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# F-crystals on schemes with constant log structure

*Dedicated to Frans Oort on the occasion of his 60th birthday*

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**Abstract.** Let $f: X \to Z$ be a log smooth and proper morphism of log schemes in characteristic $p$ such that the monoid $M_Z/\mathcal{O}_Z^*$ is constant. We prove that if $E$ is a locally free crystal on $X$ with a (weak) Frobenius structure, then for every affine PD thickening $T$ of $Z$, the crystalline cohomology $R\Gamma(X/T) \otimes \mathbf{Q}$ depends only on the monoid $M_T$, and not on the map $M_T \to \mathcal{O}_T$. This result generalizes the theorem of Hyodo and Kato comparing De Rham and crystalline cohomologies in the context of semi-stable reduction, as well as Christol's "transfer theorem" on $p$-adic differential equations with regular singularities.

## 1. Introduction

Let $V$ be a discrete valuation ring of mixed characteristic $p$, with fraction field $K$ and perfect residue field $k$; let $\overline{K}$ be an algebraic closure of $K$, $W(k)$ be the Witt ring of $k$, and $K_0$ the fraction field of $W(k)$. If $Y/K$ is a smooth and proper $K$-scheme, an important source of information about $Y/K$ is the arithmetic structure on the cohomology of $\overline{Y}/\overline{K}$. For example, its $\ell$-adic étale cohomology admits a continuous action of $\mathrm{Gal}(\overline{K}/K)$. When $\ell$ is not $p$, the inertial part of this action is quasi-unipotent, and the $\ell$-adically continuous operation of $\mathrm{Gal}(\overline{K}/K)$ amounts to an algebraic representation of the so-called Weil-Deligne group [4]. If $\ell = p$, the action seems to be so complex that it is best studied by crystalline methods, via the "mysterious functor" and what Illusie has called the "hidden structure" on the De Rham cohomology of $\overline{Y}/\overline{K}$.

When $Y/K$ has good reduction, this hidden structure on De Rham cohomology comes from the crystalline cohomology of the special fiber, which provides us with a $K_0$-form $H_{\text{cris}}$ of $H_{DR}$ on which there is a Frobenius-linear automorphism $\Phi$. These data can be conveniently expressed in terms of an action of the so-called crystalline Weil group $W_{\text{cris}}(\overline{K})$, i.e. the group of all automorphisms of $\overline{K}$ which act as some integral power of the Frobenius endomorphism of $\overline{k}$ [2, 4.1]. In the general case, Jannsen has conjectured the existence of a "$\Phi$ and $N$" acting on a canonical form $H_{ss}$ of the De Rham cohomology over the maximal unramified extension of $K(k)$. Strictly speaking, $H_{ss}$, $\Phi$, and $N$ are not canonical, and depend on an additional choice of a uniformizer and a valuation of $V$, but again the entire package (along with the implicit dependencies) can be naturally described by means of an action of a crystalline analog of the Weil-Deligne group. This can be though of as the semidirect product:

$$0 \to \overline{K}(1) \to \mathbf{W}_{\text{cris}}(\overline{K}) \to W_{\text{cris}}(\overline{K}) \to 0,$$

where the inner action of $\mathbf{W}_{\text{cris}}$ is via the usual action of $W_{\text{cris}}(\overline{K})$ on $K(1)$. (Note: $\overline{K}(1)$ is just $\overline{K}$ with a twisted action of $W_{\text{cris}}(\overline{K})$: if $c \in \overline{K}(1)$ and $\psi \in W_{\text{cris}}(\overline{K})$, this twisted action $\rho_{K(1)}$ is given in terms of the usual action by $\rho_{K(1)}(\psi)(c) = p^{-\deg\psi}\psi(c)$. )

CONJECTURE 1 (Fontaine-Jannsen). Suppose $Y/K$ is smooth and proper. Then each $H^q_{DR}(\overline{Y}/\overline{K})$ admits a canonical continuous semilinear action of $\mathbf{W}_{\text{cris}}(\overline{K})$, and this action, together with the Hodge filtration on $H^q_{DR}(Y/K)$, determines the action of $\mathrm{Gal}(\overline{K}/K)$ on $H^q_{\acute{e}t}(\overline{Y}, \mathbf{Q}_p)$.

We should point out that Grothendieck's proof of the local monodromy theorem applies here too: the restriction of any semilinear continuous action of $\mathbf{W}_{\text{cris}}$ on a finite dimensional vector space to the subgroup $K(1)$ is necessarily algebraic, i.e. given by $\lambda \mapsto e^{\lambda N}$ for some nilpotent operator $N$.

The Fontaine-Jannsen conjecture has been partially proven by Hyodo-Kato [8] (which establishes the existence of the "hidden structure"), Kato [11] (which sketches the comparison theorem), and Faltings [6] (who discusses cohomology of a curve with twisted coefficients), using the technique of logarithmic crystalline cohomology. The most difficult part of [8] is a comparison between crystalline cohomology of the special fiber of a semi-stable scheme over Spec $V$ endowed with two different log structures. In this paper we attempt to elucidate this comparison theorem by providing a new proof, using the point of view of F-crystals. In fact, our method generalizes the result in [8] to the case of coefficients in an F-crystal, or, even more generally, to an $F^\infty$-span (15). We show that the result is, at least philosophically, a consequence of Christol's transfer theorem in the theory of $p$-adic differential equations with regular singular points [3]. Our approach can perhaps be viewed as a crystalline analog of Schmidt's nilpotent orbit theorem for abstract variations of Hodge structures over the punctured unit disc.

Here is a slightly more detailed summary of the manuscript. In the first section we investigate the general properties of schemes $X$ endowed with a "constant" log structure $\alpha_X\colon M_X \to \mathcal{O}_X$, i.e. a log structure such that the associated sheaf of monoids $M_X/\mathcal{O}_X^*$ is locally constant. If $X$ is reduced, we see that this is the case if and only if for every $x \in X$ and every nonunital local section $m$ of $M_{X,x}$, $\alpha_X(m) = 0$. We call log structures with this (stronger) property "hollow." The justification for this terminology is that, if $t$ is a section of $\mathcal{O}_X$, then the log structure obtained by adjoining a formal logarithm of $t$ tells us to regard $X$ as a partial compactification of the complement of the zero set of $t$. Thus, adjoining the logarithm of zero removes all the points of $X$, rendering it "hollow." We also discuss the Frobenius morphism for log schemes. In particular, if $X \to \underline{X}$ is the canonical mapping from a log scheme $X$ to the same scheme with the trivial log structure, we show (Lemma 10) that there is a map $X \to X^{(p)}$ which behaves like a relative Frobenius map $F_{X/\underline{X}}$, even if the characteristic is zero. This construction is the key geometric underpinning of our proof of the main comparison theorem (Theorem 1) in the next section. The third section rephrases the main comparison theorem in the language of convergent crystals (Theorem 4) and explains the relationship between the Hyodo-Kato isomorphism and Christol's theorem. In particular, we show (Proposition 33) that the Hyodo-Kato isomorphism is uniquely determined by the logarithmic connection associated to a logarithmic degeneration, and we give an "explicit formula" (Claim 35) for it. The last section is devoted to a logarithmic construction of our crystalline analog of the Weil-Deligne group.

At this point I would like to acknowledge the influence of Luc Illusie on this work. He was of course one of the original creators of the notion of logarithmic structures used here, which he patiently explained to me during the spring of 1991. It was he who turned my attention to the difficult points of [8] and it was he who suggested that I try to find an alternative approach. I also benefitted from discussions with W. Messing, M. Gros, and W. Bauer, and I particularly thank P. Berthelot for a careful criticism of a preliminary version of this manuscript. The National Security Association provided partial, but generous, support of the research summarized in this article.

## 2. Constant and hollow log structures

We begin with a review of some terminology and notation. If $M$ is a monoid, we let $\lambda_M: M \to M^g$ denote the universal map from $M$ into a group. We say $M$ is "integral" if $\lambda_M$ is injective, and from now on all the monoids we consider will be assumed to have this property. We let $M^*$ denote the set of invertible elements of $M$, which forms a submonoid (in fact a subgroup) of $M$. Unless explicitly stated otherwise, our monoids will also be commutative, and in this case the group $M^*$ acts on $M$ and the orbit space $\overline{M}$ has a natural structure of a monoid; furthermore $\overline{M}^* = 0$. It is immediate to check that the natural map $\overline{M}^g \to M^g/M^*$ is bijective, so that we have a commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M^* & \longrightarrow & M & \longrightarrow & \overline{M} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle \lambda_M} & & \downarrow{\scriptstyle \lambda_{\overline{M}}} & & \\
0 & \longrightarrow & M^* & \longrightarrow & M^g & \longrightarrow & \overline{M}^g & \longrightarrow & 0
\end{array}$$

The bottom row of this diagram is an exact sequence of abelian groups.

In general, we say that a pair $(\iota, \pi)$ of monoid morphisms forms a short exact sequence of monoids if $\iota$ is an injective morphism from an abelian group $G$ into a monoid $M$ and $\pi$ induces an isomorphism between the orbit space $M/G$ and the target $P$ of $\pi$. Thus, the top row of the diagram above is an exact sequence of monoids, and we denote it by $\Xi_M$. We also say that $(\iota, \pi)$ forms an "extension of $P$ by $G$;" these form a category in the usual way which we denote by $\mathrm{EXT}^1(P, G)$.

A monoid $M$ is said to be "saturated" if any element $y$ of $M^g$ such that $ny \in M$ for some $n \in$ bf N already lies in $M$. If $M$ is saturated so is $\overline{M}$. Furthermore, $M$ is said to be "fine" if it is finitely generated. Then if $M$ is fine and saturated, so is $\overline{M}$, and furthermore $\overline{M}^g$ is a finitely generated free abelian group.

LEMMA 2. *Suppose that $P$ is a monoid and $G$ is an abelian group. Then the category* $\mathrm{EXT}^1(P, G)$ *is a groupoid, and is naturally equivalent to the groupoid* $\mathrm{EXT}^1(P^g, G)$. *In particular, the automorphism group of any element of this category is naturally isomorphic to* $\mathrm{Hom}(P^g, G)$. *If $P$ is fine and saturated and $P^* = 0$, every object of* $\mathrm{EXT}^1(P, G)$ *is split.*

*Proof.* It is easy to see that every morphism in $\mathrm{EXT}^1(P, G)$ is an isomorphism. We claim that the functor from $\mathrm{EXT}^1(P, G)$ to $\mathrm{EXT}^1(P^g, G)$ which takes $M$ to $M^g$ is an equivalence of categories. The main point is that, when $M$ is integral, the right hand square of the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & G & \longrightarrow & M & \longrightarrow & P & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle \lambda_M} & & \downarrow{\scriptstyle \lambda_P} & & \\ 0 & \longrightarrow & G & \longrightarrow & M^g & \longrightarrow & P^g & \longrightarrow & 0 \end{array}$$

is Cartesian (i.e. the map $M \to P$ is "exact" in the terminology of [10]). Thus the top sequence is the pullback of the bottom, and this gives an inverse to our functor. Let us also make explicit the isomorphism $\mathrm{Hom}(P^g, G) \to \mathrm{Aut}(\Xi_M)$. If $\theta \in \mathrm{Hom}(P^g, G) \cong \mathrm{Hom}(P, G)$, define $\tilde{\theta}\colon M \to M$ by the formula

$$\tilde{\theta}(m) =: m + \iota(\theta(\pi(\overline{m}))). \tag{1}$$

Finally, if $P$ is fine and $P^* = 0$, then $P^g = \overline{P}^g$ is a finitely generated free abelian group, so the sequence $0 \to G \to M \to P^g \to 0$ splits.

A subset $I$ of $M$ such that $a + b \in I$ whenever $a$ or $b \in I$ is called an "ideal of $M$"; the set $M^+$ of nonunits is the unique maximal ideal of $M$. An ideal $I$ is "prime" if $a + b \in I$ implies that $a$ or $b \in I$. A morphism $\beta\colon P \to M$ is said to be "local" if $\beta(P^+) \subseteq M^+$.

If $S \subseteq M$ is a submonoid, there is a universal map $\lambda_S\colon M \to M_S$ such that $\lambda_S(S) \subseteq M_S^*$. The monoid $M_S$, called the "localization of $M$ by $S$," is constructed in general in the usual way as the set of equivalence classes of pairs $(m, s) \in M \times S$, with $(m, s) \equiv (m', s')$ if and only if $m + s' + t = m' + s + t$ for some $t \in S$. Since our $M$ is integral, $\lambda_S$ is injective and we will often write $m - s$ for the equivalence class of $(m, s)$. If $S$ is a submonoid of a monoid $N$ and $\theta$ is a morphism $N \to M$ (resp. $M \to N$), we shall also allow ourselves to write $M_S$ to mean the localization of $M$ by the image (resp. inverse image) of $S$ under $\theta$, and if $s \in M$ is any element, we write $M_s$ for the localization of $M$ by the submonoid of $M$ generated by $s$.

If $S$ is a submonoid of $M$, we let $\tilde{S}$ denote the set of elements $\tilde{s}$ of $M$ such that $\tilde{s} + t \in S$ for some $t \in M$. Then $\tilde{S}$ is a submonoid of $M$ containing $S$, and the natural map $M_S \to M_{\tilde{S}}$ is an isomorphism. Notice that $\tilde{\tilde{S}} = \tilde{S}$, so that $\tilde{}$ is a closure operation. Finally, observe that the submonoids of $M$ which are closed under $\tilde{}$ are precisely the subsets whose complements are prime ideals.

LEMMA 3. *Suppose that $S$ is a submonoid of an (integral) monoid $M$ such that $M^g$ is finitely generated. Then there exists a finitely generated submonoid $S'$ of $S$ such that the natural map $M_{S'} \to M_S$ is an isomorphism, and if $M$ is finitely generated, so is $M_S$.*

*Proof.* Since $M$ is integral we have $S^g \subseteq M^g$, necessarily a finitely generated subgroup. Suppose that it is generated by $(s_1 - t_1), \ldots (s_n - t_n)$, with $s_i$ and $t_i$ in $S$. Let $S'$ be the submonoid of $S$ generated by the $\{s_i, t_i\}$. Then any element of $S$ can be written as $s = \sum_i n_i s_i' - \sum_i m_i t_i'$ with $s_i'$ and $t_i'$ in $S'$, and so we have $s + t' \in S'$ for some $t' \in S'$. This implies that $\tilde{S} = \tilde{S}'$, so $M_{S'} \cong M_S$.

If $X$ is a topological space we can work with a sheaf of monoids on $X$ instead of a single monoid and perform the analogous constructions. (Of course, the quotient space $\overline{M}$ has to be computed in the category of sheaves, not presheaves, and in Lemma 2 we can conclude only that the sequence $\Xi_M$ is locally, not globally, split.) Notice that a global section of $M$ is a unit if and only if each of its stalks is a unit. Furthermore, if $\theta\colon M \to N$ is a morphism of sheaves of monoids, then $\theta(X)\colon M(X) \to N(X)$ is local if each map of stalks $M_{X,x} \to N_{X,x}$ is, and in this case we just say that $\theta$ is local. Clearly this is true if and only $\theta(U)$ is local for every open set $U$ in $X$. Let $M_\theta$ denote the sheaf associated to the presheaf which assigns to each $U$ the localization of $M(U)$ by the inverse image of $N^*(U)$; then the map $M \to N$ factors uniquely through $M_\theta$, and the map $M_\theta \to N$ is local. We say that a morphism $\theta\colon M \to N$ is "strictly local" if it is local and the induced map $M^* \to N^*$ is an isomorphism, equivalently, if $\theta^{-1}N^* = M^*$.

If $\gamma\colon P \to N$ is any morphism, we can form the pushout diagram:

$$\begin{array}{ccc}
\gamma^{-1}N^* & \longrightarrow & P \\
\downarrow & & \downarrow \\
N^* & \longrightarrow & P^a
\end{array}$$

and the induced map $P^a \to N$ is strictly local. Notice that if $\gamma$ is local, $\gamma^{-1}N^*$ is a subgroup of $P^*$ and in this case it is especially easy to construct the pushout. In particular, when $\gamma$ is local, the map $\overline{P} \to \overline{P^a}$ is an isomorphism. For this reason it is often helpful to construct $P^a$ in two steps: first localize $\gamma$, then form the pushout.

If $P$ is a monoid and $X$ is a topological space, we also write $P$ to denote the locally constant sheaf of monoids associated to $P$ on $X$. If $N$ is a sheaf of monoids on $X$, a "chart for $N$" is a morphism of sheaves of monoids $P \to N$ such that the associated map $P^a \to N$ is an isomorphism. If, locally on $X$, $N$ admits a chart, we say that $N$ is "quasi-coherent." If in addition the monoids $P$ in the charts are (integral and) finitely generated, we say that $N$ is "fine."

Let $(X, \mathcal{O}_X)$ be a locally ringed space. A "prelogarithmic structure on $X$" is morphism of sheaves of monoids $\alpha_X\colon (M_X, +) \to (\mathcal{O}_X, \cdot)$, and such an $\alpha_X$ is a "logarithmic structure" if it is strictly local. If this is the case we can identify $M_X^*$ and $\mathcal{O}_X^*$, and we obtain an injective monoid morphism:

$$\lambda_X\colon (\mathcal{O}_X^*, \cdot) \to (M_X, +).$$

In fact, $\alpha_X$ should be thought of as an exponential map, and if $f$ is a section of $\mathcal{O}_X$, $\alpha_X^{-1}(f)$ as the (possibly empty) set of branches of $\log f$ defined by the logarithmic structure. If $(X, \mathcal{O}_X)$ is a formal scheme with logarithmic structure $\alpha_X$, we say that the data $(X, \mathcal{O}_X, M_X, \alpha_X)$ form a "logarithmic formal scheme" if $M_X$ is quasicoherent. One defines morphisms of spaces with prelogarithmic structure in the obvious way. If $f\colon X \to Y$ is a morphism of locally ringed spaces and $\alpha_Y\colon M_Y \to \mathcal{O}_Y$ is a logarithmic structure on $Y$, then the logarithmic structure associated to $f^{-1}M_Y \to \mathcal{O}_X$ is denoted by $\alpha_X\colon f^*M_Y \to \mathcal{O}_X$.

DEFINITION 4. If $f\colon X \to Y$ is a morphism of locally ringed spaces with logarithmic structure, we say that $f$ is "solid" if the map $f^*M_Y \to M_X$ is an isomorphism. A prelogarithmic structure $\alpha_X\colon M_X \to \mathcal{O}_X$ on a locally ringed space $X$ is "hollow" if for every $x \in X$, the map $\alpha_{X,x}\colon M_{X,x}^+ \to \mathcal{O}_{X,x}$ is zero, and is "constant" if the sheaf of monoids $\overline{M}_X$ is locally constant.

We sometimes just say that "$X$ is a constant log scheme" (or formal scheme) to mean that $X$ is a (formal) scheme endowed with a constant log structure.

PROPOSITION 5. *Suppose that $(T, M_T, \alpha_T)$ is a fine logarithmic formal scheme, and let $Z$ be its spine, i.e., the reduced subscheme of a subscheme of definition. Then the following conditions are equivalent:*

1. *The sheaf of monoids $\overline{M}_T$ is locally constant.*
2. *The sheaf of abelian groups $\overline{M}_T^g$ is locally constant.*
3. *Whenever $t$ and $\tau$ are points of $T$ and $t$ is a specialization of $\tau$, the natural map $\overline{M}_{T,t}^g \to \overline{M}_{T,\tau}^g$ is injective.*
4. *The logarithmic structure induced by $\alpha_T$ on $Z$ is hollow.*
5. *Locally on $T$, there exist charts $P \to M_T$ such that $P^+ \to \mathcal{O}_T \to \mathcal{O}_Z$ is the zero map.*

*Proof.* It is clear that (1) implies (2) and that (2) implies (3). Assume (3) holds. If $M_Z \to \mathcal{O}_Z$ is the logarithmic structure induced on $Z$, $\overline{M}_T \to \overline{M}_Z$ is bijective, so we may as well replace $Z$ by $T$, and assume that $T$ itself is a reduced scheme. If $t$ is a specialization of $\tau$, consider the commutative diagram

$$\begin{array}{ccc} \overline{M}_{T,t} & \longrightarrow & \overline{M}_{T,\tau} \\ \big\downarrow & & \big\downarrow \\ \overline{M}_{T,t}^g & \longrightarrow & \overline{M}_{T,\tau}^g \end{array}$$

The vertical arrows are injective because our monoids are integral, and (3) tells us that the bottom horizontal arrow is injective. This implies that the horizontal arrow on the top is also injective. Then the map $M_{T,t} \to M_{T,\tau}$ is local, and it follows that the composite

$$M_{T,t} \to M_{T,\tau} \to \mathcal{O}_{T,\tau}$$

is also local. Thus, if $m$ is any element of $M^+_{T,t}$, its image in $\mathcal{O}_{T,\tau}$ lies in the maximal ideal of $\mathcal{O}_{T,\tau}$, i.e. $\alpha_{T,t}(m)$ lies in every prime ideal of $\mathcal{O}_{T,t}$, and consequently is nilpotent, hence zero.

Suppose that (4) holds. Without loss of generality, we may assume that $T$ is affine and admits a chart: $P \to M_T$. If $T \cong \operatorname{Spf} A$, we find a map $\gamma\colon P \to M_T(T) \to A$. Suppose that $t$ is a point of $T$, and let $S \subseteq P$ be the inverse image of $M^*_{T,t}$ via the map $P \to M_{T,t}$. According to Lemma 3, we can find a finitely generated submonoid $S'$ of $S$ such that the map $P_{S'} \to P_S$ is an isomorphism. Since the elements of $S'$ map to units in $\mathcal{O}_{T,t}$, we may replace $A$ by its formal localization by $\gamma(S')$ and $P$ by $P_{S'}$. Thus, we may as well assume that the map $P \to \mathcal{O}_{T,t}$ is local. Then by (4) the image of every element of $P^+$ lies in the stalk of $I_Z$ at $t$. Observe that the ideal of the monoid algebra $\mathbf{Z}[P]$ generated by $P^+$ is finitely generated because $\mathbf{Z}[P]$ is noetherian, and it follows that the ideal $I$ of $A$ generated by $\gamma(P^+)$ is also finitely generated. Since $I$ is finitely generated, we may replace $A$ by a localization in which every element of $P^+$ lies in $I_Z$. This proves (5). To show that (5) implies (1), we show that if $P \to M_T$ is a chart as in (5) and if $U$ is a nonempty affine open set in $T$, then the map $\overline{P} \to \overline{M}_T(U)$ is an isomorphism. In fact, if $f$ is any element of $A$ which is not nilpotent modulo $I$, the image of $P^+ \to A_f$ is topologically nilpotent and hence does not meet $A^*_f$, so that $P \to A_f$ is still local. For each $f \in A$, let $P^a_f$ denote the pushout:

$$\begin{array}{ccc}
(\lambda\gamma)^{-1}A^*_f & \longrightarrow & P \\
\downarrow & & \downarrow \\
A^*_f & \longrightarrow & P^a_f
\end{array}$$

Since $P \to A_f$ is local, the induced map $\overline{P} \to \overline{P}^a_f$ is an isomorphism. Passing to the associated sheaves, we see that $\overline{P} \cong \overline{M}$.

PROPOSITION 6. *Suppose that $M$ is a sheaf of monoids on $X$ such that $\overline{M}$ is locally constant, and suppose that we are given an isomorphism $\iota\colon M^* \cong \mathcal{O}^*_X$ Then there is a unique hollow log structure $M_X \to \mathcal{O}_X$ extending $\iota$.*

*Proof.* Note first that for each open subset $U$ of $X$ and each section $m$ of $M(U)$, the set $U_m$ of all $x \in X$ such that $m_x \in M^*_x$ is both open (this is always true) and closed (because $\overline{M}_X$ is locally constant). The restriction of $m$ to $U_m$ is a unit of $M(U_m)$, and we can define $\alpha(m) \in \mathcal{O}_X(U)$ to be $\iota(m_{|U_m})$ on $U_m$ and to be zero on $U \setminus U_m$. Clearly this is the unique hollow log structure extending $\iota$.

REMARK 7. We shall denote the log scheme constructed above by $X_{M,\iota}$, or, if there is no danger of confusion, by $X_M$. If $P$ is a monoid with $P^* = 0$, we let $X_P$ denote the log scheme associated to the prelog structure sending $P^+$ to 0; this is canonically isomorphic to $X_{\mathcal{O}_X^* \oplus P}$. If $X =: (X, M_X, \alpha_X)$ is any scheme or formal scheme with constant log structure, then since $\overline{M}_X$ is locally constant, we can define a hollow log structure $\alpha_X^\natural : M_X \to \mathcal{O}_X$. We shall denote the corresponding log scheme by $X^\natural$ and call it the "hollowing out" of $X$.

If $X$ is a scheme and $P$ is a fine saturated monoid with $P^* = 0$, the set of isomorphism classes of hollow log structures on $X$ with $\overline{M}_X = P$ is naturally bijective with $\operatorname{Hom}(P^g, \operatorname{Pic} X)$. If $X$ is affine and noetherian, the sets of such log structures on $X$ and on $X_{\text{red}}$ are the same, because $\operatorname{Pic} X \cong \operatorname{Pic} X_{\text{red}}$. In general, the set of isomorphism classes of hollow log structures $\alpha_X : M_X \to \mathcal{O}_X$ with $\overline{M}_X = P$ is given by the set of isomorphism classes of extensions $1 \to \mathcal{O}_X^* \to M_X \to P \to 0$, i.e. by $\operatorname{Ext}^1(P^g, \mathcal{O}_X^*)$. If $P$ is fine, then $P^g$ is a free abelian group, and this extension group can be identified with $H^1(X, \underline{\operatorname{Hom}}(P^g, \mathcal{O}_X^*)) \cong \operatorname{Hom}(P^g, \operatorname{Pic} X)$.

If $X$ is a constant log scheme which is not hollow, the identity map does not correspond, of course, to a map of log schemes $X \to X^\natural$, but nevertheless we shall see that there are natural commutative diagrams:

$$\begin{array}{ccc} \mathcal{O}_X & \xrightarrow{d^\natural} & \Omega^1_{X^\natural/W} \\ \downarrow{\scriptstyle \text{id}} & & \downarrow{\scriptstyle \theta} \\ \mathcal{O}_X & \xrightarrow{d} & \Omega^1_{X/W} \end{array} \qquad \begin{array}{ccc} M_X & \xrightarrow{d^\natural} & \Omega^1_{X^\natural/W} \\ \downarrow{\scriptstyle \text{id}} & & \downarrow{\scriptstyle \theta} \\ M_X & \xrightarrow{d} & \Omega^1_{X/W} \end{array}$$

CLAIM 8. *There is a unique morphism*

$$\theta : \Omega^1_{X^\natural/W} \to \Omega^1_{X/W}$$

*such that $\theta d^\natural(m) = dm$ for every section $m$ of $M_X$ and $\theta(d^\natural a) = da$ for every section $a$ of $\mathcal{O}_X$.*

*Proof.* Since $d^\natural$ is the universal logarithmic derivation of $X^\natural$, it suffices to prove that $d$ is also a logarithmic derivation with respect to $\alpha^\natural$. Thus, it suffices to show that $d\alpha^\natural(m) = \alpha^\natural(m)dm$ for all sections $m$ of $M_X$. We check this on the stalks. If $m$ is a unit, then we have $m = \lambda(u)$ for a unit $u \in \mathcal{O}_X^*$, $\alpha^\natural(m) = \alpha(m) = u$, and the equality is clear. If $u$ is not a unit, then $\alpha^\natural(m)$ vanishes, and the equality is trivial.

Of course, the map $\Omega^1_{X^\natural/W} \to \Omega^1_{X/W}$ has a geometric interpretation. Let $X \hat{\times} X$ denote the exact formal completion of $X \times X$ along the the diagonal. Then the projection maps induce maps $(X \hat{\times} X)^\natural \to X^\natural$, and in fact we have a commutative diagram:

$$\begin{array}{ccc} (X \hat{\times} X)^\natural & \longrightarrow & X^\natural \hat{\times} X^\natural \\ \uparrow{\scriptstyle i} & & \uparrow{\scriptstyle j} \\ X^\natural & \xrightarrow{\text{id}} & X^\natural \end{array} \tag{2}$$

Here $X^\natural \hat{\times} X^\natural$ is the exact formal completion of the diagonal, so that $i$ and $j$ are exact closed immersions. Furthermore, $\Omega^1_{X^\natural/W}$ is the conormal bundle of $j$ and $\Omega^1_{X/W}$ is the conormal bundle of $i$. Then our map $\theta$ of (8) is induced from the map on conormal bundles in the above diagram.

If $Z$ is a logarithmic formal scheme we let $\underline{Z}$ denote the underlying formal scheme of $Z$ with the trivial log structure, and similarly for morphisms. Then it is easy to see that the natural map $\overline{M}^g_Z \otimes \mathcal{O}_Z \to \Omega^1_{Z/\underline{Z}}$ is surjective, and is bijective if $Z$ is hollow. The differentials of the complex $\Omega^{\cdot}_{Z/\underline{Z}}$ vanish, and the dual space $T_{Z/\underline{Z}}$ is a commutative Lie algebra. In characteristic $p$, it also has the structure of a restricted Lie algebra [14, Sect. 1], coming from its interpretation as the set of logarithmic derivations. Note that $\partial^{(p)} = \partial$ for $\partial \in \mathrm{Hom}(\overline{M}^g_Z, \mathbf{F}_p)$. Thus when $Z$ is hollow, $T_{Z/\underline{Z}} \cong \mathrm{Hom}(\overline{M}^g_Z, \mathcal{O}_Z)$ and its structure of a restricted Lie algebra is compatible with the "mod $p$ unit root F-crystal structure" corresponding to the $\mathbf{F}_p$-form of $T_{T/\underline{T}}$ provided by $\mathrm{Hom}(\overline{M}^g_Z, \mathbf{F}_p)$.

Suppose that $P$ is a locally constant sheaf of fine monoids on a scheme (or possibly a log scheme) $Z$ and that $P^* = 0$. Then the monoid algebras $\mathcal{O}_Z[P]$ and $\mathcal{O}_Z[P^g]$ are quasicoherent over $\mathcal{O}_Z$, and hence define schemes $\mathbf{A}_P$ and $\mathbf{G}_P$ over $Z$, with $\mathbf{G}_P \subseteq \mathbf{A}_P$. The obvious inclusion mapping $\alpha_P: P \to \mathcal{O}_Z[P]$ is a prelog structure, and in fact is the universal one with source $P$. The scheme $\mathbf{G}_P$ is a torus over $\underline{Z}$, with character group $P^g$. Notice that $\mathbf{G}_P$ represents the functor on affine $Z$-schemes which takes $\operatorname{Spec} A$ to the set of homomorphisms $(P^g, +) \to (A^*, \cdot)$. On the other hand, $\mathbf{T}_P =: \mathbf{V}(\mathcal{O}_Z \otimes P^g)$ represents the functor which takes $\operatorname{Spec} A$ to the set of homomorphisms $P^g \to (A, +)$. If $Y$ is a scheme with constant log structure, we let $\mathbf{G}_Y$ be the log scheme over $Y$ obtained by endowing $\mathbf{G}_{\overline{M}}$ with the log structure induced from $Y$.

LEMMA 9. *Suppose that $Y$ is a scheme with a fine saturated and hollow log structure. Let $\hat{Y}_{\underline{Y}}(1)$ denote the exact formal completion of the diagonal embedding $Y \to Y \times_{\underline{Y}} Y$, and let $\hat{\mathbf{G}}_Y$ denote the formal completion of $\mathbf{G}_Y$ along the identity section. Then there is a natural isomorphism of formal log schemes over $Y$ : $\hat{\mathbf{G}}_Y \to \hat{Y}_{\underline{Y}}(1)$.*

*Proof.* It is simplest to construct an isomorphism of functors on the category of exact nilpotent immersions $i: S \to T$ of logarithmic $Y$-schemes. An element of $\hat{Y}_{\underline{Y}}(1)(i)$ is a pair $(f_1, f_2)$ of morphisms $T \to Y$ which agree on $S$ and such that $\underline{f}_1 = \underline{f}_2 := \underline{f}$. Thus it suffices to look at the corresponding morphisms of monoids $f_i^*: \underline{f}^{-1} M_Y \to M_T$. These two morphisms agree when composed with the map $M_T \to M_S$, and since we have an exact sequence

$$1 \to (1 + I_T) \to M_T \to M_S \to 0,$$

they "differ" by a unique map $\underline{f}^{-1} M_Y \to 1 + I_T$. Furthermore, since $f_1$ and $f_2$ agree on $\underline{Y}$, this difference map factors through a map $\delta: \underline{f}^{-1} \overline{M}^g_Y \to 1 + I_T$. The pair $(f_1, \delta)$, which evidently determines $(f_1, f_2)$, is an element of $\hat{\mathbf{G}}_Y(i)$. On the other hand, given such a pair $(f_1, \delta)$, we can define a new morphism of monoids by setting $f_2^*(m) =: f_1^*(m) + \lambda(\delta(\overline{m}))$; this defines a morphism of log schemes because $Y$ is hollow.

Let us now fix a positive integer $p$ (usually a prime number). We shall see that there is a sort of relative Frobenius morphism, relative to $p$, on any logarithmic scheme, even in characteristic zero, and that this morphism can be used to push a constant logarithmic structure until it becomes hollow.

If $M$ is a sheaf of integral monoids on a topological space, there exists a commutative diagram:

$$\begin{array}{ccccccccc}
0 & \to & M^* & \to & M & \to & \overline{M} & \to & 0 \\
 & & \downarrow{\scriptstyle p^n} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle \mathrm{id}} & & \\
0 & \to & M^* & \to & M^{(n)} & \to & \overline{M} & \to & 0 \\
 & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle p^n} & & \\
0 & \to & M^* & \to & M & \to & \overline{M} & \to & 0
\end{array} \tag{3}$$

with exact rows in which the top square on the left is cocartesian and the bottom square on the right is Cartesian. Furthermore $h \circ g = g \circ h = p^n$, $\Xi_{M^{(n)}} \cong p^n \Xi_M$ in $\mathrm{EXT}^1(M^*, \overline{M})$, and $h$ is strictly local. To verify these claims, note first that because $M^*$ is a sheaf of groups, it is easy to form the pushout $M_Z^{(n)}$: just take the quotient of $M^* \oplus M$ by the action of $M^*$ given by $u(v, m) =: (v - p^n u, u + m)$. If $[v, m]$ is the equivalence class of $(v, m)$, $h[v, m] = v + p^n m$, and it is clear that $h$ is strictly local.

Now suppose that $\alpha: M \to \mathcal{O}_T$ is a (pre)log structure on $T$. Then $\alpha^{(n)} =: \alpha \circ h$ is also a (pre)log structure. In fact, if $\alpha$ is a log structure on $T$, then $\alpha^{(n)}$ is the log structure associated to the prelog structure $\alpha \circ p^n$. Furthermore, we have a commutative diagram of sheaves on $T$:

$$\begin{array}{ccc}
M^{(n)} & \xrightarrow{h} & M \\
\downarrow{\scriptstyle \alpha \circ h} & & \downarrow{\scriptstyle \alpha} \\
\mathcal{O}_T & \xrightarrow{\mathrm{id}} & \mathcal{O}_T
\end{array}$$

If we denote by $T^{(n)}$ the (pre)log scheme corresponding to $\alpha \circ h$, then the preceding diagram defines a canonical morphism

$$F_{T/\underline{T}}^{(n)}: T \to T^{(n)}. \tag{4}$$

In general there is no map $T^{(n)} \to T$, however.

If $T \to S$ is a morphism of log schemes, the natural map $M_T \to \Omega^1_{T/S}$ induces a map $\overline{M}^g_T \to \Omega^1_{T/\underline{T}}$, and we find a commutative diagram:

$$\begin{tikzcd}
\overline{M}^g \arrow[r, phantom, "\cong"] \arrow[dr, "p^n"'] & \overline{M}^{(n)g} \arrow[r] \arrow[d] & \Omega^1_{T^{(n)}/\underline{T}} \arrow[d] \\
& \overline{M}^g \arrow[r] & \Omega^1_{T/\underline{T}}
\end{tikzcd}$$

If $Z$ is a logarithmic scheme in characteristic $p$, we can interpret $F_{Z/\underline{Z}} =: F^{(1)}_{Z/\underline{Z}}$ as the relative Frobenius morphism of the morphism $Z \to \underline{Z}$. To see this, recall that the absolute Frobenius endomorphism of a log scheme $Z$ in characteristic $p$ is given by the commutative diagram:

$$\begin{tikzcd}
M_Z \arrow[r, "p"] \arrow[d, "\alpha_Z"] & M_Z \arrow[d, "\alpha_Z"] \\
\mathcal{O}_Z \arrow[r, "F^*_Z"] & \mathcal{O}_Z
\end{tikzcd}$$

The relative Frobenius morphism $F_{Z/\underline{Z}}$ is obtained by considering the diagram:

$$\begin{tikzcd}
Z \arrow[r, "F_{Z/\underline{Z}}"] \arrow[dr] & Z^{(1)} \arrow[r, "\pi_{Z/\underline{Z}}"] \arrow[d] & Z \arrow[d] \\
& \underline{Z} \arrow[r, "F_Z"] & \underline{Z}
\end{tikzcd}$$

in which the square is Cartesian and the map $F_{Z/\underline{Z}}$ is the unique one making the diagram commute.

LEMMA 10. *Suppose that $(Z, M_Z, \alpha_Z)$ is a log scheme in characteristic $p$, and consider the diagram:*

$$\begin{tikzcd}
\mathcal{O}^*_Z \arrow[r, "\lambda_Z"] \arrow[d, "p"] & M_Z \arrow[r, "\alpha_Z"] \arrow[d, "g"] & \mathcal{O}_Z \arrow[d, "F^*_Z"] & \pi^*_{Z/\underline{Z}} \\
\mathcal{O}^*_Z \arrow[r, "\lambda_{Z^{(1)}}"] \arrow[d, "\mathrm{id}"] & M^{(1)}_Z \arrow[r, "\alpha^{(1)}_Z"] \arrow[d, "h"] & \mathcal{O}_Z \arrow[d, "\mathrm{id}"] & \\
\mathcal{O}^*_Z \arrow[r, "\lambda_Z"] & M_Z \arrow[r, "\alpha_Z"] & \mathcal{O}_Z & F^*_{Z/\underline{Z}}
\end{tikzcd}$$

*in which the upper left square is a pushout and $\alpha^{(1)}_Z$ is the unique map making the diagram commute. Then $\alpha^{(1)}_Z$ defines a log structure on $Z$, and the corresponding log scheme $Z^{(1)}$ is the pullback of $Z \to \underline{Z}$ by $F_{\underline{Z}}$. If $Z$ is fine and saturated, so is $Z^{(1)}$, and if $\gamma\colon P \to M_Z$ is a chart of $Z$, then $\gamma \circ p$ is a chart of $Z^{(1)}$. The relative*

*Frobenius morphism $F_{Z/\underline{Z}}$ is induced by the unique map $h$: $M_Z^{(1)} \to M_Z$ such that $h \circ g = p$ and $h \circ \lambda_{Z^{(1)}} = \lambda_Z$.*

*Proof.* The morphism $h$ is just the morphism of the same name in (3), and in particular it is strictly local. This implies that $\alpha_Z^{(1)}$ is also strictly local, and hence is a logarithmic structure.

Now suppose that $\underline{Z} = \operatorname{Spec} A$ and that $\gamma: P \to M_Z(Z)$ is a chart of $Z$; let $\beta =: \alpha \circ \gamma$, and consider the diagram:

$$\begin{array}{ccc} \beta^{-1}(A^*) & \longrightarrow & P \\ \downarrow & & \downarrow{\scriptstyle \beta} \\ A^* & \longrightarrow & A \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle F_A^*} \\ A^* & \longrightarrow & A \end{array}$$

It is clear that the large rectangle is Cartesian. The pushout of the upper square is $P^a$, and now the diagram:

$$\begin{array}{ccccc} \beta^{-1}(A^*) & \longrightarrow & P & & \\ \downarrow & & \downarrow & \searrow^{\beta} & \\ A^* & \longrightarrow & P^a & \longrightarrow & A \\ \downarrow{\scriptstyle p} & & \downarrow & & \downarrow{\scriptstyle F_A^*} \\ A^* & \longrightarrow & P^{a\prime} & \longrightarrow & A \end{array}$$

shows that the large rectangle on the left is also a pushout, so that $P^{a\prime}$ is indeed the log structure associated to $\gamma \circ p$.

If $Z$ is a scheme we write $e(Z)$ for the smallest integer $e$ such that $a^e = 0$ for every nilpotent section of $\mathcal{O}_Z$, if such an integer exists (e.g. if $Z$ is noetherian).

LEMMA 11. *Suppose that $Z$ is a scheme in characteristic $p$ with constant log structure and that $p^r \geqslant e(Z)$. Then $Z^{(r)}$ is hollow, and the map*

$$\pi_{Z/\underline{Z}}^{(r)}: Z^{(r)} \to Z$$

*factors through $Z_{\mathrm{red}}$.*

*Proof.* If $z \in Z$ and and $m \in M_{Z,z}^+$, then $\alpha_Z(m)$ is nilpotent, by (5), and it follows that $\alpha_Z(m)^e = 0$. Then $\alpha_{Z^{(r)}}(g(m)) = \alpha_Z(m)^{p^r} = 0$. Since the ideal $M_{Z^{(r)}}^+$ is generated by $g(M_{Z,z}^+)$, we see that $\alpha_{Z^{(r)}}$ annihilates it. Furthermore, the map $\underline{\pi}_{Z/\underline{Z}}^{(r)}$ is just $F_Z^r$, which evidently annihilates the nilradical of $\mathcal{O}_Z$. Since $Z_{\mathrm{red}}$

has the log structure induced from that of $Z$, it follows that $\pi^{(r)}_{Z/\underline{Z}}$ factors through $Z_{\mathrm{red}}$.

## 3. The main comparison theorem

For simplicity we shall work over the Witt ring $W$ of a perfect field $k$ of characteristic $p$, where $k$ and $W$ are both endowed with the trivial logarithmic structure.

Let $Z/k$ be a fine saturated logarithmic $k$-scheme of finite type. We shall be working with various notions of crystals on $Z/k$, which take their values on various kinds of thickenings of $Z$. A "PD-thickening" of an open subset $U$ of $Z$ is an exact closed immersion $U \rightarrow T$ of f-s log schemes, together with a divided power structure $\gamma$ on the ideal of $U$ in $T$ which is compatible with the standard PD-structure on $(p)$. If $p^n \mathcal{O}_T = 0$, we say that $T$ is an object of Cris $(Z/W_n)$, and if this is true for some $n$ we say that $T$ is an object of Cris $(Z/W)$. We will also allow $T$ to be a formal scheme for the $p$-adic topology, in which case we say that it is an object of Cris $(Z/W)_\infty$.

An "enlargement of $Z/W$" is a pair $(T, z_T)$, where $T$ is a $p$-torsion free locally noetherian logarithmic formal scheme $T$ (with the $p$-adic topology), and $z_T$ is a solid (Definition 4) morphism from a subscheme of definition $Z_T$ of $T$ to $Z$. We let $I_T$ denote the ideal defining $Z_T$; if $I_T$ is the ideal $(p)$, we say that $T$ is a "$p$-adic enlargement," and if $I_T$ is a PD-ideal we say $T$ is a "PD-enlargement." (Note that since $\mathcal{O}_T$ is $p$-torsion free, the PD-structure will be unique if it exists. In particular, a $p$-adic enlargement is a PD-enlargement in a unique way.) If $T = (T, z_T)$ is an enlargement of $Z$, we write $e(T)$ for the smallest integer $e$ such that $f^e \in p\mathcal{O}_T$ for every local section $f$ of $I_T$. Morphisms and coverings of PD-thickenings and enlargements are defined in the obvious way, and one can form categories and sites without difficulty [13].

Recall that a "crystal of $\mathcal{O}_{Z/W}$-modules on Cris $(Z/W)$" assigns to each object $T$ of Cris $(Z/W)$ a sheaf of $\mathcal{O}_T$-modules $E_T$ and to each morphism $f: T' \rightarrow T$ an isomorphism $\theta_f: f^*(E_T) \rightarrow E_{T'}$, such that the standard cocycle condition is satisfied. Similarly, a "($p$-adically) convergent isocrystal on $Z/W$" assigns to each ($p$-adic) enlargement $(T, z_T)$ a sheaf of $\mathcal{O}_T \otimes \mathbf{Q}$-modules $E_T$, with morphisms of enlargements inducing corresponding isomorphisms of sheaves.

REMARK 12. If $Z$ is a local complete intersection and $\underline{Y}/W$ is smooth then the ($p$-adically completed) divided power envelope $D_Z(Y)$ of $Z$ in $Y$ is $p$-torsion free, but in general this is not true and we do not know much about the structure of the $p$-torsion – for example, we do not even know if it is closed in the $p$-adic topology, or if it is contained in the ideal $J_Z$ of $Z$ in $D_Z(Y)$. However, it is true that the $p$-torsion forms an ideal of $\mathcal{O}_D$ which is compatible with the divided powers on $J_Z$, and consequently the same is true of its closure. Furthermore, the torsion is independent of the choice of embedding $i: Z \rightarrow Y$: if $\pi: Y \rightarrow Y'$ is a smooth morphism and $Z \rightarrow Y'$ is $\pi \circ i$, then the $p$-torsion of $D_Z(Y)$ is obtained from that of $D_Z(Y')$ by base change. Now if $T$ is any affine object of Cris $(Z/W)$, we can choose a morphism $T \to Y$ and hence $T \to D_Z(Y)$, and it follows that the image in $\mathcal{O}_T$ of the $p$-torsion ideal of $D_Z(Y)$ is independent of the choice of $Y$ and of the map $T \to Y$. We denote the subscheme of $T$ defined by this ideal by $T_{tf}$, and the corresponding subscheme of $Z$ by $Z_{tf}$; so that $T_{tf}$ is an object of Cris $(Z_{tf}/W)$.

EXAMPLE 13. Perhaps the most important example to keep in mind is the following. Let $T$ to be the spectrum of a discrete valuation ring $V$ of mixed characteristic $p$ with residue field $k$ and fraction field $K$; let $\eta =:$ Spec $K$ and let $\xi =:$ Spec $k$, all endowed with the trivial log structures. If $Y/T$ is any scheme, we let $Y^\times$ denote the log scheme obtained by endowing $Y$ with the direct image [10, 1.4] of the trivial log structure on $Y_\eta$ via the open immersion $Y_\eta \to Y$. In particular, the logarithmic structure $\alpha_{T^\times}: M_{T^\times} \to \mathcal{O}_T$ of $T^\times$ is given on global sections by the inclusion mapping $V' \subseteq V$, where $V'$ is the monoid of nonzero elements of $V$. Let us normalize the valuation $v$ on $V$ so that $v(p) = 1$; then $v$ identifies $\overline{V}'$ with $e^{-1}$bf N, the set of all nonnegative rational numbers $q$ such that $eq \in$ bf N, where $e$ is the absolute ramification index of $V$. We have a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \to & V^* & \to & V' & \to & e^{-1}\text{bf N} & \to & 0 \\ & & \downarrow\cong & & \downarrow\cong & & \downarrow\cong & & \\ 0 & \to & \Gamma(T_V^\times, M_T^*) & \to & \Gamma(T^\times, M_{T^\times}) & \to & \Gamma(T^\times, \overline{M}_{T^\times}) & \to & 0 \end{array} \tag{5}$$

Let $\xi_V^\times$ denote the reduction of $T^\times$ modulo the maximal ideal of $V$; then we have a similar diagram:

$$\begin{array}{ccccccccc} 0 & \to & k^* & \to & V'/U_V^* & \to & e^{-1}\text{bf N} & \to & 0 \\ & & \downarrow\cong & & \downarrow\cong & & \downarrow\cong & & \\ 0 & \to & \Gamma(\xi_V^\times, M^*_{\xi_V^\times}) & \to & \Gamma(\xi_V^\times, M_{\xi_V^\times}) & \to & \Gamma(\xi_V^\times, \overline{M}_{\xi_V^\times}) & \to & 0 \end{array} \tag{6}$$

where $U_V^*$ is the kernel of the map $V^* \to k^*$. When $e = 1$, this sequence, like the sequence (5), has a splitting which is more canonical than others, provided by $p$. In general, in order to split (5) we need to choose a uniformizer $\pi$ for $V$, and two uniformizers determine the same splitting of (6) if and only if their ratio belongs to $U_V^*$. If $R$ is another complete DVR with the same residue field and absolute ramification index as $V$, then the log schemes $\xi_V^\times$ and $\xi_R^\times$ are isomorphic, but not canonically so, and this is why we insist on the subscript $V$ in our notation. Notice in particular that if $V/W$ is Galois, then Gal$(V/W)$ acts on $\xi_V^\times$, and the action of the inertia group $I(V/W)$ is not trivial, since it acts nontrivially on the extension (6). Using the identification of the automorphism group of this extension with $\mathrm{Hom}(\overline{V}^g, k^*)$ given by formula (1) of Lemma 2 and the fact that an element of $I(V/W)$ acts trivially on $M_{\xi_W^\times}$, we obtain a homomorphism

$$I(V/W) \to \mathrm{Hom}(\overline{V}^g/\overline{W}^g, k^*) \cong \mu_e(k),$$

and it is clear that this homomorphism is none other than the tame character of $I(V/W)$.

The log structure of the scheme $T^\times$ is not constant, but on the formal scheme $\hat{T}^\times$ it becomes so, and we can also consider the the corresponding hollow log formal scheme $\hat{T}^\natural$ (Remark 7). Similarly,the log structure on the reduction $Z^\times$ of $T^\times$ modulo $p$ is not hollow if $e > 1$, but it is constant, and we can also consider its hollowing out $Z^\natural$. We have arrows

$$\hat{T}^\natural \hookleftarrow Z^\natural \hookleftarrow \xi_V^\natural = \xi_V^\times \hookrightarrow Z^\times \hookrightarrow \hat{T}^\times.$$

These arrows make $\hat{T}^\natural$ a $p$-adic enlargement of $Z^\natural$ and an enlargement of $\xi^\times$, and make $\hat{T}^\times$ a $p$-adic enlargement of $Z^\times$ and an enlargement of $\xi_V^\times$.

The underlying scheme $Z$ of $Z^\times$ is just Spec $V/pV$, and a choice of a uniformizer $\pi$ of $V$ induces an isomorphism $V/pV \cong k[\epsilon]$, where $\epsilon^e = 0$. Thus $Z$ is a $k$-scheme in a natural way, and it is clear that there is a corresponding logarithmic map $Z^\times \to \xi^\times$ if and only if $e = 1$. We have a morphism of exact sequences of monoids:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (V/pV)^* & \longrightarrow & V^\times/(1+pV) & \longrightarrow & e^{-1}\mathrm{bf\,N} & \longrightarrow & 0 \\ & & \downarrow f & & \downarrow g & & \downarrow \mathrm{id} & & \\ 0 & \longrightarrow & k^* & \longrightarrow & V^\times/U_V & \longrightarrow & e^{-1}\mathrm{bf\,N} & \longrightarrow & 0 \end{array}$$

The morphism $f$ has a canonical splitting, and so a choice of a uniformizer $\pi$ (modulo $p$) induces a splitting of $g$, i.e. of the map $M_{Z^\times} \to M_{\xi^\times}$. If $e > 1$ this splitting does not correspond to a morphism $Z^\times \to \xi^\times$, but it does give us a morphism

$$s_\pi \colon Z^\natural \longrightarrow \xi_V^\natural. \tag{7}$$

We can also consider the logarithmic Witt scheme of $\xi_V^\times$ constructed by Hyodo and Kato [8, 3.1]. Namely, we observe that the Teichmuller mapping $k^* \to W(k)$ prolongs uniquely to a (hollow) map $M_{\xi_V^\times} \to W$, and the associated log structure on $W(k)$ defines a hollow logarithmic formal scheme $S_V^\natural$, which is an object of Cris $(\xi_V^\times/W)$. We have a canonical isomorphism

$$M_{S_V^\natural} \cong V'/U_V^* \oplus U_W^*. \tag{8}$$

Then morphisms of enlargements $T^\natural \to S^\natural$ correspond to splittings $\sigma$ of the natural projection $V' \to V'/U_V^*$ such that $\sigma$ agrees with the Teichmuller lifting when composed with the natural map $k^* \to V'/U_V^*$. It is clear that a choice of a uniformizer $\pi$ determines such a splitting $\sigma$ and that two uniformizers $\pi$ and $\pi'$ determine the same splitting if and only if their ratio is the Teichmuller lifting of an element of $k^*$. In particular, the set of all morphisms $T^\natural \to S^\natural$ is a torsor under $\mathrm{Hom}(\overline{V}^g, U_V^*)$.

LEMMA 14. *If $T$ is a thickening of $Z/W$, then with the notations of diagram (3), $T^{(n)}$ is a thickening of $Z^{(n)}$, and $F_{T/\underline{T}}^{(n)}$ is a morphism of thickenings over $F_{Z/\underline{Z}}^{(n)}$. If $Z$ is hollow and $T$ is a PD thickening of $Z$, then the reduction of $T^{(n)}$ modulo $(p^n!)$ is hollow, and if $T$ is an enlargement of $Z$, the reduction of $T^{(n)}$ modulo $(p^j)$ is hollow, if $j$ is an integer less than or equal to $p^n/e(T)$.*

*Proof.* The first statements are clear, since $\underline{F}_{Z/\underline{Z}}^{(n)}$ is just the identity map. Suppose that $Z$ is hollow and that $z$ is a point of $Z$; then for any $m \in M_{Z,z}^+$, $\alpha_Z(m) = 0$. If $T$ is a thickening of $Z$ and $t$ is a point of $Z_T$ lying over $z$, the ideal $M_{Z_T,t}^+$ is generated by the image of $M_{Z,z}^+$ (since $z_T$ is solid), and it follows that $\alpha_{T,t}$ takes $M_{T,t}^+$ into $I_T$. If $T$ is a PD thickening, we see that $\alpha_{T^{(n)}}(g(m)) = \alpha_T(m)^{p^n} = p^n!\alpha_T(m)^{[p^n]}$, which is zero modulo $(p^n!)$. If $T$ is an enlargement and $j$ is less than $p^n/e(T)$, then since $\alpha_T(m)^{e(T)}$ is divisible by $p$, $\alpha_T(m)^{p^n}$ is divisible by $p^j$.

DEFINITION 15. An "F-crystal of width $m$" is a triple $(E, \Phi, V)$, where $E$ is a crystal of $\mathcal{O}_{Z/W}$-modules and

$$\Phi: F_{Z/W}^* E \to E \quad \text{and} \quad V: E \to F_{Z/W}^* E$$

are morphisms such that $\Phi \circ V$ and $V \circ \Phi$ are multiplication by $p^m$. An "$F^\infty$-span of width $m$" is a sequence of crystals $E^n$ and maps

$$F_X^* E^{n+1} \xrightarrow{\Phi_n} E^n \quad \text{and} \quad E^n \xrightarrow{V_n} F_X^* E^{n+1}$$

such that $\Phi_n \circ V_n$ and $V_n \circ \Phi_n$ are multiplication by $p^m$. If $(E', \Phi', V')$ and $(E, \Phi, V)$ are $F^\infty$-spans, then a "morphism of level $\ell$" from $E'$ to $E$ is a sequence of morphisms of crystals $\{h^n: E'^n \to E^n : n \in \text{bf N}\}$ such that

$$\Phi_n \circ F_{Z/W}^*(h^{n+1}) = p^\ell h_n \circ \Phi_n \quad \text{and} \quad F_{Z/W}^*(h^{n+1}) \circ V_n = p^\ell V_n \circ h_n.$$

An F-crystal of width $m$ gives rise to an $F^\infty$-span of width $m$, in the obvious way. By pulling back and composing the maps of an $F^\infty$-span we also get maps

$$F_X^{d*} E^{n+d} \xrightarrow{\Phi_n^{(d)}} E^n \quad \text{and} \quad E^n \xrightarrow{V_n^{(d)}} F_X^{d*} E^{n+d}.$$

Of course, the composition of these two in either direction is multiplication by $p^{dm}$.

REMARK 16. It is clear that a crystal of $\mathcal{O}_{Z/W}$-modules on Cris $(Z/W)$ defines a $p$-adically convergent isocrystal. Moreover, by modifying the procedure of [12, 2.8], we can associate to any $F^\infty$-span $E$ on Cris $(Z/W)$ a convergent isocrystal (with an $F^\infty$-span structure). Namely, for each enlargement $T =: (T, z_T)$ of $Z/W$, let us choose an integer $r$ such that $p^r \geqslant e(T)$. Then if $T_1$ is the subscheme of definition of $T$ defined by $p$, there is a unique map $g: T_1 \to Z_T$ whose composition with the inclusion mapping $Z_T \to T_1$ is $F^r_{T_1}$, and $T^r =: (T, z_T \circ g)$ is a $p$-adic enlargement of $Z/W$. Thus we can set $E^n_{(T,z_T)} =: \mathbf{Q} \otimes E^{n+r}_{T^r}$. To eliminate the dependence on $r$, we note that if $d \geqslant 0$ and $g'$ is defined with $r' = r + d$ in place of $g$, then $F^d_Z \circ z_T \circ g = z_T \circ g'$, and hence $E^{n+r'}_{T^{r'}} \cong (F^{d*}_Z E^{n+r+d})_{T^r}$. Then we can use our given isogeny $F^{d*}_Z E^{n+r+d} \to E^{n+r}$ to identify $\mathbf{Q} \otimes E^{n+r'}_{T^{r'}}$ with $\mathbf{Q} \otimes E^{n+r}_{T^r}$.

Suppose that $X$ and $Z$ are fine saturated log schemes of finite type over $k$ and that $f: X \to Z$ is a perfectly smooth ([14, 1.2]) morphism of relative dimension $d$. If $T = (T, z_T)$ is a PD-enlargement of $Z$, then the pullback $X_T$ of $X$ to $Z_T$ is still perfectly smooth, and since $Z_T \to T$ is defined by the PD-ideal $I_T$, we can form the crystalline cohomology $R\Gamma(X_T/T, z_T^* E)$ of any crystal of $\mathcal{O}_{X/W}$-modules. We usually denote this just by $R\Gamma(X/T, E)$.

THEOREM 1. *Suppose that $f: X \to Z$ is a perfectly smooth ([14, 1.2]) morphism of relative dimension $d$ of f-s log schemes of finite type over $k$ and $(E, \Phi, V)$ is an $F^\infty$-span on $X/W$ of level $m$, flat over $\mathcal{O}_{Z/W}$. Suppose that $T$ is an affine PD-enlargement of $Z$ and that the log structure on $Z$ is hollow. Then there is a family of isogenies*

$$\rho =: \{\rho^n_T: R\Gamma(X/T, E^n) \otimes \mathbf{Q} \to R\Gamma(X/T^\natural, E^n) \otimes \mathbf{Q}\}$$

*Furthermore, $\rho$ is compatible with the base change isomorphisms corresponding to morphisms of PD-enlargements and with morphisms of $F^\infty$-spans (of arbitrary level).*

We begin by describing the "twisted inverse limit" construction on which our proof relies.

Let $M. =: \{(M_n, \pi_n) : n \in \text{bf N}\}$ be an inverse system of abelian groups indexed by the natural numbers. Fix a prime number $p$, and for each natural number $m$, let

$$\varprojlim{}^m M. =: \{y. \in M_n : \pi_n(y_{n+1}) = p^m y_n \text{ for all } n \in \text{bf N}\}. \tag{9}$$

If $m' \geqslant m$, there is a natural map

$$\xi^{m,m'}: \varprojlim{}^m M. \to \varprojlim{}^{m'} M.$$

which sends a family $x.$ to the family $y.$, where $y_n =: p^{(m'-m)n} x_n$. We usually just write $\xi^m$ for $\xi^{0,m}$.

PROPOSITION 17. *Let $R$ be a noetherian ring, separated and complete for the $p$-adic topology and let $C$ be a complex of finitely generated $p$-torsion free $R$-modules. Suppose that $m$ is a natural number and $\nu.$ is a sequence of natural numbers such that $\nu_n - nm$ is eventually increasing and* $\lim \nu_n - nm = \infty$. *Then the natural maps*

$$H^0(C) \otimes \mathbf{Q} \longrightarrow \left(\varprojlim H^0(C/p^{\nu_n}C)\right) \otimes \mathbf{Q} \xrightarrow{\xi^m} \left(\varprojlim{}^m H^0(C/p^{\nu_n}C)\right) \otimes \mathbf{Q}$$

*are isomorphisms. If $y. \in \varprojlim^m H^0(C/p^{\nu_n}C)$, then modulo $p^\infty$-torsion, $p^{\mu+t}y.$ lies in the image of $H^0(C)$, where*

$$\mu =: \sup\{nm + m - \nu_n : n \in bfN\}$$

*and $p^t$ kills the $p^\infty$-torsion of $H^1(C)$.*

*Proof.* If $M$ is an abelian group, $p$ is a prime number, and $k$ is an integer, then for any element $x$ of $M \otimes \mathbf{Q}$, we shall say that $\mathrm{ord}_p(x) \geqslant k$ if and only if $p^{-k}x$ lies in the image of the natural map $M \to M \otimes \mathbf{Q}$. The essential part of the proof is contained in the following lemma.

LEMMA 18. *Suppose that $M$ is an abelian group whose $p^\infty$-torsion subgroup is bounded* (i.e. *killed by some power of $p$.) Let $m$ be a nonnegative integer and $\{\nu_n : n \in bfN\}$ a sequence of natural numbers such that $\nu_n - nm$ is eventually increasing and* $\lim(\nu_n - nm) = \infty$. *Let $M_n =: M/p^{\nu_n}M$, with the obvious transition maps, forming an inverse system $M.$. Then*

$$\xi^m\colon \left(\varprojlim M.\right) \otimes \mathbf{Q} \to \left(\varprojlim{}^m M.\right) \otimes \mathbf{Q}$$

*is an isomorphism. Furthermore, if $y. \in \varprojlim^m M.$ and if $\mu =: \sup\{nm + m - \nu_n : n \in bfN\}$, then modulo torsion $p^\mu y.$ lies in the image of $\varprojlim M.$.*

*Proof.* First let us suppose that $M$ is torsion free. If $\hat{M} =: \varprojlim M/p^{\nu_n}M$, then $\hat{M}$ is $p$-adically separated and complete, and the maps $M_{\nu_n} \to \hat{M}_{\nu_n}$ are isomorphisms. Thus we may as well (and shall) assume that $M$ is itself separated and complete. Suppose that $y. \in \varprojlim^m M.$. For each $n \in$ bf N, let $\tilde{y}_n \in M$ be any lifting of $y_n$ and let $\tilde{x}_n =: p^{-nm}\tilde{y}_n \in M \otimes \mathbf{Q}$. I claim that $\{\tilde{x}_n\}$ is a Cauchy sequence. In fact, the compatibility condition satisfied by the sequence $y.$ implies that $p^{nm+m}(\tilde{x}_n - \tilde{x}_{n+1}) = p^m\tilde{y}_n - \tilde{y}_{n+1}$ maps to zero modulo $p^{\nu_n}M$ and hence that $\mathrm{ord}(p^{nm+m}(\tilde{x}_n - \tilde{x}_{n+1})) \geqslant \nu_n$. It follows that

$$\mathrm{ord}(\tilde{x}_n - \tilde{x}_{n+1}) \geqslant \nu_n - nm - m, \tag{10}$$

and as this approaches infinity, our sequence is indeed Cauchy. Let $x \in M \otimes \mathbf{Q}$ be its limit. In order to show that $x$ maps to the image of $y$ in $(\varprojlim^m M.) \otimes \mathbf{Q}$, choose any $N$ such that $\nu_j - jm \geqslant \nu_n - nm$ whenever $j \geqslant n \geqslant N$. Then for such $j$ and $n$,

$$\operatorname{ord}(p^{nm}\tilde{x}_j - p^{nm}\tilde{x}_{j+1}) \geqslant nm + \nu_j - jm - m \geqslant \nu_n - m,$$

and hence by induction on $j \geqslant n$ we can prove that

$$\operatorname{ord}(p^{nm}\tilde{x}_j - \tilde{y}_n) \geqslant \nu_n - m$$

for all $j \geqslant n$. It follows that the same is true in the limit, and hence

$$\operatorname{ord}(p^{nm+m}x - p^m\tilde{y}_n) \geqslant \nu_n.$$

This tells us that in fact $p^m x$ maps to $p^m y_n$ in $M_{\nu_n}$, and so $x$ maps to $y.$ in $\varprojlim^m M. \otimes \mathbf{Q}$.

For the uniqueness, suppose that $x \in M$ maps to zero in $(\varprojlim^m M.) \otimes \mathbf{Q}$. Then there exists a $t \in \mathrm{bf\,N}$ such that $\operatorname{ord}(p^{t+nm}x \geqslant \nu_n)$ for all $n \in \mathrm{bf\,N}$. Then $\operatorname{ord}(x) \geqslant \nu_n - t - nm$ for all large $n$, and it follows that $x = 0$. Finally, observe that $\operatorname{ord}(x_0) \geqslant 0$, and use (10) and induction to prove that $\operatorname{ord}(p^\mu \tilde{x}_n) \geqslant 0$ for all $n$, and hence that the same is true in the limit.

To prove the general case, let $M_f$ denote the quotient of $M$ by its $p^\infty$-torsion subgroup $M_t$ Because the latter is killed by a power of $p$, it is clear that the kernels of the maps $\varprojlim M. \to \varprojlim M_f.$ and $\varprojlim^m M. \to \varprojlim^m M_f.$ are precisely the $p^\infty$-torsion subgroups. Furthermore, for each $n$ we have an exact sequence:

$$0 \to M_t/p^{\nu_n}M \cap M_t \to M/p^{\nu_n}M \to M_f/p^{\nu_n}M_f \to 0,$$

and since the maps of the inverse system $M_t/p^{\nu_n}M \cap M_t$ are surjective, so is the map $\varprojlim M. \to \varprojlim M_f..$ Then the commutative diagram

$$\begin{array}{ccc}
\varprojlim M. \otimes \mathbf{Q} & \xrightarrow{\cong} & \varprojlim M_f. \otimes \mathbf{Q} \\
\downarrow & & \downarrow \cong \\
\varprojlim^m M. \otimes \mathbf{Q} & \longrightarrow & \varprojlim^m M_f. \otimes \mathbf{Q}
\end{array}$$

shows that the lemma is true in general.

REMARK 19. The assumption that $M$ have bounded $p$-torsion is an important limitation of our method. In particular, if we start with an $M$ which is assumed to be $p$-adically separated and complete, then our method only gives us an element in the quotient of $M$ by the closure of the $p$-torsion subgroup tensored with $\mathbf{Q}$, not in $M \otimes \mathbf{Q}$.

To prove the proposition, start with the exact sequence

$$0 \to H^0(C)/p^{\nu_n}H^0(C) \to H^0(C/p^{\nu_n}C) \to \operatorname{Tor}_1^R(H^1(C), R_{\nu_n}) \to 0.$$

Now $\operatorname{Tor}_1^R(H^1(C), R_{\nu_n})$ is just the kernel of multiplication by $p^{\nu_n}$ on $H^1(C)$, and the union over all $\nu_n$ of these is a submodule of $H^1(C)$. Because $H^1(C)$ is noetherian, this submodule is finitely generated, and so there exists an integer $t$ such that $p^t$ annihilates it. If $y. \in \varprojlim^m H^0(C/p^{\nu_n}C)$, then each $p^t y_n$ belongs in fact to $H^0(C)/p^{\nu_n}H^0(C)$, and the sequence of elements thus constructed lies in $\varprojlim^m H^0(C)/p^{\nu_n}H^0(C)$. As is well-known, $H^0(C)$ is finitely generated over $R$ and $p$-adically separated and complete, and the map $H^0(C) \to \varprojlim H^0(C/p^{\nu_n}c)$ is bijective. Thus the proposition follows from the lemma.

*Proof of Theorem 1.* To simplify the notation we just give the proof for $n = 0$, writing $E$ for $E^0$. Suppose that $\underline{T} = \operatorname{Spf} A$, and recall from [1, 7.24.3] that $R\Gamma(X/T, E)$ is a perfect complex of $A$-modules. Let us consider the following diagram, constructed so that the top squares are Cartesian:

$$\begin{tikzcd}
X \arrow[r, "F_{X/Z}"] \arrow[dr, "f"'] & X^{\sharp 1} \arrow[r] \arrow[d, "f^\sharp"] & X^{\flat 1} \arrow[r] \arrow[d, "f^\flat"] & X \arrow[d, "f"] \\
& Z \arrow[r, "F_{Z/\underline{Z}}"] \arrow[d] & Z^{(1)} \arrow[r, "\pi_{Z/\underline{Z}}"] \arrow[d] & Z \\
& T \arrow[r, "F_{T/\underline{T}}"] & T^{(1)} &
\end{tikzcd}$$

Denoting the pullbacks of $E^1$ to $X^{\flat 1}$ and $X^{\sharp 1}$ by the obvious superscripts and using the base changing theorem in crystalline cohomology, we see that there is an isomorphism:

$$LF^*_{T/\underline{T}} R\Gamma(X^\flat/T^{(1)}, E^{\flat 1}) \cong R\Gamma(X^{\sharp 1}/T, E^{\sharp 1}).$$

The map $\underline{F}_{T/\underline{T}}$ is the identity, and hence the $LF^*_{T/\underline{T}}$ can be omitted. Thus we have an isomorphism:

$$R\Gamma(X^{\flat 1}/T^{(1)}, E^{\flat 1}) \cong R\Gamma(X^\sharp/T, E^{\sharp 1}). \tag{11}$$

We also have an isomorphism $F^*_{X/Z}E^{\sharp 1} \cong F^*_X E^1$, and hence $\Phi$ induces a map $F^*_{X/Z}E^{\sharp 1} \to E$. Since $X/Z$ is perfectly smooth, $F_{X/Z}$ is the *exact* Frobenius morphism, and so we have a relative F-span in the sense of [14, 5.2.1]. In particular, by [14, 7.3.7], we have morphisms:

$$R\Gamma(X^{\sharp 1}/T, E^{\sharp 1}) \to R\Gamma(X/T, E) \quad \text{and} \quad R\Gamma(X/T, E) \to R\Gamma(X^{\sharp 1}/T, E^{\sharp 1})$$

such that the composite in either direction is multiplication by $p^{m+d}$. Composing these arrows with (11) and iterating, we see that there are maps:

$$\begin{aligned}
\phi_n &: R\Gamma(X^{\flat n}/T^{(n)}, E^{\flat n}) \longrightarrow R\Gamma(X/T, E) \\
v_n &: R\Gamma(X/T, E) \longrightarrow R\Gamma(X^{\flat n}/T^{(n)}, E^{\flat n})
\end{aligned} \tag{12}$$

such that $v_n \circ \phi_n$ and $\phi_n \circ v_n$ are multiplication by $p^{n(m+d)}$. There are similar diagrams and morphisms relative to $S =: T^{\natural}$.

Now we see from Lemma 14 that the reduction of $T^{(n)}$ modulo $(p^n!)$ is hollow, and hence canonically isomorphic to the reduction of $S^{(n)}$ modulo $(p^n!)$. Thus we have a natural isomorphism

$$\begin{aligned}\epsilon_n\colon & R\Gamma(X^{\flat n}/T^{(n)}, E^{\flat n}) \otimes^{\mathbf{L}} W/p^n!W \\ & \to R\Gamma(X^{\flat n}/S^{(n)}, E^{\flat n}) \otimes^{\mathbf{L}} W/p^n!W. \qquad (13)\end{aligned}$$

We define $\rho_n$ by means of the following commutative diagram:

$$\begin{array}{ccc} R\Gamma(X/T, E) \otimes^{\mathbf{L}} W/p^n!W & \xrightarrow{v_n} & R\Gamma(X^{\flat n}/T^{(n)}, E^{(n)}) \otimes^{\mathbf{L}} W/p^n!W \\ \downarrow{\scriptstyle \rho_n} & & \downarrow{\scriptstyle \epsilon_n} \\ R\Gamma(X/S, E) \otimes^{\mathbf{L}} W/p^n!W & \xleftarrow{\phi_n} & R\Gamma(X^{\flat n}/S^{(n)}, E^{(n)}) \otimes^{\mathbf{L}} W/p^n!W \end{array}$$

One verifies easily that the reduction of $\rho_{n+1}$ modulo $(p^n!)$ is $p^{m+d}\rho_n$ and that all our constructions are compatible with base change and Frobenius maps. Then the existence of $\rho$, as well as its inverse, follows from Proposition 17 applied to $C =: \mathrm{Hom}(R\Gamma(X/T, E), R\Gamma(X/T^{\natural}, E))$ and $\nu_n =: \mathrm{ord}_p(p^n!) = (p^n - 1)/(p-1)$.

It is clear that $\rho$ is compatible with the base change maps corresponding to morphisms of enlargements. To check compatibility with morphisms of $F^{\infty}$-spans, suppose that $h\colon E' \to E$ is such a morphism, of level $\ell$. Then $h$ induces morphisms on $R\Gamma$, which we denote by the same letter. It is easy to verify that we have

$$\phi_n \circ h_n^{\flat} = p^{n\ell} h_0 \circ \phi_n' \quad \text{and} \quad h_n^{\flat} \circ v_n' = p^{n\ell} v_n \circ h_0$$

for all $n$. Then we compute modulo $(p^n!)$

$$\begin{aligned} p^{n\ell} h_0 \circ \rho_n' &= p^{n\ell} h_0 \circ \phi_n' \circ \epsilon_n' \circ v_n' \\ &= h_0 \circ p^{n\ell} \phi_n' \circ \epsilon_n' \circ v_n' \\ &= \phi_n \circ h_n^{\flat} \circ \epsilon_n' \circ v_n' \\ &= \phi_n \circ \epsilon_n \circ h_n^{\flat} \circ v_n' \\ &= \phi_n \circ \epsilon_n \circ p^{n\ell} v_n \circ h_0 \\ &= p^{n\ell} \rho_n \circ h_0 \end{aligned}$$

This shows that $\{h_0 \circ \rho_n'\}$ and $\{\rho_n \circ h_0\}$ have the same image under the map $\xi^{m+d, m+d+\ell}$, and it follows that they correspond to the same element in $R\,\mathrm{Hom} \otimes \mathbf{Q}$.

REMARK 20. If $\mu =: m + \sup\{nm - (p^n - 1)/(p-1)\}$ and $p^t$ kills the torsion of $R^1\,\mathrm{Hom}(\mathrm{R}\Gamma(X/T, E), \mathrm{R}\Gamma(X'/T', E'))$, then we see that $p^{\mu+t}\rho$ comes from an element of $\mathrm{Hom}(\mathrm{R}\Gamma(X/T, E), \mathrm{R}\Gamma(X'/T', E')$ in Theorem 1. If $T$ and

$T'$ are $p$-adic enlargements, then in place of $\mu$ we can use the slightly better $m + \sup\{nm - p^n\}$.

We next give some variations and generalizations of our main comparison Theorem 1. In the next section we use the language of convergent crystals to package some of these results in a more systematic form.

DEFINITION 21. We shall say that two logarithmic schemes $Z$ and $Z'$ are "almost the same" if $\underline{Z} = \underline{Z}'$ and additionally the sheaves of monoids $M_Z$ and $M_Z{}'$ are the same. We say that thickenings $T$ of $Z$ and $T'$ of $Z'$ are "almost the same" if the underlying thickenings of schemes are the same and $M_T = M_{T'}$. If $T$ is almost the same as $T'$ and $S$ is almost the same as $S'$, then we say that two morphisms $f\colon S \to T$ and $f'\colon S' \to T'$ are "the same" if they comprise the same morphisms of underlying schemes and of monoids.

For example, if $Z$ is hollow and $f\colon S \to T$ is any morphism of enlargements of $Z$, then $f^\natural\colon S^\natural \to T^\natural$ is "the same" as $f$.

THEOREM 2. *Suppose that $Z/k$ and $Z'/k$ are affine f-s log schemes which are almost the same, and suppose that their log structures are constant. (This implies that $Z_{\mathrm{red}} = Z'_{\mathrm{red}}$.) Suppose that $f\colon X \to Z$ and $f'\colon X' \to Z'$ are perfectly smooth morphisms of relative dimension d, whose restrictions to $Z_{\mathrm{red}}$ are the same. Suppose also that we are given locally free $F^\infty$-spans $E$ and $E'$ on $X/W$ and $X'/W$, respectively, which become the same over $Z_{\mathrm{red}}$. If $T$ and $T'$ are affine objects of* Cris $(Z/W)_\infty$ *which are almost the same, there is a canonical isogeny:*

$$R\Gamma(X/T_{tf}, E) \otimes \mathbf{Q} \xrightarrow{\rho_{T,T'}} R\Gamma(X'/T'_{tf}, E') \otimes \mathbf{Q}.$$

*The family of isogenies $\rho$ satisfies the obvious cocycle conditions and is compatible with base change: if $f\colon S \to T$ and $f'\colon S' \to T'$ are the same then the diagram*

$$\begin{array}{ccc} Lf^*R\Gamma(X/T_{tf}, E) & \xrightarrow{\theta_f} & R\Gamma(X/S_{tf}, E) \\ \Big\downarrow {\scriptstyle f^*(\rho_{T,T'})} & & \Big\downarrow {\scriptstyle \rho_{S,S'}} \\ Lf^*R\Gamma(X/T'_{tf}, E) & \xrightarrow{\theta_{f'}} & R\Gamma(X/S'_{tf}, E) \end{array}$$

*commutes.*

*Proof.* The first difficulty we have to overcome is that the enlargements we were considering in Theorem 1 are noetherian, which is not necessarily the case here. We begin by considering the case in which $f = f'$ and $Z = Z'$ is hollow; in this case we may and shall further suppose without loss of generality that $T' = T^\natural$. Since $Z$ is affine and its log structure is constant, we can find an exact closed immersion of $Z$ into an affine log smooth formal scheme $Y/W$. Since $Y$ is affine and formally smooth, the absolute Frobenius endomorphism of $Y_k$ lifts to $Y$. Let $\lambda_D\colon D_Z(Y) \to Y$ be the logarithmic divided power envelope of $Z$ in $Y$, and let $T_Z(Y)$ be the universal $p$-adic enlargement of $Z$ in $Y$ [12, 2.3]. There is a commutative diagram:

$$\begin{array}{ccccc} D_Z(Y)_{tf} & \xrightarrow{\tilde{F}} & T_Z(Y) & \xrightarrow{\tilde{\tau}} & D_Z(Y)_{tf} \\ \downarrow{\scriptstyle \lambda_{tf}} & & \downarrow{\scriptstyle \tau} & \swarrow{\scriptstyle \lambda_{tf}} & \\ Y & \xrightarrow{F_Y} & Y & & \end{array}$$

Since $(F_Y \circ \lambda_D)^*$ takes the ideal of $Z \subseteq Y$ into $p\mathcal{O}_D$, $F_Y \circ \lambda_{tf}$ makes $D_Z(Y)_{tf}$ into a $p$-adic enlargement of $Z$, and hence we find the map $\tilde{F}$ of the diagram above. Similarly, $\tilde{\tau}$ is the PD-morphism coming from the universal property of $D_Z(Y)$. Since $\tilde{F}$ is automatically a PD-morphism, we can conclude that $\tilde{\tau} \circ \tilde{F} = F_{D_{tf}}$, the map $D_{tf} \to D_{tf}$ induced by $F_Y$.

Pulling back the isomorphism $\rho_T$ provided by Proposition 1 provides us with an isomorphism

$$R\Gamma(X/T_Z(Y), E^1) \otimes \mathbf{Q} \xrightarrow{\rho_T} R\Gamma(X/T_Z^\natural(Y), E^1) \otimes \mathbf{Q}.$$

We can assemble the pullback of this map by $\tilde{F}$, the base change maps of cohomology, and the Frobenius morphisms into the following diagram:

$$\begin{array}{ccc} L\tilde{F}^* R\Gamma(X/T_Z(Y), E^1) \otimes \mathbf{Q} & \xrightarrow{\tilde{F}^*\rho_T} & L\tilde{F}^* R\Gamma(X/T_Z^\natural(Y), E^1) \otimes \mathbf{Q} \\ \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \cong} \\ R\Gamma(X^\sharp/D_Z(Y)_{tf}, F_X^* E^1) & \longrightarrow & R\Gamma(X^\sharp/D_Z(Y)_{tf}^\natural, F_X^* E^1) \\ \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \phi^\natural} \\ R\Gamma(X/D_Z(Y)_{tf}, E) \otimes \mathbf{Q} & & R\Gamma(X/D_Z(Y)_{tf}^\natural, E) \otimes \mathbf{Q} \end{array}$$

As we have seen, the arrows $\phi$ and $\phi^\natural$ are isogenies, and hence we can use the diagram to define an isogeny $\rho_D$ along the bottom. It is clear that if $g: Y' \to Y$ is a morphism of log smooth schemes and liftings of Frobenius, then $\rho_D$ pulls back to $\rho_{D'}$.

Now suppose that $f$ and $f'$ are as in the statement of the theorem. The previous paragraph tells us that the result is true if we restrict to $Z_{\mathrm{red}} = Z'_{\mathrm{red}}$. By Lemma 11, for large $r$ $Z^{(r)}$ and $Z'^{(r)}$ are hollow and hence are the same, and furthermore the maps

$$\pi^{(r)}_{Z/\underline{Z}}: Z^{(r)} \to Z \quad \text{and} \quad \pi^{(r)}_{Z'/\underline{Z}'}: Z'^{(r)} \to Z'$$

factor through $Z_{\mathrm{red}}$. Thus the case we have already proved applies to give us the map $\rho^{(r)}$ in the diagram below. The arrows $\phi_r$ and $v_r$ come from diagram (12), and we use the diagram to define $\rho$.

$$\begin{array}{ccc} R\Gamma(X/T, E) \otimes \mathbf{Q} & \xrightarrow{v_r} & R\Gamma(X^{\flat r}/T^{(r)}, E^{\flat r}) \otimes \mathbf{Q} \\ \downarrow{\scriptstyle p^{mr+dr}\rho} & . & \downarrow{\scriptstyle \rho^{(r)}} \\ R\Gamma(X/S, E) \otimes \mathbf{Q} & \xleftarrow{\phi_r} & R\Gamma(X^{\flat r}/S^{(r)}, E^{\flat r}) \otimes \mathbf{Q} \end{array} \tag{14}$$

This diagram defines the morphism $\rho$. It is easy to verify that it is independent of the choice of $r$ and satisfies the claimed compatibilities.

Let us return to the situation of Example 13 and explain how to deduce the isomorphism of Hyodo-Kato [8] from the above theorem. If $Y \to T$ has semi-stable reduction, then $Y^\times \to T^\times$ is perfectly smooth, and if $Y/T$ is (classically) smooth then $Y^\times \to T^\times$ is solid. In fact, let $Y^\times \to T^\times$ be any proper and perfectly smooth morphism of logarithmic schemes, and let $X^\times$ (resp. $Y_{Z^\times}$) denote the pullback of $Y^\times/\hat{T}^\times$ to $\xi_V^\times$ (resp. to $Z^\times$). Choose a uniformizer $\pi$ of $V$, and let $X^\natural/Z^\natural$ denote the pullback of $X^\times$ to $Z^\natural$ via the corresponding projection $s_\pi \colon Z^\natural \to \xi_V^\times$. The choice of $\pi$ (modulo $k^*$) also determines a morphism $T^\natural \to S_V^\natural$, and by the base change theorem for crystalline cohomology we obtain an isomorphism

$$R\Gamma(X^\times/S_V^\natural) \otimes V \cong R\Gamma(X^\natural/T^\natural).$$

According to Theorem 2, we also have an isomorphism

$$R\Gamma(X^\natural/T^\natural) \otimes \mathbf{Q} \cong R\Gamma(Y_{Z^\times}^\times/\hat{T}^\times) \otimes \mathbf{Q}.$$

Finally, we can compute crystalline cohomology using the De Rham cohomology of the given lifting $Y^\times/T^\times$ to obtain an isomorphism:

$$R\Gamma(\hat{Y}_{Z^\times}^\times/\hat{T}^\times) \cong R\Gamma(Y^\times, \Omega^\cdot_{\hat{Y}^\times/\hat{T}^\times}) \cong R\Gamma(Y^\times, \Omega^\cdot_{Y^\times/T^\times}).$$

Assuming that the log structure on $Y_\eta^\times$ is trivial, we can assemble these to get the Hyodo-Kato isomorphism:

$$R\Gamma(X^\times/S_V^\natural) \otimes K \cong R\Gamma(Y/K, \Omega^\cdot_{Y_\eta/K}) \tag{15}$$

In particular, we have:

THEOREM 3 (Hyodo-Kato). *Suppose that $Y/T$ is proper and with semi-stable reduction. Then there is a canonical isomorphism*

$$R\Gamma(X^\times/S_V^\natural) \otimes K \cong R\Gamma(Y/K, \Omega^\cdot_{Y_\eta/K})$$

In the next section we shall show how to generalize this isomorphism to the case of coefficients, and how to express it in a somewhat more canonical way, using the language of convergent crystals.

## 4. Crystals and isocrystals

In this section we study the meaning of the isomorphism $\rho$ of Theorems 1 and 2 in the context of crystals, and we attempt to explain how it is related to Christol's transfer theorem [3].

We begin with a brief review of the theory of the residue mapping (log of monodromy) for logarithmic crystals. Suppose that $E$ is a crystal of $\mathcal{O}_{Z/W}$-modules on a fine saturated log scheme $Z/k$. For each object $T$ of Cris $(Z/W)$, we have sheaves $\overline{M}^g_T$ and $\Omega^1_{T/\underline{T}}$ of abelian groups on $T$, and as $T$ varies these define sheaves $\overline{M}^g$ and $R_{Z/W}$ on Cris $(Z/W)$. Furthermore, we have a surjective morphism of sheaves of $\mathcal{O}_{Z/W}$-modules:

$$\mathcal{O}_{Z/W} \otimes \overline{M}^g \to R_{Z/W}. \tag{16}$$

If the log structure of $Z$ is constant, then $\mathcal{O}_{Z/W} \otimes \overline{M}^g$ is quasicoherent and in fact forms a crystal of $\mathcal{O}_{Z/W}$-modules. We write $R^f_{Z/W}$ for the quotient of $R_{Z/W}$ by its $p$-torsion.

For each $T \in$ Cris $(Z/W)$, $\Omega^1_{T/\underline{T}}$ can be identified with the ideal of $T$ in the exact first infinitesimal neighborhood $P^1_{T/\underline{T}}$ of the diagonal embedding $T \to T \times_{\underline{T}} T$. We may endow $\Omega^1_{T/\underline{T}}$ with the trivial PD-structure (which is automatically compatible with the PD-structure on $J_T$), and then $P^1_{T/\underline{T}}$ becomes a PD-thickening $P$ of $Z/W$. If $E$ is a crystal of $\mathcal{O}_{Z/W}$-modules, we have canonical identifications $p_2^* E_T \cong E_P \cong p_1^* E$, and the difference $p_2^* - p_1^*$ then induces a morphism $E_T \to E_T \otimes \Omega^1_{T/\underline{T}}$. This morphism is $\mathcal{O}_T$-linear, and as $T$ varies it defines a morphism of sheaves of $\mathcal{O}_{Z/W}$-modules:

$$R\colon E \to E \otimes R^1_{Z/W},$$

which we call the "residue mapping" of $E$. As usual, we can prolong $R$ to a sequence of maps

$$R^q\colon E \otimes R^q_{Z/W} \to E \otimes R^{q+1}_{Z/W}$$

for all $q$, and $R^{q+1} \circ R^q = 0$. For each $\mathcal{O}_T$-algebra $A$ and each

$$\partial \in \mathbf{T}_{T/\underline{T}}(A) =: \mathrm{Hom}(\Omega^1_{T/\underline{T}}, A),$$

we get an $A$-linear endomorphism $R(\partial)$ of $E \otimes A$. If $E$ is a convergent isocrystal on $Z/W$, then there is a similar theory for each enlargement $T$ of $Z/W$. We shall say that $R$ is nilpotent (resp. that $\mathbf{Q} \otimes R$ is nilpotent) if $R(\partial)$ is nilpotent for every $\mathcal{O}_T$-algebra $A$ (resp., for every $\mathbf{Q} \otimes \mathcal{O}_T$-algebra) and every $\partial$.

DEFINITION 22. Let $E$ be a crystal on $Z/W$. A "transfer structure" on $E$ is a family of isogenies $\rho =: \{\rho_{T,T'}\colon \mathbf{Q} \otimes E_T \to \mathbf{Q} \otimes E_{T'}\}$, indexed by the set of pairs $(T, T')$ of thickenings of $Z/W$ which are almost the same (Definition 21), satisfying the following conditions:

1. If $T, T'$, and $T''$ are almost the same, then $\rho_{T',T''} \circ \rho_{T,T'} = \rho_{T,T''}$; furthermore $\rho_{T,T} = \mathrm{id}$.
2. The family $\rho$ is compatible with pullbacks: if $f\colon S \to T$ and $f'\colon S' \to T'$ are the same, then $\rho_{S,S'} \circ \theta_f = \theta_{f'} \circ f^*(\rho_{T,T'})$ (c.f. Theorem 2).

The category of "crystals with transfer structure" is defined in the obvious way, and in particular its morphisms are required to be compatible with the transfer structure. There are obvious analogies for convergent isocrystals and $p$-adically convergent isocrystals.

When $Z$ is hollow, we can give a more explicit description of transfer structures. Consider the full subcategory $\mathrm{hCris}(Z/W)$ of $\mathrm{Cris}\,(Z/W)$ consisting of those PD-thickenings whose log structure is also hollow. (This is a special case of Kato's "narrow crystalline site," c.f. [9]). There is the obvious analog for enlargements, and the notion of a crystal on $\mathrm{hCris}(Z/W)$ (resp. on $\mathrm{hEnl}(Z/W)$) is defined in the obvious way. If $T$ is any object of $\mathrm{Cris}\,(Z/W)$, we let $h(T)$ or $T^\natural$ denote the hollowing out (Remark 7) of $T$, which is an object of $\mathrm{hCris}(Z/W)$. If $E$ is a crystal on $\mathrm{hCris}(Z/W)$ we can define a crystal $h^*E$ on $\mathrm{Cris}\,(Z/W)$ by letting $h^*E_T =: E_{T^\natural}$ for any object $T$ of $\mathrm{Cris}\,(Z/W)$. Then $h^*E_T$ has an obvious transfer structure: if $S$ is almost the same as $T$, then $S^\natural = T^\natural$, and we let $\rho_{S,T}$ be the identity map $E_{S^\natural} \to E_{T^\natural}$. There is also an obvious restriction functor $E \mapsto E^\natural$ which takes crystals on $\mathrm{Cris}\,(Z/W)$ to crystals on $\mathrm{hCris}(Z/W)$. Then it is clear that we can view a transfer structure on $E$ as an isogeny $\rho\colon h^*(E^\natural) \to E$.

When $Z$ is a *split* hollow log scheme, we can make everything even more explicit. Let $Q =: \overline{M}_Z$, and recall (Remark 7) that the splitting of $Z$ allows us to identify $Z$ with $\underline{Z}_Q$.

PROPOSITION 23. *Suppose that $\underline{Z}_Q$ is a split hollow log scheme. Then there is an equivalence between the category of crystals on* $\mathrm{hCris}(Z/W)$ *and the category of pairs $(\underline{E}, \underline{R})$, where $\underline{E}$ is a crystal on $\underline{Z}/W$ and $\underline{R}\colon \underline{E} \to \underline{E} \otimes \underline{R}^1_{Z/W}$ is a morphism of crystals of $\mathcal{O}_{Z/W}$-modules such that for every $A$ and every $\partial \in \mathbf{T}_{Z/\underline{Z}}(A)$, $\rho(\partial)(\rho(\partial)-1)\cdots(\rho(\partial)-n)$ tends to zero p-adically.*

*Proof.* For any object $T$ of $\mathrm{Enl}(\underline{Z}/W)$, $T_Q$ is an object of $\mathrm{hCris}(Z/W)$, and so if $E$ is a crystal on $\mathrm{hCris}(Z/W)$, we can define $\underline{E}_T =: E_{T_Q}$. It is clear that this construction defines a crystal on $\mathrm{Enl}(\underline{Z}/W)$, and that $E \mapsto \underline{E}$ is functorial. In particular, we obtain a morphism of crystals $\underline{R}\colon \underline{E} \to \underline{E} \otimes \underline{R}_{Z/W}$.

We show that our functor is an equivalence by working locally on $Z$, with the aid of a closed immersion of $\underline{Z}$ in a smooth $\underline{Y}/W$. Let $Y =: \underline{Y}_Q$, so that we have an exact closed immersion $Z \to Y$, let $D$ be the PD-envelope of $Z$ in $Y$, and let $D(1)$ be the (exact) PD-envelope of $Z$ in $Y \times Y$. Then if $T$ is any other object of $\mathrm{hCris}(Z/W)$, locally on $T$ there exists a map (not unique) $T \to D$. Then the usual pattern applies, and we see that to give a crystal on $\mathrm{hCris}(Z/W)$ is the same as giving its value $E_D$ on $D$ and an isomorphism $\epsilon\colon p_2^*E_D \to p_1^*E_D$ on $D(1)$, satisfying the cocycle condition. Furthermore, the data of $\epsilon$ is equivalent to that of a connection

$$E \xrightarrow{\nabla_D} E \otimes \Omega^1_{Y/W} \cong \left(E \otimes \Omega^1_{\underline{Y}/W}\right) \oplus \left(E \otimes \Omega^1_{Y/\underline{Y}}\right).$$

Thus we can view $\nabla_D$ as a pair $(\underline{\nabla}, R)$, where $\underline{\nabla}$ defines a crystal on $\underline{Z}/W$ and $R$ is the horizontal endomorphism with the convergence property described. (See [10] or [14, 1.1.8] for details.)

REMARK 24. Instead of choosing a splitting of $M_Z$, we could try to choose an embedding of $\underline{Z}$ in a smooth $\underline{Y}$ over $W$ and an extension of $P$ by $\mathcal{O}^*_Y$ extending $M_Z$. Then if we let $Y$ be the hollow log scheme obtained from $Y$ and this extension, the universal $D_Z(Y)$ is an object of $\mathrm{hCris}(Z/W)$, and taking the value of $E$ on this object we find an $\mathcal{O}_D$-module $E_D$ together with an integrable connection $\nabla\colon E_D \to E_D \otimes \Omega^1_{Y/W}$. There is an exact sequence

$$0 \to \Omega^1_{\underline{Y}/W} \to \Omega^1_{Y/W} \to \Omega^1_{Y/\underline{Y}} \to 0,$$

and in particular $\nabla$ induces a morphism $R_D\colon E_D \to E_D \otimes \Omega^1_{Y/\underline{Y}}$. However, there is no canonical splitting of the above sequence, in general, and hence no way to construct a connection $E_D \to E_D \otimes \Omega^1_{\underline{Y}/W}$ without choosing a splitting of $M_Z$. Of course, in the special case in which $\underline{Z} = \operatorname{Spec} k$, the datum of such a connection is empty, and so in this case a splitting of $M_Z$ is not necessary.

COROLLARY 25. *Suppose that $E$ is a convergent or $p$-adically convergent isocrystal on $Z/W$, where $Z$ is hollow. Then for every hollow enlargement $T$ of $Z/W$, $E_T$ is a flat $\mathbf{Q} \otimes \mathcal{O}_T$-module.*

*Proof.* This question is local on $T$, so we may as well assume that $M_T$ is split, i.e. $T \cong \underline{T}_{\overline{M}}$ and $E_T \cong \underline{E}_{\underline{T}}$. According to [12, 2.9], every $p$-adically convergent crystal on $\underline{Z}/W$ is flat, so $\underline{E}_{\underline{T}}$ is flat over $\mathbf{Q} \otimes \mathcal{O}_T$.

It easy to describe the "logarithmic part of the monodromy" of a crystal on $\mathrm{hCris}(Z/W)$. Namely, if $T$ is an object of $\mathrm{hCris}(Z/W)$ and $g \in \mathrm{Aut}(T/\underline{T})$, then the crystal structure gives us an isomorphism

$$\theta_g\colon \underline{g}^* E_T \to E_T;$$

since $\underline{g}$ is the identity, $\theta_g$ is just a linear automorphism of $E_T$. The cocycle condition says that that $\theta_{gh} = \theta_h \circ h^*(\theta_g) = \theta_h \circ \theta_g$, so $g \mapsto \theta_g$ defines a right action of $\mathrm{Aut}(T/\underline{T})$ on $E_T$. Now $\mathrm{Aut}(T/\underline{T})$ is just the set of automorphisms of the extension

$$1 \to \mathcal{O}^*_T \to M^g_T \to \overline{M}^g_T \to 0$$

which restrict to the identity on $Z$. Thus

$$\mathrm{Aut}(T/\underline{T}) \cong \hat{\mathbf{G}}_M(T) =: \mathrm{Ker}\Big(\mathbf{G}_M(T) \to \mathbf{G}_M(Z)\Big).$$

Thanks to the divided powers on $J_T$, we have a morphism $\log\colon 1 + J_T \to J_T$ which induces a morphism

$$\log\colon \hat{\mathbf{G}}_M(T) \to \hat{\mathbf{T}}_M(T), \tag{17}$$

where $\hat{\mathbf{T}}_M(T) =: \mathrm{Ker}\Big(\mathbf{T}_M(T) \to \mathbf{T}_M(Z)\Big)$. When the divided powers on $J_T$ are nilpotent, log is an isomorphism, with inverse given by the usual exponential series exp. The following formula is due to Hyodo and Kato [8, Sect. 5]; it follows easily from the formula for $\epsilon: p_2^*E \to p_1^*E$; *c.f.* also [14, (1.1.8)].

PROPOSITION 26. *Suppose that $R_T: E_T \to E_T \otimes \overline{M}^g$ is integrable and that its reduction modulo p has nilpotent p-curvature. Then for any $u \in \hat{\mathbf{G}}_M(T)$,*

$$u^{R_T} =: \sum_{n=0}^{\infty}(u-1)^{[n]}R_T(R_T-1)(R_T-2)\cdots(R_T-n+1)$$

*converges. If $R_T$ is the connection matrix of a crystal on* $\mathrm{hCris}(Z/W)$*, then the monodromy action $\theta_T$ of $\hat{\mathbf{G}}_M(T)$ is given by $\theta_T(u) = u^{R_T}$. If $R_T(\partial)$ is nilpotent for all $\partial \in \mathbf{T}_M$, then $\theta$ factors through the logarithm morphism (17) and extends to an action of the algebraic group $\mathbf{T}_M$, in terms of which we can write:*

$$\theta_T(u) = \exp(R_T \log u).$$

REMARK 27. There are evident analogs of the above results for convergent isocrystals, in which the convergence condition is replaced by the statement that the monodromy map (26) should converge in the open tube of radius one.

We shall also need to give an explicit description of the functor $h^*$ on a split hollow log scheme $\underline{Z}_P$. Again we suppose that $\underline{Z} \subseteq \underline{Y}$ is a closed immersion, where $\underline{Y}/W$ is smooth. Let $\alpha_P: P \to \mathcal{O}_{\underline{Y}}[P]$ denote the monoid algebra associated to $P$. Then $\alpha_P$ defines a prelog structure on $\underline{X} =: \mathrm{Spec}_{\underline{Y}}(\mathcal{O}_{\underline{Y}}[P])$, and we let $X$ denote the corresponding log scheme (again writing $\alpha_P$ for the corresponding log structure). Then $X/\underline{Y}$ and $X/W$ are log smooth. Because $P^* = 0$ there is a section $\underline{Y} \to \underline{X}$ which gives an exact closed immersion $Y \to X$, where $Y =: \underline{Y}_P$. Let $\hat{X}$ denote the formal completion of $X$ along $Y$, and notice that the log structure of $\hat{X}$ is constant, but not hollow. We find a morphism of log schemes $\hat{X}^\natural \to Y$, and correspondingly morphisms of complexes:

$$\Omega^{\cdot}_{Y/W} \xrightarrow{\pi^*} \Omega^{\cdot}_{\hat{X}^\natural/W} \xrightarrow{\theta} \Omega^{\cdot}_{\hat{X}/W}, \tag{18}$$

where $\pi^*$ is the map induced from the morphism $X^\natural \to Y$ and $\theta$ is the map constructed in Claim 8. Now if $D$ is the divided power envelope of $Z$ in $X$, it follows that a crystal on $Z$ is determined by a sheaf $E_D$ of $\mathcal{O}_D$-modules endowed with an integrable connection

$$\nabla_D: E_D \to \Omega^1_{X/W} \otimes E_D.$$

Similarly, if $T$ is the divided power envelope of $Z$ in $Y$, a crystal on $\mathrm{hCris}(Z/W)$ is determined by a sheaf $E_T$ of $\mathcal{O}_T$-modules endowed with an integrable connection

$$\nabla_T: E_T \to \Omega^1_{Y/W} \otimes E_T.$$

CLAIM 28. *If $E$ is a crystal on* $\mathrm{hCris}(Z/W)$, *then*

$$h^* E_D \cong \mathcal{O}_D \otimes_{\mathcal{O}_T} E_T,$$

*and $\nabla_D$ is the unique extension of $\nabla_T$ compatible with the map (18).*

*Proof.* The definition of $h^*E$ says that $h^*E_D = E_{D^\natural}$, and since there is a log map $D^\natural \to T$, we can identify $E_D$ with the pullback of $E_T$. To see that the connection is as claimed, we note that $(h^*E)_{D(1)} \cong E_{D(1)^\natural}$, where $D(1)$ is the universal enlargement of $Z$ in $X \times X$. We can thus identify $(h^*E)_{D(1)}$ with the pull back of $E_{T(1)}$ via the natural map $D(1)^\natural \to T(1)$. Now the compatibility with the connections follows from diagram (2) in the proof of Claim 8.

If $\underline{Z}$ is smooth over $k$, then the statement and proof of Proposition 23 show that crystals are determined by their values on local liftings (which are noetherian) together with a stratification. Then the method of [13, 0.7.2] proves:

PROPOSITION 29. *Suppose that $Z/k$ is a hollow log scheme and that $\underline{Z}/k$ is smooth. Then every convergent isocrystal on* $\mathrm{hEnl}(Z/W)$ *admits an integral structure of a crystal on $Z/W$.*

The nilpotence of the residue mapping for F-crystals is well-known. In fact, we shall see that it holds more generally, for $F^\infty$-spans.

PROPOSITION 30. *If $R$ is the residue mapping of an $F^\infty$-span on $Z/W$, then* $\mathrm{id}_{\mathbf{Q}} \otimes R$ *is nilpotent. The same is true for convergent $F$-isospans, at least if $\underline{Z}/k$ is smooth.*

*Proof.* We shall show that $R$ is, up to conjugacy, infinitely divisible by $p$. This follows from the following lemma.

LEMMA 31. *The map $F^*_{Z/\underline{Z}}\colon R_{Z^{(1)}/W} \to R^f_{Z/W}$ is divisible by $p$. Consequently, if $E'$ is a crystal of $\mathcal{O}_{Z/W}$-modules on $Z^{(1)}/k$, the residue mapping $R\colon E \to E \otimes \mathrm{R}^f_{Z/W}$ of $E =: F^*_{Z/\underline{Z}}E'$ is divisible by $p$.*

*Proof.* Although $\mathcal{O}_T \otimes \overline{M}^g$ is not necessarily quasicoherent, we have a commutative diagram with exact rows:

$$\begin{tikzcd}
\mathrm{Ker}^{(1)} \arrow[r] \arrow[d] & \mathcal{O}_{T^{(1)}} \otimes \overline{M}^g_T \arrow[r, "d^{(1)}"] \arrow[d, "\iota\otimes p"] & \Omega^1_{T^{(1)}/\underline{T}} \arrow[r] \arrow[d, "F^*_{T/\underline{T}}"] & 0 \\
\mathrm{Ker} \arrow[r] & \mathcal{O}_T \otimes \overline{M}^g_T \arrow[r, "d^f"] & \Omega^{1f}_{T/\underline{T}} \arrow[r] & 0
\end{tikzcd}$$

Here $T^{(1)}$ is the log scheme appearing in the relative Frobenius diagram (3); recall that the natural map $\iota\colon \mathcal{O}^{(1)}_T \to \mathcal{O}_T$ is an isomorphism. Note that $pd^f \circ (\iota \otimes \mathrm{id}) =$

$d^f \circ (\iota \otimes p) = F^*_{T/\underline{T}} \circ d^{(1)}$ annihilates $\mathrm{Ker}^{(1)}$, and since $\Omega^{1f}_{T/\underline{T}}$ is $p$-torsion free, so does $d^f \circ (\iota \otimes \mathrm{id})$. Thus there is a map

$$\psi_{T/\underline{T}}\colon \Omega^1_{T^{(1)}/\underline{T}} \to \Omega^{1f}_{T/\underline{T}}$$

such that $\psi_{T/\underline{T}} \circ d^{(1)} = d^f(\iota \otimes \mathrm{id})$. It is clear that $F^*_{T/\underline{T}} = p\psi_{T/\underline{T}}$. If $E'$ and $E$ are as described, we can identify $E_T$ with $E'_{T^{(1)}}$, and with this identification,

$$R = (\mathrm{id} \otimes F^*_{T/\underline{T}}) \circ R' = p(\mathrm{id} \otimes \psi) \circ R'.$$

Now suppose that $E$ is an $F^\infty$-span. We have for every $n > 0$ a commutative diagram

$$\begin{array}{ccc} (F^*_Z E^n)_T & \xrightarrow{p^n R^{(n)}_T(\partial)} & (F^*_Z E^n)_T \otimes \Omega^{1f}_{T/\underline{T}} \\ \downarrow{\scriptstyle \Phi^{(n)}} & & \downarrow{\scriptstyle \Phi^{(n)}} \\ E_T & \xrightarrow{R_T(\partial)} & E_T \otimes \Omega^{1f}_{T/\underline{T}} \end{array}$$

where $\Phi^{(n)}$ is a $p$-isogeny. If, on the other hand, $E$ is an $F^\infty$-isospan, then for every $n \geqslant 0$ we are given a convergent isocrystal $E^n$ on $Z/W$ and an isomorphism $F^*(E^n) \to E^{n-1}$. We can restrict everything to $\mathrm{hEnl}(Z/W)$ and, if $\underline{Z}/k$ is smooth, we can apply Proposition 29 to see that for each $n$ there is an integral structure $E^n_0$ on $E^n$. Then for each $n$ there is a $p$-isogeny $F^{*n}\colon E^n_0 \to E_0$, say of level $m_n$, and we have a similar diagram. Thus in either case, Proposition 30 will follow from the following:

LEMMA 32. *Suppose that $T =\colon \mathrm{Spf}\, B$ is an affine $p$-adic formal scheme, flat and of finite type over $W$, $E$ and $\Omega$ are finitely generated $p$-torsion free $B$-modules, and $R\colon E \to \Omega \otimes E$ is a linear map. Suppose that for every integer $n$ there exist a finitely generated $B$-module $E^n$, a map $R_n\colon E^n \to \Omega \otimes E^n$ and a $p$-isogeny $\Phi_n\colon E^n \to E$ such that $R\Phi_n = p^n \mathrm{id}_\Omega \otimes \Phi_n R_n$. Then for every $\mathbf{Q} \otimes B$ algebra $A$ and every $B$-linear map $\partial\colon \Omega \to A$, the corresponding endomorphism $R(\partial)$ of $E \otimes_B A$ is nilpotent.*

*Proof.* It suffices to consider the universal case, in which $A =\colon \mathbf{Q} \otimes S^\cdot(\Omega)$ and $\partial$ is the "inclusion." Let $r\colon E \otimes S^\cdot(\Omega) \to E \otimes S^\cdot(\Omega)$ denote $\mathrm{R}(\partial)$ and let $G$ denote the graded $S^\cdot(\Omega)$-subalgebra of the graded ring of graded endomorphisms of $\mathbf{Q} \otimes E \otimes S^\cdot(\Omega)$ generated by $r$. This ring is commutative and finitely generated over $B$, and its $p$-adic completion $\hat{G}$ is topologically finitely generated over $W$. By [16, 4.5], if $Q$ is any maximal ideal of $\mathbf{Q} \otimes \hat{G}$, the quotient $(\mathbf{Q} \otimes \hat{G})/Q$ is a finite extension $K'$ of $K$, and the valuation $v$ of $K$ extends uniquely to $K'$. Let $\theta\colon G \to K'$ denote the corresponding homomorphism; which in fact maps $G$ into the valuation ring of $K'$.

For each $n$, $\Phi_n\colon E^n \to E$ is a $p$-isogeny, so that there exist $m_n$ and $V_n\colon E \to E^n$ such that $\Phi_n V_n = V_n \Phi_n = p^{m_n}$. Then $p^{m_n} R(\partial)^i = p^{ni} \Phi_n R_n(\partial)^i V_n$ for all $i$, and $p^{m_n} r^i = p^{ni} r_{i,n}$, where $r_{i,n} =\colon \Phi_n R^i_n(\partial) V_n \in \mathrm{End}_S(E \otimes S^\cdot(\Omega))$. Obviously this element commutes with $r$ modulo torsion, and, since it stabilizes the faithful $G$-module $E$, it is integral over $G$. We can therefore extend $\theta$ to the ring generated by $G$ and $r_{i,n}$, and $\theta(r_{i,n})$ will belong to the valuation ring of $K'$. Since $m_n v(p) + iv(\theta(r)) = ni + v(\theta(r_{i,n}))$, we find that $v(\theta(r)) \geqslant n - m_n v(p)/i$ for all $i$, and taking the limit as $i \to \infty$ shows that $v\theta((r)) \geqslant n$. Since this is true for all $n$, we see that $\theta(r) = 0$. Thus, the image $\hat{r}$ of $r$ in $\mathbf{Q} \otimes \hat{G}$ belongs to every maximal ideal, and by [16, 5.3], it follows that $\hat{r}$ is nilpotent. Then there exists $n > 0$ such that $r^n$ is infinitely divisible by $p$. But $r^n$ is homogeneous of degree $n$, and since each homogeneous piece of $G$ is $p$-adically separated, it follows that $r^n = 0$.

In Christol's transfer theorem, the differential equation determines the transfer structure uniquely. We show that this is true in our context as well, provided that the monoid $\overline{M}_Z$ is locally free.

PROPOSITION 33. *Suppose that $Z$ is a hollow log scheme such that $\overline{M}_Z$ is locally free, and $E$ is a crystal or a convergent isocrystal on $Z/W$ with nilpotent residue mapping. Then if $E$ admits a transfer structure, it is unique. In particular, the transfer structure on an $F^\infty$-span is unique. If $\underline{Z}/W$ is smooth, the same is true for $F^\infty$-isospans.*

*Proof.* We can view a transfer structure on $E$ as an isogeny

$$\rho\colon h^*(E^\natural) \to E.$$

Note that there is a natural identification: $(h^*(E^\natural))^\natural \cong E^\natural$, and with this identification $\rho^\natural = \mathrm{id}$, by the cocycle condition (22.1). If we let $H =: \mathrm{Hom}(h^*(E^\natural), E)$, then we can view $\rho$ as a global section of $\mathbf{Q} \otimes H$, and hence it will suffice to prove that the natural map

$$\mathbf{Q} \otimes \Gamma(\mathrm{Enl}(Z/W), H) \to \mathbf{Q} \otimes \Gamma(\mathrm{hEnl}(Z/W), H^\natural)$$

is injective. It is clear that $H$ also admits a transfer structure, and hence is isogenous to $h^*(H^\natural)$, so it suffice to prove the injectivity with $h^*(H^\natural)$ in place of $H$. Furthermore $H$ also has nilpotent residue mapping. Changing our notation, we see that it will suffice to prove:

LEMMA 34. *Suppose that $E$ is an isocrystal on* $\mathrm{hEnl}(Z/W)$*, or a crystal on* $\mathrm{hCris}(Z/W)$*, with nilpotent residue mapping. Then the natural map*

$$\mathbf{Q} \otimes \Gamma(\mathrm{Enl}(Z/W), h^*E) \to \mathbf{Q} \otimes \Gamma(\mathrm{hEnl}(Z/W), E)$$

*is injective.*

*Proof.* Suppose that $e$ is a member of the kernel and that $T$ is a thickening of $Z$. We have to prove that $e_T = 0$, and it suffices to do this locally on $T$. We may and shall assume that $T$ is affine, that $M_T$ is split, and that $P =: \overline{M}_Z$ is free. A splitting $P =: \overline{M}_Z \to M_T$ defines a prelog structure $\beta_T\colon P \to \mathcal{O}_T$. Let use the notation of Claim (28). Thus, we suppose that $\underline{T} = \operatorname{Spf} B$, we let $B[P]$ denote the monoid algebra over $B$, and we let $X =: \operatorname{Spec} B[P]$, with its canonical prelog and log structures. Because $P$ is free, $B[P]$ is just a polynomial algebra over $B$. We have exact closed immersions $j: T \to X$ and $j^\natural: T^\natural \to X$, and since $Z$ is hollow, the maps $Z \to X$ obtained by composing $Z_T \to T$ with $j$ and $j^\natural$ are the same. Let $D$ be the universal thickening of $Z$ in $X$; since $X$ is a polynomial algebra, the natural map $\mathcal{O}_D \to (\mathbf{Q} \otimes B)[[P]]$ is injective. Since $e_T = j^*(e_D)$, it suffices to prove that $e_D = 0$, and our assumption on $e$ implies that $e_{T^\natural} = j^{\natural *}(e_D) = 0$. Furthermore, from the fact that $e$ is horizontal, it follows that $\nabla_D(e_D) = 0$, where $\nabla_D: h^* E_D \to h^* E_D \otimes \Omega^1_{X/\underline{T}}$. Now $e_D$ can be written uniquely as a formal sum

$$e = \sum_{p \in P} e_p \otimes \alpha(p), \qquad \text{where } e_p \in E_T.$$

By Claim 28,

$$\nabla(e_D) = \sum_{p \in P} R_T(e_p) \otimes \alpha(p) + e_p \otimes \alpha(p) dp.$$

Of course, $e_0 = j^\natural(e_D) = 0$ by assumption. If $p \neq 0$, there is a map $\partial: P^g \to \mathbf{Z}$ such that $\partial(p) \neq 0$, and we find that $R_T(\partial)(e_p) + \partial(p) e_p = 0$. Since $R_T(\partial)$ is nilpotent and $\partial(p)$ is a nonzero element of $\mathbf{Q}$, it follows that $e_p = 0$.

Proposition 33 suggests that it should be possible to give an "explicit formula" for the transfer structure on an isocrystal with nilpotent residue map. If we give ourselves some "coordinates," this is indeed possible. Let us suppose that the log structure on $Z$ is hollow and split, let $P =: \overline{M}$, and let $P \to M_Z$ be a splitting. Suppose we are also given an embedding $\underline{Z} \to \underline{Y}$, where $\underline{Y}/W$ is smooth. Let $T$, $D$, $X$, and $Y$ be as in the proof of Lemma 34. Then an isocrystal $E$ on $Z$ will have a value $E_D$ on $D$, and $E_D$ has an integrable connection $\nabla_D: E_D \to E_D \otimes \Omega^1_{X/W}$. Furthermore, we have a natural map $d: P^g \to \Omega^1_{X/W}$, which induces an isomorphism $\mathcal{O}_X \otimes P^g \to \Omega^1_{X/Y}$. Define $E_D^{\mathrm{nil}}$ to be the set of all sections $e$ of $E_D$ such that $\partial^n(e) = 0$ for all $\partial \in \operatorname{Hom}(P^g, \mathbf{Z})$ and all $n$ sufficiently large. It is clear that $E_D^{\mathrm{nil}}$ is an $\mathcal{O}_T$-submodule of $E$ and that it inherits a connection

$$\nabla^{\mathrm{nil}}: E^{\mathrm{nil}} \to E^{\mathrm{nil}} \otimes \Omega^1_{\underline{T}/W}.$$

Furthermore, $\nabla_D$ induces a nilpotent $\Omega^1_{T/\underline{T}}$-valued endomorphism $R$ of $E^{\mathrm{nil}}$. Tensoring with $\mathcal{O}_X$, we see that $R$ and $\nabla^{\mathrm{nil}}$ induce a connection

$$\mathcal{O}_X \otimes E^{\mathrm{nil}} \to \mathcal{O}_X \otimes E^{\mathrm{nil}} \otimes \Omega^1_{X/W}$$

CLAIM 35. *Suppose that $E$ is a crystal on $Z/W$ with nilpotent residue map and which admits a transfer structure. The maps*

$$(E^{\mathrm{nil}}, R, \nabla^{\mathrm{nil}}) \to (E^\natural_{T_P}, R_T, \nabla_T) \ \textit{and} \ (\mathcal{O}_D \otimes_{\mathcal{O}_T} E^{\mathrm{nil}}, \nabla^{\mathrm{nil}}) \to (E_D, \nabla_D)$$

*are isogenies.*

*Proof.* Since $E$ is assumed to have a transfer structure, we may assume that $E = h^*(E^\natural)$. The explicit description (Claim 28) of the functor $h^*$ makes it clear that the map $E^{\mathrm{nil}} \to E^\natural$ is surjective, and Lemma 34 implies that it is injective, so the first statement is clear. The second statement now follows, using again the description of $h^*$.

It is convenient to express the main results of the previous section in the following way.

THEOREM 4. *Suppose that $f: X \to Z$ is a perfectly smooth morphism of fine saturated log schemes over $k$ and that the log structure on $Z$ is constant. Then if $E$ is an $F^\infty$-span on $X/W$ and $q \in bfN$, there is a convergent $F^\infty$-isospan $E^q$ of flat $\mathbf{Q} \otimes \mathcal{O}_{Z/W}$-modules such that for each $p$-adic enlargement $T$ of $Z/W$, $E^q_T \cong \mathbf{Q} \otimes R^q f_{X/T*}E$. Furthermore the value of $E^q$ on each enlargement $(T, z_T)$ depends, up to canonical isomorphism, only on $(\underline{T}, M_T)$, not on $\alpha_T: M_T \to \mathcal{O}_T$. If $\underline{Z}/k$ is smooth, the residue mapping of $E^q$ is nilpotent.*

The main new difficulty is the flatness; recall that in general, even in characteristic zero, coherent sheaves with integrable logarithmic connection need not be flat. (Of course, in characteristic zero, the analogous flatness statement of the above theorem should certainly hold, and in fact a special case was proved years ago with a different language by Steenbrink [15, 2.18].) The flatness here is a consequence of the fact that $E_T$ is independent of $\alpha_T$.

The key technical point is the following:

LEMMA 36. *Suppose that $f: X \to Z$ is as in Theorem 4 and $T$ is a $p$-adic enlargement of $Z/W$. Then $\mathbf{Q} \otimes R^q f_*(X_T/T)$ is a flat sheaf over $\mathbf{Q} \otimes \mathcal{O}_T$.*

*Proof.* We follow the method of [12, 3.1]. This question is local on $T$ and on $Z$, and so we may and shall assume that both are affine. We may also assume that $M_Z$ is split and that $\overline{M}_Z$ is a constant monoid $P$.

Suppose first that $Z$ is hollow, so that $Z \cong Z_P$. Choose an embedding of $\underline{Z}$ in a smooth $\underline{Y}/W$. Then we have an exact closed immersion $Z \subseteq Y$, where $Y =: \underline{Y}_P$. Let $Y(1) =: Y \times_W Y$, and let $Z \to Y(1)$ be the embedding via the diagonal (which is no longer exact). Let $\hat{Y}$ and $\hat{Y}(1)$ denote the exact formal completions of $Y$ and $Y(1)$ along $Z$, respectively. Note that $\underline{Y}(1) \cong \underline{Y} \times_W \underline{Y}$, but the process of exactification changes the underlying scheme, and in fact $\underline{\hat{Y}(1)} \cong \underline{Y}(1) \times \hat{G}_P$. Observe that the projection map $\hat{Y}(1) \to \hat{Y}$ is flat. If we let $T$ (resp. $T(1)$) denote the exact universal $p$-adic enlargement of $Z$ in $Y$ (resp., in $Y(1)$), it follow that the maps $T(1) \to T$ are also flat. Thus, formation of the cohomology sheaves $R^q f_{X/T*}E$ commutes with pullback to $T(1)$, and we have isomorphisms:

$$\mathbf{Q} \otimes p_2^* R^q f_{X/T*}E \cong \mathbf{Q} \otimes R^q f_{X/T(1)*}E \cong \mathbf{Q} \otimes p_1^* R^q f_{X/T*}E.$$

It is clear that these isomorphisms satisfy the cocycle condition. This tells us that $\mathbf{Q} \otimes R^q f_{X/T*}E$ admits a stratification, and hence defines a $p$-adically convergent crystal $E^q$ on hEnl$(Z/W)_p$. Corollary 25 implies that $E^q_T$ is a flat sheaf of $\mathbf{Q}\otimes\mathcal{O}_T$-modules, and it follows that the formation of $\mathbf{Q} \otimes H^q(X_T, E)$ commutes with all base change $T' \to T$. But for any hollow $p$-adic enlargement $T'$ of $Z/W$, locally on $T'$ there exists a map $T' \to T$, and it follows that each $\mathbf{Q} \otimes H^q(X/T', E)$ is flat over $\mathbf{Q} \otimes \mathcal{O}_{T'}$. More generally, since the log structure on any $p$-adic enlargement $T'$ of $Z/W$ is constant, we can apply Theorem 1 to find an isomorphism

$$\mathbf{Q} \otimes H^q(X/T', E) \cong \mathbf{Q} \otimes H^q(X/T'^{\natural}, E),$$

so $\mathbf{Q} \otimes H^q(X/T', E)$ is also flat.

Now if $Z$ is not hollow, consider the diagram (14) used in the proof of Theorem 2. Suppose that $(T, z_T)$ is a $p$-adic enlargement of $Z/W$. From that proof we see that we have an isogeny

$$R\Gamma(X^{br}/T^{(r)}, E^{br}) \to R\Gamma(X/T, E).$$

Now for $n$ large enough, $Z^{(r)}$ is hollow, so we conclude that $\mathbf{Q} \otimes H^q(X/T, E)$ is also flat over $\mathbf{Q} \otimes \mathcal{O}_T$.

*Proof of Theorem 4.* Using the lemma and the base changing formula for crystalline cohomology, we see that $R^q f_* E$ forms a $p$-adically convergent isocrystal on Enl$(Z/W)$. It is clear that it inherits an $F^\infty$-isospan structure, and the method of Remark 16 allows us to prolong $E^q$ to a convergent crystal. Finally, the fact that $E^q_T$ does not depend on $\alpha_T$ follows from Theorem 2.

## 5. The logarithmic Weil group

The structure of a convergent $F^\infty$-isospan on a fine and saturated log scheme $Z/k$ can be elucidated by enlarging the category Enl$(Z/W)$ slightly. We define $W_{\text{cris}}(Z/W)$ to be the category of "Weil enlargements", having the same collection of objects as Enl$(Z/W)$ but such that the set of morphisms $T' \to T$ is the set of pairs $(\psi_T, d)$, where $\psi_T: T' \to T$ is a morphism of formal schemes such that there is a commutative diagram

$$\begin{array}{ccc}
T' & \xrightarrow{\psi_T} & T \\
\uparrow & & \uparrow \\
Z_{T'} & \xrightarrow{\psi_Z} & Z_T \\
\downarrow{\scriptstyle z_{T'}} & & \downarrow{\scriptstyle z_T} \\
Z & \xrightarrow{F_Z^d} & Z
\end{array}$$

The composition $(\psi_T, d)\circ(\psi'_{T'}, d')$ is defined to be $(\psi_T\circ\psi'_{T'}, d+d')$. We often write $\psi$ for $(\psi_T, d)$ and deg$(\psi)$ for $d$. If $E$ is a convergent $F^\infty$-isospan on $Z/W$, then for each morphism $(\psi_T, d): T' \to T$ of Weil enlargements we have $\psi_T^*(E_T^{n+d}) \cong (F_Z^{d*}E^{n+d})_{T'}$, and hence there is an isomorphism:

$$\Phi_\psi: \psi_T^*(E_T^{n+d}) \to E_{T'}^n$$

induced by $\Phi_n^{(d)}: F_Z^{d*}E^{n+d} \to E^n$. In particular, when $E$ is an F-isocrystal, we find isomorphisms

$$\Phi_\psi: \psi_T^*(E_T) \to E_T,$$

extending the usual transition maps indexed by morphisms of enlargements. Thus, an F-isocrystal can be viewed as a crystal on the category of Weil enlargements.

If $T$ and $T'$ are Weil enlargements of $Z/W$, then the set of morphisms $T' \to T$ will be denoted by $W_{\text{cris}}^{\geqslant}(T', T)$, and we write $W_{\text{cris}}^{\geqslant}(T)$ for the (not necessarily commutative) monoid $W_{\text{cris}}^{\geqslant}(T, T)$. We note that if $E$ is a convergent F-isocrystal, $W_{\text{cris}}^{\geqslant}(T, T)$ acts semilinearly on the right on $E_T$. There is a natural map

$$W_{\text{cris}}^{\geqslant}(T', T) \to W_{\text{cris}}^{\geqslant}(\underline{T}', \underline{T}).$$

We let $W_{\text{cris}}(T)$ be the group associated to the monoid $W_{\text{cris}}^{\geqslant}(T)$. Recall that $\xi_T$ stands for the isomorphism class of the extension of monoids

$$0 \to \mathcal{O}_T^* \to M_T \to \overline{M}_T \to 0,$$

and that $\overline{M}_T \cong \underline{z}_T^*(\overline{M}_Z)$.

LEMMA 37. *If $T$ is hollow, we have an exact sequence:*

$$1 \to \hat{\mathbf{G}}_{\overline{M}}(\underline{T}') \to W_{\text{cris}}^{\geqslant}(T', T) \longrightarrow W_{\text{cris}}^{\geqslant}(\underline{T}).$$

*An element $\underline{\psi}$ of $W_{\text{cris}}^{\geqslant}(\underline{T}', \underline{T})$ lies in the image of $W_{\text{cris}}^{\geqslant}(T', T)$ if and only if $\underline{\psi}^*\xi_{T'} = p^{\deg \underline{\psi}}\xi_T$. In particular, this is always the case if $T'$ is affine or if $\xi_{T'} = 0$.*

*Proof.* If $\psi_1$ and $\psi_2$ are two elements of $W_{\text{cris}}^{\geqslant}(T', T)$ with the same image in $W_{\text{cris}}^{\geqslant}(\underline{T}', \underline{T})$, then in particular they have the same degree $d$. Thus they both act as multiplication by $p^d$ on $M_Z$ and hence also on $\overline{M}_{Z_T}$, since the map $Z_T \to Z$ is solid. Since also $\underline{\psi}_{1,T} = \underline{\psi}_{2,T}$, they share the same action on $\mathcal{O}_{Z_T}^*$, and so their actions on $M_{Z_T}$ differ by a map $\mu: \overline{M}_{Z_T} \to \mathcal{O}_{Z_{T'}}^*$. But they agree on $M_Z$, and the map $\overline{M}_Z \to \overline{M}_{Z_T}$ is an isomorphism, and it follows that $\mu$ vanishes, so $\psi_{1,Z}$ and $\psi_{2,Z}$ agree on all of $M_{Z_T}$. Thus, $\psi_1$ and $\psi_2$ agree on the log scheme $Z_T$, and hence Lemma 9 tells us that they differ by an element of $\hat{\mathbf{G}}_T(T') \cong \hat{\mathbf{G}}_{\overline{M}}(\underline{T})$. This proves the exactness of the sequence. For the characterization of the image of $W_{\text{cris}}^{\geqslant}(T', T)$, just observe that, since $T$ is hollow, there is a logarithmic morphism lying over an element $\underline{\psi}$ of $W_{\text{cris}}^{\geqslant}(\underline{T}', \underline{T})$ if and only if there is a commutative diagram of monoid morphisms:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathcal{O}_T^* & \xrightarrow{\lambda_T} & M_T & \longrightarrow & \overline{M}_T & \longrightarrow & 0 \\
 & & \big\downarrow \underline{\psi}^* & & \big\downarrow \psi^* & & \big\downarrow \overline{\psi}^* & & \\
1 & \longrightarrow & \mathcal{O}_{T'}^* & \xrightarrow{\lambda_{T'}} & M_{T'} & \longrightarrow & \overline{M}_{T'} & \longrightarrow & 0
\end{array}$$

But $\overline{\psi}^*$ is multiplication by $p^d$, and we see that this diagram exists if and only if there is an isomorphism of extensions $\underline{\psi}^*\xi_{T'} = p^d\xi_T$.

It seems natural to introduce the following analog of the Weil-Deligne group. Composition with the $p$-adic logarithm defines a map

$$\hat{\mathbf{G}}_M(T') \to \mathbf{T}_M(T') \otimes \mathbf{Q}.$$

We define $\mathbf{W}_{\text{cris}}(Z/W)$ to be the category whose objects are the hollow enlargements of $Z/W$, and in which the morphisms $\mathbf{W}_{\text{cris}}^{\geq}(T', T)$ are taken to be the pushout:

$$\mathbf{W}_{\text{cris}}^{\geq}(T', T) =: \mathbf{T}_M(T') \otimes \mathbf{Q} \oplus_{\hat{\mathbf{G}}_M(T')} W_{\text{cris}}^{\geq}(T', T) \tag{19}$$

Composition is defined using the addition law on $\mathbf{T}_M$ in the obvious way. Using the same notation for the associated groups, we have an exact sequence

$$1 \to \mathbf{T}_M(T) \otimes \mathbf{Q} \to \mathbf{W}_{\text{cris}}(T) \to W_{\text{cris}}(\underline{T}) \to 0 \tag{20}$$

Recall that $\mathbf{T}_M(T) \cong \operatorname{Hom}(\overline{M}^g, \mathcal{O}_T)$, on which there is a natural semilinear action of $W_{\text{cris}}(T)$: if $\psi \in W_{\text{cris}}(T)$ and $\tau \in \mathbf{T}_M(T)$,

$$\psi(\tau) =: \psi \circ \tau \circ \psi^{-1} = p^{-\deg \psi}\psi \circ \tau. \tag{21}$$

This action of $W_{\text{cris}}(T)$ can also be thought of as a Tate twist of the action of $W_{\text{cris}}(\underline{T})$. We should also point out that it follows immediately from formula (1) of Lemma 2 that this action is also the action of $W_{\text{cris}}(T)$ on $\mathbf{T}_M \otimes \mathbf{Q}$ by inner automorphism corresponding to the exact sequence (20).

We can summarize this viewpoint as follows.

PROPOSITION 38. *Suppose that $Z/k$ is a hollow log scheme and that $E$ is a convergent $F$-isocrystal on $Z/W$. Then $E$ determines a crystal on $\mathbf{W}_{\text{cris}}(Z/W)$. In particular, if $T$ is any hollow enlargement of $Z/W$, then there is a natural semilinear action of $\mathbf{W}_{\text{cris}}(T)$ on $E_T$ (on the right). The residue mapping $R_T \in \overline{M}^g \otimes_{\mathbf{Z}} \operatorname{End}(E_T)$ is nilpotent, and the action $\theta$ of an element $\tau \in \mathbf{T}_M(T)$ is given by $\theta_\tau = \exp(R_T\tau)$. Furthermore, $R_T \in \operatorname{End}(E) \otimes \overline{M}^g$ commutes with the action of $\mathbf{W}_{\text{cris}}(T)$:*

$$R_T \circ \theta_\psi = (\theta_\psi \otimes \psi_M) \circ R_T = p^{\deg \psi}\theta_\psi \circ \psi^*(R_T).$$

Let us now look at the specific case of $Z =: \xi_V^\times$, the "punctured point" associated to a complete discrete valuation ring as described in Example 13. Because $Z = \operatorname{Spec} k^\times$ is affine, we let $\mathbf{W}_{\text{cris}}(k_V^\times)$ denote the *opposite* of the category $\mathbf{W}_{\text{cris}}(\xi_V^\times)$. If $\tilde{V}$ is a finite extension of $V$, we can endow its formal spectrum $\tilde{T}$ with the log structure obtained by pulling back the log structure of $T^\natural$ via the natural map $\tilde{T} \to \hat{T}$. Let $\tilde{\xi}_V^\times =: \operatorname{Spec} \tilde{k}_V^\times$ denote the log scheme obtained by using the induced log structure on the residue field of $\tilde{V}$. Then the diagram:

$$\begin{array}{ccc} \tilde{\xi}_V^\times & \longrightarrow & \tilde{T}^\natural \\ \downarrow & & \\ \xi_V^\times & & \end{array}$$

makes $\tilde{T}^\natural$ a hollow enlargement of $\xi_V^\times$; we let $\tilde{V}^\natural$ denote the corresponding object of $\mathbf{W}_{\text{cris}}(k_V^\times)$. Then if $E$ is a convergent F-isocrystal on $\xi_V^\times$, we see that $\mathbf{W}_{\text{cris}}(\tilde{V}^\natural) = \mathbf{W}_{\text{cris}}(\tilde{T}^\natural)^{op}$ operates semilinearly on the *left* on $E_{\tilde{V}^\natural}$. A valuation of $V$ determines a map $\tau\colon \overline{M}^g =: K^*/V^* \to \mathbf{Q}$, which is in fact an element of $\mathbf{T}_M(\mathbf{Q})$, and in fact we have a bijection between the set of valuations and $\mathbf{T}_M(\mathbf{Q})$. If we identify an element $\tau$ of $\mathbf{T}_M(\mathbf{Q})$ with the corresponding automorphism of $\mathbf{W}_{\text{cris}}(\tilde{V}^\natural)$, then $\psi \circ \tau = \tau$, so formula (21) tells us that for any $\psi \in \mathbf{W}_{\text{cris}}(\tilde{V}^\natural)$,

$$\tau\psi = p^{\deg\psi}\psi\tau.$$

The logarithmic Witt scheme $S_V^\natural$ described in Example 13 is an especially interesting enlargement, and we denote the corresponding object of $\mathbf{W}_{\text{cris}}(k_V^\times)$ by $W_V^\natural$. In particular, if $E$ is a convergent F-isocrystal on $\xi_V^\times$, then its value on $S_V^\natural$ is a $K_0$-vector space endowed with a linear endomorphism $N =: R_T(\tau)$ and an $F_{K_0}^*$-linear endomorphism $\Phi =: \theta_{\phi_S}$, satisfying $N\Phi_S = p\Phi_S N$.

Working in the other direction, we see that the formal spectrum $T'$ of any finite extension of $V$ (with the log structure induced from $T^\natural$) becomes a hollow enlargement of $\xi_V^\times$. Passing to the limit, we find a group extension:

$$0 \to \mathbf{T}_M(\overline{K}) \to \mathbf{W}_{\text{cris}}(\overline{V}^\natural) \to W_{\text{cris}}(\underline{V}) \to 0 \tag{22}$$

which acts naturally on $E_{\overline{T}^\natural} = \overline{K} \otimes E_{T^\natural}$. A choice of a uniformizer $\pi$ of $V$ induces a morphism $f_\pi\colon T^\natural \to S_V^\natural$. It also compatibly splits the sequence $\Xi_{T'^\natural}$ for every $T'$, and these splittings define a section of (22) and hence a map $\mathbf{W}_{\text{cris}}(\overline{T}^\natural) \to \mathbf{W}_{\text{cris}}(S_V^\natural)$. It is clear that we have an isomorphism

$$\rho_\pi\colon f_\pi^* E_{S_V^\natural} \cong E_{\overline{T}^\natural},$$

compatible with these actions. Thus, $\rho_\pi$, $\Phi_S$, and $R_S$ completely determine $E_{\overline{T}^\natural}$. We can also reverse this determination. Namely, we have an exact sequence:

$$1 \longrightarrow \mathrm{Gal}(\overline{K}/K_0) \longrightarrow W_{\text{cris}}(\overline{V}) \xrightarrow{\deg} \mathbf{Z} \longrightarrow 0$$

and we can use the splitting $\sigma_\pi$ of (22) to define a semilinear action of this group on $E_{\overline{T}^\natural}$. The invariants of $\mathrm{Gal}(\overline{K}/K_0)$ are a $K_0$-form of $E_{\overline{T}^\natural}$ which is stable under the action of $\mathbf{T}_M(K_0)$ (this gives $R_S$) and endowed with an action of $W_{\text{cris}}/I_{\text{cris}} \cong \mathbf{Z}$ (this gives $\Phi_S$).

To summarize:

PROPOSITION 39. *If we choose an algebraic closure $\overline{K}$ of $K$, the category of convergent F-isocrystals on $\xi_V^\times$ becomes equivalent to the category of finite dimensional $\overline{K}$-vector spaces equipped with a semilinear action of the group $\mathbf{W}_{\text{cris}}(\overline{V}^\natural)$. If we choose a uniformizer and a nontrivial valuation of $V$, the category becomes equivalent to the category of finite dimensional $K_0$-vector spaces, equipped with a nilpotent endomorphism $N$ and a Frobenius-linear endomorphism $\Phi$, satisfying $N\Phi = p\Phi F^*N$.*

If $Y/V$ has semi-stable reduction, then $Y^\times/T^\times$ is perfectly smooth, and if $Y/V$ is (classically) smooth then $Y^\times \to T^\times$ is solid. Moreover, $\hat{Y}^\times$ is an enlargement of its special fiber $X^\times$, so an F-crystal $E$ on $X^\times/W$, has a value $E_{\hat{Y}^\times}$ on $\hat{Y}^\times$. By the existence theorem for a proper morphism, $E_{\hat{Y}^\times}$ extends to a coherent sheaf $E_{Y_K}$ on $Y_K$, and the crystal structure induces an integrable connection on this sheaf. Furthermore, we have

$$H^q_{DR}((E_Y, \nabla_Y)/K) \cong H^q_{DR}((E_{\hat{Y}}, \nabla_Y)/K) \cong E^q_{\hat{T}^\times} \cong E^q_{\hat{V}^\natural}$$

in the notation of Theorem 4. Thus we can restate the theorem of Hyodo and Kato as follows:

THEOREM 5. *Suppose that $Y/V$ is proper and has semistable reduction and $E$ is an F-crystal on the (logarithmic) special fiber $X^\times/k_V^\times$. Then the De Rham cohomology $H^q_{DR}((E_Y, \nabla_Y)/K) \otimes \overline{K}$ admits a canonical action of $\mathbf{W}_{\text{cris}}(\overline{K}^\natural)$. In particular, $H^q_{DR}(Y_K/K)$ admits such an action.*

Warning: It is conjectured, but not known in general, that this action is functorial for maps of the generic fiber.

REMARK 40. Suppose that we start with a smooth proper scheme $Y/K$ with "potentially log good reduction;" i.e. such that there exists a finite extension $K'/K$ and a perfectly smooth and proper $Y'/T$ such that $Y'_{K'} \cong Y_{K'}$. Then in addition to the a semilinear action $\rho_{\text{cris}}$ of $\mathbf{W}_{\text{cris}}(\overline{K}^\natural)$ on $H_{DR}(Y/\overline{K})$, we have also a semilinear action $\rho_{DR}$ of $\text{Gal}(\overline{K}/K)$, coming from the isomorphism $H_{DR}(Y/K) \otimes \overline{K} \cong H_{DR}(Y/\overline{K})$. Let $\mathbf{W}_{\text{cris}}(\overline{K}/K)$ denote the inverse image of $\text{Gal}(\overline{K}/K)$ in $\mathbf{W}_{\text{cris}}(\overline{K}^\natural)$. Then we can define a *linear* action $\rho_p$ of the algebraic $\mathbf{W}(\overline{K}/K)$ on $H^q_{DR}(Y'/K')$ by combining these two actions. That is, we set $\rho_p(\gamma) =: \rho_{DR}(\gamma) \circ \rho_{\text{cris}}(\gamma)^{-1}$. The group $\mathbf{W}(\overline{K}/K)$ is just the usual Weil-Deligne group constructed in [5], and for each prime $\ell \neq p$ there is also an action of this group on $\ell$-adic cohomology. It is natural to conjecture that these two actions are compatible, in the sense of *op. cit.*. For a much more detailed discussion of this conjecture (from a different point of view), *c.f.* [7].

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