COMPOSITIO MATHEMATICA

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*Compositio Mathematica*, tome 98, nº 2 (1995), p. 205-217
<http://www.numdam.org/item?id=CM_1995__98_2_205_0>

*Compositio Mathematica* **98:** 205–217, 1995.



# Deformation of Hilbert Schemes of Points on a Surface

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Received 27 July 1993; accepted in final form 12 July 1994

**Abstract.** Let $S$ be a smooth projective surface over the complex numbers; let $S^{(r)}$ be its $r$-fold symmetric product and $S^{[r]}$ the Hilbert scheme of 0-dimensional subschemes of length $r$.

In case $K_S$ is trivial, the deformation theory of $S^{[r]}$ has been studied by Beauville and Fujiki in order to construct examples of higher-dimensional symplectic manifolds. In that case $S^{[r]}$ has deformations which are not Hilbert schemes of points on a surface.

We prove that under suitable hypotheses (e.g. if $S$ is of general type) this cannot happen; every (small) deformation of $S^{(r)}$ and $S^{[r]}$ is induced naturally by a deformation of $S$ (in particular, all deformations of $S^{(r)}$ are locally trivial).

## 0. Introduction

For any smooth complex projective variety $X$ and any positive integer $r$, let $X^{(r)}$ be the $r$-fold symmetric product and $X^{[r]}$ the Hilbert scheme parametrizing zero-dimensional subschemes of $X$ of length $r$. If $Y$ is a compact complex space, let $D_Y$ be the functor of formal deformations of $Y$ and $D'_Y$ its subfunctor of locally trivial formal deformations.

As $X$ is the quotient of a smooth variety (namely $X^r$) via the action of a finite group (the symmetric group on $r$ letters), all deformations of $X^r$ to which the action of the symmetric product extends (cfr. [C1], [C2] and the proof of theorem 0.1) induce locally trivial deformations of $X^{(r)}$. On the other hand any deformation of $X$ induces a deformation of $X^{[r]}$ by taking the relative Hilbert scheme. Therefore we have natural maps $D_X \to D'_{X^{(r)}} (\subset D_{X^{(r)}})$ and $D_X \to D_{X^{[r]}}$.

If $X$ is a curve, the symmetric product is smooth and coincides with the Hilbert scheme, so that the two maps above coincide. Kempf [Ke] proved that $D_X \to D_{X^{(r)}}$ is an isomorphism if $X$ is nonhyperelliptic; in fact it is an isomorphism if and only if the genus of $X$ is zero or at least 3 (see [F1]).

If $X$ has higher dimension, the Hilbert scheme is no longer smooth as soon as $r > 3$; the map $D_X \to D_{X^{(r)}}$ is an isomorphism if (for instance) $X$ is of general type (see [F2]).

If $S$ is a smooth complex projective surface, $S^{(r)}$ is a normal variety which admits $S^{[r]}$ as a natural desingularization. Beauville and Fujiki have considered the case where $S$ has trivial canonical bundle. They proved that if $K_S$ is trivial, the $S^{[r]}$'s have a structure of symplectic varieties and that for $r \geqslant 2$ they can be deformed to varieties which are not Hilbert schemes of surfaces any longer.

In this paper we want prove an analogue of Kempf's theorem for surfaces, using techniques of deformations of singular varieties.

Let $S$ be a smooth complex projective surface, $r \geqslant 2$ an integer. Our main results are the following:

THEOREM 0.1. *If either $h^0(S, \theta_S)$ or $h^1(S, \mathcal{O}_S)$ vanish, then the natural map $D_S \to D'_{S(r)}$ is an isomorphism.*

THEOREM 0.2. *If $h^0(S, \mathcal{O}_S(-2K_S))$ vanishes, then the natural map $D'_{S(r)} \to D_{S(r)}$ is an isomorphism.*

THEOREM 0.3. *if $h^0(S, \mathcal{O}_S(-K_S))$ vanishes, then the natural map $D'_{S(r)} \to D_{S([r]}$ is an isomorphism.*

In particular if $S$ is a surface of general type, all deformations of $S^{(r)}$ and $S^{[r]}$ are induced by deformations of $S$.

As one can see from the proof, the results also hold in case $S$ is non projective if one replaces the Hilbert scheme with the corresponding Douady space. See also Remark 4.4 for a brief discussion of the sharpness of the results obtained.

Relations between the singularities of a variety and the deformation theory of a desingularisation have previously been studied by Fujiki [Fu] in the case of suitable partial desingularizations of symplectic V-manifolds. Without the symplectic hypothesis, there are results by Burns and Wahl [B-W] on the deformations of surfaces with rational double points and of their minimal resolutions; M.-H. Saito [Sai] generalized this to the case of varieties having a transversal $A_1$ singularity obtained as a double cover.

The paper goes as follows: in Section 1 we set some notation and recall some standard definitions; in Section 2 we collect some preliminary results on deformation theory and local cohomology; in Section 3 we give some lemmas on transversal $A_1$ singularities and in Section 4 we put everything together and prove Theorems 0.1–0.3.

## 1. Definitions and notation

We will always work over the complex numbers. If $f: X \to Y$ is a morphism, we will denote by $\Omega_{X/Y}$ (resp. $\theta_{X/Y}$) the relative cotangent (resp. tangent) sheaf.

Let $X$ be any topological space, $Z \subset X$ locally closed, $\mathcal{F}$ a sheaf of abelian groups on $X$. By $H^i_Z(X, \mathcal{F})$ (resp. $\mathcal{H}^i_Z(X, \mathcal{F})$) we devote the cohomology groups (resp. sheaves) with support on $Z$. For more details we refer to [Gr]. If $\mathcal{F}, \mathcal{G}$ are sheaves on a scheme $X$, we denote by $\operatorname{Ext}^i(\mathcal{F}, \mathcal{G})$ (resp $\mathcal{E}xt^i(\mathcal{F}, \mathcal{G})$) the Ext groups (resp. sheaves).

Let $X$ be a complex space: $X$ will be called a V-manifold if it is normal and has only quotient singularities. Here we refer the reader to [St]. In particular if $j: U \to X$ is the inclusion of the smooth locus and $r: Y \to X$ is a desingularization, we have $\theta_X = j_*\theta_U = r_*\theta_Y$.

Let $X$ be a locally Noetherian scheme, $\mathcal{F}$ a coherent sheaf on $X$, $Z$ a closed subscheme. We define $\mathrm{depth}_Z\mathcal{F}$ as in [Gr]. We remark that if $X$ is locally Cohen–Macaulay (and thus in particular if $X$ is smooth) and $Z$ is irreducible we have $\mathrm{depth}_Z\mathcal{O}_X$ =codim $(Z \subset X)$ (see e.g. [Ka], theorem 136).

If $X$ is a reduced compact complex space we will denote by $D_X$ the formal deformation functor, and by $D'_X$ the subfunctor of locally trivial deformations (cfr. [G-K]).

## 2. Deformations, singularities and local cohomology

Let $X$ be a reduced compact complex space; it is well known (see for instance [G-K]) that $D_X$ (resp. $D'_X$) has as tangent space $\mathrm{Ext}^1(\Omega_X, \mathcal{O}_X)$ (resp. $H^1(X, \theta_X)$) and as obstruction space $\mathrm{Ext}^2(\Omega_X, \mathcal{O}_X)$ (resp. $H^2(X, \theta_X)$). The two functors obviously coincide for $X$ smooth; in general from the local to global spectral sequence of Ext we get an exact sequence

$$\begin{aligned} 0 &\to H^1(X, \theta_X) \to \mathrm{Ext}^1(\Omega_X, \mathcal{O}_X) \to H^0(X, \mathcal{E}xt^1(\Omega_X, \mathcal{O}_X)) \\ &\to H^2(X, \theta_X) \to \mathrm{Ext}^2(\Omega_X, \mathcal{O}_X), \end{aligned}$$

where the mappings $H^i(X, \theta_X) \to \mathrm{Ext}^i(\Omega_X, \mathcal{O}_X)$ for $i = 1, 2$ induce the inclusion of functors $D'_X \subset D_X$.

LEMMA 2.1. *A sufficient condition for $D'_X \subset D_X$ to be an isomorphism is*

$$H^0(X, \mathcal{E}xt^1(\Omega_X, \mathcal{O}_X)) = 0.$$

*Proof.* The hypothesis implies that there is an isomorphism between the tangent spaces and an injection between the obstruction spaces, a standard criterion for isomorphism. For a sketch of proof see for instance [R1], criteria 0.1 and 0.2. □

Let $p: Y \to X$ be a resolution of singularities, and assume $p_*(\theta_Y) = \theta_X$. Then there is a natural mapping of functors $D'_X \to D_Y$ (see for instance [B-W] where it is defined for $X$ a surface with Du Val singularities).

The spectral sequence $H^p(X, R^q p_*\theta_Y) \Rightarrow H^k(Y, \theta_Y)$ gives an exact sequence

$$\begin{aligned} 0 &\to H^1(X, \theta_X) \to H^1(Y, \theta_Y) \to H^0(X, R^1 p_*\theta_Y) \\ &\to H^2(X, \theta_X) \to H^2(Y, \theta_Y) \end{aligned}$$

where the mappings $H^i(X, \theta_X) \to H^i(Y, \theta_Y)$ for $i = 1, 2$ define this mapping of functors.

LEMMA 2.2. *A sufficient condition for $D'_X \to D_Y$ to be an isomorphism is*

$$H^0(X, R^1 p_* \theta_Y) = 0.$$

*Proof.* As in Lemma 2.1. □

We now state some results we will use and extend them slightly as needed.

LEMMA 2.3. (Schlessinger [Schl]). *If $\mathcal{G}$ is any coherent sheaf on a variety $X$ and $Y$ is a subscheme of $X$ such that* $\mathrm{depth}_Y \mathcal{O}_X \geqslant 2$, *then* $\mathcal{H}^0_Y(\mathcal{G}^\vee) = \mathcal{H}^1_Y(\mathcal{G}^\vee) = 0$. □

LEMMA 2.4. *Let $X$ be any projective variety, and let $W$ be a subscheme of $X$ such that* $\mathrm{depth}_W \mathcal{O}_X \geqslant 3$. *Then*

$$\mathcal{H}^0_W(\mathcal{E}xt^1(\Omega_X, \mathcal{O}_X)) = \mathcal{H}^2_W(\theta_X).$$

*Proof.* Embed $X$ in a smooth variety $Y$, and let $i: X \to Y$ be the immersion. Then we have an exact sequence

$$0 \to \theta_X \to i^*\theta_Y \to \mathcal{N} \to \mathcal{E}xt^1(\Omega_X, \mathcal{O}_X) \to 0,$$

where $\mathcal{N}$ is the normal bundle of $X$ in $Y$, i.e. the dual of $\mathcal{I}_X/\mathcal{I}_X^2$. We break this into two short exact sequences:

$$0 \to \theta_X \to i^*\theta_Y \to \mathcal{N}' \to 0,$$
$$0 \to \mathcal{N}' \to \mathcal{N} \to \mathcal{E}xt^1(\Omega_X, \mathcal{O}_X) \to 0.$$

By applying $\mathcal{H}_W$ to the second one, and as by Lemma 2.3 $\mathcal{H}^0_W(\mathcal{N}) = \mathcal{H}^1_W(\mathcal{N}) = 0$, we get

$$\mathcal{H}^0_W(\mathcal{E}xt^1(\Omega_X, \mathcal{O}_X)) = \mathcal{H}^1_W(\mathcal{N}').$$

Now we apply the same functor to the first exact sequence, and we recall that

$$\mathcal{H}^1_W(i^*\theta_Y) = \mathcal{H}^2_W(i^*\theta_Y) = 0$$

because $i^*(\theta_Y)$ is locally free. Therefore

$$\mathcal{H}^0_W(\mathcal{E}xt^1(\Omega_X, \mathcal{O}_X)) = \mathcal{H}^2_W(\theta_x).$$ □

LEMMA 2.5. *If $X$ is a $V$-manifold and $W$ is an analytic subspace of codimension $r$ then* $\mathrm{depth}_W \mathcal{O}_X \geqslant r$. *Moreover, in this case* $\mathrm{depth}_W \theta_X \geqslant r$.

*Proof.* The statement is local, so we may assume that $X$ is the quotient of a smooth manifold $Y$ via a finite group $G$, acting freely in codimension 1. Let $p\colon Y \to X$ be the quotient map. We have $\mathcal{O}_x = p_*^G \mathcal{O}_Y$, and $\theta_X = p_*^G \theta_Y$ by [St]; arguing as in Corollary 1 of [Schl] we prove that $\mathcal{O}_X$ (resp. $\theta_X$) is a direct summand of $p_* \mathcal{O}_Y$ (resp. $p_* \theta_Y$). So if $W' = f^{-1}(W)$, then $\mathrm{depth}_W \mathcal{O}_X \geqslant \mathrm{depth}_{W'} \mathcal{O}_Y$, and the same for $\theta$ by [Gr] page 73. However $p$ is finite, so the codimension of $W$ in $X$ is the same as the codimension of $W'$ in $Y$. □

LEMMA 2.6. *Let $p\colon Y \to X$ a resolution of singularities, $E \subset Y$ the exceptional locus, and assume $\theta_X = j_* \theta_V = p_*(\theta_Y) = \theta_X$, where $j\colon V \to X$ is the inclusion of the smooth locus. (This is true in particular if $X$ is a $V$-manifold). Then we have a natural inclusion $p_*(\mathcal{H}_E^1(\theta_Y)) \to R^1 p_* \theta_Y$.*

*Proof.* Let $U = Y \backslash E$, and $i\colon U \to Y$ the inclusion. On $Y$ we have the exact sequence

$$0 \to \theta_Y \to i_* \theta_U \to \mathcal{H}_E^1(\theta_Y) \to 0.$$

Applying $p_*$ yields

$$0 \to p_* \theta_Y \to p_* i_* \theta_U \to p_* \mathcal{H}_E^1(\theta_Y) \to R^1 p_* \theta_Y \to R^1 p_*(j_* \theta_U).$$

This gives the required natural map. Now by hypothesis $p_* \theta_Y = p_* i_* \theta_U = \theta_X$; so the map is injective. □

## 3. The case of transversal $A_1$ singularities

In this section $X$ will be a (not necessarily compact) V-manifold with only $A_1$ transversal singularities; that is, the singular locus $Z$ of $X$ is smooth and the couple $(Z, X)$ is locally analytically isomorphic to $(\mathbb{C}^k \times \{0\}, \mathbb{C}^k \times C)$ where $C$ is the cone in $\mathbb{C}^3$ of equation $xy = z^2$ and $0 \in C$ is the origin.

The results we collect here are a local restatement of those obtained by Saito [Sai] in the case where $X$ can be seen as a double cover of a smooth compact variety branched over the union of two smooth irreducible divisors intersecting transversally. We will mainly be interested in the case where $X$ is a quotient of a smooth variety $X'$ via an involution having as fixed set a smooth codimension 2 subvariety.

LEMMA 3.1. *Let $X$ be a $V$-manifold whose only singularity is a transversal $A_1$ singularity $Z$. Then $\mathcal{E}xt^1(\Omega_X, \mathcal{O}_X)$ is a line bundle on $Z$, and $\mathcal{E}xt^r(\Omega_X, \mathcal{O}_X) = 0$ for $r \geqslant 2$.*

*Proof.* The fact that a given sheaf be locally free of a given rank is a local property, so to prove our claim we may assume that $X$ is indeed a product $Z \times C$. But then the result follows in a straightforward way from the case where $X$ is a surface, where it is well known. □

LEMMA 3.2. *Let $X$ be as in Lemma 3.1. If $p\colon Y \to X$ is the desingularization given by blowing up $Z$, and $E$ is the exceptional divisor, then $p_*(\mathcal{O}_E(nE) \otimes \theta_Y)$ is zero for $n \geqslant 2$, and $p_*(\mathcal{O}_E(E) \otimes \theta_Y) = p_*(\mathcal{O}_E(E) \otimes \theta_{E/Z})$ is a line bundle on $Z$.*

*Proof.* There is a natural inclusion

$$p_*(\mathcal{O}_E(nE) \otimes \theta_{E/Z}) \to p_*(\mathcal{O}_E(nE) \otimes \theta_Y)$$

induced by the inclusion $\theta_{E/Z} \to \theta_Y \otimes \mathcal{O}_E$. So it is enough to prove the result locally; i.e. we can assume that $X$ is a product $C \times \mathbb{C}^k$. By [Ha], Corollary III.12.9 it is enough to prove that $h^0(Y_x, (\mathcal{O}_E(nE) \otimes \theta_Y)_{|Y_x})$ is zero for $n \geqslant 2$ and is equal to $h^0(Y_x, \mathcal{O}_E(E) \otimes \theta_{E/Z}) = 1$ for $n = 1$. Now $Y_x$ is isomorphic to $\mathbb{P}^1$, and on $Y_x$ we have exact sequences $0 \to \theta_E \to \theta_Y \to N_{E/Y} \to 0$ and $0 \to p^*\theta_Z \to \theta_{E/Z} \to 0$. As $N_{E/Y} = \mathcal{O}_E(E)$ and its restriction to $Y_x$ is $\mathcal{O}(-2)$, the restriction of $\theta_{E/Z}$ is $\mathcal{O}(2)$ and $p^*\theta_Z$ restricts on a fiber to a direct sum of copies of the trivial line bundle, the claimed result is straightforward. □

Arguments similar to those in the proof of next lemma can be found in the case where $X$ is a surface (resp. in the case where $X$ is a double cover of a smooth variety branched over a divisor with two smooth irreducible components meeting transversally) in [B-W] (resp. [Sai]).

LEMMA 3.3. *Under the same hypotheses as Lemma 3.2 we have*

$$R^1 p_*(\theta_Y) = p_*((\theta_{E/Z}) \otimes \mathcal{O}_E(E)).$$

*Proof.* From lemma 2.6 we have an inclusion $p_*\mathcal{H}^1_E(\theta_Y) \to R^1 p_*(\theta_Y)$. Therefore if they are locally isomorphic they are isomorphic. But locally the isomorphism follows from the two-dimensional case, proven in [B-W]. So $p_*\mathcal{H}^1_E(\theta_Y) = R^1 p_*(\theta_Y)$.

Now by [Gr], Theorem 2.8 we have

$$\mathcal{H}^1_E(\theta_Y) = \operatorname{limdir} \mathcal{E}xt^1_{\mathcal{O}_Y}(\mathcal{O}_{nE}, \theta_Y),$$

where $\mathcal{O}_{nE} = \mathcal{O}_Y / \mathcal{I}^n_E$; we recall that taking the direct image commutes with direct limits. We have as usual an exact sequence

$$0 \to \mathcal{O}_Y(-nE) \to \mathcal{O}_Y \to \mathcal{O}_{nE} \to 0;$$

applying $\mathcal{H}om(\cdot, \theta_Y)$ we get that

$$\mathcal{E}xt^1(\mathcal{O}_{nE}, \theta_Y) = \theta_Y \otimes \mathcal{O}_{nE}(nE),$$

and so also their direct images coincide. Tensoring the exact sequence

$$0 \to \mathcal{O}_{(n-1)E}((n-1)E) \to \mathcal{O}_{nE}(nE) \to \mathcal{O}_E(nE) \to 0$$

with $\theta_Y$ and applying $p_*$, from the fact (Lemma 3.2) that $p_*(\mathcal{O}_E(nE) \otimes \theta_Y)$ vanishes for $n \geqslant 2$, we get

$$p_*\mathcal{H}^1_E(\theta_Y) = p_*(\mathcal{O}_E(E) \otimes \theta_Y);$$

the claimed result now follows from Lemma 3.2. □

The following lemma is essentially due to Saito, although he only states it in the case where $X$ is a double cover.

LEMMA 3.4. *Assume $X$ has a transversal $A_1$ singularity, and that it is a divisor in a smooth manifold $W$. Then $\mathcal{E}xt^1(\Omega_X, \mathcal{O}_X)$ is the restriction to $Z$ of the normal bundle to $X$ in $W$, i.e. of $\mathcal{O}_X(X)$.*

*Proof.* We have a locally free resolution of $\Omega_X$ given by

$$0 \to \mathcal{O}_X(-X) \to \Omega_{W|X} \to \Omega_X \to 0.$$

We can use it to compute $\mathcal{E}xt^1(\Omega_X, \mathcal{O}_X)$; we get an exact sequence

$$\theta_{W|X} \to \mathcal{O}_X(X) \to \mathcal{E}xt^1(\Omega_X, \mathcal{O}_X) \to 0.$$

Therefore $\mathcal{E}xt^1(\Omega_X, \mathcal{O}_X)$ is isomorphic to $\mathcal{O}_{\bar{Z}} \otimes \mathcal{O}_X(X)$ where $\bar{Z}$ is the subscheme of $X$ whose ideal is locally generated by the entries of a $\dim W \times 1$ matrix representing the map $\theta_{W|X} \to \mathcal{O}_X(X)$ (in other words, this ideal is the 0th Fitting ideal sheaf of $\mathcal{E}xt^1(\Omega_X, \mathcal{O}_X)$).

It's now enough to verify that $Z = \bar{Z}$; this can easily be done by choosing local (analytic) coordinates $(x, y, z, u_1, \ldots, u_k)$ in $W$ such that $X$ has equation $xy - z^2 = 0$. The map $\theta_{W|X} \to \mathcal{O}_X(X)$ is dual to $\mathcal{O}_X(-X) \to \Omega_{W|X}$, and therefore the matrix representing one of them is the transpose of the matrix representing the other; but the latter is given by the vector of partial derivatives of the equation defining $X$, hence by $(y, x, -2z, 0, \ldots, 0)$. Hence the ideal of $\bar{Z}$ is generated locally by $(x, y, z)$, and therefore $Z = \bar{Z}$. □

LEMMA 3.5. (Fujiki). *Given a smooth manifold $X'$ with an involution having a smooth codimension 2 submanifold $Z'$ as fixed locus, let $X$ be the quotient of $X'$ via the involution. In particular $X$ is a complex space having a transversal $A_1$ singularity $Z$. Then, under the natural identification of $Z$ with $Z'$, $\mathcal{E}xt^1(\Omega_X, \mathcal{O}_X)$ as a sheaf on $Z$ coincides with $(\Lambda^2 N_{Z'|X'})^{\otimes 2}$.*

*Proof.* This is part of the proof of Lemma 5.1 in [Fu]. There it is stated for the dual of this bundle, but it is in fact proven for it (in Fujiki's case the bundle is anyway trivial). □

LEMMA 3.6. *In the same hypotheses of Lemma 3.5, let $p: Y \to X$ be the desingularization of $X$ obtained by blowing up $Z$. Let $E$ be the exceptional locus. Then $R^1 p_*(\theta_Y)$ is supported on $Z$; as a sheaf on $Z$ it is the line bundle $\Lambda^2 N_{Z'|X'}$.*

*Proof.* We consider the blowup $p': Y' \to X'$ of $X'$ along $Z'$; $Y'$ is naturally a double cover of $Y$ branched over the exceptional divisor $E'$, and $E$ and $E'$ are naturally isomorphic (also compatibly with the maps to $Z$ and $Z'$). By Lemma 3.3 $R^1 p_*(\theta_Y) = p_*(\theta_{E/Z} \otimes \mathcal{O}_E(E))$; via the natural identifications, the latter is just $p'_*(\theta_{E'/Z'} \otimes \mathcal{O}'_E(2E'))$. The result is now an easy exercise (see [Ha], exercise III.8.4). □

REMARK 3.7. Let $X$ be a complex space with a transversal $A_1$ singularity $Z$ and let $Y$ the desingularization given by blowing up $Z$. Then we have $\mathcal{E}xt^1(\Omega_X, \mathcal{O}_X) = (R^1 p_* \theta_Y)^{\otimes 2}$. This follows from Lemmas 3.5 and 3.6 if $X$ is the quotient of a smooth variety by an involution. The general case, which we do not need here (including results about other kinds of transversal rational double points) will be proven in [F2].

Saito [Sai] essentially proves this result in the case $X$ is a double cover of a smooth variety branched over the union of two smooth divisors intersecting transversally.

## 4. Deformations of Hilbert schemes of surfaces

We introduce some more notation, and recall some well-known facts. Let $S$ be a (smooth projective) surface; denote by $S^r$ the product of $r$ copies of $S$, by $S^{(r)}$ the symmetric product (i.e. the quotient of $S^r$ by the permutation group on $r$ elements) and by $S^{[r]}$ the Hilbert scheme parametrizing subschemes of length $r$ of $S$. $S^{(r)}$ is smooth exactly on the subset corresponding to $r$ distinct points; the singular locus $S^{(r)}_{\text{sing}}$ is irreducible, and its smooth locus is the set $Z$ where exactly two points coincide. Let $W$ be $S^{(r)}_{\text{sing}} \backslash Z$; $S^{(r)}_{\text{sing}}$ (hence also $Z$) has codimension 2 in $S^{(r)}$ and $W$ has codimension 2 in $S^{(r)}_{\text{sing}}$. The variety $S^{(r)} \backslash W$ has only transversal $A_1$ singularities along $Z$.

LEMMA 4.1. *With the previous notation, $\mathcal{E}xt^1(\Omega_{S^{(r)}}, \mathcal{O}_{S^{(r)}})$ restricted to $Z$ is isomorphic to $q^*(\mathcal{O}_S(-2K_S))$ where $q: Z \to S$ is the natural map sending the $n$-tuple $\{P, P, Q_1, \ldots, Q_{r-2}\}$ to $P$.*

*Proof.* We have a natural map $S^2 \times S^{(r-2)} \to S^{(r)}$ given by

$$((P_1, P_2)\{Q_1, \ldots, Q_{r-2}\}) \to \{P_1, P_2, Q_1, \ldots, Q_{r-2}\}.$$

We consider its restriction to the points whose image does not lie in $W$. This map induces an isomorphism between $Z$ and its inverse image $Z'$; by taking suitable neighbourhoods of $Z$ and $Z'$ we can assume it's just the quotient of the natural involution of $S^2$.

Applying Lemma 3.5 gives the claimed result, as it is easy to see that for any $Y$ the normal bundle to the diagonal $\Delta_Y$ in $Y \times Y$ is isomorphic to the tangent bundle. □

THEOREM 0.2. *If $h^0(S, \mathcal{O}_S(-2K_S))$ vanishes, then the natural map $D'_{S^{(r)}} \to D_{S^{(r)}}$ is an isomorphism.*

*Proof.* Let $X = S^{(r)}$. We want to prove that

$$H^0(X, \mathcal{E}xt^1(\Omega_X, \mathcal{O}_X))$$

vanishes. Let $Z$ and $W$ in $X$ be as before. Let $Y = S^{r-1}$, and let $f\colon Y \to S^{(r)}_{\text{sing}}$ be given by $f(P_1, \ldots, P_{r-1}) = \{P_1, P_1, P_2, \ldots, P_{r-1}\}$. It is clear that $f$ is surjective and that $f^{-1}(W)$ has codimension 2 in $Y$. Moreover out of $f^{-1}(W)$ the pullback of $\mathcal{E}xt^1(\Omega_X, \mathcal{O}_X)$ coincides with $p_1^*(-2K_S)$, where $p_1$ is the projection on the first factor. So take any section of $\mathcal{E}xt^1(\Omega_X, \mathcal{O}_X)$; this gives a section of $p_1^*(-2K_S)$ out of $f^{-1}(W)$ which by Hartogs can be extended to all of $Y$. But then it must be zero by the hypothesis on $S$ (otherwise restricting it to some closed subset of the form $S \times (P_2, \ldots, P_{r-1})$ we would get a nonzero section of $\mathcal{O}_S(-2K_S)$, a contradiction).

So any section of $\mathcal{E}xt^1(\Omega_X, \mathcal{O}_X)$ must vanish on $Z$; hence

$$H^0(\mathcal{E}xt^1(\Omega_X, \mathcal{O}_X)) = H^0_W(\mathcal{E}xt^1(\Omega_X, \mathcal{O}_X)).$$

But now by Lemma 2.4 $\mathcal{H}^0_W(\mathcal{E}xt^1(\Omega_X, \mathcal{O}_X)) = \mathcal{H}^2_W(\theta_X)$ and the latter is zero by Lemma 2.5, because the codimension of $W$ is 4. □

PROPOSITION 4.2. *If either $H^0(S, \theta_S)$ or $H^1(S, \mathcal{O}_S)$ vanishes, then*

$$H^1(S^{(r)}, \theta_{S^{(r)}}) = H^1(S, \theta_S).$$

*Proof.* By Künneth formula with these hypotheses we have

$$H^1(S^r, \theta_{S^r}) = \otimes_{i=1}^r H^1(S, \theta_S).$$

Now by [Fu], p. 84, $H^1(S^{(r)}, \theta_{S^{(r)}})$ is equal to the elements of $H^1(S^r, \theta_{S^r})$ which are invariant under the action of the permutation group; these are easily seen to be the image of $H^1(S, \theta_S)$ via the diagonal embedding. □

THEOREM 0.1. *If either* $h^0(S, \theta_S)$ *or* $h^1(S, \mathcal{O}_S)$ *vanish, then the natural map* $D_S \to D'_{S^{(r)}}$ *is an isomorphism.*

*Proof.* It is clear that Remark 4.2 implies that there is an isomorphism on tangent spaces; now from [Fu] p. 84 we can derive that there is injection on obstruction spaces ($H^2(S, \theta_S)$ is in a natural way a direct summand of $H^2(S^{(r)}, \theta_{S^{(r)}})$). So the usual criterion can be applied. □

THEOREM 0.3. *If* $h^0(S, \mathcal{O}_S(-K_S))$ *vanishes, then the natural map* $D'_{S^{(r)}} \to D_{S^{[r]}}$ *is an isomorphism.*

*Proof.* Let $p: S^{[r]} \to S^{(r)}$ be the natural map. We want to prove that

$$H^0(S^{(r)}, R^1 p_*(\theta_{S^{[r]}})) = 0.$$

The technique is very much the same as in the proof of Theorem 0.2, and we use the same notation.

The sheaf $R^1 p_*(\theta_{S^{[r]}})$ is supported on $S^{(r)}_{\text{sing}}$ and its restriction to $Z$ is a line bundle. Arguing as before we prove that every section has to vanish on $Z$ (as there $R^1 p_* \theta_{S^{[r]}} = q^*(\mathcal{O}_S(-K_S))$. We are left to show the vanishing to $H^0_W(S^{(r)}, R^1 p_*(\theta_{S^{[r]}}))$. By [Gr], Proposition 4.5 we have a spectral sequence

$$H^p_W(S^{(r)}, R^q p_*(\theta_{S^{[r]}})) \Rightarrow H^k_{p^{-1}(W)}(S^{[r]}, \theta_{S^{[r]}}).$$

Hence it is enough to show that

$$H^1_{p^{-1}(W)}(S^{[r]}, \theta_{S^{[r]}}) \quad \text{and} \quad H^2_W(S^{(r)}, p_* \theta_{S^{[r]}}) = H^2_W(S^{(r)}, \theta_{S^{(r)}})$$

both vanish. This follows from Lemma 2.5 and the well-known fact that the codimension of $p^{-1}(W)$ in $S^{[r]}$ is 2. □

COROLLARY 4.3. *If $S$ is a surface of general type or a regular surface of Kodaira dimension one, then formal deformations of $S$ are the same as formal deformations of* $S^{(r)}$ *and of* $S^{[r]}$.

*Proof.* This follows immediately from the previous results; in fact in these hypotheses $H^0(\mathcal{O}_S(-K_S)) = H^0(\mathcal{O}_S(-2K_S)) = 0$, so $D_{S^{[r]}}$ is naturally isomorphic to $D'_{S^{(r)}}$ (by Theorem 0.2), and the latter is isomorphic to $D_{S^{(r)}}$ by Theorem 0.3; now if $S$ is of general type $H^0(S, \theta_S) = 0$ and regularity means exactly that $H^1(S, \mathcal{O}_S) = 0$, so Theorem 0.1 applies and $D_S$ is naturally isomorphic to $D'_{S^{(r)}}$. □

REMARK 4.4. Theorem 0.1 is clearly sharp, as if its conditions are not fulfilled then the natural map $D_S \to D'_{S^{(r)}}$ is injective but not surjective on the tangent spaces.

As for Theorem 0.3, we have Beauville's results on the deformations of $S^{[r]}$ when $K_S$ is trivial; for instance if $S$ is a $K_3$ surface the tangent spaces to this to functors have respective dimensions 20 and 21. One might still ask oneself whether the Kuranishi families are set-theoretically the same (e.g. by one or both of them having a non-reduced structure); however Beauville proves that in this case both families are smooth, so that we have a proper inclusion (in fact there are general results about unobstructedness of deformations for varieties with no infinitesimal automorphisms and trivial dualizing sheaf, see e.g. [R2] and [Kw]).

Theorem 0.2 also fails if $K_S$ is trivial. In fact $S^{(r)}$ has canonical singularities, hence rational; therefore the 'extra' deformations of $S^{[r]}$ constructed by Beauville can be blow down to deformations of $S^{(r)}$ by an easy extension of the argument in Proposition 2.3 of [B-W], yielding non-locally trivial deformations of $S^{(r)}$. Actually it is easy to see that we obtain in this way all deformations of $S^{(r)}$; the latter are therefore also unobstructed (this does not follow from any of the known criteria of unobstructedness for varieties with trivial canonical bundle, as these criteria always assume smoothness in codimension 2). It would be interesting to understand whether the deformations of $S^{(r)}$ obtained in this way are smoothings (this is easy to prove for $r = 2$).

EXAMPLE 4.5. Let $S = \mathbb{P}^2$ and $r = 2$; then $X = S^{(2)}$ can be identified with the secant variety of the Veronese surface in $\mathbb{P}^5$. This is known to be the symmetric determinantal cubic hypersurface, and has therefore a family of non-locally trivial deformations (the cubic hypersurfaces). In particular Theorem 0.2 does not hold for $S = \mathbb{P}^2, r = 2$.

Using the results of this paper, we can easily describe the Kuranishi family of $X$. As $X$ has only transversal $A_1$ singularities (or more generally hypersurface singularities) we have $\mathcal{E}xt^i(\Omega_X, \mathcal{O}_X) = 0$ for $i \geqslant 2$ (Lemma 3.1); moreover $\mathcal{E}xt^1(\Omega_X, \mathcal{O}_X) = \mathcal{O}_{\mathbb{P}^2}(-2K_{\mathbb{P}^2})$, where $\mathbb{P}^2 \subset X$ is the Veronese surface. Therefore we have an exact sequence

$$\begin{aligned} 0 &\to H^1(X, \theta_X) \to \mathrm{Ext}^1(\Omega_X, \mathcal{O}_X) \to H^0(X, \mathcal{E}xt^1(\Omega_X, \mathcal{O}_X)) \to \\ &\to H^2(X, \theta_X) \to \mathrm{Ext}^2(\Omega_X, \mathcal{O}_X) \to H^1(X, \mathcal{E}xt^1(\Omega_X, \mathcal{O}_x)) \to \\ &\to H^3(X, \theta_X). \end{aligned}$$

Via the standard exact sequences

$$\begin{aligned} &0 \to \theta_X \to \theta_{\mathbb{P}^5} \otimes \mathcal{O}_X \to \mathcal{O}_X(3) \to 0, \\ &0 \to \mathcal{O}_{P^5} \to \mathcal{O}_{P^5}(1)^{n+1} \to \theta_{\mathbb{P}^5} \to 0, \\ &0 \to \mathcal{O}_{\mathbb{P}^5}(-3) \to \mathcal{O}_{\mathbb{P}^5} \to \mathcal{O}_X \to 0; \end{aligned}$$

it is easy to see that $H^i(X, \theta_X) = 0$ for $i = 1, 2, 3$ (use Kodaira vanishing on $\mathbb{P}^5$ and diagram chasing). Therefore we get

$$\begin{aligned} &\mathrm{Ext}^1(\Omega_X, \mathcal{O}_X) = H^0(\mathbb{P}^2, -2K_{\mathbb{P}^2}), \\ &\mathrm{Ext}^2(\Omega_X, \mathcal{O}_X) = 0. \end{aligned}$$

Hence the Kuranishi family of $X$ is smooth of dimension 28. This can also be proven in a different way. First of all one checks (by computing a suitable Kodaira-Spencer map) that all deformations of $X$ are cubics in $\mathbb{P}^5$. Now the family of all cubics in $\mathbb{P}^5$ has dimension 55; on this family we have a natural action of $\mathbb{P}$GL(6), which has dimension 35. $\mathbb{P}$GL(6) has a 8-dimensional subgroup (isomorphic to $\mathbb{P}$GL(3) and induced by automorphisms of the Veronese surface) which sends $X$ to itself, acting faithfully on $X$. Therefore the Kuranishi family is smooth of dimension

$$55 - (35 - 8) = 28,$$

as expected.

We remark that in this case the family stays versal but not semiuniversal for a generic small deformation of $X$; in fact the Kuranishi family of a generic cubic hypersurface in $\mathbb{P}^5$ has dimension 20. The difference is just given by the difference of dimension of the automorphism group between the special and the general fibre.

## Acknowledgements

I would like to heartily thank Fabrizio Catanese for suggesting this problem to me, and for many discussions which helped me to overcome mathematical and nonmathematical obstacles.

An earlier version of the paper was the subject of a talk in the L'Aquila conference 'Classification of algebraic varieties' of May 1992; I want to thank the participants, and especially H. Flenner, M. Reid and J. Wahl, for useful suggestions and stimulating discussions (as well as the organizers who made this possible by providing us with a nice environment).

I thank the referee for pointing out a mistake in a previous version of the manuscript.

The author is a member of GNSAGA of CNR and was partially supported by the European Science Program contract no. SCI-0398-C(A).

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