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Relative generic singularities of the Exponential Map

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Abstract. We investigate generic properties of the Exponential Map defined as $\text{Exp}(v) = h_1^v$, for a vector field $v \in \Gamma(g)$ (where $\Gamma(g)$ denotes the Lipschitz sections of a subsheaf g of vector subspaces of the sheaf of all smooth vector fields on a smooth manifold M and h_t^v is the flow generated by v). We study restrictions of Exp to a suitable class of germs of submanifolds of M , and find necessary and sufficient conditions for a subsheaf $h \subset g$ such that for a generic vector field $v \in \Gamma(h)$ the singularities of the flow of v arise as singularities of the flow of a generic vector field belonging to $\Gamma(g)$. Applications of these results to Riemannian and sub-Riemannian geometry are presented and the context is chosen to include a theorem of A. Weinstein concerning the Riemannian Exponential Map.

1. Introduction

The main motivation of this paper lies in understanding the theorem of A. Weinstein [12]; in fact, the paper is just a slight generalization, with an easy direct proof, of that theorem. For a smooth n -dimensional manifold X , we consider the space \mathcal{G} of all smooth complete Riemannian metrics on X , endowed with C^∞ -Whitney topology. For each $g \in \mathcal{G}$ and $q \in X$, $\exp(g)|_q: T_q X \rightarrow X$ is the smooth map called the classical exponential map. To each $v \in T_q X$ it assigns the end point of the unique geodesic curve $\gamma: [0, 1] \rightarrow X$, $\gamma(0) = q$, $\dot{\gamma}(0) = v$. Let us write $g_{ij}(q) = \langle \frac{\partial}{\partial q_i}|_q, \frac{\partial}{\partial q_j}|_q \rangle$ for the entries of the matrix of the metric g and $g^{ij}(q)$ for the inverse matrix of $g_{ij}(q)$ then the function $H(q, p) = \frac{1}{2} \sum_{i,j=1}^n g^{ij}(q) p_i p_j$ on $T^* X$, defines a Hamiltonian vector field whose trajectories in $T^* X$ project onto geodesics in X (cf. [2]). The exponential map is a Lagrangian map, i.e. $\text{Exp}(g)_q = \pi_X \circ h_1^{vH}|_{T_q X}$, where h_t^{vH} is the flow of the Hamiltonian vector field defined by H and $h_i^{vH}|_{T_q X}$ is a Lagrangian immersion (cf. [1]). Recall that any germ of a Lagrangian immersion can be obtained in the above way by taking for H a suitable function on $T^* X$, (not necessarily quadratic with respect to p). By \mathfrak{h} we denote the space of quadratic Hamiltonians, and by \mathfrak{g} the space of all smooth Hamiltonians. It was stated by [12] (cf. [11]) that for a generic metric on X , i.e. for a generic $H \in \mathfrak{h}$ the map $\text{Exp}(g)|_q$ has only the singularities which are generic

for Lagrangian maps, i.e. for a generic Hamiltonian $H \in \mathfrak{g}$. Now we consider the problem in a more general way.

Let g be a subsheaf of vector subspaces of the sheaf of all smooth vector fields on a smooth manifold M . There are two natural questions to ask.

- (1) Does g admit a subsheaf $h \subset g$ (we call it accessible) such that for a generic $v \in \Gamma(h)$ (where $\Gamma(h)$ denotes the Lipschitz sections of h) the singularities of the flow of v are the same as the singularities of the flow of a generic vector field $w \in \Gamma(g)$.
- (2) What are necessary and sufficient conditions for the existence of such pairs g, h ?

In attempting to answer these questions we define the Exp-map as $\text{Exp}(v) = h_1^v$ for $v \in \Gamma(g)$, where h_t^v is the flow generated by v , and we study the restrictions of Exp to a suitable class of germs of submanifolds W . We impose some rather natural restrictions on g , e.g. we say assume that any vector field on X , which is a “piecewise section of g ” can be approximated by sections of g . Then we find necessary and sufficient conditions, which answer our second question.

The paper is organized in three sections. In section 2 we formulate the problem and describe the assumed properties of the sheaf g . Then we give examples of sheaves satisfying these properties: the sheaf of all smooth vector fields, the sheaf of Hamiltonian vector fields and the sheaf of Hamiltonian vector fields with quadratic Hamiltonians. Section 3 contains the main results. We prove that the image of the r -jet of the Exp-map

$$E^r: \Gamma(g) \times W \rightarrow J^r(W, X)$$

is a submersive submanifold, and that a subspace h of g is accessible if and only if $E^r|_{\Gamma^*(h) \times W}$ is a submersion. The necessary condition (α -property) and the sufficient condition (β -property) for $E^r|_{\Gamma^*(h) \times W}$ to be a submersive map are found using the perturbation technique for the differential equation $\dot{x} = v(x)$. The last section of the paper contains applications to the Riemannian and sub-Riemannian cases, which were most interesting to us. As a consequence, a shorter proof of the standard genericity theorem for the Exp-map on a Riemannian manifold is presented (cf. [12, 5]) and an obstruction to the genericity of the Exp-map regarded as a family of Lagrangian maps is indicated. Analogous genericity results are obtained for sub-Riemannian Hamiltonians. In that case the image of the Exp-map is an isotropic submanifold and the generic properties of the sub-Riemannian Exp-map are reduced to those of isotropic submanifolds in the cotangent bundle.

2. Formulation of the problem

Let M be a locally trivial fiber bundle over X , $\pi: M \rightarrow X$. Let g be a subsheaf of vector subspaces of the sheaf of all smooth vector fields, $g \subset \Xi(M)$ on M . By

$\Gamma(g)$ we denote the space of Lipschitz sections of g over M . Let $v \in \Gamma(g)$ and let $t \rightarrow h_t^v: M \rightarrow M$ be a flow on M generated by v . Suppose that at each point $x \in M$ we are given a space of germs \mathcal{M}_x of a class of submanifolds of M through x .

Let

$$\mathcal{M} = \bigcup_{x \in M} \mathcal{M}_x.$$

We shall assume that for every $v \in \Gamma(g)$ and $W_x \in \mathcal{M}_x$, $h_t^v(W_x) \in \mathcal{M}_{h_t^v(x)}$.

Let $W \in \mathcal{M}$, we define the Exp-map in the following way ([6]);

$$\text{Exp}_v: W \rightarrow X, \quad \text{Exp}_v = \pi \circ h_1^v|_W. \tag{1}$$

By $J^r = J^r(W, X)$ we denote the space of r -jets of smooth mappings $W \rightarrow X$. Let $\pi_r: J^r \rightarrow X$ denote the canonical projection onto the image space of the mapping. We have a natural map

$$E^r: \Gamma(g) \times W \rightarrow J^r(W, X),$$

we write also

$$E = E^r: \Gamma(g) \rightarrow C^\infty(W, J^r), \quad E^r(v); W \rightarrow J^r(W, X),$$

where $E^r(v)$ is the r -jet extension of Exp_v .

Let h be a sheaf of vector subspaces of g . Let A^r be a submanifold of the jet space $J^r(W, X)$.

DEFINITION 2.1. We say that $A^r \subset J^r(W, X)$ is *typical for E^r* if there is a residual subset $\Gamma'(g)$ of $\Gamma(g)$ such that for every $v \in \Gamma'(g)$ the corresponding jet-extension $E^r(v)$ is transversal to A^r .

In what follows we are interested in finding the subspaces of g which retain the typicality property for E^r .

DEFINITION 2.2. We say that the subsheaf $h \subset g$ is *accessible* if for every submanifold A^r , which is typical for E^r , there exists an open and dense subset $\Gamma'(h)$ in $\Gamma(h)$, such that for every $w \in \Gamma'(h)$ the corresponding jet extension $E^r(w)$ is transversal to A^r .

In what follows we fix $x_0 \in M$ with $\pi(x_0) = 0 \in X \cong \mathbb{R}^n$. Let $W = W_{x_0} \in \mathcal{M}_{x_0}$. We denote

$$J^* = \pi_r^{-1}(\mathbb{R}^n - \{0\}), \quad J_g^* = E^r(\Gamma(g))(W) \cap J^*;$$

clearly $\Gamma^*(g) = \{v \in \Gamma(g); v(x_0) \neq 0\}$, is an open subset of $\Gamma(g)$.

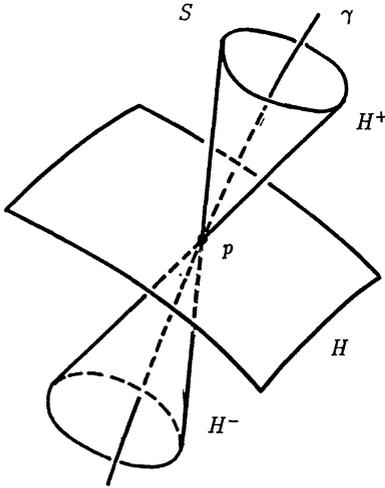


Fig. 1.

2.1. PROPERTIES OF THE SHEAF g

By g_{x_0} we denote the space of germs at x_0 of the sheaf g . Without loss of generality we introduce two important assumptions which have to be satisfied by our sheaf g .

PROPERTY 2.3. *If $w \in g_{x_0}$ and $v \in \Gamma(g)$, then*

$$(h_t^v)_* w \in g_{h_t^v(x_0)};$$

it follows that h_1^v induces an isomorphism $(h_1^v)_: g_{x_0} \rightarrow g_{x_1}$, and $x_1 = h_1^v(x_0)$.*

Let $v \in \Gamma^*(g)$; by γ we denote the integral curve of v starting at x_0 .

PROPERTY 2.4. *Let us take a point p on the curve γ and a section w of g defined in a neighbourhood of p such that $j_p^r w = 0$. Then we assume that there exist:*

1. *a hypersurface H , separating M into two half-spaces H^+ and H^- , (as illustrated in Figure 1) transversal to γ at the point $p \in H \cap \gamma$, and*
2. *a family of vector fields $P_\epsilon(v, w) \in \Gamma^*(g)$, parametrized by $\epsilon \neq 0$, depending linearly on w with the following property*

$$P_\epsilon(v, w) \xrightarrow{\epsilon \rightarrow 0} \begin{cases} v & \text{on } H^- \\ v + w & \text{on } H^+ \end{cases}.$$

P_ϵ converges uniformly together with its derivatives up to order r , outside an open neighbourhood U of p and in a cone-like neighbourhood S of the curve γ ,

$$S = \{x \in M; d(x, \gamma) < Cd(x, p)\}$$

for some positive constant C , and a metric $d(\cdot, \cdot)$ on M .

One can briefly state this assumption as follows:

Every vector field on M , which is a “piecewise section of g ” can be approximated by sections of g .

Unless otherwise stated, in what follows, we assume both of these properties hold for our sheaf g .

Now we show how these assumptions work in some special situations.

EXAMPLE 2.5. Let g be the sheaf of all smooth vector fields on M . Then we can simply define

$$P_\epsilon(v, w) = v + \varphi_\epsilon w,$$

where φ_ϵ is a smooth function on M , such that

$$|D^a \varphi_\epsilon| \leq \frac{C_a}{\epsilon^a},$$

vanishing on H^- , varying on the strip of distance ϵ from H and equal 1 on the rest of H^+ .

EXAMPLE 2.6. Let g be the sheaf of all Hamiltonian vector fields on $M = (T^*X, \omega_X)$, where ω_X is the Liouville symplectic form on the cotangent bundle T^*X . Let v, w be Hamiltonian vector fields with Hamiltonians H, K respectively, i.e. $\omega_X(v, \bullet) = -dH, \omega_X(w, \bullet) = -dK$. Let us denote the above correspondence of 1-forms and vector fields by J . We put

$$P_\epsilon(v, w) = Jd(H + \varphi_\epsilon K),$$

where φ_ϵ is defined as in Example 2.5.

EXAMPLE 2.7. Let g be the sheaf of Hamiltonian vector fields with Hamiltonians quadratic with respect to $p: H(q, p) = \sum_{i,j} g^{ij}(q)p_i p_j((q, p)$ denote the standard Darboux coordinates on T^*X). Then the hypersurface \mathcal{H} is defined by a smooth function L on $X, \mathcal{H} = \{(q, p): L(q) = 0\}$, and the function φ_ϵ depends only on q .

3. A transversality theorem

We start with a description of the image space of $(\text{Exp})_*$. Let $g_x^{(r+1)}$ be the space of germs at x of vector fields in g vanishing at x together with all derivatives up to order r . By $J^r g_x$ we denote the jet-space of vector fields

$$J^r g_x = \frac{g_x}{g_x^{(r+1)}}.$$

By π^*g we shall denote the sheaf of TX -valued vector fields on M along π ; i.e. the fields of the form

$$x \rightarrow \pi_*v(x),$$

where v is a section of g .

PROPOSITION 3.1.

J_g^* is an immersive submanifold of J^r

and the tangent space $T_zJ_g^*$ can be identified with

$$\{j_{x_1}^r w; w \in \pi^*g_{x_1}|_{W_{x_1}}\},$$

where $j_{x_1}^r w$ is the r -jet of w and z is equal to the r -jet $E^r(v)$.

Proof. As we remarked in Section 2

$$J_g^* = E^r(\Gamma^*(g))(W) \cap J^r.$$

We will show that $(E^r)_*$ has constant rank on $\Gamma^*(g)$.

Let $\xi \in \Gamma(g)$ and let $t \rightarrow v + t\xi$ be a line in $\Gamma(g)$. Consider the curve $\gamma: t \rightarrow E^r(v + t\xi) \in C^\infty(W, J^r)$. The tangent vector to γ can be thought of as

$$\left. \frac{d}{dt} E^r(v + t\xi) \right|_{t=0} = j^r \left. \frac{d}{dt} \pi_*(y(x, 1, t)) \right|_{t=0} = j^r \pi_*(u(x, 1)),$$

where $y(x, s, t)$ is a solution of the equation (with parameter t)

$$\partial_s y(x, s, t) = (v + t\xi)(y(x, s, t)). \tag{2}$$

We can view y as a perturbation of the solution of the equation $\dot{x} = v(x)$. Thus we can write y in the form

$$y(x, s, t) = y_0(x, s) + tu(x, s) + o(t).$$

From (2) we see that $u(x, s)$ satisfies the equation

$$\partial_s u(x, s) = Dv(y_0(x, s))u(x, s) + \xi(y_0(x, s)), \tag{3}$$

where $y_0(x, s)$ is a solution of the equation

$$\partial_s y_0(x, s) = v(y_0(x, s)). \tag{4}$$

We can also write $h_t^v(x) = y_0(x, t)$.

By Property 2.3 we have the following result.

LEMMA 3.2. *The linear part of the perturbation y is given by*

$$u(x, 1) = \int_0^1 (h_{1-t}^v)_* \xi(h_t^v(x)) dt \in g_{x_1}, \tag{5}$$

Proof. We write

$$u(x, s) = \int_0^s (h_{s-t}^v)_* \xi(h_t^v(x)) dt. \tag{6}$$

Obviously (6) satisfies equation (3):

$$\begin{aligned} \partial_s u(x, s) &= \xi(h_s^v(x)) + \int_0^s Dv(h_{s-t}^v(x))(h_{s-t}^v)_* \xi(h_t^v(x)) dt \\ &= \xi(h_s^v(x)) + \int_0^s \left(\frac{\partial}{\partial s} h_s^v \right)_* (h_{-t}^v)_* \xi(h_t^v(x)) dt \\ &= \xi(h_s^v(x)) + Dv(h_s^v(x)) \int_0^s (h_s^v)_* (h_{-t}^v)_* \xi(h_t^v(x)) dt, \end{aligned}$$

where

$$\int_0^s (h_s^v)_* (h_{-t}^v)_* \xi(h_t^v(x)) dt = u(x, s). \tag{7}$$

Q.E.D.

Now we prove that $T_z J_g^* \hookrightarrow J^r g$. Indeed by (7) we can approximate $u \in T_z J^* g$ by Riemann sums: each summand

$$(h_{-(1-t)}^v)_* u(x, 1) = \xi(h_t^v(x)) \tag{8}$$

belongs, by Property 2.3, to $J^r g_{x_1}$. $J^r g_{x_1}$ is a vector subspace of the (finite dimensional) vector space of all jets of vector fields and therefore closed.

To prove that $J^r g \hookrightarrow T_z J_g^*$ we have to show that for every u there exists a ξ such that (6) is satisfied. This will follow from the fact that g has, by assumption, Property 2.4. First, we can assume that $u \in g_{x_1}^{(r)}$. We choose a suitable hypersurface H (cf. Property 2.4) intersecting transversally the trajectory γ of v starting at x_0 . Then apply Property 2.4 putting $w = (h_{-\delta}^v)_* u$, and $p = h_{1-\delta}^v(x_0)$ (cf. Figure 1). Let us denote by ξ_0 the vector field equal to v over H^- and $v + w$ over H^+ . ξ_0 induces a flow $h_t^{\xi_0}$ which gives a smooth $h_1^{\xi_0}$ in a neighbourhood of x_0 and let $E^r(\xi_0)$ be its r -jet at x_0 . By Property 2.4 we have an approximating family P_ϵ for which, for sufficiently small ϵ , we have

$$E^r(P_\epsilon(v, w)) = E^r(v) + \xi + o(|\xi|) + A_\epsilon,$$

(for more details see the proof of Theorem 3.8), where $A_\epsilon \xrightarrow{\epsilon \rightarrow 0} 0$ uniformly. Since $\dim J^r g_x$ is independent of x , then $T_z J_g^*$ is independent of z , so J_g^* is an immersive submanifold. Q.E.D.

REMARK 3.3. We conjecture that the space J_g^* is a submanifold of $J(W, X)$.

The following corollary follows from the above proof.

COROLLARY 3.4. The mapping $E^r: \Gamma^*(g) \times W \rightarrow J_g^*$ is a submersion.

Let $h \subset g$, be a subspace of g . We introduce the following space of Lipschitz sections of h

$$\Gamma^*(h) = \{v \in \Gamma(h): v(x_0) \neq 0\}.$$

PROPOSITION 3.5. A subspace h of g is an accessible subspace of the space g if and only if

$$E^r|_{\Gamma^*(h) \times W}: \Gamma^*(h) \times W \rightarrow J_g^*$$

is a submersion.

Proof. First we prove that “only if” part. Let $E^r|_{\Gamma^*(h) \times W}$ be a submersion and let $A^r \subset J^r$ be a typical submanifold for E^r . Then by Thom-Abraham transversality theorem (cf. [6]), there exists an open and dense subset $\mathcal{A} \subset \Gamma^*(h)$ such that for every $a \in \mathcal{A}$ the mapping $E^r(a): W \rightarrow J_g^*$, is transversal to A^r . Thus the subspace h is accessible. To prove the “if” part, we note that the accessibility of h implies transversality of E^r to an arbitrary point from J_g^* . But this is exactly the submersivity of $E^r|_{\Gamma^*(h) \times W}$. Q.E.D.

REMARK 3.6. If $E^r|_{\Gamma^*(h) \times W}$ is a submersion then there exists a finite dimensional subspace h_0 of h , such that if $v \in \Gamma^*(h)$ and $t \rightarrow E^r(tv) \in J_g^*$ is a curve, then for every $t \neq 0$ we have

$$T_{E^r(tv)J_g^*} = \text{Im}(E^r|_{\Gamma^*(tv+h_0)})^*|_{E^r(tv)}. \tag{9}$$

So the mapping $E^r: \Gamma^*(tv + h_0) \rightarrow J_g^*$ is a submersion.

Let $v, \xi \in \Gamma^*(g)$. We introduce the following iterated bracket of v and ξ :

$$[v, \xi]_i = \underbrace{[v, [\dots, [v, \xi], \dots,]]}_{i \times} \in \Gamma^*(g), \tag{10}$$

where $[v, \xi]_0 = \xi, [v, \xi]_1 = [v, \xi]$.

Let h be a subsheaf of g .

DEFINITION 3.7. We say that h satisfies the α -property if for almost all $v \in h$, for every $w \in g_{x_0}^{(k)}$, $k \leq r$, $x_0 \in M$, and every $W_{x_0} \in \mathcal{M}$ there exists an $l \in \mathbb{N} \cup \{0\}$ and germs of vector fields $\xi_0, \xi_1, \dots, \xi_l \in h_{x_0}$, such that

$$\pi_*([v, \xi_0]_j + [v, \xi_1]_{j-1} + \dots + \xi_j)|_{W_{x_0}} \in \pi^*(g)_{x_0}^{(k+1)}|_{W_{x_0}} \tag{\alpha_j}$$

$$\left(\text{i.e. } j_{x_0}^k \left(\pi_* \sum_{i=0}^j [v, \xi_i]_{j-i}|_{W_{x_0}} \right) = 0 \right), \text{ for } j = 0, \dots, l - 1,$$

and

$$\pi_*([v, \xi_0]_l + [v, \xi_1]_{l-1} + \dots + \xi_l - w)|_{W_{x_0}} \in \pi^*(g)_{x_0}^{(k+1)}|_{W_{x_0}} \tag{\alpha_l}$$

$$\left(\text{i.e. } j_{x_0}^k \left(\pi_* \sum_{i=0}^l [v, \xi_i]_{l-i}|_{W_{x_0}} \right) = j_{x_0}^k(\pi_* w|_{W_{x_0}}) \right).$$

By $\pi^*(g)_{x_0}^{(k+1)}$ we denote the space, defined by g , of germs of sections of the induced bundle π^*TX , with zero k -jet at x_0 .

THEOREM 3.8. *Let g be the sheaf of analytic vector fields. Let $E^r|_{\Gamma^*(h) \times \mathcal{M}}$ be a submersive map. Then h satisfies the α -property.*

Proof. We know that for some finite dimensional subspace h_0 , $h_0 \subset h$, $E^r|_{\Gamma^*(tv+h_0) \times \mathcal{M}}$ is submersive. Let B_0 be a closed ball in h_0 . Then $E^r(tv + B_0)$ contains some neighbourhood of $E^r(tv)$. Making use of the assumption of analyticity of g we have that $E^r(tv + B_0)$ is an analytic subset of J_g^* . Thus we immediately obtain that (see [7]) there exists an $N \in \mathbb{N}$ such that for every $w \in \Gamma^*(g)$

$$E^r(t(v + t^{N-1}w)) \in E^r(tv + B_0). \tag{11}$$

Thus for every t there exists $\xi \in B_0$ such that

$$E^r(t(v + t^{N-1}w)) = E^r(t(v + \xi)). \tag{12}$$

Using Puiseux theorem we can assume that ξ is a convergent fractional power series:

$$\xi = \xi(t^{1/m})$$

(depending also on w).

We notice that $E^r(t(v + \xi))$ is the r -jet at x_0 of the mapping $\pi(z(x, 1, t))$, where $z(x, s, t)$ satisfies the equation

$$\partial_s z(x, s, t) = [t(v + \xi)](z(x, s, t)). \tag{13}$$

Let us denote $u(x, t) = z(x, 1, t)$. Inserting $\bar{s} = st$ into (13) we find that $u(x, \bar{s})$ (which obviously depends also on ξ), satisfies

$$\partial_{\bar{s}}u(x, \bar{s}) = (v + \xi)(u(x, \bar{s})); \tag{14}$$

thus concluding, we see that: $E^r(t(v + \xi))$ is the r -jet at x_0 of $u(x, t)$, where $u(x, \bar{s})$ is the solution of (14).

Let us look on (14) as a perturbation of the equation

$$\partial_{\bar{s}}y_0(x, \bar{s}) = v(y_0(x, \bar{s})),$$

which is clearly satisfied by $y_0(x, \bar{s}) = h_{\bar{s}}^v(x)$. We linearize (14); it is easy to see that the linear (with respect to ξ) term $y = y_{\xi}(x, \bar{s})$ satisfies

$$\partial_{\bar{s}}y = Dv(y_0)y + \xi(\bar{s}, y_0), \tag{15}$$

where $u(x, \bar{s}) = y_0(x, \bar{s}) + y_{\xi}(x, \bar{s}) + \{\text{terms of order } \geq 2 \text{ with respect to } \xi\}$ and $y(x, 0) \equiv 0, y_0(x, 0) = x$.

Now we expand ξ with respect to \bar{s}

$$\xi = \xi_0 + \xi_1\bar{s} + \dots,$$

where $\xi_i = \xi_i(x)$. Then from (15) the r -jet (with respect to x) of the linear term of the expansion of y with respect to $\bar{s} = t$ is

$$tj_{x_0}^r \xi_0(x).$$

That is just the right hand side of the equation (12). If $j_{x_0}^r \xi_0(x) \neq 0$ then similar arguments applied to the left hand side of (12) give the equality

$$t^N j_{x_0}^r w = tj_{x_0}^r \xi_0. \tag{16}$$

So the α -property is satisfied for $k = r$ and $l = 0$, if we put $N = 1$.

If $j_{x_0}^r \xi_0 = 0$, then we have to consider the second term of the expansion of y with respect to \bar{s} . To do so we differentiate both sides of (15) with respect to \bar{s} , and put $\bar{s} = t = 0$.

Now we have

$$\partial_{\bar{s}}^2 y|_{\bar{s}=0} = D\xi_0(x)v(x) + \xi_1(x). \tag{17}$$

Since $j_{x_0}^r \xi_0 = 0, j_{x_0}^r [v, \xi_0] = j_{x_0}^r D\xi_0 v$, so we get

$$j_{x_0}^r \partial_{\bar{s}}^2 y|_{\bar{s}=0} = j_{x_0}^r ([v, \xi_0] + \xi_1). \tag{18}$$

and

$$j_{x_0}^r y(x, t) = \left(\frac{t^2}{2}\right) j_{x_0}^r([v, \xi_0] + \xi_1) + o(t^2). \tag{19}$$

Comparing both sides of (12) we obtain the α -property satisfied for $k = r$ and $l = 1$, provided $j_{x_0}^r([v, \xi_0] + \xi_1) \neq 0$.

If $j_{x_0}^r([v, \xi_0] + \xi_1) = 0$ then we continue the above procedure. Finally we arrive at the following statement:

Let $l \in \mathbb{N}$ be the smallest number for which

$$j_{x_0}^r \left(\sum_{i=0}^l [v, \xi_i]_{l-i} \right) \neq 0.$$

Then

$$j_{x_0}^r y(x_0, t) = \left(\frac{t^l}{l!}\right) j_{x_0}^r \left(\sum_{i=0}^l [v, \xi_i]_{l-i} \right) + o(t^l).$$

Passing to vector fields along π we see that this statement ends the proof of Theorem 3.8, i.e. we have got that the α -property is a necessary condition for the submersivity of $E^r|_{\Gamma^*(h) \times \mathcal{M}}$. Q.E.D.

Now we are going to state the corresponding sufficient condition. For this purpose we assume: at each fiber h_x of h , equipped with the inverse limit topology ($h_x = \varprojlim_{U \ni x} h_U$, where U denotes an open neighbourhood of x), there is a distinguished open set $h_x^0 \subset h_x$. We take v to be a Lipschitz section of h^0 and write $v \in \Gamma(h^0)$, i.e. for any $x \in M$, $v(x) \in h_x^0$.

DEFINITION 3.9. We say that h satisfies the β -property if for every $v \in \Gamma(h^0)$, for every $w \in g_{x_0}^{(k)}$, $k \leq r$, $x_0 \in M$, and every $W_{x_0} \in \mathcal{M}$ there exists an $l \in \mathbb{N}$ and the germ of a vector field $\xi \in h_{x_0}^0$, such that

$$\pi_*([v, \xi]_j|_{W_{x_0}}) \in \pi^*(g)_{x_0}^{(k+1)}|_{W_{x_0}} \tag{\beta_j}$$

(i.e. $j_{x_0}^k(\pi_*[v, \xi]_j|_{W_{x_0}}) = 0$), for $j = 0, \dots, l - 1$,

and

$$\pi_*([v, \xi]_{l-w}|_{W_{x_0}}) \in \pi^*(g)_{x_0}^{(k+1)}|_{W_{x_0}} \tag{\beta_l}$$

(i.e. $j_{x_0}^k(\pi_*([v, \xi]_{l-w}|_{W_{x_0}})) = 0$).

THEOREM 3.10. *Let a subsheaf h of g satisfy the (β) -property. Then $E^r|_{\Gamma^*(h) \times \mathcal{M}}$ is a submersion.*

Proof. Let $v \in \Gamma^*(h)$, and h_t^v be the flow corresponding to v . We denote

$$g_{x_1} = \bigoplus_{k \leq r} \frac{g_{x_1}^{(k)}}{g_{x_1}^{(k+1)}} \oplus g^{(r+1)}, \quad x_1 = h_1^v(x_0),$$

and by

$$pr_k: g_{x_1}^{(k)} \rightarrow \frac{g_{x_1}^{(k)}}{g_{x_1}^{(k+1)}}$$

we denote the canonical projection. We choose $w_{ki} \in g_{x_1}^{(k)}$, $i \in I_k \subset \mathbf{N}$, such that $\{pr_k(w_{ki})\}_{i \in I_k}$ form a basis of the vector space $g_{x_1}^{(k)}/g_{x_1}^{(k+1)}$. The set of all elements of the form $\{pr_k(w_{ki}), 0 \leq k \leq r, i \in I_k\}$ gives a basis of the space $g_{x_1}/g_{x_1}^{(r+1)}$.

Take any element $\bar{w} = w_{ki}$ of our basis. Let $w = \pi_* \bar{w}|_{W_{x_1}}$. We show that for $\lambda \in \mathbf{R}$ sufficiently close to zero, there exists a $\xi_\lambda \in \Gamma^*(h)$, such that the tangent vector $(d/dt)(E^r(v + t\xi_\lambda))|_{t=0}$ to the curve $t \rightarrow E^r(v + t\xi_\lambda)$ is equal to $\lambda w + o(\lambda)$. In fact, let ξ be a section of h , defined in a neighbourhood U of x_1 such that

$$\pi_*([v, \xi]_l - \bar{w})|_{W_{x_1}} \in \pi^*(g)_{x_1}^{(k+1)}|_{W_{x_1}},$$

where

$$\pi_*([v, \xi]_j)|_{W_{x_1}} \in \pi^*(g)_{x_1}^{(k+1)}|_{W_{x_1}}, \quad \text{for } j = 0, \dots, l - 1.$$

Let us take a $\delta > 0$. We consider $(h_{-\delta}^v)_*(\xi) \in h_{x_\delta}$, $x_\delta = h_{1-\delta}^v(x_0)$, i.e. the vector field $\xi \in g_{x_1}^{(k+1)}$, moved to the point x_δ . We assume that δ is so small that $x_1 \in h_{-\delta}^v(U)$. Let us take a suitable hypersurface H transversal to the trajectory $\gamma: [0, 1] \ni \rightarrow h_t^v(x_0)$ at the point x_δ , (cf. Property 2.4). We consider two parts H^- and H^+ of an open neighbourhood of the trajectory (see Figure 2 below).

Now we consider the following (cf. Property 2.4),

$$\xi_0 = \begin{cases} v & \text{on } H^- \\ v + \xi & \text{on } H^+ \end{cases}.$$

Let

$$\mathcal{E}_t \left(v + \begin{cases} 0 & \text{on } H^- \\ \xi & \text{on } H^+ \end{cases} \right)$$

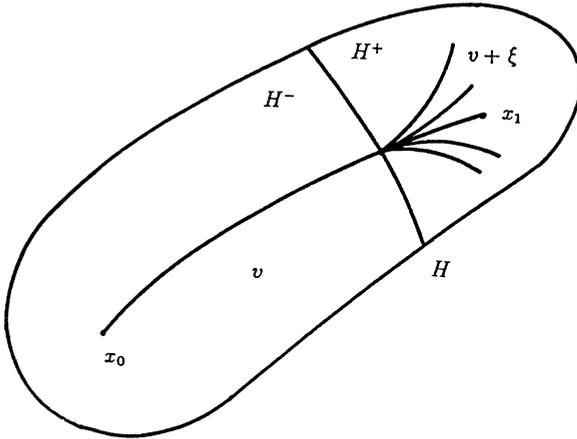


Fig. 2.

denote the flow induced by ξ_0 on M . By the transversality of H to γ we know that

$$\mathcal{E}_1 \left(v + \begin{cases} 0 & \text{on } H^- \\ \xi & \text{on } H^+ \end{cases} \right)$$

is smooth in a neighbourhood of x_0 . Thus we can have its r -jet,

$$\mathcal{E}^r \left(v + \begin{cases} 0 & \text{on } H^- \\ \xi & \text{on } H^+ \end{cases} \right).$$

As before, we denote by

$$E^r \left(v + \begin{cases} 0 & \text{on } H^- \\ \xi & \text{on } H^+ \end{cases} \right)$$

the r -jet of $\mathcal{E}_1(\xi_0)$ at x_0 .

We compute the linear terms of

$$E^r \left(v + \begin{cases} 0 & \text{on } H^- \\ \xi & \text{on } H^+ \end{cases} \right)$$

with respect to ξ . Essentially we repeat the proof of Theorem 3.8, where we studied the equation (15). In the present case we obtain that the solution of (15) has the expansion

$$j^k \pi_* y(x, \delta) = \left(\frac{\delta^l}{l!} \right) j^k \pi_* ([v, \alpha]) + o(\delta^l) = \left(\frac{\delta^l}{l!} \right) w + o(\delta^l),$$

which proves that

$$E^r \left(v + \begin{cases} 0 & \text{on } H^- \\ \xi & \text{on } H^+ \end{cases} \right) = E^r(v) + \left(\frac{\delta^l}{l!} \right) w + o(\delta^l).$$

The field ξ is $(k+1)$ -flat at x_δ so we use the Property 2.4 and write the approximating family $P_\epsilon(v, \xi)$. Obviously we have

$$E^r(P_\epsilon(v, \xi)) = E^r \left(v + \begin{cases} 0 & \text{on } H^- \\ \xi & \text{on } H^+ \end{cases} \right) + E_\epsilon(v, \xi),$$

where $E_\epsilon \xrightarrow{\epsilon \rightarrow 0} 0$.

Then for sufficiently small ϵ we have

$$E^r(P_\epsilon(v, \xi)) = E^r(v) + \left(\frac{\delta^l}{l!} \right) w + o(\delta^l) + A_\epsilon,$$

where $A_\epsilon \xrightarrow{\epsilon \rightarrow 0} 0$. We take ϵ so small that $|A_\epsilon| < o(\delta^l)$, $\lambda = \frac{\delta^l}{l!}$ and $\xi_\lambda = P_\epsilon(v, \xi) - v$. Q.E.D.

Our results can be briefly recapitulated as follows.

COROLLARY 3.11. *If h is accessible then h satisfies the α -property. If h satisfies the β -property then h is accessible.*

4. Genericity of Exp for Riemannian and sub-Riemannian metrics

Let M be the cotangent bundle; $M = T^*X$, $\dim X = n$, with $\pi = \pi_X: T^*X \rightarrow X$ the canonical bundle projection. By g we denote the sheaf of Hamiltonian vector fields on T^*X . Let h be a subsheaf of g and let \mathcal{M} denote a class of germs of submanifolds of M ; unspecified for the moment by \bar{g} and \bar{h} we denote the corresponding sheaves of local Hamiltonians on T^*X ; thus \bar{g} is the sheaf of germs of functions f , such that Jdf is a section of \bar{g} . For $W \in \mathcal{M}$ and $\bar{p}_0 \in W$ we denote by $\mathcal{F}_{W_{\bar{p}_0}}$ the space of germs of functions on W , at p_0 ; $\mathcal{F}_{W_{\bar{p}_0}}^{(k)}$ are the germs vanishing up to order $k - 1$ at \bar{p}_0 .

Let $v_{f_1}, v_{f_2} \in \Gamma^*(g)$, $v_{f_i} = Jdf_i$, where f_1, f_2 are the corresponding Hamiltonians. We have

$$[v_{f_1}, v_{f_2}] = v_{\{f_1, f_2\}},$$

where $\{.,.\}$ denotes the standard Poisson bracket on T^*X . In the Darboux coordinates (q, p) on T^*X we have, of course

$$(\pi_X)_* v_f = \partial_p f,$$

where by $\partial_p f$ we denote the n -tuple $(\partial_{p_1} f, \dots, \partial_{p_n} f) \in (\bar{g}_{\bar{p}_0})^n$.

PROPOSITION 4.1. *The α -property for the subsheaf $h \subset g$ is equivalent to the following condition:*

(α): *For almost all $f \in \bar{h}_{\bar{p}_0}$, for every $\eta \in \bar{g}_{\bar{p}_0}^{(k)}$, $k \leq r$ and $W_{\bar{p}_0} \in \mathcal{M}_{\bar{p}_0}$, there exist: an $l \in \mathbf{N} \cup \{0\}$ and $f_0, f_1, \dots, f_l \in \bar{h}_{\bar{p}_0}$ such that*

$$\sum_{i=0}^j \partial_p \{f, f_i\}_{j-i} \Big|_{W_{\bar{p}_0}} \in (\mathcal{F}_{W_{\bar{p}_0}}^{(k+1)})^n,$$

$$\left(\text{i.e. } j_{\bar{p}_0}^k \left(\sum_{i=0}^j \partial_p \{f, f_i\}_{j-i} \right) \Big|_{W_{\bar{p}_0}} = 0 \right),$$

for $j = 0, \dots, l - 1$, and

$$\left(\partial_p \eta - \sum_{i=0}^l \partial_p \{f, f_i\}_{l-i} \right) \Big|_{W_{\bar{p}_0}} \in (\mathcal{F}_{W_{\bar{p}_0}}^{(k+1)})^n.$$

$$\left(\text{i.e. } j_{\bar{p}_0}^k \partial_p \eta|_{W_{\bar{p}_0}} = j_{\bar{p}_0}^k \sum_{i=0}^l \partial_p \{f, f_i\}_{l-i} \Big|_{W_{\bar{p}_0}} \right).$$

By a straightforward modification of the above α -property we obtain the β -property expressed in terms of Hamiltonians.

Let $\mathcal{L}_{\bar{p}}$, $\bar{p} \in T^*X$ denote the space of germs at \bar{p} of Lagrangian submanifolds in (T^*X, ω_X) . If $v \in \Gamma(g)$ then the class $\mathcal{L} = \cup_{\bar{p} \in T^*X} \mathcal{L}_{\bar{p}}$ is obviously preserved by the flow of symplectomorphisms h_t^v . From now on we take \mathcal{L} as \mathcal{M} .

Let us fix a germ at \bar{p} of a Lagrangian submanifold $W_{\bar{p}} \in \mathcal{L}_{\bar{p}}$. We shall study $E|_{W_{\bar{p}}}$, or more formally

$$\text{Exp}_v|_{W_{\bar{p}}} = \pi_X \circ h_1^v|_{W_{\bar{p}}}, \quad q = \pi_X(\bar{p}),$$

(in the most interesting and classical case $W_{\bar{p}}$ is the germ of the fibre T_q^*X). We notice that in standard terminology $h_1^v|_{W_{\bar{p}}}$ is a Lagrangian embedding and $\text{Exp}_v|_{W_{\bar{p}}}$ is the corresponding Lagrangian projection.

We shall now discuss the genericity property of the Exp-map in the Riemannian geometry.

Let $h \subset g$ denote the subsheaf of Hamiltonian vector fields with quadratic Hamiltonians with respect to p . We look on $\Gamma(h)$ as the space of geodesic vector fields on X with the families of quadratic nondegenerate forms on T^*X playing the role of nondegenerate Hamiltonians. In what follows we assume $\det(h^{ij}) \neq 0$

for our Hamiltonian vector fields $v_H \in \Gamma(h^0)$, $H = \Sigma h^{ij} p_i p_j$, (see Definition 3.9).

THEOREM 4.5. *Let $\bar{p} \in T^*V$. There exists an open and dense set of Riemannian metrics $h' \subset \Gamma(\bar{h})$, such that for every $f \in h'$, the Exp-map $\text{Exp}_{v_f}|_{W_{\bar{p}}}$ has only the singularities appearing in generic Lagrangian projections.*

Proof. First we prove the accessibility of h . It is enough to check the sufficient condition, i.e. the β -property for our sheaf h ; Without loss of generality we take $\bar{p}_0 = (0, p_0)$; let $W_{\bar{p}_0} = W_{p_0}$ be an element of $\mathcal{L}_{\bar{p}_0}$. We denote $\mathcal{F}_{p_0} = \mathcal{F}_{W_{p_0}}$ the space of germs at p_0 of analytic functions on W_{p_0} . Thus the β -property reads as follows:

Let $f \in \bar{h}_{\bar{p}_0}$ be a nondegenerate Hamiltonian at $q = 0$. Let $k \in \mathbf{N}$. For every $\bar{\eta} \in g_{\bar{p}_0}^{(k)}$, there exists an $l \in \mathbf{N} \cup \{0\}$ and $f_0 \in \bar{h}_{\bar{p}_0}$ such that

(β_j) :

$$\partial_p \{f, f_0\}_j|_{W_{p_0}} \in (\mathcal{F}_{p_0}^{(k+1)})^n, \quad \text{for } j = 0, \dots, l - 1$$

and

(β_l) :

$$(\partial_p \eta - \partial_p \{f, f_0\}_l)|_{W_{p_0}} \in (\mathcal{F}_{p_0}^{(k+1)})^n.$$

We write

$$f = \sum_{ij} g^{ij}(q) p_i p_j.$$

Let us take

$$f_0 = \sum_{ij} h^{ij}(q) p_i p_j \in \bar{g}_{\bar{p}_0}^{(N)}. \tag{20}$$

Then

$$\{f, f_0\}_l = \partial_{v_f}^l f_0 = \sum_{ijkrs} \left(g^{ij}(q) p_j \frac{\partial}{\partial q^i} - g^{ij}(q) p_i p_j \frac{\partial}{\partial p_k} \right)^l h^{rs}(q) p_r p_s.$$

Let $i_{W_{p_0}}$ be a Lagrangian embedding of W_{p_0} into T^*X . First assume that $i_{W_{p_0}}(W_{p_0})$ can be described by

$$q = \phi(p), \quad \phi(p) = (\phi_1(p), \dots, \phi_n(p)). \tag{21}$$

Then the (β_l) -condition can be written in the form

$$\partial_p \eta(q, p)|_{\{q=\phi(p)\}}$$

$$= \partial_p \sum_{i_1, \dots, j_{l+2}} g^{i_3 j_3}(q) \dots g^{i_{l+2} j_{l+2}}(q) h_{i_3 \dots j_{l+2}}^{i_1 i_2}(q) p_{i_1} \dots p_{i_{l+2}}|_{\{q=\phi(p)\}},$$

for the lowest degree terms in (q, p) .

Let us define $I = \langle \phi_1, \dots, \phi_n \rangle$, the ideal in $\mathcal{F}_{p_0} (= \mathcal{F}_{W_{p_0}})$ generated by $\phi_1(p), \dots, \phi_n(p)$. We consider \mathcal{F}_{p_0} as a graded ring with respect to $I^s, s \in \mathbf{N}$. Let $\eta \in \bar{g}_{p_0}$. We put

$$w = \partial_p \eta|_{q=\phi(p)} \in (\mathcal{F}_{p_0}^{(N)})^n.$$

Let s be the biggest integer such that

$$w \in (I^s)^n. \tag{22}$$

The matrix $g^{ij}(0)$ is invertible, so we can find $l \in \mathbf{N} \cup \{0\}$ and $f_0 = \sum_{i,j} h^{ij}(q) p_i p_j$ (i.e. a matrix $h^{ij}(q)$), such that

$$\begin{aligned} & \sum_{i_1, \dots, j_{\kappa+2}} g^{i_3 j_3}(\phi(p)) \dots g^{i_{\kappa+2} j_{\kappa+2}}(\phi(p)) h_{i_3 \dots j_{\kappa+2}}^{i_1 i_2} \\ & \times (\phi(p)) \partial_p (p_{i_1} \dots p_{i_{\kappa+2}}) \in (I^{s+1})^n, \end{aligned}$$

for $\kappa = 0, \dots, l - 1$, and

$$\begin{aligned} & \sum_{i_1, \dots, j_{l+2}} g^{i_3 j_3}(\phi(p)) \dots g^{i_{l+2} j_{l+2}}(\phi(p)) h_{i_3 \dots j_{l+2}}^{i_1 i_2} \\ & \times (\phi(p)) \partial_p (p_{i_1} \dots p_{i_{l+2}}) - w \in (I^{s+1})^n. \end{aligned}$$

But from (22) we have $I^{s+1} \subset \mathcal{F}_{p_0}^{(N+1)}$. Thus we obtain the (β) -condition, i.e.

$$\partial_p \{f, f_0\}_\kappa|_{q=\phi(p)} \in (\mathcal{F}_{p_0}^{(N+1)})^n, \quad \text{for } \kappa = 0, \dots, l - 1,$$

and

$$\partial_p \{f, f_0\}_l|_{q=\phi(p)} - w \in (\mathcal{F}_{p_0}^{(N+1)})^n.$$

Let us discuss briefly what modifications should be done if (21) is not satisfied. Suppose for a moment that W_{p_0} is transversal to the fibres of T^*X . In this case we write $i_{W_{p_0}}(q) = (q, \psi(q))$ and the (β) -condition is satisfied immediately. In fact, for any $w = (w^1, \dots, w^n) \in (\mathcal{F}_{p_0}^{(N)})^n$ (elements of \mathcal{F}_{p_0} are parametrized by q) there exists an $f_0 = \sum_{i,j} h^{ij}(q) p_i p_j$ such that

$$w^k(q) = \sum_i h^{ik}(q) \psi_i(q) \pmod{(\mathcal{F}_{p_0}^{(N+1)})}.$$

This is obvious; taking $p_0 = (1, 0, \dots, 0)$ we find $h^{1k}(q) = w^k(q)$.

The most general “ IJ ”-case, $I \cup J = \{1, \dots, n\}$, $I \cap J = \emptyset$, where

$$i_{W_{p_0}}(q_I, p_J) = (q_I, \phi_J(q_I, p_J), \psi_I(q_I, p_J), p_J)$$

can be treated in a similar way by mixing both methods.

Q.E.D.

REMARK 4.3. A slightly stronger genericity result is true also for the map $h_1^v|_{W_{p_0}} : W_{p_0} \rightarrow T^*X$. In this case the (β) -condition is fulfilled and one can state the result as follows:

Let $\bar{p} \in T^*X$. There exists an open and dense subset h' of quadratic hamiltonians h such that for every $f \in \Gamma(h')$ the Lagrangian embedding $h_1^{vf}|_{W_{p_0}}$ is generic in the space of all mappings $h_1^v|_{W_{p_0}}$ induced by general Hamiltonian vector fields $v \in \Gamma(g)$. (This extends the Theorem 1 in [11], p. 735).

REMARK 4.4. Let us choose a family of germs $W_{\bar{p}(q)}$ of fibers of T^*X defined by a section $X \rightarrow T^*X$, $X \ni q \rightarrow \bar{p}(q) \in T^*X$. Then we define

$$\text{Exp}_q = \text{Exp}_v|_{W_{\bar{p}(q)}}$$

and look on it as a family of maps parametrized by $q \in X$. C.T.C. Wall ([11], Conjecture 2, p. 735) conjectured that for a generic metric on X , i.e. $f = \sum_{i,j} g^{ij} p_i p_j$, $\text{Exp} = \{\text{Exp}_q : q \in X\}$ is a generic n -parameter family of Lagrangian projections (cf. [1]). A straightforward calculation shows that the necessary condition (α) is not fulfilled. In this case we have $W = T^*X$. Thus the “Wall Conjecture” is not true (cf. [5, 3]). In fact there is an infinite number of constraints resulting from the obvious formula:

$$(h_t^v)_* v = v, \quad v = Jdf. \tag{23}$$

Since f is quadratic with respect to p , this implies that $(h_t^v)_* v$ must be linear with respect to v , and this is a strong constraint for h_t^v . To be more explicit, let $(p, Q) \rightarrow G_t(p, Q)$, $\det((\partial^2 G_t / \partial p \partial Q)(p, Q)) \neq 0$, be a generating function for h_t^v , (for a standard notation see [2]). By $(q, p) \rightarrow Q_t(q, p)$ we denote the solution of the equation $q - (\partial G_t / \partial p)(p, \bullet) = 0$. We define

$$\Phi^j(q, p) = \partial_{q_l} Q_t^j(q, p) \sum_i g^{il}(q) p_i - \partial_{p_s} Q_t^j(q, p) \sum_{ik} g_{,s}^{ik}(q) p_i p_k.$$

By a simple verification we find

$$\partial_p^\alpha (\Phi^j \circ (h_t^v)^{-1})(q, p) \equiv 0, \quad \text{for } |\alpha| > 1.$$

One can conjecture that all constraints satisfied by the family Exp arise from the one above simply by differentiation.

Now let us pass to the Exp-map in the sub-Riemannian geometry (cf. [9]). Let $M = T^*X$ be as above, and $\dim X = n + 1$. By $V \subset TX$ we denote a smooth distribution of hyperplanes on X , i.e. a subbundle of TX . All our arguments are valid for any dimension of V , however for simplicity of notation, we shall assume that $\text{codim } V = 1$. Locally V is annihilated by a 1-form,

$$\omega = dq^{n+1} + \sum_{i=1}^n A_i(q) dq^i. \tag{24}$$

Let $H: T^*X \rightarrow R$ be a smooth function. We say that the Hamiltonian vector field v_H with hamiltonian H is *horizontal* if

$$(\pi_X)_* v_H|_{\bar{p}} \in V_{\pi_X(\bar{p})}, \quad \text{for all } \bar{p} \in T^*X.$$

By k we denote the sheaf of horizontal Hamiltonian vector fields. An easy check shows that if $v_f \in k$, then

$$f = \bar{f}(q, p'_1, \dots, p'_n),$$

for some smooth function \bar{f} and $p'_i = p_i - A_i(q)p_{n+1}$.

Let V be equipped with a quadratic nondegenerate form $\langle \cdot, \cdot \rangle$ varying smoothly on $q \in X$. By g^{ij} we denote the inverse matrix to that defined by $\langle \cdot, \cdot \rangle$. Analogously to the usual Riemannian case we have the sheaf (subsheaf of k) h of horizontal geodesic vector fields defined by the quadratic Hamiltonians

$$f(q, p) = \sum_{ij=1}^n g^{ij}(q)(p_i - A_i(q)p_{n+1})(p_j - A_j(q)p_{n+1}).$$

Both these sheaves are subsheaves of the sheaf of Hamiltonian vector fields g . By \bar{h} we denote the space of Hamiltonians quadratic in $p' = (p'_1, \dots, p'_n)$.

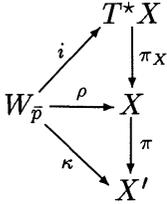
Let $X_i = (\partial/\partial q^i) - A_i(q)(\partial/\partial q^{n+1})$, $i = 1, \dots, n$; they give a basis of sections of V . Let $q \in X$. If the basic vector fields X_i , along with all their commutators span $T_q X$ then the distribution V is said to satisfy the *Hörmander condition* at q . If this condition is fulfilled at every $q \in X$ then V is called also a *non-holonomic distribution* (cf. [9, 10]).

Let I be a submanifold of T^*X , $\dim I \leq n + 1$. If $\omega_X|_I = 0$, then we call I an isotropic submanifold. Let $\mathcal{I}_{\bar{p}}$, $\bar{p} \in T^*X$ denote the space of germs at \bar{p} , of isotropic submanifolds of dimension n in (T^*X, ω_X) . We see that $\mathcal{I} = \cup_{\bar{p} \in T^*X} \mathcal{I}_{\bar{p}}$ is preserved by the flow of symplectomorphisms h_t^v , $v \in \Gamma(g)$.

In what follows we assume that we are given a fibering π on X , $\pi: X \rightarrow X'$, such that $\ker \pi_* \oplus V = TX$. Locally $\pi(q) = (q')$, $q = (q', q^{n+1})$, $q' = (q^1, \dots, q^n)$. Let $W_{\bar{p}} \in \mathcal{I}_{\bar{p}}$. The inclusion $i_{W_{\bar{p}}}: W_{\bar{p}} \rightarrow T^*X$ is called an isotropic immersion of $W_{\bar{p}}$, and $\rho_{W_{\bar{p}}} = \pi_X \circ i_{W_{\bar{p}}}$ an *isotropic projection*. Now we define

$$\kappa_{W_{\bar{p}}}: W_{\bar{p}} \rightarrow X', \quad \kappa_{W_{\bar{p}}} = \pi \circ \rho_{W_{\bar{p}}};$$

these maps we will call *subisotropic maps*. A smooth mapping $\kappa: W_{\bar{p}} \rightarrow X'$ is subisotropic if and only if there exists an isotropic immersion $i: W_{\bar{p}} \rightarrow T^*X$ such that the following diagram commutes



Let $f \in \mathfrak{h}$, and Exp_{v_f} be the corresponding Exp-map. We denote $\tilde{\text{Exp}}_{v_f} = \pi \circ \text{Exp}_{v_f}$ and we call $\tilde{\text{Exp}}_{v_f}$ a *sub-Exp-map*. Adapting the proof of the Theorem 4.2 we obtain the following result.

THEOREM 4.5. *Let $\bar{p} \in T^*X$. Then there exists an open and dense set of quadratic Hamiltonians $\mathfrak{h}' \subset \mathfrak{h}$, such that if $f \in \mathfrak{h}'$, then the sub-Exp-map $\tilde{\text{Exp}}_{v_f}|_{W_{\bar{p}}}$, for any projection π , has only the singularities appearing in generic subisotropic maps.*

Now let $q_0 \in X$ and let H^{n-1} be an isotropic subspace of $T_{q_0}^*X$, $\bar{p}_0 \in H^{n-1}$, $\bar{p}_0 \neq 0$. What is interesting for us now is the description of all possible generic singularities of germs

$$(H, \bar{p}_0) \xrightarrow{\text{exp}(g)} X,$$

where g varies over the space of all Riemannian metrics on V . To get such a description we apply Theorem 4.5. Thus if $\pi: X \rightarrow X'$ is any projection (as in the statement of the Theorem) then all generic singularities of

$$\tilde{E} = \pi \circ \text{exp}(g)$$

are of the form

$$p = \frac{\partial F}{\partial q}(\lambda, q)|_{\Sigma_F},$$

where

$$\Sigma_F = \left\{ (\lambda, q): \frac{\partial F}{\partial \lambda_i} \Big|_{H^{n-1} \times X} = 0 \right\},$$

with the generating family F having only the generic singularities (see [4]).

To get $\text{exp}(g)$ from $\pi \circ \text{exp}(g)$ we remark that every line l in H^{n-1} passing through 0 is sent by $\text{exp}(g)$ into a geodesic, and thus is a horizontal curve with

a non-zero tangent vector, which is projected by π onto a non-zero vector. Let l_p be the line passing through 0 and p ; let $\xi_0 = \bar{p}_0$ be considered as a vector in $T_{p_0}l_{\bar{p}_0}$, $x'_0 = \tilde{E}(\bar{p}_0)$, and

$$v'_0 = (\tilde{E}_*)_{\bar{p}_0}(\xi_0).$$

Let $N' \subset X'$ be any germ of a submanifold at x'_0 transversal to v'_0 and of codimension one. Let $N \subset X$ be any submanifold of codimension 2, such that $\pi|_N: N \rightarrow N'$ is a diffeomorphism and N is “generic” with respect to V . Now we lift \tilde{E} to a map E into X such that N lies in the image of E and the curves $\tilde{E}(l_p)$ (p close to \bar{p}_0) are lifted into horizontal curves. We shall illustrate this procedure a little bit later, in the 2-dimensional case.

Now we investigate the generic subisotropic maps. We denote

$$D = \{p \in W_{\bar{p}}: \text{Im}(\rho_*)_p \text{ is not transversal to } V_{\rho(p)}\}$$

and

$$\Delta = \kappa(D).$$

We will call Δ a horizontal set of ρ . Δ is the set of points of the image manifold $S = \rho(W_{\bar{p}})$, in which S is tangent to the distribution V . Let Γ denote the set of critical points of κ and $\Sigma = \kappa(\Gamma)$ denote the set of its critical values.

LEMMA 4.6. *Let V be a contact distribution (V satisfies the strong bracket generating hypothesis [9]), then for a generic subisotropic map κ , D is a curve.*

Proof. We show this fact for $n = 2, \dim X = 3$. $\pi: X \rightarrow X', \pi(q^1, q^2, q^3) = (q^1, q^2)$. The general case is straightforward. Distribution V is annihilated by

$$\omega = dq^3 + \sum_{i=1}^2 A_i(q^1, q^2) dq^i. \tag{25}$$

$\rho(W_{\bar{p}})$ is covered by geodesics. Without loss of generality we restrict our considerations to the vertical $W_{\bar{p}} \subset T_{\pi_X(\bar{p})}^*X$. We choose the parameterization $\{u_1, u_2\}$ of $W_{\bar{p}}$, such that u_1 parameterizes the geodesics (obviously horizontal with respect to V). Then

$$\frac{\partial \rho^3}{\partial u_1} + \sum_{i=1}^2 A_i \circ \kappa \frac{\partial \rho^i}{\partial u_1} \equiv 0.$$

The second equation

$$F(u_1, u_2) = \left(\frac{\partial \rho^3}{\partial u_2} + \sum_{i=1}^2 A_i \circ \kappa \frac{\partial \rho^i}{\partial u_2} \right) (u_1, u_2) = 0,$$

on maximal smooth strata of S defines a smooth curve. We see that $\nabla F \neq 0$. In fact

$$\frac{\partial F}{\partial u_1} = \left(\frac{\partial A_2}{\partial q^1} - \frac{\partial A_1}{\partial q^2} \right) \left(\frac{\partial \rho_1}{\partial u_2} \frac{\partial \rho_2}{\partial u_1} - \frac{\partial \rho_1}{\partial u_1} \frac{\partial \rho_2}{\partial u_2} \right) \neq 0$$

outside of the set of critical points of κ , because

$$[X_1, X_2] = \left(\frac{\partial A_2}{\partial q^1} - \frac{\partial A_1}{\partial q^2} \right) \frac{\partial}{\partial q^3} \neq 0.$$

Q.E.D.

Now we assume, $\dim X = 3$. For a contact distribution we have the following result.

THEOREM 4.7. *Let V be a contact distribution on R^3 , annihilated by $\omega = dq^3 + \sum_{i=1}^2 A_i(q^1, q^2) dq^i$, $\pi: R^3 \rightarrow R^2: (q^1, q^2, q^3) \rightarrow (q^1, q^2)$, is the projection. Then for a generic subisotropic map;*

- (1) ρ is an immersion, $\Delta = \emptyset$ and κ is a diffeomorphism, fold or cusp-map.
- (2) ρ is an immersion, $\Gamma = \emptyset$ and Δ is smooth curve.
- (3) ρ is a singular map of corank 1, right-left equivalent (\mathcal{A} -equivalent, [8]) to Whitney's Cross-cap (S_0) with horizontal and critical sets tangent with the second order tangency.

Proof. For generic isotropic map, the corresponding map κ is one of the Whitney's stable cases of smooth mappings $R^2, 0 \rightarrow R^2, 0$ provided S is a smooth hypersurface of R^3 . If S is the remaining stable case – the Cross-cap (cf. [8]), then the subisotropic map κ is a fold-map. Distribution V , defined by ω is transversal to the fibers of π , so we easily see that, in the smooth case of S , the horizontal (Δ) and critical (Γ) sets are disjoint, which proves the first two cases.

For the lifting of κ we can write

$$z(u_1, u_2) = - \int_0^{u_1} x(s, u_2) \frac{\partial y}{\partial s}(s, u_2) ds + \phi(u_2),$$

where we use the notation $(q) = (x, y, z)$. Thus for x, y, z we write the following expansions

$$\begin{aligned} x &= \beta_1 u_1 + \beta_2 u_2 + \beta_{11} u_1^2 + \beta_{12} u_1 u_2 + \beta_{22} u_2^2 + \mathbf{m}^3, \\ y &= \alpha_1 u_1 + \alpha_2 u_2 + \alpha_{11} u_1^2 + \alpha_{12} u_1 u_2 + \alpha_{22} u_2^2 + \mathbf{m}^3, \\ z &= -\frac{2}{3} \alpha_{11} \beta_1 u_1^3 - \left(\frac{1}{2} \beta_1 \alpha_{12} + \alpha_{11} \beta_2\right) u_1^2 u_2 - \alpha_{12} \beta_2 u_1 u_2^2 \\ &\quad + \gamma_2 u_2 + \gamma_{22} u_2^2 - \alpha_1 \int_0^{u_1} x(s, u_2) ds + \mathbf{m}^3, \end{aligned}$$

where \mathfrak{m} denotes the maximal ideal in the space of germs of smooth functions of variables u_1, u_2 .

The cases $(\beta_1\alpha_2 - \alpha_1\beta_2) = 0$, and $\alpha_1 = 0$ may appear transversally, then generically $\gamma_2 \neq 0$ and we have the smooth case of S . So we have to assume $\alpha_1 = (\partial_y/\partial u_1) \neq 0$. In this case we define new coordinates of $W_{\bar{p}}$

$$(u_1, u_2) \rightarrow (y, u_2) = (y(u_1, u_2), u_2).$$

Now the equation $(\partial z/\partial u_1) + x(\partial y/\partial u_1) \equiv 0$ is transformed into $(\partial z/\partial y)(\partial y/\partial u_1) + x(\partial y/\partial u_1) \equiv 0$, which finally is equivalent to

$$\frac{\partial z}{\partial y}(y, u_2) = -x(y, u_2).$$

Thus

$$z(y, u_2) = - \int_0^y x(s, u_2) ds + \phi(u_2).$$

By [4] $\kappa: (y, u_2) \rightarrow (x(y, u_2), y)$ is a fold-map, so $(\partial x/\partial u_2) = 0$ and $(\partial^2 x/\partial u_2^2) \neq 0$. Writing

$$x(y, u_2) = \beta_1 y + \beta_{11} y^2 + \beta_{12} y u_2 + \beta_{22} u_2^2 + \mathfrak{m}^3$$

we have

$$z(y, u_2) = \gamma_2 u_2 - \frac{1}{2} \beta_1 y^2 + \gamma_{22} u_2^2 - \frac{1}{3} \beta_{11} y^3 - \frac{1}{2} \beta_{12} y^2 u_2 - \beta_{22} y u_2^2 + \gamma_{222} u_2^3 + \mathfrak{m}^4.$$

Finally

$$\rho: (y, u_2) \rightarrow (\beta_1 y + \beta_{11} y^2 + \beta_{12} y u_2 + \beta_{22} u_2^2 + \mathfrak{m}^3, y, \gamma_2 u_2 - \frac{1}{2} \beta_1 y^2 + \gamma_{22} u_2^2 + \mathfrak{m}^3), \quad \beta_{22} \neq 0,$$

and the singular case: $\gamma_2 = 0$ and $\gamma_{22} \neq 0$ may happen generically.

By a left coordinate change we obtain

$$j^2 \rho(0) = (\beta_{12} y u_2 + \beta_{22} u_2^2, y, \gamma_{22} u_2^2).$$

The coordinate change

$$U_2 = u_2 + \frac{\beta_{12}}{2\beta_{22}} y$$

now transforms ρ to a map germ whose 2-jet is following

$$j^2 \rho(0) = \left(\beta_{22} u_2^2 - \frac{\beta_{12}^2}{4\beta_{22}} y^2, y, \gamma_{22} \left(u_2 - \frac{\beta_{12}}{2\beta_{22}} y \right)^2 \right).$$

The left coordinate change

$$X = \bar{X} - \frac{\beta_{12}^2}{4\beta_{22}} Y^2,$$

$$Z = \bar{Z} + \frac{\gamma_{22}\beta_{12}^2}{4\beta_{22}^2} Y^2 + \frac{\gamma_{22}}{\beta_{22}} \bar{X}$$

gives

$$j^2\rho(0) = \left(\beta_{22}u_2^2, y, -\frac{\gamma_{22}\beta_{12}}{\beta_{22}}u_2y \right),$$

which is 2-determined and describes Whitney's Cross-cap.

Now we easily check that the equation of horizontal points D is transformed, in new coordinates, to the following one,

$$\frac{\partial z}{\partial u_2}(y, u_2) = 0,$$

i.e.

$$2\gamma_{22}u_2 - \frac{1}{2}\beta_{12}y^2 + 2\beta_{22}yu_2 + 3\gamma_{222}u_2^2 + \mathbf{m}^3 = 0.$$

Thus for D we obtain

$$u_2 = \frac{\beta_{12}}{4\gamma_{22}}y^2 + \mathbf{m}^3.$$

Analogously for the set of critical points of κ

$$\Gamma: u_2 = -\frac{\beta_{12}}{2\beta_{22}}y + \mathbf{m}^2.$$

So the order of the tangency of Δ and Σ is given by the formula

$$x = \frac{\beta_{12}^2}{4\beta_{22}}y^2 + \mathbf{m}^3.$$

Q.E.D.

REMARK 4.8. One can explicitly calculate the Exp-map in the case of Heisenberg group $\mathbf{H} = \mathbf{R}^3$ (cf. [9]), equipped with the distribution V annihilated by

$$\omega = dz + \frac{1}{2}(y dx - x dy),$$

and Hamiltonian

$$H(p, q) = \frac{1}{2}(p_1 - \frac{1}{2}yp_3)^2 + \frac{1}{2}(p_2 + \frac{1}{2}xp_3)^2.$$

One computes

$$\begin{aligned} h_1^{vH}|_{\{q=0\}}(p) = & \left(\frac{1}{2}(p_1(\cos p_3 + 1) - p_2 \sin p_3), \frac{1}{2}(p_2(\cos p_3 + 1) \right. \\ & \left. - p_1 \sin p_3), p_3, \frac{1}{p_3}(p_2(\cos p_3 - 1) + p_1 \sin p_3), \right. \\ & \left. \frac{1}{p_3}(-p_1(\cos p_3 - 1) \right. \\ & \left. + p_2 \sin p_3), (p_1^2 + p_2^2) \left(\frac{p_3 - \sin p_3}{2p_3^2} \right) \right). \end{aligned}$$

We take $\pi(x, y, z) = (x, y)$. Then we have $\exp(g): W_{\bar{p}} \rightarrow X$, and

$$\begin{aligned} \tilde{E}|_{W_{\bar{p}}} = & \left(\frac{1}{p_3}(p_2(\cos p_3 - 1) + p_1 \sin p_3), \frac{1}{p_3}(-p_1(\cos p_3 - 1) \right. \\ & \left. + p_2 \sin p_3), (p_1^2 + p_2^2) \left(\frac{p_3 - \sin p_3}{2p_3^2} \right) \right) \Big|_{W_{\bar{p}}}, \end{aligned}$$

where

$$W_{\bar{p}} = \{(p_1, p_2, p_3) : Ap_1 + Bp_2 + Cp_3 = 0\}.$$

By simple check we find that if

$$\bar{p} \in \{(p_1, p_2, p_3); p_3 = 2k\pi\},$$

and

$$p_3 = ap_1 + bp_2 + 2k\pi,$$

then $\exp(g)$ is not generic. In other cases it is an immersion.

The set of singular values of $\pi_X \circ h_1^{vH}$ (the usual caustic of Exp-map) is formed by the family of rotationally invariant paraboloids

$$z = (x^2 + y^2) \frac{2a - \sin 2a}{4(1 - \cos 2a)},$$

and the line $x = 0$, $y = 0$, where a is a solution of the equation $\operatorname{tg} x = x$. Simplifying the system by isometry $(x, y, z) \rightarrow (x, y, z - \frac{1}{2}xy)$, we obtain the generating family for the isotropic map $h_1^{vH}|_{\{q=0\}}$, namely

$$F(x, y, z, \lambda) = x\lambda_1 \cos \lambda_3 + y\lambda_2 \cos \lambda_3 + z\lambda_3 - \frac{\sin 2\lambda_3}{4\lambda_3} + \frac{\lambda_1\lambda_2}{\lambda_3}(\cos \lambda_3 - 1) \cos \lambda_3.$$

Note added in proof. 1. We note that the integral formula (7) implies the accessibility criterion. 2. In the Hamiltonian case, for $p_0 \neq 0$, condition (β) is not satisfied. However the straightforward proof of Theorem 4.5 follows from the integral formula.

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