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Extensions with restricted ramification and duality for arithmetic schemes

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Let X be a $(d+1)$-dimensional smooth arithmetic scheme lying smoothly and properly over an open subset U of $\mathrm{Spec}(\mathcal{O}_K)$ for some algebraic number field K with geometrically connected fibres of dimension d. It is well-known, that there is a duality pairing for the sheaf cohomology of X, which connects the etale cohomology groups of a locally constant constructible sheaf F with the cohomology with compact support of the sheaf $F^D(d)$, where (d) means the d-fold Tate-twist (see [6]).

In this paper we will prove a duality theorem, which is exclusively formulated in terms of the ordinary sheaf cohomology, not using cohomology groups with compact support. One could also view the result as a computation of the cohomology with compact support in terms of ordinary sheaf cohomology using a dualizing sheaf.

With the above notations let $U = \mathrm{Spec}(\mathcal{O}_{K,S})$ where S is a finite set of places of K including the archimedean places. We write S^f for the set of finite places in S and S_∞ for the set of archimedean places of K, thus $S = S^f \cup S_\infty$. As usual we denote the maximal extension of K, unramified outside S by K_S. For an integer n and an abelian group A we denote the kernel of the n-multiplication map by ${}_nA$. Let $C_{S^f}(K_S)$ be the S^f-idele class group of the field K_S (for a definition see below) and we put $I_{n,U} := {}_nC_{S^f}(K_S)$. The $\mathrm{Gal}(K_S/K)$-module $I_{n,U}$ is the direct limit of its finite submodules and hence defines a sheaf of $\mathbb{Z}/n\mathbb{Z}$-modules on U_{et}. Then the following holds.

THEOREM 1. *Let U be an open subscheme of* $\mathrm{Spec}(\mathcal{O}_K)$, *where K is a finite extension of $\mathbb{Q}$. Further let*

$$\pi : X \longrightarrow U$$

be a smooth and proper morphisms of schemes with fibres of pure dimension d. Let n be an integer invertible on U and assume K to be totally imaginary if n is even. Then for every locally constant, constructible sheaf F of $\mathbb{Z}/n\mathbb{Z}$-modules on X_{et} the cup-product

$$H^i_{\text{et}}(X, F) \times H^{2d+2-i}_{\text{et}}(X, \mathcal{H}om(F, \pi^* I_{n,U}(d))) \xrightarrow{\cup} H^{2d+2}_{\text{et}}(X, \pi^* I_{n,U}(d)) \xrightarrow{\sim} \mathbb{Z}/n\mathbb{Z}$$

defines a perfect pairing of finite abelian groups for all i.

Using generalized Poincaré-duality the heart of the proof of Theorem 1 is the case $d = 0$, which can be reformulated in terms of Galois groups with restricted ramification of algebraic number fields. The proof combines a result of Wingberg on pro-p-extensions with restricted ramification with a theorem of Washington on the behaviour of the prime-to-p part of ideal class groups in $\mathbb{Z}_p$-extensions of algebraic number fields.

For a profinite group G and a prime number p we denote the maximal pro-p factor group of G by $G(p)$. Using the language of higher etale homotopy groups we prove the following theorem for the case of arithmetic surfaces.

THEOREM 2. *Assume that the fibres of* $\pi: X \longrightarrow U$ *are smooth, projective curves of nonzero genus and let* p *be a prime number, which is invertible on* U. *Then one has:*

$$\pi_i^{\text{et}}(X)(p) = 0 \text{ for all } i \geqslant 2.$$

Combining Theorems 1 and 2 we conclude:

THEOREM 3. *Assume that the fibres of* $\pi: X \longrightarrow U$ *are smooth, projective curves of genus* g *and let* p *be a prime number invertible on* U. *Assume that* p *is odd or that* K *is totally imaginary. Then the following holds for the etale fundamental group* $G := \pi_1^{\text{et}}(X)$*:*

(i) *If* $g \geqslant 1$ *then* G *is a profinite duality group at* p *of dimension 4.*
(ii) *If* $g = 0$ *then* G *is a profinite duality group at* p *of dimension 2.*

REMARK. (a) If $p = 2$ and K is not totally imaginary the theorem remains true, if we replace the word duality group by virtual duality group, i.e. there is an open subgroup being a duality group.

(b) Under certain restrictions to X and U it was shown in [8], that the maximal pro-p factor group of G has similar duality properties.

1. In this section we prove a duality theorem for Galois groups with restricted ramification of algebraic number fields which will imply Theorem 1 for the case $d = 0$.

Let K be a number field and let S be a finite set of places of K which contains all archimedian places. By K_S we denote the maximal extension of K, which is unramified outside S and we denote $Gal(K_S/K)$ by G_S.

For an intermediate field $K \subset L \subset K_S$ we use the following notations:

$S(L)$	the set of places of L lying above S,
$E_{L,S}$	the group of $S(L)$-units of L,
J_L	the idele group of L,
$J_{L,S}$	the S-idele group of L,
$C_S(L)$	the S-idele class group of L,
$Cl_S(L)$	the S-ideal class group of L.

We further use similar notations for the set of finite primes S^f in S. The definitions are the same word-by-word, however we could not find them in the literature and therefore we give the definitions. Let L be a finite extension of K in K_S:

J_{L,S^f}	$\{(a_w) \in J_L \mid a_w = 1 \text{ for } w \notin S^f\} \cong \prod_{v \in S^f} L_v^\times$,
E_{L,S^f}	$= E_{L,S}$-the group of S^f-units,
C_{L,S^f}	$J_{S^f}(L)/E_{S^f}(L)$.

Further we denote:

$$J_{S^f} = \varinjlim J_{L,S^f}, \quad E_{S^f} = \varinjlim E_{L,S^f}, \quad C_{S^f} = \varinjlim C_{L,S^f},$$

where the limit runs through all finite subextensions L of K in K_S.

REMARK: If K is totally imaginary, every extension of K is automatically unramified at the archimedian places. In this situation the pair (G_S, C_{S^f}) is a P-class formation for the set P of prime numbers l with $l^\infty | \# G_S$ having analogous properties like the usual P-class formation (G_S, C_S) (see Proposition 10 below). Each element of $C_S(K)$ which can be represented by an idele with support in the set of archimedian places is an universal norm for (G_S, C_S). The advantage of (G_S, C_{S^f}) is that it is free of this 'redundancy at infinity', i.e. the group of universal norms in $C_{S^f}(K)$ is much smaller.

Let p be a prime number and we assume K to be totally imaginary if $p = 2$. By $S_p(K)$ we denote the set of places of K dividing p. For an abelian group A we will denote the subgroup of elements which are annihilated by a power of p by $\mathrm{Tor}_p(A)$.

THEOREM 4. *If $S \supseteq S_p(K) \cup S_\infty(K)$ then G_S is a duality group at p of dimension 2 with p-dualizing module $I = \mathrm{Tor}_p(C_{S^f}(K_S))$, i.e. for every finite discrete p-primary G_S-module M and all i, the cup-product*

$$H^i(G_S, M) \times H^{2-i}(G_S, \mathrm{Hom}(M, I)) \xrightarrow{\cup} H^2(G_S, I) \xrightarrow{\sim} \mathbb{Q}_p/\mathbb{Z}_p$$

defines a perfect pairing of finite groups.

We will prove Theorem 4 in several steps.

PROPOSITION 5. *Under the above assumptions one has* $cd_p(G_S) = 2$. *For a finite discrete p-primary* G_S*-module* M *the cohomology groups* $H^i(G_S, M)$ *are finite for all* i. *The p-dualizing module of* G_S *is isomorphic to* $\mathrm{Tor}_p(C_{S^f}(K_S))$.

Proof. The statement about the cohomological dimension as well as the finiteness statement on the cohomology can be found at various places in the literature (see [6], [4]). Following the description of the dualizing module of an arbitrary profinite group in [10], [11] it holds for the p-dualizing module I of G_S:

$$I = \varinjlim_{\substack{L,cor^* \\ n}} H^2(\mathrm{Gal}(K_S/L), \mathbb{Z}/p^n\mathbb{Z})^*,$$

where L runs through the finite subextensions of K in K_S.

Since $\mu_{2p} \in K_S$ we can take the limit over all L which are totally imaginary. Using Tate's long exact sequence ([6] I Sect. 4 Thm. 4.10) we obtain the following exact sequence:

$$0 \longrightarrow H^0(\mathrm{Gal}(K_S/L), \mu_{p^n}) \longrightarrow \prod_{v \in S^f(L)} H^0(\mathrm{Gal}(\bar{K}_v/L_v), \mu_{p^n}) \rightarrow$$
$$\rightarrow H^2(\mathrm{Gal}(K_S/L), \mathbb{Z}/p^n\mathbb{Z})^* \longrightarrow H^1(\mathrm{Gal}(K_S/L), \mu_{p^n}).$$

Going to the limit over L and n we obtain:

$$I \cong \mathrm{coker}\left(\mu_{p^\infty} \overset{\mathrm{diag}}{\hookrightarrow} \prod_{v \in S^f(K)} (\mathrm{Ind}_{G_v}^{G_S} \mu_{p^\infty}) \right),$$

where G_v denotes the decomposition group in G_S of a fixed extension of v to K_S. Using the exact sequence

$$0 \longrightarrow E_{S^f}(K_S) \longrightarrow J_{S^f}(K_S) \longrightarrow C_{S^f}(K_S) \longrightarrow 0$$

and the fact, that $E_{S^f}(K_S)$ is p-divisible we obtain the exact sequence

$$0 \longrightarrow \mathrm{Tor}_p(E_{S^f}(K_S)) \longrightarrow \mathrm{Tor}_p(J_S(K_{S^f})) \longrightarrow \mathrm{Tor}_p(C_S(K_{S^f})) \longrightarrow 0.$$

As $\mu_{p^\infty} \subset E_{S^f}(K_S)$ and by the definition of J_{S^f} we get the exact sequence

$$0 \longrightarrow \mu_{p^\infty} \overset{\mathrm{diag}}{\longrightarrow} \prod_{v \in S^f(K)} (\mathrm{Ind}_{G_v}^{G_S} \mu_{p^\infty}) \longrightarrow \mathrm{Tor}_p(C_{S^f}(K_S)) \longrightarrow 0,$$

which proves the proposition. □

In order to prove Theorem 4 we have to verify the vanishing of the following limit for $i = 0, 1$ (cf. [10], [11]):

$$D_i(\mathbb{Z}/p\mathbb{Z}) \overset{\mathrm{def}}{=} \varinjlim_{\substack{U \subset G_S \\ \mathrm{cor}^*}} H^i(U, \mathbb{Z}/p\mathbb{Z})^*,$$

where the limit runs through the open subgroups of U and the transition maps are the duals of the corestriction maps.

It is easy to see that $D_0(\mathbb{Z}/p\mathbb{Z}) = 0$ since $p^\infty | \# G_S$. In order to prove the vanishing of $D_1(\mathbb{Z}/p\mathbb{Z})$ the following theorem is crucial.

THEOREM 6. *If $S \supset S_p \cup S_\infty$ then $C_S(K_S)$ is p-divisible.*

REMARK. It is easily seen, that the p-divisibility of $C_S(K_S)$ is equivalent to the fact, that the local field $(K_S)_v$ is a p-closed local field for all $v \in S(K_S)$. If S omits only finitely many primes of K this is an easy consequence of the theorem of Grunwald-Hasse-Wang (see [7]). However if S is finite this is non-trivial. If S does not contain S_p one even does not know, whether the supernatural order of G_S is divisible by p^∞.

In [13] Wingberg investigates the similar pro-p situation for Theorem 6. In order to refer to his result we introduce the following notations:

$K_S(p)$	the maximal pro-p subextension of K in K_S,
$G_S(p)$	the Galois group of $K_S(p)/K$,
K_v	the completion of the number field K at the prime v,
$K_v(p)$	the maximal pro-p extension of K_v,
$\mathcal{G}_v$	the Galois group of $K_v(p)/K_v$,
$\mathcal{T}_v$	the inertia group of v in $\mathcal{G}_v$,
G_v	the decomposition group of v in $G_S(p)$,
$H_1 \star H_2$	the free pro-p product of the pro-p groups H_1 and H_2.

Using a theorem of Kuz'min, Wingberg proved the following theorem see [13]:

THEOREM 7. (Wingberg [13]). *Assume $\mu_{2p} \subset K$ and $S \supset S_p(K) \cup S_\infty$ and suppose that $C_S(K_S(p))$ is not p-divisible. Then there exists a prime $v \in S_p(K)$ such that the decomposition group G_v of v in the extension $K_S(p)/K$ is the full group $G_S(p)$ and the following holds:*

There exists a finite set T of primes of K containing S such that the homomorphism

$$\phi : \mathop{\bigstar}_{\substack{v' \in S \setminus S_\infty \\ v' \neq v}} \mathcal{G}_{v'} \star \mathop{\bigstar}_{v' \in T \setminus S} (\mathcal{G}_{v'}/\mathcal{T}_{v'}) \xrightarrow{\sim} G_S(p)$$

induced by the maps: $\phi_{v'} : \mathcal{G}_{v'} \hookrightarrow \mathrm{Gal}(K(p)/K) \overset{\mathrm{can}}{\twoheadrightarrow} G_S(p)$ is an isomorphism.

COROLLARY 8. *If $C_S(K_S(p))$ is not p-divisible then there exists a prime $v \in S_p(K)$ such that the following inequality holds*

$$[K_v : \mathbb{Q}_p] \geqslant \sum_{\substack{v' \in S_p(K) \\ v' \neq v}} [K_{v'} : \mathbb{Q}_p].$$

Proof of the corollary: As $\mu_p \subset K$ the following (in)equality holds by [9] II Section 5 Thm. 4 for a prime $v' \in S_p(K)$:

$$\operatorname{rank} G_{v'} \leqslant \operatorname{rank} \mathcal{G}_{v'} = [K_{v'} : \mathbb{Q}_p] + 2.$$

Now assume that $\#S_p(K) > 1$ (otherwise the statement of the corollary is trivial) and assume that $C_S(K_S(p))$ is not p-divisible. By Theorem 7 there exists a prime $v \in S_p(K)$ such that $G_S = G_v$ and the free product decomposition of G_S yields the following inequalities:

$$\begin{aligned}
[K_v : \mathbb{Q}_p] &= \operatorname{rank} \mathcal{G}_v - 2 \\
&\geqslant \operatorname{rank} G_v - 2 = \operatorname{rank} G_S - 2 \\
&\geqslant \left(\sum_{\substack{v' \in S_p(K) \\ v' \neq v}} \operatorname{rank} \mathcal{G}_{v'} \right) - 2 = \left(\sum_{\substack{v' \in S_p(K) \\ v' \neq v}} ([K_{v'} : \mathbb{Q}_p] + 2) \right) - 2 \\
&\geqslant \sum_{\substack{v' \in S_p(K) \\ v' \neq v}} [K_{v'} : \mathbb{Q}_p],
\end{aligned}$$

which proves the corollary. □

In order to prove Theorem 6 it is obviously sufficient to prove that the group $C_S(L_S(p))$ is p-divisible for a cofinal set of finite extensions L of K in K_S. Since $\mu_{2p} \subset K_S$ we assume without loss of generality that $\mu_{2p} \subset K$, in particular K is totally imaginary, containing the imaginary abelian field $\mathbb{Q}(\zeta_{2p})$. Now what we need is a method of leaving the bad situations described in theorem 7. For this we use a result of Washington, where k_n denotes the unique subextension of degree p^n in the cyclotomic $\mathbb{Z}_p$-extension of a number field k and h_n denotes the class number of k_n.

THEOREM 9. (Washington [12]). *Let k be an imaginary abelian numberfield and let*

$$H = \{l \mid l \textit{ prime number}, l \textit{ divides } h_n \textit{ for some } n\}.$$

Then H is infinite.

Now returning to our situation assume that L is a finite subextension of K in K_S and that $C_S(L_S(p))$ is not p-divisible.

CLAIM: There is a finite extension L'/L contained in K_S such that $C_S(L'_S(p))$ is p-divisible.

Proof of the claim: Following Theorem 9, choose a prime number $l > p$ and n such that:

(i) $l | h(\mathbb{Q}(\zeta_{p^n}))$ and
(ii) $(l, [L : \mathbb{Q}]) = 1$.

By class field theory there exists a finite cyclic unramified extension $F/\mathbb{Q}(\zeta_{p^n})$ of degree l. As the only prime of $\mathbb{Q}(\zeta_{p^n})$ which divides p is principal it completely splits in F, hence there are l different primes dividing p in F. By condition (ii) we see that F and $L(\zeta_{p^n})$ are linearly disjoint over $\mathbb{Q}(\zeta_{p^n})$. Hence every prime v of $L(\zeta_{p^n})$ dividing p splits into l different primes in LF. Therefore the field LF has the property that for every prime dividing p there are at least $l - 1$ other primes having the same absolute local degree. By Corollary 8 we obtain that the group $C_S((LF)_S(p))$ is p-divisible which proves the claim.

Thus we have proved Theorem 6. □

Now we are able to prove the vanishing of D_1 (see above).

$$\begin{aligned} D_1(\mathbb{Z}/p\mathbb{Z}) &= \varinjlim_L H^1(\mathrm{Gal}(K_S/L), \mathbb{Z}/p\mathbb{Z})^* \\ &= \varinjlim_L \mathrm{Gal}(K_S/L)^{ab}/p \\ &= \varinjlim_L C_S(L)/p \\ &= C_S(K_S)/p = 0. \end{aligned}$$

Thus the proof of Theorem 4 is complete. □

In order to get a better understanding of Theorem 4 we give the following proposition for the case of totally imaginary K. (Compare [6] I, Sect. 4, 4.2, 4.5, 4.6.)

PROPOSITION 10. *Let K be a totally imaginary number field and let $S \supset S_\infty(K)$ be a finite set of primes. Then the following holds.*

(i) *The pair (G_S, C_{Sf}) is a P-class formation for the set P of primes l with $l^\infty | \# G_S$.*
(ii) *The reciprocity map:* rec: $C_{Sf}(K) \xrightarrow{\text{rec}} G_S^{ab}$ is surjective with divisible kernel.
(iii) *If p is a prime number such that all primes dividing p are in S and if M is a finite p-primary G_S-module then the cup-product defines isomorphisms:*

$$\alpha^r\colon \operatorname{Ext}^r_{G_S}(M, C_{Sf}) \xrightarrow{\sim} H^{2-r}(G_S, M)^*$$

for all r.

Proof. Using the definition of C_S and C_{Sf} we get the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \operatorname{Ind}^{G_S}\mathbb{C}^\times & = & \operatorname{Ind}^{G_S}\mathbb{C}^\times & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & E_S & \longrightarrow & J_S & \longrightarrow & C_S & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & E_{Sf} & \longrightarrow & J_{Sf} & \longrightarrow & C_{Sf} & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0. & &
\end{array}$$

Hence we obtain for all $i \geqslant 1$ isomorphisms: $H^i(G_S, C_S) \xrightarrow{\sim} H^i(G_S, C_{Sf})$. Since (G_S, C_S) is a P-class formation the same follows for (G_S, C_{Sf}). We call the reciprocity maps associated to these class formations by rec and rec_f and we denote the kernel of rec (rec_f) by $D_S(K)$ ($D_{Sf}(K)$). It is well-known ([6] I, Sect. 4, 4.5), that rec is surjective and that $D_S(K)$ is divisible. Hence we get the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \prod_{v\in S_\infty(K)} \mathbb{C}^\times & \longrightarrow & D_S(K) & \longrightarrow & D_{Sf}(K) & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \prod_{v\in S_\infty(K)} \mathbb{C}^\times & \longrightarrow & C_S(K) & \longrightarrow & C_{Sf}(K) & \longrightarrow & 0 \\
 & & & & \downarrow{\scriptstyle \mathrm{rec}} & & \downarrow{\scriptstyle \mathrm{rec}_f} & & \\
 & & & & G_S^{ab} & = & G_S^{ab} & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

This proves (ii). By Theorem 1.13 of [6] I Sect. 1 and by (ii) we obtain that α^r is an isomorphism for $r \geqslant 1$ (cf. [6] I Sect. 4 Thm. 4.6.(a)). The statement for $r = 0$ follows from Proposition 5. For $r < 0$ both groups are zero. □

REMARK. Now we can conclude Theorem 4 from Proposition 10 (iii) and from Theorem 6: C_{Sf} is a quotient of C_S, hence p-divisible. Therefore the spectral sequence

$$E_2^{pq} = H^r(G_S, \operatorname{Ext}^q_{\mathbb{Z}}(M, C_{Sf})) \Longrightarrow \operatorname{Ext}^{p+q}_{G_S}(M, C_{Sf})$$

degenerates to a sequence of isomorphisms: $H^p(G_S, \mathrm{Hom}(M, C_{Sf})) \xrightarrow{\sim} \mathrm{Ext}^p_{G_S}(M, C_{Sf})$, which by Proposition 10(iii) implies Theorem 4.

2. In this section we prove Theorem 1.

Let K be a number field, $U \subset \mathrm{Spec}(\mathcal{O}_K)$ an open subscheme and let S be the finite set of places of K, containing the archimedian places and such that $U = \mathrm{Spec}(\mathcal{O}_{K,S})$. As it is well known every finite discrete G_S-module M defines a locally constant constructible sheaf on U_{et}. If M is a discrete G_S module, annihilated by some integer n, then M is the direct limit of its finite submodules and therefore defines a sheaf of $\mathbb{Z}/n\mathbb{Z}$-modules on U_{et}, which is the direct limit of locally constant constructible sheaves. The following proposition is well-known.

PROPOSITION 11. *Assume that n is invertible on U. Then for all i:*

$$H^i_{\mathrm{et}}(U, M) \cong H^i(G_S, M).$$

Proof. If M is finite this is [6] II, Sect. 2, 2.9. The general case follows since etale cohomology as well as Galois cohomology commute with direct limits. □

Therefore Theorem 1 for $d = 0$ is equivalent to Theorem 4 which is already proved. Further one knows that the p-dualizing module of a duality group is divisible (or else $\mathrm{Tor}_p(C_{Sf}(K_S))$ is p-divisible as $C_S(K_S)$ is p-divisible). For further use we note:

COROLLARY 12. *The stalks of the sheaf $I_{n,U}$ are injective $\mathbb{Z}/n\mathbb{Z}$-modules.*

Now let X,π,U,n be as in the introduction. By $D(X, \mathbb{Z}/n\mathbb{Z})$ resp. $D(U, \mathbb{Z}/n\mathbb{Z})$ we denote the derived category of $\mathbb{Z}/n\mathbb{Z}$-module sheaves on X_{et} resp. U_{et}. As π and n fulfill the conditions of generalized Poincaré-duality we get the following isomorphism of objects in $D(U, \mathbb{Z}/n\mathbb{Z})$ (see Theorem 3.2.5. in [1] XVIII):

$$R\pi_*\mathcal{H}om_{D(X,\mathbb{Z}/n\mathbb{Z})}(F, \pi^* I_{n,U}(d)[2d]) \cong \mathcal{H}om_{D(U,\mathbb{Z}/n\mathbb{Z})}(R\pi_* F, I_{n,U}). \quad (*)$$

Here (d) is the d-fold Tate-twist and $[2d]$ is the shift by $2d$.

By Corollary 12 the stalks of $I_{n,U}$ (resp. $\pi^* I_{n,U}$) are injective $\mathbb{Z}/n\mathbb{Z}$-modules. Therefore we have an isomorphism

$$\mathcal{H}om_{D(X,\mathbb{Z}/n\mathbb{Z})}(F, \pi^* I_{n,U}(d)[2d]) \cong \mathcal{H}om_{D(X,\mathbb{Z}/n\mathbb{Z})}(F, \pi^* I_{n,U}(d))[2d].$$

Applying $R\Gamma(U, -)$ on the left hand side of $(*)$ we get a complex of abelian groups whose r^{th} cohomology group is isomorphic to

$$H^{r+2d}_{\mathrm{et}}(X, \mathcal{H}om(F, \pi^* I_{n,U}(d))).$$

By the proper-smooth base change theorem [5] VI, Sect. 4, 4.2) the sheaves $R^i\pi_*F$ are locally constant and constructible on U. Applying $R\Gamma(U,-)$ on the right hand side of $(*)$ we get a complex of abelian groups whose r^{th} cohomology group by theorem 1 for $d=0$ is isomorphic to

$$\operatorname{Hom}(H^{2-r}_{\text{et}}(X,F),\mathbb{Z}/n\mathbb{Z}).$$

This proves Theorem 1. □

3. In this section we prove Theorems 2 and 3. For a scheme X we denote by X_{et} its etale homotopy type, i.e. a pro-simplicial set. The etale homotopy groups of X are by definition the homotopy groups of X_{et} and it is well-known that these pro-groups are pro-finite, whenever the scheme X is noetherian, connected and geometrically unibranch ([2] Theorem 11.1). By $\tilde{X}_{\text{et}}$ we denote the universal covering of X_{et}. If p is a prime number and $Y.$ is a pro-simplicial set, we denote the pro-p completion of $Y.$ by $Y.^{\wedge p}$. The maximal pro-p factor group of a pro-group G is denoted by $G(p)$. For the following we need

PROPOSITION 13. *Assume that $Y.$ is simply connected (i.e. $\pi_1(Y.)=0$) and that $\pi_i(Y.)$ is pro-finite for all $i\geqslant 2$. Then we have isomorphisms for all i:*

$$\pi_i(Y.)(p)\to\pi_i(Y.^{\wedge p}).$$

Proof. If G is an abelian pro-finite group, the canonical surjection: $G\to G(p)$ has a kernel with trivial p-Sylow subgroup, i.e. the supernatural order of the kernel is prime to p. Therefore G is a p-good pro-group in the sense of [2], i.e. for every finite p-primary $G(p)$-module M the canonical homomorphism: $H^i(G(p),M)\xrightarrow{infl}H^i(G,M)$ is an isomorphism for all i. Hence Proposition 13 follows by induction on i from Theorem 6.7. of [2]. □

Now let K be a number field, let $U\subset\operatorname{Spec}(\mathcal{O}_K)$ be an open subscheme and let p be a prime number invertible on U. Then the following holds:

PROPOSITION 14. *The higher etale homotopy groups of U have no p-part, i.e.*

$$\pi_i^{\text{et}}(U)(p)=0 \quad \text{for } i\geqslant 2.$$

Further the canonical morphism:

$$U_{\text{et}}^{\wedge p}\to K(\pi_1^{\text{et}}(U)(p),1)$$

with $\pi_1^{\text{et}}(U)(p)$ the maximal pro-p factor group of the etale fundamental group of U is a weak homotopy equivalence.

Proof. Let S be the finite subset of places of K, containing the archimedian places and such that $U = \mathrm{Spec}(\mathcal{O}_{K,S})$. Then $G_S(K) \cong \pi_1^{\mathrm{et}}(U)$ and Proposition 11 implies that the universal covering $\tilde{U}_{\mathrm{et}}$ has no cohomology with values in p-primary coefficient groups. Hence the pro-p completion of $\tilde{U}_{\mathrm{et}}$ is contractible and therefore the first statement of the proposition follows from Proposition 13. By a theorem of O. Neumann (see [4] Prop. 22) for every finite p-primary $G_S(p)$-torsion module M the canonical homomorphism $H^i(G_S(p), M) \xrightarrow{infl} H^i(G_S, M)$ is an isomorphism for all i. The same arguments as for the first statement then also show the second. □

Similar arguments apply in the geometric situation. Here we denote the genus of a curve C by $g(C)$.

PROPOSITION 15. *Let k be a field and let C be a connected, smooth curve over k. Assume either that C is incomplete or $g(C) > 0$. Then*

$$\pi_i^{\mathrm{et}}(C) = 0 \quad \text{for } i \geqslant 2,$$

i.e. the canonical morphism

$$C_{\mathrm{et}} \longrightarrow K(\pi_1^{\mathrm{et}}(C), 1)$$

is a weak homotopy equivalence. If k is separably closed and p is an arbitrary prime number then also the canonical morphism

$$C_{\mathrm{et}}^{\wedge p} \longrightarrow K(\pi_1^{\mathrm{et}}(C)(p), 1)$$

is a weak homotopy equivalence.

Proof. First, if necessary, we replace k by a suitable extension, such that C is geometrically connected. We denote the base change of C to an separable closure k^s of k by C_{k^s}. Since the morphism $C_{k^s} \to C$ is pro-etale the canonical morphism:

$$\tilde{C}_{k^s,\mathrm{et}} \xrightarrow{\mathrm{can}} \tilde{C}_{\mathrm{et}}$$

from the universal covering of $C_{k^s,\mathrm{et}}$ to the universal covering of C_{et} is a weak homotopy equivalence. Therefore we can assume k to be separably closed also in the first statement. In order to show the first statement we show that the higher homotopy groups of C_{et} (which are abelian profinite groups) have no p-part for an arbitrary prime number p. If either C is incomplete or $p = \mathrm{char}(k)$ then $H^i_{\mathrm{et}}(C, \mathbb{Z}/p\mathbb{Z}) = 0$ for $i \geqslant 2$. Therefore $H^i(\tilde{C}_{\mathrm{et}}, \mathbb{Z}/p\mathbb{Z}) = 0$ for $i \geqslant 1$, which implies $\pi_i^{\mathrm{et}}(C)(p) = 0$ for $i \geqslant 2$. If C is complete and $p \neq \mathrm{char}(k)$ then $H^i_{\mathrm{et}}(C, \mathbb{Z}/p\mathbb{Z}) = 0$ for $i > 2$ and Poincaré-duality [5] V, Sect. 2, Thm. 2.1) implies an isomorphism:

$H^2_{\text{et}}(C,\mathbb{Z}/p\mathbb{Z}) \cong H^0_{\text{et}}(C,\mathbb{Z}/p\mathbb{Z})^* = \mathbb{Z}/p\mathbb{Z}$. If C' is a finite etale cover of C of degree n then the following diagram is commutative:

$$\begin{array}{ccc} H^2_{\text{et}}(C,\mathbb{Z}/p\mathbb{Z}) & \xrightarrow{\sim} & \mathbb{Z}/p\mathbb{Z} \\ \downarrow{\scriptstyle \text{can}} & & \downarrow{\scriptstyle \cdot n} \\ H^2_{\text{et}}(C',\mathbb{Z}/p\mathbb{Z}) & \xrightarrow{\sim} & \mathbb{Z}/p\mathbb{Z} \end{array} .$$

Therefore $H^2(\tilde{C}_{\text{et}},\mathbb{Z}/p\mathbb{Z}) = 0$ if and only if there are etale covers of C of degree divisible by arbitrary high powers of p. This is obviously the case if and only if $g(C) > 0$. This proves the first statement. The same arguments also prove the second statement. □

Now let $\pi: X \to U$ be smooth, proper with fibres which are curves of nonzero genus. We denote by $U(i) = \tilde{U}$ the pro-scheme representing the universal covering of U. By $X(i)$ we denote the pro-scheme $X \times_U U(i)$. Fixing a (geometric) base point u of U we denote $X \times_U u$ by X_u. Further let p be a prime number invertible on U. Then by Theorem 11.5. of [3] and Propositions 14, 13 we get isomorphisms for all $r \geqslant 1$:

$$\pi_r((X_u)^{\wedge p}_{\text{et}}) \xrightarrow{\sim} \pi_r(X(i)^{\wedge p}_{\text{et}}).$$

By Proposition 15 we obtain $\pi_r(X(i)^{\wedge p}_{\text{et}}) = 0$ for $r \geqslant 2$, in particular we have $H^r_{\text{et}}(X(i),\mathbb{Z}/p\mathbb{Z}) = 0$ for $r \geqslant 2$. If Y is a finite etale cover of X, the same argument applies for $Y(i)$ (possibly Y is defined over an etale cover of U). Going to the limit we obtain for the universal cover $\tilde{X}_{\text{et}}$ of X_{et}: $H^r(\tilde{X}_{\text{et}},\mathbb{Z}/p\mathbb{Z}) = 0$ for $r \geqslant 1$. Since the groups $\pi^{\text{et}}_r(X)$ are profinite they are p-good for $r \geqslant 2$ and we obtain $0 = \pi_r(\tilde{X}_{\text{et}})(p) = \pi^{\text{et}}_r(X)(p)$ for $r \geqslant 2$. This proves Theorem 2. □

Now Theorem 3 part (i) is an easy consequence of Theorem 2 and Theorem 1: For every finite p-primary $\pi^{\text{et}}_1(X)$-module M the canonical homomorphism

$$H^r(\pi^{\text{et}}_1(X), M) \longrightarrow H^r_{\text{et}}(X, M)$$

is an isomorphism for all r by Theorem 2. Hence Theorem 1 implies the duality statement (i). Part (ii) is even easier. As the fibres are simply connected we have an isomorphism $\pi^{\text{et}}_1(X) \xrightarrow{\sim} \pi^{\text{et}}_1(U)$ which proves the assertion in view of Theorem 1 for U and proposition 14. □

References

1. Artin, M., Grothendieck, A. and Verdier, J. L.: *Theorie des Topos et Cohomologie Etale des Schemas*(SGA4) LNM 269, 270, 305.