum of functions in the same space transforming under a character of $T = (\mathcal{O}^\times)^m$. Thus in analyzing our functions we may as well restrict ourselves to those functions transforming under a fixed character of T. If we choose a uniformizer ϖ, such functions are determined in turn by the following functions on the cone $(\mathbb{Z}^+)^m$:

$$\Phi(z_1, z_2, \ldots, z_m) = \phi(\varpi^{z_1}, \varpi^{z_2}, \ldots, \varpi^{z_m}).$$

This space of functions, call it $\mathcal{U}$, has the following property. For $x \geqslant 0$, let again $\rho_i(x)$ be the translation operator defined by:

$$\rho_i(x)\Phi(z_1, \ldots, z_i, \ldots, z_m) = \Phi(z_1, \ldots, x + z_i, \ldots, z_m).$$

Then for each i, there are $\lambda_{i,j,\xi,m} \in \mathbb{C}$ and integers $y_{i,j,\xi,m} \geqslant 0$ such that, for any Φ

$$\rho_i(x)\Phi(z) - \sum \lambda_{i,j,\xi,m} \xi^x x^m \rho_i(y_{i,j,\xi,m})\Phi(z) = 0$$

when $z_i \geqslant M_i$. The integer M_i depends on x and on Φ. However, it does not depend on the z_j with $j \neq i$. As written the sum is over all $\xi \in \mathbb{C}^\times$ and all integers $m \geqslant 0$. However, only finitely many of the scalars λ_* arenon zero and the integers $y_{i,j,\xi,m}$ are $\geqslant 0$ (and do not depend on x). Now we choose x larger than all the integers $y_{i,j,\xi,m}$. Then the above equation is a *non-trivial* difference equation, which a given Φ satisfies for $z_i \geqslant M_i(\Phi)$ and $z_j \geqslant 0$ if $j \neq i$. We stress that for lower values of x the difference equation could be tautological. Now define a Schwartz-Bruhat function Φ on $\mathbb{Z}^+$ as being a function which is constant (possibly 0) for large values of the variable. A Schwartz-Bruhat function on $(\mathbb{Z}^+)^m$ is a sum of tensor products of Schwartz-Bruhat functions in one variable. Solving the above system of independent difference equations (for instance in terms of the formal Mellin transform) we find that the functions in $\mathcal{U}$ have the form:

$$\sum_{\chi \in X} \chi(z)\Phi_\chi(z),$$

where X is a finite set of finite functions on $\mathbb{Z}^m$ and the Φ_χ are Schwartz-Bruhat functions on $(\mathbb{Z}^+)^m$. If follows that the functions in $\mathcal{V}$ have the required forms. Thus the functions $l(\pi(a)v)$ have the required form, except that the set X may depend on the vector v. At any rate the set X is not uniquely determined since some of the projections of the support of a function ϕ on some factor may be contained in a compact subset of $k^{\times}$. However, one may choose the χ to be exponents of the representation π (see [JS] and [JPS]2) which are finite in number. At any rate for our purposes, this is not a critical point. □

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