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Selfdual and non-selfdual 3-dimensional Galois representations

Dedicated to Frans Oort on the occasion of his 60th birthday

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1. Introduction

In [vG-T], an example of a (compatible system of λ-adic) 3-dimensional $G_{\mathbb{Q}} = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-representation(s) was constructed. This representation ρ is non-selfdual. By definition, this means that the contragredient ρ^* is not isomorphic to $\rho(2)$. The (2) here denotes a Tate twist; it is needed because the absolute values of eigenvalues of the image of a Frobenius element at p under ρ have absolute value p. Hence for ρ^* one finds absolute values $1/p$, so comparing ρ and ρ^* is of interest only after the Tate twist above, which results in a $\rho(2)$ yielding absolute values $1/p$ as well. We note here in passing that in concrete cases, it is normally rather easy to verify that a given representation is non-selfdual. Namely, one may compare traces of ρ and $\rho^*(-2)$. The latter trace is for representations such as ours just the complex conjugate of the former. Hence if some trace of ρ is not real, then the representation is non-selfdual. Moreover, if some trace divided by its complex conjugate is not a root of unity, then ρ is non-selfdual in the stronger sense that even when multiplied by a Dirichlet character, it still is not isomorphic to $\rho^*(-2)$. This is used explicitly in 5.11.

Our interest in such 3-dimensional $G_{\mathbb{Q}}$-representations was motivated by a question of Clozel. The question was if one could explicitly construct such a Galois representation and a modular form on $\mathrm{GL}(3)$ such that their local L-factors are the same for all primes. In the $\mathrm{GL}(2)$-case, a procedure for associating Galois representations to cusp forms is well known. For $\mathrm{GL}(3)$ and non-selfdual Galois representations however, such a relation remains completely conjectural. Our paper [vG-T] may be regarded as some partial affirmative evidence for Clozel's question; an example is exhibited where the L-factors coincided at least for all odd primes less then 71. The Galois representation constructed turned out to be irreducible,

non-selfdual, and ramified only at the prime 2 ([vG-T, Sect. 3, Remark 2]). Two more examples of Galois– and automorphic representations of this kind for which several L-factors coincide are given in [GKTV].

While there exists an algorithm to compute the modular forms on GL(3) (see [AGG]), there is no straightforward way to find three dimensional Galois representations. Of course one can consider the Sym^2 of two dimensional Galois representations, but such representations are selfdual (their image lies in the group of similitudes of a quadratic form). As explained in [vG-T], one place to look for non-selfdual Galois representations is in the H^2 (etale cohomology) of a surface S with $\dim H^{2,0} = 2$ which has an automorphism σ of order 4, defined over $\mathbb{Q}$. In case the Néron-Severi group NS of S has an orthogonal complement W of rank 6 which is $G_{\mathbb{Q}}$-stable and if σ acts with three eigenvalues i on W, then the eigenspaces V_ℓ, V'_ℓ of σ on $W \otimes \overline{\mathbb{Q}}_\ell$ are likely candidates for a non-selfdual 3-dimensional $G_{\mathbb{Q}}$-representation.

$$H^2(S_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell) = NS_{\mathbb{Q}_\ell} \oplus W_{\mathbb{Q}_\ell}, \quad W \otimes \overline{\mathbb{Q}}_\ell = V_\ell \oplus V'_\ell.$$

(The two eigenspaces of σ are now each isotropic for the intersection form on H^2, so there is no obvious reason why the associated representations should be selfdual.)

In [vG-T] we used a one parameter family of such surfaces (constructed by Ash and Grayson) to find the example cited above. The two new examples also arise in this family. The present paper is concerned with a different construction of 3-dimensional Galois representations. As before, these are found in the H^2 of surfaces. The surfaces S under consideration will be degree 4 cyclic base changes of elliptic surfaces $\mathcal{E}$ with base $\mathbb{P}^1$. By taking the orthogonal complement to a large algebraic part in H^2 together with all cohomology coming from the intermediate degree 2 base change, one obtains (see 2.4 below) a representation space for $G_{\mathbb{Q}}$. This comes equipped with the action of the automorphism of order 4 defining the cyclic base change. Taking an eigenspace of this action finally yields the representation we wish to study.

Our main technical result is a formula for the traces of Frobenius elements on this space in terms of the number of points on $\mathcal{E}$ and S over a finite field (Theorem 3.5). This formula allows us to compute the characteristic polynomial of Frobenius in many cases. Using it we succeed in proving that certain examples obtained yield selfdual representations, while others do not. For some of the selfdual cases we can actually exhibit 2-dimensional Galois representations whose symmetric square seems to coincide with the 3-dimensional Galois representation (see (5.5) and (5.6)).

We also find many non-selfdual representations; some examples are given in Section 5.11. Thus far, these have not been related to modular forms on GL(3) probably because the conductor of these Galois representations seems rather large.

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2. The construction

2.1. In this section, a general method for constructing representation spaces for Galois groups is explained. This is done by first considering cyclic quotients of $\mathbb{P}^1$, then base changing surfaces defined over the quotient, and finally taking pieces in the cohomology of the base changed surface. We will now describe all these steps of the construction in detail.

2.2. On $\mathbb{P}^1$, define the automorphism σ by $\sigma(z) = (z+1)/(-z+1)$. One checks that σ^2 is given by $z \mapsto -1/z$. Moreover, the function $u = (z^2-1)/z$ is σ^2-invariant and transforms under σ as $\sigma^* u = -4/u$. Hence the quotient $\mathbb{P}^1/\langle\sigma\rangle$ is described as

$$\mathbb{P}^1_z \xrightarrow{j} \mathbb{P}^1_z/\langle\sigma^2\rangle = \mathbb{P}^1_u \xrightarrow{h} \mathbb{P}^1_z/\langle\sigma\rangle = \mathbb{P}^1_t,$$

in which $j\colon z \mapsto u = (z^2-1)/z$ and $h\colon u \mapsto t = (u^2-4)/u$ and $h \circ j\colon z \mapsto t = (z^4 - 6z^2 + 1)/z(z^2-1))$. The quotient map is totally ramified over $t = \pm 4i$ and this is all the ramification.

To have this ramification over points $r \pm si$ (with $s \neq 0$) use the additional automorphism $g_{r,s}\colon t \mapsto r + st/4$. The composition

$$\pi_{r,s} = g_{r,s} \circ h \circ j$$

now defines the cyclic degree 4 cover of $\mathbb{P}^1$ that we will consider. It is totally ramified above $r \pm si$ and unramified elsewhere.

2.3. Let $\mathcal{E} \to \mathbb{P}^1_t$ denote a stable, minimal elliptic surface with base $\mathbb{P}^1$. By pull back, two other elliptic surfaces are derived from $\mathcal{E}$ as

$$\begin{array}{ccccc} S_0 & \longrightarrow & X_0 & \longrightarrow & \mathcal{E} \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{P}^1_z & \xrightarrow{j} & \mathbb{P}^1_u & \xrightarrow{g_{r,s}\circ h} & \mathbb{P}^1_t. \end{array}$$

Let X, S denote the (again stable) minimal models of X_0, S_0 over $\mathbb{P}^1_u, \mathbb{P}^1_z$ respectively. Explicitly, if $\mathcal{E}$ corresponds to a Weierstrass equation

$$y^2 + a_1(t)xy + a_3(t) = x^3 + a_2(t)x^2 + a_4(t)x + a_6(t),$$

then X and S are obtained from

$$\begin{aligned} &y^2 + a_1(g_{r,s}(h(u)))xy + a_3(g_{r,s}(h(u)))y \\ &\quad = x^3 + a_2(g_{r,s}(h(u)))x^2 + a_4(g_{r,s}(h(u)))x + a_6(g_{r,s}(h(u))) \end{aligned}$$

and

$$\begin{aligned} &y^2 + a_1(\pi_{r,s}(z))xy + a_3(\pi_{r,s}(z))y \\ &\quad = x^3 + a_2(\pi_{r,s}(z))x^2 + a_4(\pi_{r,s}(z)x + a_6(\pi_{r,s}(z)), \end{aligned}$$

respectively. The model given by such an affine Weierstrass minimal model (including the fiber over infinity) will be denoted by $\mathcal{E}_{\text{aff}}, X_{\text{aff}}, S_{\text{aff}}$.

If the morphisms on $\mathbb{P}^1$ under consideration map $z_0 \mapsto u_0 \mapsto t_0$, and if moreover $\pi_{r,s}$ is unramified over t_0 (which means $t_0 \neq r \pm si$), then the fibre of S over z_0 is the same as the fiber of X over u_0 and the fiber of $\mathcal{E}$ over t_0. In case there is ramification over t_0, we have by the assumption that $\mathcal{E}$ is stable that the fibre over t_0 is of type I_ν with $0 \leqslant \nu$. Then the fibre of X over u_0 is of type $I_{2\nu}$ and that of S over z_0 is of type $I_{4\nu}$.

Note finally that by construction, one has an automorphism $\sigma\colon S \to S$ which is on the model given by the (x, y, z)-coordinates given by

$$\sigma\colon (x, y, z) \mapsto (x, y, (z+1)/(-z+1)).$$

The order of σ equals 4.

2.4. Let K be any perfect field. We will construct ℓ-adic representations of $G_K = \mathrm{Gal}(\overline{K}/K)$ associated to the situation above. To this end, we require that $\pi = \pi_{r,s}\colon \mathbb{P}^1 \to \mathbb{P}^1$ and $\mathcal{E} \to \mathbb{P}^1$ are defined over K. Then the constructed S and X are elliptic surfaces defined over K as well. Moreover, the automorphism σ on S is defined over K. Consider the cohomology group $H^2(S_{\overline{K}}, \mathbb{Q}_\ell)$.

By pullback we may regard $H^2(X_{\overline{K}}, \mathbb{Q}_\ell)$ as a G_K-invariant subspace of $H^2(S_{\overline{K}}, \mathbb{Q}_\ell)$. In fact, because $X = S/\sigma^2$ one knows $H^2(X_{\overline{K}}, \mathbb{Q}_\ell) = H^2(S_{\overline{K}}, \mathbb{Q}_\ell)^{\sigma^2 = I}$. So this subspace is σ-invariant as well, and moreover it is generated by the ± 1-eigenspaces of σ in $H^2(S_{\overline{K}}, \mathbb{Q}_\ell)$. A second subspace to be considered is $A_\ell(S)$, the $\mathbb{Q}_\ell$-subspace in H^2 spanned by all components of bad fibers of $S \to \mathbb{P}^1$. This $A_\ell(S)$ is also both σ- and G_K-invariant.

Define

$$W_\ell = H^2(S_{\overline{K}}, \mathbb{Q}_\ell) / \left(H^2(X_{\overline{K}}, \mathbb{Q}_\ell) + A_\ell(S)\right).$$

The space W_ℓ comes equipped with a σ- and a G_K-action. Since the automorphism σ on S is defined over K, the two actions commute. Fix a 4th primitive root of unity $i \in \overline{\mathbb{Q}}_\ell$. By what is said above, the only eigenvalues of σ on W_ℓ are $\pm i$. After extending scalars from $\mathbb{Q}_\ell$ to $\mathbb{Q}_\ell(i)$, the G_K-representation W_ℓ splits according to the eigenvalues $\pm i$ of σ. Put

$$V_\ell = \text{ the } i\text{-eigenspace of } \sigma \text{ in } W_\ell.$$

Our goal will be to study for the case $K = \mathbb{Q}$ the $G_{\mathbb{Q}}$-representation on V_ℓ. In particular, we will answer the question how to compute traces of Frobenius elements in $G_{\mathbb{Q}}$ for such a representation.

3. The trace formula

We will use the notations and the construction from Section 2 above. Without loss of generality we will assume that the original surface $\mathcal{E} \to \mathbb{P}^1$ is obtained from an affine Weierstrass minimal model. Any fiber E_t of this model is either an affine smooth elliptic curve given by a Weierstrass equation, or it is an affine cubic equation with a double point. This E_t hence differs from the fiber of $\mathcal{E}$ over t in at most two ways: firstly, on E_t we have ignored the point at infinity. Secondly, in case the fiber of $\mathcal{E} \to \mathbb{P}^1$ over t is an n-gon with $n \geqslant 2$, then it has more components than E_t. Recall the notation $\mathcal{E}_{\text{aff}}, X_{\text{aff}}, S_{\text{aff}}$ for the union of these affine fibers (including the fiber over infinity). In order to compute traces, we will for a moment require that the above setup with $\mathcal{E}, X, S, E_t$ etc. is all defined over a finite field $\mathbb{F}_p$ with p an odd prime. Put $q = p^n$ and write F_q for the qth power map in $G_{\mathbb{F}_q}$. If $a \in \mathbb{F}_q^*$, then write $\left(\frac{a}{q}\right) = 1$ in case a is a square in $\mathbb{F}_q^*$, and $\left(\frac{a}{q}\right) = -1$ otherwise.

PROPOSITION 3.1. *Let $S \to \mathbb{P}^1$ be an elliptic surface constructed by pull back from $\mathcal{E} \to \mathbb{P}^1$ as in Section 2. For the associated representation space W_ℓ one has the formula*

$$\begin{aligned}\operatorname{trace}(F_q|W_\ell) &= \#S_{\text{aff}}(\mathbb{F}_q) - \#X_{\text{aff}}(\mathbb{F}_q)\\ &= 2\#E_\infty(\mathbb{F}_q) + \sum_{\substack{t\in\mathbb{F}_q\\ t^2+16\neq 0}} \left(\frac{u_t^2+4}{q}\right)\left(\left(\frac{t^2+16}{q}\right)+1\right)\#E_{r+st/4}(\mathbb{F}_q),\end{aligned}$$

in which u_t denotes a root in $\mathbb{F}_q$ of $X^2 - tX - 4 = 0$ and E_t is the fiber over t of $\mathcal{E} \to \mathbb{P}^1$.

REMARK 3.2. Note that the above formula makes sense: a u_t as required exists whenever the factor $\left(\frac{t^2+16}{q}\right) + 1$ is non-zero. Moreover, in that case the symbol $\left(\frac{u_t^2+4}{q}\right)$ is independent of the choice of u_t.

The explicit formula for the trace here shows that it is expressed completely in terms of the number of points in the fibres of the original surface $\mathcal{E} \to \mathbb{P}^1$.

In what follows, and in particular in the proof of Proposition 3.1, the following general lemma will be useful.

LEMMA 3.3. *Suppose $Y \to \mathbb{P}^1$ is a minimal, stable elliptic surface defined over $\mathbb{F}_q$, corresponding to a minimal affine Weierstrass model Y_{aff}, with stable fibers over each point in $\mathbb{P}^1$.*

Put $A_\ell(Y) \subset H^2(Y_{\overline{\mathbb{F}_q}}, \mathbb{Q}_\ell)$ to be the subspace spanned by a fiber, the zero section, and all components of singular fibers not meeting the zero section. Let $H^2_{\text{tr}}(Y)$ be the orthogonal complement of $A_\ell(Y)$ in $H^2(Y_{\overline{\mathbb{F}_q}}, \mathbb{Q}_\ell)$.

Then

$$\text{trace}(F_q | H^2_{\text{tr}}(Y)) = \#Y_{\text{aff}}(\mathbb{F}_q) - q^2 - q.$$

Proof. By the Lefschetz trace formula one knows

$$\#Y(\mathbb{F}_q) = 1 + q^2 + \text{trace}(F_q | H^2_{\text{tr}}(Y)) + \text{trace}(F_q | A_\ell(Y)).$$

The lemma will follow once we have shown that moreover

$$\#Y(\mathbb{F}_q) = \text{trace}(F_q | A_\ell(Y)) + \#Y_{\text{aff}}(\mathbb{F}_q) - q + 1.$$

This will be done by describing the contributions to each of these terms coming from the various (components of) fibers and the zero section.

Firstly, there is a contribution $q + q = 2q$ from the fiber and the zero section to $\text{trace}(F_q | A_\ell(Y))$. Also, all smooth fibers contribute to each of $\#Y_{\text{aff}}(\mathbb{F}_q)$ and $\#Y(\mathbb{F}_q)$ the same number of affine points, plus one point at infinity in $Y(\mathbb{F}_q)$.

What remains are the singular fibers. If Y has such a fiber over a point t, then by assumption it is an n-gon for $n \geqslant 2$. If t is *not* $\mathbb{F}_q$-rational, then because Y is defined over $\mathbb{F}_q$, the Frobenius map F_q interchanges the fiber at t with possibly several others in a cyclic way, and one concludes from this that the total contribution obtained from this to any of $\text{trace}(F_q | A_\ell(Y))$, $\#Y_{\text{aff}}(\mathbb{F}_q)$ and $\#Y(\mathbb{F}_q)$ is 0.

On the other hand, suppose we have a singular fiber over a rational point. In Y_{aff}, this fiber is represented by a cubic curve with a node, given by an affine Weierstrass equation. We distinguish the two possibilities where the tangent lines at the node are $\mathbb{F}_q$-rational (split multiplicative) or non-$\mathbb{F}_q$-rational (non-split). In case of a split multiplicative fiber, all components of the n-gon we consider in Y are defined over $\mathbb{F}_q$. Hence this contributes $(n-1)q$ to $\text{trace}(F_q | A_\ell(Y))$, and $n(q+1) - n = nq$ rational points to $\#Y(\mathbb{F}_q)$, and $q - 1$ to $\#Y_{\text{aff}}(\mathbb{F}_q)$.

In case of a non-split multiplicative fiber one finds $q + 1$ points in $Y_{\text{aff}}(\mathbb{F}_q)$. To compute the contributions to $\#Y(\mathbb{F}_q)$ and to $\text{trace}(F_q | A_\ell(Y))$, we treat the cases n odd and n even separately. The points on the zero-component where other components of the n-gon intersect it, are interchanged by F_q in case of a non-split fiber. From this one concludes that if n is odd, then the zero-component is the only component defined over $\mathbb{F}_q$. The intersection point of the two components 'farthest away' from the zero-component is $\mathbb{F}_q$-rational as well, hence the contribution to $\#Y(\mathbb{F}_q)$ equals $q + 2$ and that to $\text{trace}(F_q | A_\ell(Y))$ is 0 for odd n. For n even, the component 'opposite' the zero-one is defined over $\mathbb{F}_q$ as well, and this yields a contribution $2(q+1)$ to $\#Y(\mathbb{F}_q)$ and q to $\text{trace}(F_q | A_\ell(Y))$.

Combining all these possible contributions now gives the relation between $\#Y(\mathbb{F}_q)$, $\#Y_{\mathrm{aff}}(\mathbb{F}_q)$ and $\mathrm{trace}(F_q|A_\ell(Y))$ stated above. This proves the lemma. □

Proof of Proposition 3.1. The second equality is a straightforward application of the definitions given in Section 2. We will prove the first equality using Lemma 3.3. With notations as in the lemma, we have

$$\#S_{\mathrm{aff}}(\mathbb{F}_q) - \#X_{\mathrm{aff}}(\mathbb{F}_q) = \mathrm{trace}(F_q|H^2_{\mathrm{tr}}(S)) - \mathrm{trace}(F_q|H^2_{\mathrm{tr}}(X)).$$

By pull back, we may regard $H^2_{\mathrm{tr}}(X)$ as a subspace of $H^2_{\mathrm{tr}}(S)$ which is F_q-invariant. Hence what remains to be proven is that

$$H^2_{\mathrm{tr}}(S)/H^2_{\mathrm{tr}}(X) \cong H^2(S_{\overline{\mathbb{F}_q}}, \mathbb{Q}_\ell)/\left(H^2(X_{\overline{\mathbb{F}_q}}, \mathbb{Q}_\ell) + A_\ell(S)\right)$$

as $\mathbb{Q}_\ell[F_q]$-modules. Using the following commutative diagram with exact rows and columns

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A_\ell(X) & \longrightarrow & H^2(X_{\overline{\mathbb{F}_q}}, \mathbb{Q}_\ell) & \longrightarrow & H^2_{\mathrm{tr}}(X) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A_\ell(S) & \longrightarrow & H^2(S_{\overline{\mathbb{F}_q}}, \mathbb{Q}_\ell) & \longrightarrow & H^2_{\mathrm{tr}}(S) & \longrightarrow & 0,
\end{array}$$

this is immediate. □

3.4. Fix from now on $i \in \overline{\mathbb{Q}_\ell}$ with $i^2 + 1 = 0$. Proposition 3.1 allows us to compute traces of Frobenius on the space W_ℓ. However, what we are actually interested in, is the subspace $V_\ell \subset W_\ell \otimes_{\mathbb{Q}_\ell} \overline{\mathbb{Q}_\ell}$ defined as the i-eigenspace of the automorphism σ acting on W_ℓ. Recall that the only eigenvalues of σ on W_ℓ are $\pm i$, hence $V_\ell = (I - i\sigma)W_\ell$. Moreover, $I - i\sigma = 2I$ on V_ℓ. Now reason as in [vG-T, Sect. 3.8]; it follows that

$$\begin{aligned} 2\,\mathrm{trace}(F_q|V_\ell) &= \mathrm{trace}(F_q|(I - i\sigma)W_\ell) \\ &= \mathrm{trace}(F_q|W_\ell) - i\,\mathrm{trace}(F_q\sigma|W_\ell), \end{aligned}$$

the latter trace is precisely the trace of F_q acting on a quartic twist $\tilde{W}_\ell$ of W_ℓ. Define

$$\tilde{S} = \left(S \times_{\mathrm{Spec}(\mathbb{F}_q)} \mathrm{Spec}(\mathbb{F}_{q^4})\right)/\langle \sigma \times F_q \rangle;$$

geometrically this is the same surface as S, but the F_q-action is different. Now $\tilde{W}_\ell$ is by definition the subspace of $H^2(\tilde{S}_{\overline{\mathbb{F}_q}}, \mathbb{Q}_\ell)$ that corresponds geometrically with W_ℓ.

Since $\tilde{S} = S$ geometrically, we can in principle calculate $\mathrm{trace}(F_q\sigma|W_\ell) = \mathrm{trace}(F_q|\tilde{W}_\ell)$ using Proposition 3.1. In practice, we need to understand $\tilde{S}_{\mathrm{aff}}(\mathbb{F}_q)$

and analogously $\tilde{X}_{\mathrm{aff}}(\mathbb{F}_q)$. By the definition of $\tilde{S}$, these are the points P in $S_{\mathrm{aff}}(\mathbb{F}_{q^4})$ which satisfy $\sigma^{-1}(P) = F_q(P)$. Note that under the map

$$S_{\mathrm{aff}} \to \mathcal{E}_{\mathrm{aff}},$$

such points map to $\mathbb{F}_q$-rational points in $\mathcal{E}_{\mathrm{aff}}$. Thus to find the points in $\tilde{S}_{\mathrm{aff}}(\mathbb{F}_q)$, we need to know the points in $\mathcal{E}_{\mathrm{aff}}(\mathbb{F}_q)$, and the Galois action on the points above them in the covering $S_{\mathrm{aff}} \to \mathcal{E}_{\mathrm{aff}}$. This covering is completely determined by the covering on the bases $\pi_{r,s}\colon \mathbb{P}^1_z \to \mathbb{P}^1_t$.

Similarly, the intermediate surface $\tilde{X} = \tilde{S}/\sigma^2$ is the quadratic twist over $\mathbb{F}_{q^2}/\mathbb{F}_q$ of X, using the automorphism of order 2 on X induced by σ. From this description one deduces that $\mathbb{F}_q$-rational points in $\tilde{X}_{\mathrm{aff}}(\mathbb{F}_q)$ correspond precisely to points in $X_{\mathrm{aff}}(\mathbb{F}_{q^2})$ which are not $\mathbb{F}_q$-rational, but which map to points in $\mathcal{E}_{\mathrm{aff}}(\mathbb{F}_q)$. Again, this is completely given in terms of fibers in $\mathcal{E}_{\mathrm{aff}}$ over rational points, together with the covering $\mathbb{P}^1_u \to \mathbb{P}^1_t$. The explicit trace formula obtained in this way is as follows:

THEOREM 3.5. *With notations as above,*

$$\begin{aligned}
\mathrm{trace}(F_q|V_\ell) &= \frac{\#S_{\mathrm{aff}}(\mathbb{F}_q) - \#X_{\mathrm{aff}}(\mathbb{F}_q)}{2} - i\frac{\#\tilde{S}_{\mathrm{aff}}(\mathbb{F}_q) - \#\tilde{X}_{\mathrm{aff}}(\mathbb{F}_q)}{2} \\
&= \#E_\infty(\mathbb{F}_q) + \sum_{\substack{t\in\mathbb{F}_q \\ t^2+16\in\mathbb{F}_q^{*2}}} \left(\frac{u_t^2+4}{q}\right)\#E_{r+st/4}(\mathbb{F}_q) \\
&\quad - i\sum_{\substack{t\in\mathbb{F}_q \\ t^2+16\in\mathbb{F}_q^{*}\setminus\mathbb{F}_q^{*2}}} \epsilon(t)\#E_{r+st/4}(\mathbb{F}_q)
\end{aligned}$$

in which $\epsilon(t) = \begin{cases} +1 & \text{if } X^4 - tX^3 - 6X^2 + tX + 1 \\ & \qquad \text{divides } X^{q+1} + X^q - X + 1\,; \\ -1 & \text{otherwise.} \end{cases}$

Proof. The first equality follows from the discussion above. To prove the second, we consider the fibers of $\mathcal{E}_{\mathrm{aff}}$ over all rational points $r + st/4$ in $\mathbb{P}^1$. Recall that the maps on the bases we consider are given by

$$\begin{aligned}
z \mapsto u = (z^2-1)/z \mapsto t &= (u^2-4)/u \\
&= (z^4 - 6z^2 + 1)/z(z^2-1) \mapsto r + st/4.
\end{aligned}$$

The fiber over infinity in $\mathcal{E}_{\mathrm{aff}}$ occurs twice in X_{aff} (over $u = 0, \infty$) and 4 times in S_{aff} (over $z = \pm 1, 0, \infty$). Since these fibers are all over rational points, this does not contribute to the imaginary part of $\mathrm{trace}(F_q|V_\ell)$, and one obtains $(4-2)/2 = 1$ times $\#E_\infty(\mathbb{F}_q)$ as contribution to the real part. Next, contributions from fibers over t with $t^2 + 16 = 0$ are 0, because these t are precisely the ramification points of the map described above, so they give the same contribution to any of S_{aff}, X_{aff}.

Is $t \in \mathbb{F}_q$ not a ramification point, then the points u_t in $\mathbb{P}^1_u$ mapped to t satisfy $(u_t^2 - 4)/u_t = t$. These points are $\mathbb{F}_q$-rational precisely when $t^2 + 16 \in \mathbb{F}_q^{*2}$. In that case, the fiber of X_{aff} over each of the two u_t is identical to the fiber $\mathcal{E}_t$ of $\mathcal{E}$ over t. Moreover, we obtain this fiber not in $S_{\text{aff}}(\mathbb{F}_q)$ in case there is no rational z with $(z^2 - 1)/z = u_t$ (which is equivalent to $(\frac{u_t^2+4}{q}) = -1$). The other possibility is that $(\frac{u_t^2+4}{q}) = 1$, which means precisely that all 4 values of z over the u_t's are rational (and over each of them, the same fiber $\mathcal{E}_t$ is obtained). If, on the other hand, $(\frac{u_t^2+4}{q}) = -1$ then we obtain 4 copies of $\mathcal{E}_t$ in $\tilde{S}_{\text{aff}}(\mathbb{F}_q)$ precisely in the case that $F_q(z) = \sigma^{-1}(z)$, or equivalently $z^q = (z-1)/(z+1)$, for (any of the) z satisfying $z \mapsto u_t \mapsto t$. Written out, this means that the roots of $X^4 - tX^3 - 6X^2 + tX + 1 = 0$ in $\overline{\mathbb{F}_q}$ are required to satisfy $X^{q+1} + X^q - X + 1 = 0$. This proves Theorem 3.5. □

In Section 5.1 below, the following corollary will be important.

COROLLARY 3.6. *With notations as in Theorem 3.5, suppose that for each $t \in \mathbb{F}_q$ the two fibers $E_{r\pm st/4}$ are isomorphic over $\mathbb{F}_q$.*

Then $\operatorname{trace}(F_q|V_\ell) \in \mathbb{Z}$.

Proof. We have to show that the imaginary part of $\operatorname{trace}(F_q|V_\ell)$ is 0. By Theorem 3.5 this imaginary part is a sum over $t \in \mathbb{F}_q, t^2 + 16 \in \mathbb{F}_q^* \setminus \mathbb{F}_q^{*2}$. We will show that in this sum, the contributions from $\pm t$ cancel. By assumption $\#E_{r+st/4}(\mathbb{F}_q) = \#E_{r-st/4}(\mathbb{F}_q)$. Hence what remains to be proven is that $\epsilon(t) = -\epsilon(-t)$ for t as above.

Take any such t. The 4 values of z which are mapped to t can be written as $\{z, \sigma(z), \sigma^2(z), \sigma^3(z)\}$. Since $t^2 + 16 \in \mathbb{F}_q^* \setminus \mathbb{F}_q^{*2}$, none of these are equal, hence we have either $z^q = \sigma(z)$ or $z^q = \sigma^{-1}(z)$. By definition, $\epsilon(t) = -1$ in the former case and $\epsilon(t) = 1$ in the latter. Furthermore, note that if z is mapped to t, then $-z$ is mapped to $-t$, and $\sigma(z) = -\sigma^{-1}(-z)$. This shows that $z^q = \sigma(z)$ precisely when $(-z)^q = \sigma^{-1}(-z)$. In other words, $\epsilon(t) = -\epsilon(-t)$, as claimed. □

4. 3-dimensional examples

4.1. Throughout, the notations introduced in Section 2 will be used. We will now indicate a general construction of examples where the desired representation V_ℓ has dimension 3. Two explicit families of such examples will be given.

To this end, we will assume that the surface $\mathcal{E} \to \mathbb{P}^1$ is of the simplest possible kind: a rational elliptic surface. For such surfaces, it is known that $h^2(\mathcal{E}) = 10 =$ the rank of the Néron-Severi group; cf. [S, Proof of Lemma 10.1]. We will investigate under what conditions the degree 4 cyclic base change under $\pi_{r,s} : \mathbb{P}^1 \to \mathbb{P}^1$ will yield a 3-dimensional V_ℓ, or equivalently, a 6-dimensional W_ℓ.

PROPOSITION 4.2. *Suppose the representation space V_ℓ in Section 2 is constructed with as starting point a stable, rational elliptic surface $\mathcal{E} \to \mathbb{P}^1$. Let* $\operatorname{rank}(\mathcal{E})$ *denote the rank of the group of sections of $\mathcal{E} \to \mathbb{P}^1$. Then one has*

$$\dim\ V_\ell = 4 + \operatorname{rank}\mathcal{E}$$

in case $\mathcal{E}$ has smooth fibres over the ramification points of the cyclic map $\pi_{r,s}$, and

$$\dim\ V_\ell = 2 + \operatorname{rank}\mathcal{E}$$

in case the fibres over these ramification points are both singular of the same type.

Proof. Recall that the base changed surface is denoted S, and the intermediate surface under the degree 2 base change is called X. By definition of W_ℓ (cf. 2.4) one has:

$$\dim W_\ell = h^2(S) - h^2(X) - (\dim A_\ell(S) - \dim A_\ell(X))$$

with A_ℓ the subspace of H^2 spanned by all components of bad fibers.

Since $\mathcal{E}$ is rational and stable and $\pi_{r,s}$ is composed of two morphisms of degree 2, one can compute the dimensions $h^2(X)$ and $h^2(S)$ using general theory of (elliptic) surfaces. In fact, note that by [K, Sect. 12] our surfaces have $c_1^2 = 0$ and $c_2 = 12\chi(\mathcal{O}) = 12$ for $\mathcal{E}$ and $= 24$ for X and $= 48$ for S. Hence reasoning as in [vG-T, (3.1)] it follows that $h^2(X) = 22$ and $h^2(S) = 46$. In fact, the well known argument from loc. cit. shows that any elliptic surface with base $\mathbb{P}^1$ has $h^1 = 0$, hence $h^{0,2} = \chi(\mathcal{O}) - 1$. This implies $h^{0,2}(X) = 1$ (and X is a K3 surface) and $h^{0,2}(S) = 3$. So one concludes that the Hodge structure corresponding to V_ℓ has Hodge numbers $h^{0,2} = h^{2,0} = 1$ and $h^{1,1} = \dim V_\ell - 1$.

The space $A_\ell(\mathcal{E})$ has dimension $1 + n = 10 - 1 - \operatorname{rank}(\mathcal{E})$, with $n =$ the number of components of bad fibers which do not meet the zero section. We will distinguish two cases now, depending on the fibers of $\mathcal{E}$ over the ramification points $r \pm si$ of $\pi_{r,s}$.

In case $\mathcal{E} \to \mathbb{P}^1$ has smooth fibers over $r \pm si$, each of the bad fibers of $\mathcal{E} \to \mathbb{P}^1$ occurs twice in the double cover X and 4 times in the degree 4 cover S. This gives all the bad fibers of X and S, hence

$$\dim\ A_\ell(X) = 1 + 2n \quad \text{and} \quad \dim\ A_\ell(S) = 1 + 4n.$$

The conclusion in this case is that

$$\begin{aligned} \dim W_\ell &= 46 - 22 - 2n \\ &= 24 - 16 + 2\operatorname{rank}(\mathcal{E}) \\ &= 8 + 2\operatorname{rank}(\mathcal{E}). \end{aligned}$$

The second case to consider is that the fibers of $\mathcal{E}$ over $r \pm si$ are of type I_ν for some $\nu \geqslant 1$. Then $\dim\ A_\ell(\mathcal{E}) = 1 + 2(\nu - 1) + m = 9 - \operatorname{rank}(\mathcal{E})$, where $m =$ the

number of components of bad fibers different from the ones over $r \pm si$ which do not meet the zero section. In X we obtain instead of the two I_ν-fibers two $I_{2\nu}$-ones, and similarly in S two $I_{4\nu}$-fibers. It follows that

$$\dim A_\ell(X) = 1 + 2(2\nu - 1) + 2m \quad \text{and} \quad \dim A_\ell(S) = 1 + 2(4\nu - 1) + 4m,$$

hence

$$\begin{aligned} \dim W_\ell &= 46 - 22 - (4\nu + 2m) \\ &= 24 - 2(10 - \operatorname{rank}(\mathcal{E})) \\ &= 4 + 2\operatorname{rank}(\mathcal{E}). \end{aligned}$$

This proves Proposition 4.2. □

4.3. The conclusion from the proposition above is, that to obtain representations V_ℓ which are 3-dimensional, one may consider rank 1 stable, rational elliptic surfaces having two bad fibers of the same type, and then base change them via a π which is ramified over the points corresponding to these fibers. In order to find rational stable elliptic surfaces of rank 1, the table in [O-S] is useful. It lists the possible configurations of bad fibers, although there is no guarantee that any surface with a given configuration really exists. By restricting ourselves to the cases where the surfaces have a non-trivial group of torsion sections as well (which makes it easier to write down Weierstrass equations), the following two examples were found.

As the referee of this paper pointed out to us, we could have saved ourselves some work here: Persson's list [P] not only gives all possible configurations of bad fibers of rational elliptic surfaces, but also provides realizations of the surfaces. The ones we describe below in Examples 4.4 and 4.7 are denoted $LE_2(8;0,0,0)$ and $Q(2;0,0,0)$ in *loc. cit.* It may be interesting to study the other examples from Persson's list of rank 1 rational stable elliptic surfaces with two singular fibers of the same type as well.

EXAMPLE 4.4. Consider $\mathcal{E}$, corresponding to the Weierstrass equation

$$y^2 = x(x^2 + 2(a + bt + ct^2)x + 1).$$

This defines a rational elliptic surface, as follows e.g. from [S, (10.14)]. Note that although three parameters appear here, geometrically this family depends on only one: using affine transformations $t := \lambda t + \mu$ and rescaling x, y one can, for $c \neq 0$, transform the equation into $Y^2 = X(X^2 + d(t^2+1)X + 1)$. The fiber over infinity of $\mathcal{E}$ can be determined by changing coordinates to $u = 1/t, \eta = y/t^3, \xi = x/t^2$. This yields the new equation

$$\eta^2 = \xi(\xi^2 + 2(c + bu + au^2)\xi + u^4).$$

To obtain a multiplicative fiber over $t = \infty$ i.e., over $u = 0$, one needs to assume

$$c \neq 0.$$

In that case, the fiber over $t = \infty$ is of type I_8. The other bad fibers occur over points where two zeroes of the cubic polynomial in x coincide. This happens when

$$(a + bt + ct^2)^2 - 1 = (a - 1 + bt + ct^2)(a + 1 + bt + ct^2) = 0.$$

The type of bad fiber depends on the multiplicity of the roots of this equation. Since we want to obtain a stable elliptic surface of rank 1, we need that all bad fibers over finite t are of type I_1. This means that all 4 roots of the above equation must be simple, i.e.

$$(b^2 - 4c(a - 1))(b^2 - 4c(a + 1)) \neq 0.$$

Under the above conditions, we indeed find a surface as required. A section of infinite order is given by

$$t \mapsto \left(x = \alpha, y = \frac{\alpha}{\sqrt{2c}}(b + 2ct)\right)$$

in which α satisfies $\alpha^2 + 2a\alpha + 1 = \frac{b^2}{2c}\alpha$.

If one takes

$$b = -2r(a + 1)/(r^2 + s^2) \quad \text{and} \quad c = (a + 1)/(r^2 + s^2),$$

one obtains a surface which will be denoted $\mathcal{E}_{a,r,s}$ which has the property that two of the I_1-fibers are over $r \pm si$.

REMARK 4.5. The above example is in fact unique in the following sense. Any stable, rational elliptic surface with a section of order 2 and with rank 1 has according to [O-S] one I_8 and four I_1 fibers. If the surface is given by a Weierstrass equation, one may assume without loss of generality that the section of order 2 corresponds to $x = 0, y = 0$ and that the I_8 fiber is located over $t = \infty$. Hence after rescaling x, y the example above is obtained.

4.6. We will now give a more geometric approach to the current example which however will not be used in the remainder of the paper. Recall from 4.4 that $\mathcal{E}$ may be defined by:

$$\mathcal{E} = \mathcal{E}_d : \quad t^2 = z(z^2 + d(y^2 + 1)z + 1),$$

where our t is the old Y, y is the old t and z is the old X. We now substitute $y := y/\sqrt{-d}$.

Each $\mathcal{E}_d$ can be viewed as a double cover of $\mathbb{P}^2$ (with coordinates $(x : y : z)$) branched over the quartic curve (note the degree of the branch curve must be even) defined by F:

$$F : z(-y^2z + x^3 + dx^2z + xz^2), \quad Z(F) = E \cup l_\infty.$$

The curve $F = 0$ thus consists of an elliptic curve E (with affine equation $y^2 = x(x^2 + dx + 1)$) and the line at infinity $z = 0$. The pencil of elliptic curves is given by the inverse images of the lines $y = \lambda x$. It has I_1 fibers over the 4 lines tangent to E (and thus the fiber over the line $x = 0$ must be an I_8 fiber), the points of tangency are thus points of order 4 on E.

The section of the family $\mathcal{E}$ over the t-line is given by putting $x = \alpha$ for a suitable number α. This corresponds to the section

$$\lambda \mapsto (x, y, s) = (\beta, \beta\lambda, i\beta\lambda), \quad \text{with} \quad \beta^2 + d\beta + 1 = 0 \quad \text{and}$$
$$\mathcal{E} : \ s^2 = -y^2 + x(x^2 + dx + 1).$$

(Note that the lines $x = \beta$ intersect the curve $F = 0$ in two points, each with multiplicity 2, thus the inverse image of such a line is reducible. Moreover, $x = \beta$ intersects $y = \lambda x$ in one point, and thus each component of the inverse image of $x = \beta$ is (the image of) a section.)

Let $\Sigma_1 = B\mathbb{P}^2$ be the blow-up of $\mathbb{P}^2$ in the base point $(0 : 0 : 1)$ of the pencil, it has a morphism $B\mathbb{P}^2 \to \mathbb{P}^1$ whose fibers are the lines in the pencil. We have a 2:1 map $\mathcal{E} \to B\mathbb{P}^2$ of fibrations over $\mathbb{P}^1$. The 4:1 cyclic cover S of $\mathcal{E}$ branched along two of the I_1 fibers is a double cover of the ruled surface Σ_4, obtained as the pull-back of $\Sigma_1 \to \mathbb{P}^1$ along the cyclic 4:1 map $\mathbb{P}^1 \to \mathbb{P}^1$. (We write Σ_n for the Hirzebruch surface of degree n ([BPV], V.5).) Similarly, the $2:1$ intermediate cover X (a K3 surface) is a double cover of Σ_2.

$$\begin{array}{ccccc}
S & \longrightarrow & X & \longrightarrow & \mathcal{E} \\
\downarrow & & \downarrow & & \downarrow \\
\Sigma_4 & \longrightarrow & \Sigma_2 & \longrightarrow & B\mathbb{P}^2 \\
\cup & & \cup & & \cup \\
C & \longrightarrow & E' & \longrightarrow & E
\end{array}$$

We consider the branch-loci of the 'vertical' 2:1 maps (the surfaces Σ_i actually have to be blown-up in the singular points of the branch loci to obtain the minimal models of the coverings). The branch locus of $\mathcal{E} \to B\mathbb{P}^2$ is the union of three divisors: the strict transform of the elliptic curve E (again denoted by E), the inverse image of $l_\infty \subset \mathbb{P}^2$ and the exceptional divisor $C_1 \subset B\mathbb{P}^1$. A fiber of $B\mathbb{P}^2 \to \mathbb{P}^1$ meets this branch locus in 4 points, 2 are on E, and the other two are on l_∞ and C_1.

The branch locus of $X \to \Sigma_2$ is the union of the inverse image E' of E (E' has two nodes and its normalization $\tilde{E}'$ is an elliptic curve, 2-isogenous to E) and two rational curves (one lying over l_∞ and the other is $C_2 \subset \Sigma_2$ lying over $C_1 \subset B\mathbb{P}^2$). The inverse image of E' in Σ_4 is a curve C with arithmetic genus 7 and geometric genus 3. Its normalization is a double cover of $\tilde{E}'$ branching over the 4 points of $\tilde{E}'$ over 2 points of order 4 on E in the branch locus of $\Sigma_2 \to B\mathbb{P}^2$. The branch locus $S \to \Sigma_4$ is the union of C and two rational curves, one lying 4:1 over $l_\infty \subset B\mathbb{P}^2$, the other is $C_4 \subset \Sigma_4$ lying over $C_2 \subset \Sigma_2$.

Equations for the curves E' and C may be obtained as follows. We can choose the pair of tangent lines to E as being defined by $y^2 = \lambda^2 x$ with $\lambda^2 = d + 2$. Then (the function field of) E' is defined by $u^2 = (y - \lambda x)/(y + \lambda x)$ (so $y = \lambda x(1 + u^2)/(1 - u^2)$) and C is defined by $v^4 = (y - \lambda x)/(y + \lambda x)$ (so $y = \lambda x(1 + v^4)/(1 - v^4)$).

EXAMPLE 4.7. Other examples of such elliptic surfaces are obtained as follows. Consider the minimal smooth surface corresponding to

$$y^2 = x^3 + a(x + t^2(t - 1))^2.$$

This only works of course when $a \neq 0$. In that case, the fiber over infinity is given by $y^2 = x^3 + a$. The minimal surface has an I_6-fiber at $t = 0$, an I_3 at $t = 1$ and three I_1-fibers over the zeroes of $27t^3 - 27t^2 - 4a$. Note that this polynomial only has three different zeroes when we demand $a(a + 1) \neq 0$, which we do from now on. A section of infinite order is then provided by

$$t \mapsto \left(x = t^2, y = t^3\sqrt{1 + a}\right).$$

To obtain two of the I_1-fibers over $r \pm si$, one may start with a (new) parameter q and take

$$s = 2q/(3q^2 - 1), \quad r = sq, \quad a = -27(r^2 + s^2)(2r - 1)/4.$$

(These formulas are found by first writing the quotient r/s as q, and then demanding that the monic polynomial in t having $r \pm si = s(q \pm i)$ as zeroes, is a divisor of $27t^3 - 27t^2 - 4a$.) The resulting surface will be denoted $\mathcal{E}_q$.

4.8. We also give a geometrical approach to the elliptic surfaces defined by the equation $y^2 = x^3 + a(x + t^2(t - 1))^2$. This equation exhibits the surface as a 2:1 cover of the x-t plane branched over a sextic curve which is singular in $(0, 0)$. Blowing up ($x := tx$), taking the strict transform and homogenizing gives a quartic curve in $\mathbb{P}^2$ (coordinates $(x : t : z)$) defined by

$$F := tx^3 + a(xz + t(t - z))^2.$$

Let $P = (0 : 0 : 1)$, $Q = (0 : 1 : 1)$, then $\mathrm{Sing}(F = 0) = \{P, Q\}$. The elliptic surface is defined by the (affine) equation:

$$\mathcal{E} : \ y^2 = tx^3 + a(x + t(t - 1))^2.$$

The elliptic pencil is given by the inverse images of the lines $t = \lambda$. The base point of the pencil is $(1 : 0 : 0)$ which lies on $F = 0$. The line $t = x$ intersects $F = 0$ in P with multiplicity 4 and thus defines two sections of $\mathcal{E}$ (there are also two sections over $x = 0$). The I_1-fibers correspond to the lines $t = \lambda$ which are tangent to $F = 0$.

The curve $F = 0$ is in fact rational (project from P for example). A nice parametrization is obtained as follows. Consider the birational isomorphism of $\mathbb{P}^2$:

$$\phi : \mathbb{P}^2 \to \mathbb{P}^2, \quad (x : t : z) \mapsto (u : v : w) := (x^2 : xt : t(t-z) + xz),$$

with birational inverse:

$$\psi : \mathbb{P}^2 \to \mathbb{P}^2, \quad (u : v : w) \mapsto (x : t : z) := (u(u-v) : v(u-v) : wu - v^2).$$

Note that $\phi^*(uv + aw^2) = F$. A parametrization of $F = 0$ is thus:

$$\begin{aligned}(p : q) \mapsto (u : v : w) &:= (p^2 : -aq^2 : pq) \mapsto (x : t : z) \\ &= (p^2(aq^2 + p^2) : -aq^2(aq^2 + p^2) : q(p^3 - a^2q^3)).\end{aligned}$$

In this parametrization, the two points with $p^2 + aq^2 = 0$ map to P and the point with $p = 0$ maps to Q. The point with $q = 0$ maps to $(1 : 0 : 0)$, the base point of the pencil.

5. Numerical examples

We will now use members of both families constructed in Examples 4.4 and 4.7 to compute traces.

5.1. In the case of Example 4.4, it is actually slightly more convenient to view it as elliptic surface over the u-line. In other words, writing t again for the parameter on the base as is done throughout this paper, we consider

$$y^2 = x(x^2 + 2(c + bt + at^2)x + t^4).$$

This does not change the conditions on a, b, c derived in 4.4. To obtain I_1-fibers over $t = r \pm si$ however, we now may put

$$b = -2r(a+1) \quad \text{and} \quad c = (r^2 + s^2)(a+1).$$

For several choices of $a, r, s \in \mathbb{Z}$ we list trace$(F_p|V_\ell)$ for small primes p in the following table. An entry $(*)$ indicates that p is a bad prime; i.e., for that choice of a, r, s our construction fails in characteristic p. This happens when $p = 2$, when the two ramification points $r \pm si$ coincide mod p, and when the two conditions on a, b, c derived in 4.4 are not satisfied mod p.

	(a,r,s):	$(0,0,1)$	$(2,0,1)$	$(2,3,4)$	$(-2,0,1)$	$(-3,0,1)$	$(1,1,1)$
	bad primes:	2	2, 3	2, 3, 5	2, 3	2	2
$p=3$		-1	$(*)$	$(*)$	$(*)$	-1	i
$p=5$		-3	3	$(*)$	1	3	$-3i$
$p=7$		-3	-11	-9	5	9	3
$p=11$		-9	-5	-11	21	-9	$-9i$
$p=13$		-11	5	-23	5	-5	$11i$
$p=17$		-13	19	13	9	-1	-13
$p=19$		-1	15	17	3	31	i
$p=23$		13	15	-13	-23	-7	-13
$p=29$		-11	3	35	-71	43	$11i$
$p=31$		33	13	33	13	33	33
$p=37$		-19	-19	-63	13	-29	$-19i$
$p=41$		-41	59	-5	33	-37	41
$p=43$		7	7	27	27	-25	$7i$
$p=47$		17	39	43	97	-47	17

The most general result we can prove about this (a,r,s)-family is the following.

PROPOSITION 5.2. *For integers a, r, s with $a \neq \pm 1$ and $s \neq 0$ and $r = 0$, the corresponding $G_{\mathbb{Q}}$-representation on V_ℓ is (possibly upto semi-simplification) selfdual.*

Proof. In case $r = 0$, the conditions $a \neq \pm 1$ and $s \neq 0$ guarantee that the surface one obtains indeed gives a 3-dimensional V_ℓ. Furthermore, the construction in this case starts from an equation of the form

$$y^2 = x(x^2 + 2(at^2 + c)x + t^4).$$

Thus the fibers over any t_0 and $-t_0$ are the same. Using Corollary 3.6 one concludes that any Frobenius element F_p at a good prime p has a characteristic polynomial on V_ℓ with coefficients in $\mathbb{Z}$. Since the dual representation in general gives complex conjugate eigenvalue polynomials, and since these continuous representations are determined by the images of sufficiently many Frobenius elements, this proves the proposition. □

REMARK 5.3. Our construction in fact gives rise to another 3-dimensional $G_{\mathbb{Q}}$-representation which is selfdual. Namely, the intermediate surface X is a K3 surface which has a 19-dimensional $G_{\mathbb{Q}}$-invariant subspace in $H^2(X)$ generated by components of fibers and the pull backs of the sections of $\mathcal{E}$. The 3-dimensional orthogonal complement U is a $G_{\mathbb{Q}}$-representation which is selfdual in view of the intersection form on it. (In fact, the traces of Frobenius acting on U_ℓ are in $\mathbb{Z}$, hence the argument in the proof above also shows that this representation is (possibly upto semi-simplification) selfdual.) Morrison [M] has even shown that such a K3

surface is isogenous to the Kummer surface of an abelian surface A. It turns out that A is either isogenous to a product $E \times E$ for an elliptic curve E, or A is (absolutely) simple and the endomorphism ring of A is an order in an indefinite quaternion algebra. Namely, the transcendental lattice of A has rank at most 3. Hence the Néron-Severi group of A has rank at least 3. This group generates the symmetric part of the endomorphism algebra of A (by [Mum, IV, Sect. 20]). Using the possible types of this algebra as given e.g. in [O, Prop. 6.1] and the possibilities for the rank of the symmetric part as listed in the table [Mum, IV, p. 202] one finds that A is of one of the two types described above. In case $A \sim E \times E$ the conclusion is that over a finite extension K of $\mathbb{Q}$, U_ℓ is isomorphic to $\mathrm{Sym}^2(H^1(E))$ as G_K-representation. On the other hand, if A is simple then the endomorphisms on A will split $H^1(A)$ into nontrivial subspaces, and one may expect that over an extension K, the representation U_ℓ is a symmetric square of a 2-dimensional subrepresentation of $H^1(A)$.

By examining some explicit examples we find that in general the representations U_ℓ seem unrelated to our V_ℓ's.

5.4. Based on the traces given in the above table (in fact a much longer table giving such traces for many more primes was used), it seems that all the examples given above exhibit a common pattern. Moreover, we did not find a single example in this family without that pattern. To explain it, start with a 2-dimensional $G_{\mathbb{Q}}$-representation T_ℓ, with the property that the characteristic polynomial of any Frobenius element at a 'good' prime p is of the form $X^2 - a_pX + p$. One obtains 3-dimensional representations from this by taking the tensor product of $\mathrm{Sym}^2(T_\ell)$ with an arbitrary Dirichlet character δ^{-1}. Then

$$\mathrm{trace}(F_p|\delta^{-1} \otimes \mathrm{Sym}^2(T_\ell)) = \delta(p)^{-1}(a_p^2 - p),$$

in other words,

$$p + \delta(p)\mathrm{trace}(F_p|\delta^{-1} \otimes \mathrm{Sym}^2(T_\ell)) = a_p^2.$$

It appears that the examples V_ℓ in the present family are all of this kind. In other words, if one multiplies the traces in the table above by values of a suitable Dirichlet character δ and then adds p to the result, one obtains numbers whose square roots all seem to be in a fixed (usually quadratic) extension of $\mathbb{Q}$. Moreover these square roots are upto sign in most cases recognizable as traces of a 2-dimensional representation. We will illustrate this by considering the above numerical examples one by one.

5.5. For $(a, r, s) = (2, 3, 4)$ one takes $\delta(p) = \left(\frac{p}{5}\right)$. Then for all primes p with $7 \leqslant p \leqslant 97$ it turns out that

$$p + \delta(p)\mathrm{trace}(F_p|V_\ell) = a_p^2,$$

in which a_p is the trace of F_p acting on the Tate module of the elliptic curve given by $y^2 = x(x^2 + 10x + 10)$.

5.6. Next, consider $(a, r, s) = (0, 0, 1)$. With δ the trivial character, one finds

$$p + \delta(p)\text{trace}(F_p|V_\ell) = p + \text{trace}(F_p|V_\ell)$$
$$= \begin{cases} \text{a square if } p \equiv \pm 1 \bmod 8; \\ \text{a square times 2 if } p \equiv \pm 3 \bmod 8, \end{cases}$$

at least when $3 \leqslant p \leqslant 97$. A 2-dimensional representation having as traces of F_p integers when $p \equiv \pm 1 \bmod 8$ and integers times $\sqrt{2}$ when $p \equiv \pm 3 \bmod 8$ can be obtained as follows. Suppose E over $\mathbb{Q}(\sqrt{2})$ is an elliptic curve which is 2-isogenous to its Galois conjugate. The restriction of scalars of this curve defines an abelian surface A over $\mathbb{Q}$, and the 2-isogeny gives a multiplication by $\sqrt{2}$ on A defined over $\mathbb{Q}$. Taking an eigenspace for this multiplication inside $T_\ell A \otimes \overline{\mathbb{Q}_\ell}$ then defines a 2-dimensional representation as required. In the present example the elliptic curve defined by

$$y^2 = x(x^2 - (4 + 4\sqrt{2})x + 2 + \sqrt{2})$$

is 2-isogenous to its conjugate, and seems at least for $3 \leqslant p \leqslant 97$ to correspond to V_ℓ in the way just described.

Similarly, for $(a, r, s) = (-3, 0, 1)$ and δ the trivial character one finds at least for $3 \leqslant p \leqslant 97$ such a relation with the elliptic curve given by

$$y^2 = x(x^2 - (4 + 4\sqrt{2})x - (2 + 2\sqrt{2})).$$

Both elliptic curves given here were found using the table in [T. p. 80].

5.7. The case $(a, r, s) = (1, 1, 1)$ seems related to $(a, r, s) = (0, 0, 1)$. The traces at all small primes differ by a quadratic character modulo 16, namely the character χ defined by $\chi(-1) = 1$ and $\chi(5) = i$.

5.8. If $(a, r, s) = (-2, 0, 1)$, put $\delta(p) = \left(\frac{3}{p}\right)$. For $5 \leqslant p \leqslant 97$ one finds that

$$p + \delta(p)\,\text{trace}(F_p|V_\ell) = \begin{cases} \text{a square if } p \equiv 1, 5, 19, 23 \bmod 24; \\ \text{a square times 2 if } p \equiv 7, 11, 13, 17 \bmod 24. \end{cases}$$

Hence here we may look for a relation with an elliptic curve over $\mathbb{Q}(\sqrt{6})$ which is 2-isogenous to its conjugate.

5.9. Finally, when $(a, r, s) = (2, 0, 1)$ one may take $\delta(p) = \left(\frac{-1}{p}\right)$. Then

$$p + \delta(p)\,\text{trace}(F_p|V_\ell) = \begin{cases} \text{a square if } p \equiv 1, 3 \bmod 8; \\ \text{a square times 2 if } p \equiv 5, 7 \bmod 8 \end{cases}$$

at least for all small good primes. So in this case we expect a relation with an elliptic curve over $\mathbb{Q}(\sqrt{-2})$ which is 2-isogenous to its conjugate.

REMARK 5.10. In case the Galois representation on V_ℓ is indeed selfdual then the isomorphism of Galois representations $\phi : V_\ell \xrightarrow{\cong} V_\ell^{\mathrm{dual}}(-2)$ defines an 'extra' Tate-class $[\phi] \in H^4((S \times S)_{\overline{\mathbb{Q}}}, \overline{\mathbb{Q}}_\ell)$. According to the Tate conjecture, there would have to be an algebraic cycle $Z \subset S \times S$, defined over $\mathbb{Q}$, whose cohomology class is $[\phi]$. It would be interesting to find such a Z using the geometry of the elliptic surface S.

5.11. We will now consider the family described in Example 4.7. It depends on a parameter q, and for an integer value of q all bad primes p satisfy $p|6q$ or $p|3q^2-1$ or $p|q^2+1$ or $p|9q^2+1$. Some numerical examples are given in the following table.

	q :	1	2	3
	bad primes:	$2,3,5$	$2,3,5,11,37$	$2,3,5,13,41$
$p=7$		$-3+2i$	$2-3i$	$6-5i$
$p=11$		$-10-5i$	$(*)$	$-4+3i$
$p=13$		$1-2i$	$-4+11i$	$(*)$
$p=17$		$-4-13i$	$-4-i$	$-8+5i$
$p=19$		$-10+5i$	$27-8i$	$-16+31i$
$p=23$		$-3-10i$	$30-19i$	$-22+11i$
$p=29$		$20-19i$	$10+17i$	$-17-2i$
$p=31$		$13-12i$	$3-46i$	$1-24i$
$p=37$		$-23-10i$	$(*)$	$-4-31i$
$p=41$		$29+48i$	$15-18i$	$(*)$
$p=43$		$16-31i$	$58-43i$	$-17-8i$
$p=47$		$-49-2i$	$-14-49i$	$-18-7i$

Since the above traces do not all differ multiplicatively from their complex conjugates by roots of unity, these representations are non-selfdual in the strong sense alluded to in the Introduction of this paper; i.e., even after multiplication by a Dirichlet character they are not isomorphic to a Tate twist of their contragredient.

REMARK 5.12. For the above examples, we have not computed the determinant of the representations involved. This determinant equals the cube of the cyclotomic character times a Dirichlet character with values in $\{\pm 1, \pm i\}$. Taking $p = 17$, it turns out that for $q = 1$ the characteristic polynomial of Frobenius equals $X^3 - (-4+13i)X^2+17(-13+30i)X+17^3 i$, resp. $X^3-(-4-i)X^2-17(1+4i)X+17^3 i$ in case $q = 2$. Hence the Dirichlet character appearing in these two cases is in fact a character of exact order 4.

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