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The generalized Thue inequality

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Abstract. Let $F(\underline{\underline{x}}) = F(x, y)$ be a form in $\mathbb{Z}[x, y]$ of degree $r \geqslant 3$ and without multiple factors. A generalization of the classical Thue inequality $|F(\underline{\underline{x}})| \leqslant h$ is the inequality $|F(\underline{\underline{x}})| \leqslant h|\underline{\underline{x}}|^\gamma$ where $|\underline{\underline{x}}|$ is the maximum norm. When $\gamma < r-2$ this inequality has only finitely many solutions in integers. The present paper deals with upper bounds for the number of such solutions.

1. Introduction

As is well known, a *Thue equation*

$$F(x, y) = h$$

has only finitely many solutions in integers. Here F is a form of degree $r \geqslant 3$ with coefficients in $\mathbb{Z}$ and without multiple factors, and $h \in \mathbb{Z}$. Upper bounds $B_1(r, h)$ for the number of solutions which depend only on r and h but are independent of the coefficients of F were given by Evertse [2] and then by Bombieri and Schmidt [1]. Clearly the *Thue inequality*

$$|F(x, y)| \leqslant h \tag{1.1}$$

also has only finitely many solutions. Upper bounds $B_2(r, h)$ for the number of solutions were given by Schmidt [7] and by Thunder [9], [10]. It is an immediate consequence of Roth's Theorem that a *generalized Thue inequality*

$$|F(x, y)| \leqslant h|\underline{\underline{x}}|^\gamma \tag{1.2}$$

where $|\underline{\underline{x}}| = \max(|x|, |y|)$ and where $\gamma < r - 2$, has only finitely many solutions. The obvious question whether there is a bound $B_3(r, \gamma, h)$ for the number of solutions of (1.2) has a negative answer, as may be seen as follows.

Given two forms F, G as above, write $F \sim G$ if there is a transformation $T \in \mathrm{SL}(2, \mathbb{Z})$ with $F(\underline{\underline{x}}) = G(T\underline{\underline{x}})$; here we use the notation $\underline{\underline{x}} = (x, y)$. We will show that given $\gamma > 0$ and given a form G there is a constant $c_1(G, \gamma) > 0$ and there are infinitely many forms $F \sim G$ such that the inequality $|F(\underline{\underline{x}})| \leqslant |\underline{\underline{x}}|^\gamma$

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has at least $c_1(G, \gamma)H(F)^{2\gamma/r^2}$ primitive solutions. Here a pair (x, y) is called *primitive* if $\gcd(x, y) = 1$, and $H(F)$ is the maximum modulus of the coefficients of F.

For natural k let T be the map $(x, y) \mapsto (X, Y)$ with $X = kx + y$, $Y = (k-1)x + y$, and set $F(\underline{X}) = G(T^{-1}\underline{X})$. Then $F \sim G$ and $H(F) \ll_G k^r$, with the implicit constant in $\ll$ depending only on G. We have $|G(\underline{x})| \ll_G |\underline{x}|^r$ for $\underline{x} \in \mathbb{R}^2$, and therefore when k is large, the inequality $|G(\underline{x})| \leqslant k^\gamma$ will have $\gg_{G,\gamma} k^{2\gamma/r}$ integer solutions $\underline{x}$. In fact it will have $\gg_{G,\gamma} k^{2\gamma/r}$ primitive solutions with $xy > 0$, and these solutions will have $|G(\underline{x})| \leqslant |kx + y|^\gamma$. When $\underline{x}$ is such a solution and $\underline{X} = T\underline{x}$, then $\underline{X}$ is again primitive and $|F(\underline{X})| = |G(\underline{x})| \leqslant |kx + y|^\gamma = |\underline{X}|^\gamma$. Thus $|F(\underline{X})| \leqslant |\underline{X}|^\gamma$ has $\gg_{G,\gamma} k^{2\gamma/r} \gg_{G,\gamma} H(F)^{2\gamma/r^2}$ primitive solutions.

Now suppose that F is a form of degree r with $s + 1$ nonzero coefficients in $\mathbb{Z}$:

$$F(x, y) = \sum_{i=0}^{s} a_i x^{r_i} y^{r-r_i}, \tag{1.3}$$

where $0 = r_0 < r_1 < \cdots < r_s = r$. We will not need to assume that F has no multiple factors. As we saw in [6], new methods can be used for the Thue inequality when F is "sparse," i.e., when $r > 2s$. It turns out that the analogous condition $r - \gamma > 2s$ works for the generalized Thue inequality. In what follows, set

$$\rho = r - \gamma. \tag{1.4}$$

THEOREM. *Let F be of the type (1.3), ρ a number with*

$$2s < \rho \leqslant r, \tag{1.5}$$

and $\gamma = r - \rho$. Then the number of primitive solutions of the generalized Thue inequality (1.2) is

$$\leqslant c_2(r, \rho)h^\kappa,$$

where

$$\kappa = \max(2/\rho, 1/(\rho - 2s)). \tag{1.6}$$

When $4s \leq \rho \leq r$, the number of primitive solutions is

$$\ll c_3(r, s, \rho)h^{2/\rho}$$

with an absolute constant in $\ll$ and

$$c_3(r, s, \rho) = s^{1+(r/\rho)} \exp(\rho^{-1}(12r + 4rs\rho^{-1} \log s + 3200s \log^3 r)).$$

In particular when $\rho \geqslant s \log s$, $r \geqslant s \log^3 s$, then $s \log^3 r \ll r$ and $c_3 \leqslant s(c_4 s)^{r/\rho}$ with an absolute constant c_4, so that the number of primitive solutions is $\ll s(c_4 s)^{r/\rho} h^{2/\rho}$. When $\rho = r$, we recover the bound $\ll s^2 h^{2/r}$ of [6, below (1.9)], at least for primitive solutions. A special case of (1.2) is

$$|ax^r - by^r| \leqslant h|\underline{\underline{x}}|^{r-\rho}.$$

The number of primitive solutions, assuming $4 \leqslant \rho \leqslant r$, is $\leqslant c_4^{r/\rho} h^{2/\rho}$.

Note that we have to restrict ourselves to primitive solutions. For there certainly are forms F for which the Thue equation $F(\underline{\underline{x}}) = 1$ has a solution $\underline{\underline{x}}_0$ with $|\underline{\underline{x}}_0|$ arbitrarily large. Then $\underline{\underline{x}} = t\underline{\underline{x}}_0$ will have $|F(\underline{\underline{x}})| \leqslant |\underline{\underline{x}}|^\gamma$ precisely when $|t|^r \leqslant |\underline{\underline{x}}_0|^\gamma |t|^\gamma$, i.e., when $|t|^\rho \leqslant |\underline{\underline{x}}_0|^\gamma$. The number of choices for t cannot be bounded in terms of r, s, ρ.

Let $f(x, y)$ be a polynomial of total degree $\gamma < r - 2s$. Suppose f has coefficients of modulus $\leqslant M$. The diophantine equation

$$F(x, y) = f(x, y) \tag{1.7}$$

yields (1.2) with $h = \binom{\gamma + 2}{2} M$. When $\underline{\underline{x}} = t\underline{\underline{x}}_0$ with x_0 primitive, then also $\underline{\underline{x}}_0$ satisfies (1.2), so that the number of possibilities for $\underline{\underline{x}}_0$ is estimated by our Theorem. Once $\underline{\underline{x}}_0$ is fixed, (1.7) gives an algebraic equation for t of degree r, hence with at most r solutions t. Hence the number of solutions of (1.7) is

$$\leqslant c_5(r) M^\kappa,$$

with κ given by (1.6). Again, under suitable conditions on r, s, ρ, good explicit bounds may be given.

Mahler [4] gave an asymptotic formula for the number $N_F(h)$ of solutions of the Thue inequality (1.1). He established that $N_F(h) \sim A_F h^{2/r}$ as $h \to \infty$, where A_F is the area of the region of $\underline{\underline{x}} \in \mathbb{R}^2$ with $|F(\underline{\underline{x}})| \leqslant 1$. We expect that for $0 \leqslant \gamma < r - 2$ there is an analogous formula for the number $N_{F,\gamma}(h)$ of solutions of the generalized Thue inequality (1.2):

$$N_{F,\gamma}(h) \sim A_{F,\gamma} h^{2/\rho} \quad \text{as} \quad h \to \infty,$$

where $A_{F,\gamma}$ is the area of the plane region $|F(\underline{\underline{x}})| \leqslant |\underline{\underline{x}}|^\gamma$. This should hold generally, i.e., for forms F not necessarily of the sparse type (1.3); but good error estimates are more likely for sparse forms.

2. The Plan of the Paper

We will follow [6] very closely – our task will be to show that the method developed in that paper for Thue inequalities extends to generalized Thue inequalities.

The Mahler height of a form

$$F(x,\ y) = a_0(x - \alpha_1 y)\dots(x - \alpha_r y) \tag{2.1}$$

is

$$M(F) = |a_0| \prod_{i=1}^{r} \max(1,\ |\alpha_i|).$$

It has the properties that $M(F(x,y)) = M(F(y,x))$, and $M(FG) = M(F)M(G)$. The Mahler height $M(\alpha)$ of an algebraic number α is the height of its homogenized defining polynomial (chosen to have coprime coefficients in $\mathbb{Z}$). If F as above has coefficients in $\mathbb{Z}$, each $M(\alpha_i) \leqslant M(F)$.

Set

$$R = e^{800 \log^3 r}, \tag{2.2}$$

$$C = (2r^{1/2}M(F))^r hR, \tag{2.3}$$

$$Y_L = C^{2/(\rho-2)}, \quad Y_S = Y_0^{1/(\rho-2s)}, \tag{2.4}$$

with

$$Y_0 = (e^6 s)^r R^{2s} h. \tag{2.5}$$

Then

$$Y_S^\rho > Y_0 > (rs)^{2s}(4e^3 s)^r h \tag{2.6}$$

since $R > rs$.

We will distinguish *large*, *medium* and *small* solutions to (1.2). Writing $\underline{\underline{x}} = (x,\ y)$, $|\underline{\underline{x}}| = \max(|x|,\ |y|)$, $\langle\underline{\underline{x}}\rangle = \min(|x|,\ |y|)$, a solution will be called

large	if	$\lvert\underline{\underline{x}}\rvert > Y_L$,
medium	if	$\lvert\underline{\underline{x}}\rvert \leqslant Y_L$ and $\langle\underline{\underline{x}}\rangle \geqslant Y_S$,
small	if	$\langle\underline{\underline{x}}\rangle < Y_S$.

PROPOSITION 1. *The number of primitive large solutions is* $\leqslant c_6(s,\ r,\ \rho)$. *When* $\rho \geqslant 4$, *this number is*

$$\ll s\left(\frac{\log r}{\log \rho}\right)^2 \left(1 + \frac{\log\log r}{\log \rho}\right). \tag{2.7}$$

PROPOSITION 2. *The number of primitive medium solutions is*

$$\ll \frac{s^2 r^2}{\rho(\rho-2)}(1 + r^{-2} \log h). \tag{2.8}$$

PROPOSITION 3. *The number of small solutions is* $\leqslant c_7(s, r, \rho)h^{\kappa}$ *with* κ *given by (1.6). When* $\rho \geqslant 4s$, *the number of small solutions is* $\ll c_3 h^{2/\rho}$ *with* $c_3 = c_3(r, s, \rho)$ *as in the Theorem.*

The Theorem follows from these propositions since the bound (2.7) is

$$\ll s(\log r/\log \rho)^3 \ll s \exp(\rho^{-1} r),$$

and the bound in (2.8) is

$$\ll s^2 (r/\rho)^2 h^{2/\rho} \ll s^2 \exp(\rho^{-1} r) h^{2/\rho}.$$

3. Large Solutions

LEMMA 1. *For every* $\underline{\underline{x}} = (x, y)$ *with (1.2) and* $y \neq 0$ *there is an* α_i *(as given in (2.1)) with*

$$\min\left(1, \left|\alpha_i - \frac{x}{y}\right|\right) \leqslant (2^{r/2} M(F))^r h |\underline{\underline{x}}|^{-\rho}.$$

Proof. This lemma corresponds to Lemma 4 of [6] and the proof is the same. In fact one just has to recall (1.4) and to substitute in Lemma 1 of [1], which essentially is already in Lewis and Mahler [3].

LEMMA 2. *There is a subset* S *of the set* $\{\alpha_1, \dots, \alpha_r\}$ *of cardinality* $|S| \leqslant 6s + 4$ *such that for every* $\underline{\underline{x}}$ *with (1.2) and* $y \neq 0$ *there is an* $\alpha_i \in S$ *with*

$$\min\left(1, \left|\alpha_i - \frac{x}{y}\right|\right) \leqslant C |\underline{\underline{x}}|^{-\rho}. \tag{3.1}$$

Proof. This corresponds to Lemma 8 of [6] and is deduced in exactly the same way.

Now if $|\underline{\underline{x}}| \geqslant Y_L$, say $y \geqslant Y_L$, we have from (2.4) that the minimum in (3.1) is $< y^{-(\rho+2)/2}$, and therefore

$$\left|\alpha_i - \frac{x}{y}\right| < y^{-(\rho+2)/2}.$$

Observe that $y \geqslant Y_L = C^{2/(\rho-2)} > M(F)^{2r/(\rho-2)} \geqslant M(F) \geqslant M(\alpha_i)$. But in [8] it was pointed out that the number of solutions of $\left|\alpha_i - \frac{x}{y}\right| < y^{-\rho}$ with $y > M(\alpha_i)$

is $\leqslant c_8(r, \rho)$, and when $\rho \geqslant 3$, it is in fact $\ll (\log r/\log \rho)^2(1+\log\log r/\log \rho)$. If we apply this with $(\rho+2)/2$ in place of ρ and note that $\log((\rho+2)/2) \gg \log \rho$, we see that for fixed α_i our number of solutions is under an analogous bound. After multiplication by $|S| \ll s$ we obtain the estimates of Proposition 1.

4. Medium Solutions

Given (1.3), let P_i for $0 \leqslant i \leqslant s$ be the point in the plane with coordinates $(r_i, -\log|a_i|)$. In [6] we defined the Newton Polygon to be the "lower boundary" of the convex hull of $P_0, \dots, P_s$, and denoted its vertices by $P_{i(0)}, P_{i(1)}, \dots, P_{i(\ell)}$; here $1 \leqslant \ell \leqslant s$. Also $\sigma(i, j)$ for $i \neq j$ was the slope of the segment P_iP_j. Further for $0 < j \leqslant \ell$ we set $\sigma(i(j)) = \sigma(i(j-1), i(j))$ and for $0 \leqslant j < \ell$ we set $\sigma^+(i(j)) = \sigma(i(j), i(j+1))$, so that $\sigma(i(j)), \sigma^+(i(j))$ are the slopes of the segments of the Newton polygon to the left and to the right of $P_{i(j)}$. For each root α of $f(x) = F(x, 1)$ we defined integers $k(\alpha), K(\alpha)$ having $0 \leqslant k(\alpha) < K(\alpha) \leqslant \ell$. Also, $H = H(f)$ was the maximum modulus of the coefficients a_i, and q an integer with $|a_q| = H$.

LEMMA 3. *Suppose (1.2) holds with $|x| \leqslant |y|$, $y \neq 0$. Let α be a root of $f(x)$ with*

$$|x - \alpha y| = \min_{1\leqslant j\leqslant r} |x - \alpha_j y|. \tag{4.1}$$

Suppose that $q < i(K)$ where $K = K(\alpha)$. Then there is a u, $1 \leqslant u \leqslant i(K)$, with

$$\left|\alpha - \frac{x}{y}\right| < \frac{1}{H^{(1/u)-(1/r)}} \left(\frac{(rs)^{2s}(2e^3s)^r h}{|y|^\rho}\right)^{1/u}. \tag{4.2}$$

Proof. Our lemma corresponds to Lemma 15 in [6]. The only difference in the proof is that in Lemma 10 of [6], h is to be replaced by $h|\underline{\underline{x}}|^\gamma = h|y|^\gamma = h|y|^{r-\rho}$, and therefore the conclusion of that Lemma is true with $|y|^\rho$ in place of $|y|^r$.

LEMMA 4. *Suppose (1.2) holds with $|x| \leqslant |y|$ and*

$$|y|^\rho \geqslant 2^r(rs)^{2s}h. \tag{4.3}$$

Let α be a root of $f(x)$ with (4.1). Suppose that $i(k) < q$ where $k = k(\alpha)$. Then there is a v, $1 \leqslant v \leqslant s - i(k)$, with

$$\left|\alpha^{-1} - \frac{y}{x}\right| < \frac{1}{H^{(\rho/rv)-(1/r)}} \left(\frac{(rs)^{2s}(4e^3s)^r h}{|x|^\rho}\right)^{1/v}. \tag{4.4}$$

Proof. The lemma corresponds to Lemma 16 of [6]. In analogy to (8.8) of that paper we now have

$$\left|\alpha - \frac{x}{y}\right| < \left(\frac{2^r(rs)^{2s}h}{\Delta^*|y|^\rho}\right)^{1/v}$$

with

$$\Delta^* = \Delta^*(\alpha,\ v) = |a_{i(k)}||\alpha|^{r_{i(k)}-v}.$$

As in [6] we have $\log(|\alpha|^v\Delta^*(\alpha,\ v)) > 0$, so that

$$\left|\alpha - \frac{x}{y}\right| < |\alpha|\left(\frac{2^r(rs)^{2s}h}{|y|^\rho}\right)^{1/v} \leqslant |\alpha|$$

by (4.3), and therefore $|x| < 2|\alpha y|$. We may infer that

$$\begin{aligned}\left|\alpha^{-1} - \frac{y}{x}\right| = \left|\frac{y}{\alpha x}\right|\left|\alpha - \frac{x}{y}\right| &< \left(\frac{2^r(rs)^{2s}h}{\Delta^*|\alpha^v y^{\rho-v}x^v|}\right)^{1/v}\\ &< \left(\frac{4^r(rs)^{2s}h}{|a_{i(k)}||\alpha|^{-(\rho-v-r_{i(k)})}|x|^\rho}\right)^{1/v}\\ &= \left(\frac{4^r(rs)^{2s}h}{\Gamma(\alpha,\ v)|x|^\rho}\right)^{1/v}\end{aligned} \tag{4.5}$$

with

$$\Gamma(\alpha,\ v) = |a_{i(k)}||\alpha|^{-(\rho-v-r_{i(k)})},$$

so that

$$\log\ \Gamma(\alpha,\ v) = -(\rho - v - r_{i(k)})\ \log\ |\alpha| + \log\ |a_{i(k)}|. \tag{4.6}$$

Case 1. $\rho - v - r_{i(k)} \geqslant 0$. We proceed as in [6]. The number $k = k(\alpha)$ has $\sigma^+(i(k)) > \log|\alpha| - \log(e^3 s)$, and on the other hand $\sigma(i(k),\ q) \geqslant \sigma^+(i(k))$ by $q > i(k)$ and by convexity considerations. Therefore

$$\begin{aligned}\log\ \Gamma(\alpha,\ v) &\geqslant -(\rho - v - r_{i(k)})\sigma(i(k),\ q) + \log|a_{i(k)}| - r\ \log(e^3 s)\\ &= -(\rho - v - r_q)\sigma(i(k),\ q) + \log|a_q| - r\ \log(e^3 s),\end{aligned}$$

for clearly $\sigma(i(k),\ q) = (\log|a_q| - \log|a_{i(k)}|)/(r_{i(k)} - r_q)$. When $\rho - v - r_q \geqslant 0$, we observe that $\sigma(i(k),\ q) \leqslant 0$ by the choice of q and that

$$\log\ \Gamma(\alpha,\ v) \geqslant \log|a_q| - r\ \log(e^3 s). \tag{4.7}$$

When $\rho - v - r_q < 0$, we observe that $\sigma(i(k), q) \geqslant \sigma(0, q)$, and

$$\begin{aligned}\log \Gamma(\alpha, v) &\geqslant -(\rho - v - r_q)\sigma(0, q) + \log|a_q| - r \log(e^3 s)\\ &= (\rho - v - r_q)((\log|a_q| - \log|a_0|)/r_q)\\ &\quad + \log|a_q| - r \log(e^3 s)\\ &\geqslant ((\rho - v)/r_q) \log|a_q| - r \log(e^3 s).\end{aligned}$$

Since by hypothesis $\rho > 2s > v$, we obtain

$$\log \Gamma(\alpha, v) \geqslant ((\rho - v)/r) \log|a_q| - r \log(e^3 s). \tag{4.8}$$

By (4.7) this holds always in Case 1, and therefore

$$\Gamma(\alpha, v) \geqslant (e^3 s)^{-r} H^{(\rho - v)/r},$$

which in conjunction with (4.5) yields (4.4).

Case 2. $\rho - v - r_{i(k)} < 0$. Then $i(k) > 0$, since $v \leqslant s < \rho$. We claim that

$$\sigma^+(i(k)) < \log|\alpha| + \log(e^3 s). \tag{4.9}$$

This is certainly true if $\sigma(s) = \sigma(i(\ell)) < \log|\alpha| + \log(e^3 s)$, and otherwise $K = K(\alpha)$ was smallest with $\sigma^+(i(K)) \geqslant \log|\alpha| + \log(e^3 s)$ (see [6, Sect. 6]). But $k < K$, so that indeed (4.9) holds. Now (4.6) gives

$$\log \Gamma(\alpha, v) > -(\rho - v - r_{i(k)})(\sigma^+(i(k)) - \log(e^3 s)) + \log|a_{i(k)}|.$$

But

$$\sigma^+(i(k)) \geqslant \sigma(0, i(k)) = (\log|a_0| - \log|a_{i(k)}|)/r_{i(k)} \geqslant -\log|a_{i(k)}|/r_{i(k)}.$$

Thus

$$\log \Gamma(\alpha, v) > ((\rho - v)/r_{i(k)}) \log|a_{i(k)}| - r \log(e^3 s). \tag{4.10}$$

We observe that $0 < i(k) < q$, therefore $\sigma(0, i(k)) < \sigma(0, q)$, and

$$\begin{aligned}r_{i(k)}^{-1} \log|a_{i(k)}| &= r_{i(k)}^{-1} \log|a_0| - \sigma(0, i(k)) > r_q^{-1} \log|a_0| - \sigma(0, q)\\ &= r_q^{-1} \log|a_q| \geqslant r^{-1} \log|a_q|,\end{aligned}$$

which together with (4.10) gives (4.8) again, and therefore (4.4).

Now if

$$Y_S \leqslant |x| \leqslant |y|, \tag{4.11}$$

we have (4.3) by (2.6), and since either $q < i(K)$ or $q > i(k)$ will certainly hold, the conclusion of Lemma 3 or Lemma 4 will hold. Moreover, the right hand sides of (4.2), (4.4) will increase in u resp. v, so that we may replace u, v by s. Combining this with Lemma 7 of [6] we obtain the following lemma, which corresponds to Lemma 17 of [6].

LEMMA 5. *There is a set S of roots of $F(x, 1)$ and a set S^* of roots of $F(1, y)$, both of cardinality $\leqslant 6s + 4$, such that every solution of (1.2) with (4.11) either has*

$$\left|\alpha - \frac{x}{y}\right| < \frac{R}{H^{(1/s)-(1/r)}}\left(\frac{(rs)^{2s}(4e^3 s)^r h}{|y|^\rho}\right)^{1/s} \tag{4.12}$$

for some $\alpha \in S$, or has

$$\left|\alpha^* - \frac{y}{x}\right| < \frac{R}{H^{(\rho/rs)-(1/r)}}\left(\frac{(rs)^{2s}(4e^3 s)^r h}{|x|^\rho}\right)^{1/s} \tag{4.13}$$

for some $\alpha^* \in S^*$.

The medium solutions to (1.2) were those with $Y_S \leqslant \langle \underline{x} \rangle$, $|\underline{x}| \leqslant Y_L$. Without loss of generality we may restrict ourselves to solutions with

$$Y_S \leqslant |x| \leqslant |y| \leqslant Y_L.$$

We will estimate such solutions with (4.13) (the case (4.12) being easier since the exponent of H is better). We have

$$\left|\alpha^* - \frac{y}{x}\right| < K/(2|x|^{\rho/s}) \tag{4.14}$$

with

$$\begin{aligned} K &= 2R(rs)^2(4e^3 s)^{r/s} h^{1/s} H^{(1/r)-(\rho/rs)} \\ &< R^2(e^5 s)^{r/s} h^{1/s} H^{-\rho/2rs} \end{aligned} \tag{4.15}$$

by (2.2) and since $\rho > 2s$. Let $y_0/x_0, \ldots, y_\nu/x_\nu$ be the solutions of (4.14) with $Y_S \leqslant x \leqslant Y_L$, ordered such that $x_0 \leqslant \cdots \leqslant x_\nu$. Then for $0 \leqslant i < \nu$,

$$\frac{1}{x_i x_{i+1}} \leqslant \left|\alpha^* - \frac{y_i}{x_i}\right| + \left|\alpha^* - \frac{y_{i+1}}{x_{i+1}}\right| < \tfrac{1}{2}K(x_i^{-\rho/s} + x_{i+1}^{-\rho/s}) \leqslant K x_i^{-\rho/s},$$

so that we have the "gap principle"

$$x_{i+1} > K^{-1} x_i^{(\rho/s)-1} \geqslant K^{-1} Y_S^{(\rho/s)-2} x_i = K^{-1} Y_0^{1/s} x_i > e^{r/s} H^{\rho/2rs} x_i$$

by (2.4), (2.5), (4.15). Therefore

$$x_\nu > (e^{r/s} H^{\rho/2rs})^\nu,$$

$$\log x_\nu > \nu((\rho/2rs) \log H + (r/s)) > (\nu\rho/2rs)(\log H + r). \tag{4.16}$$

On the other hand, by the definitions (2.2), (2.3), (2.4) of R, C, Y_L, we get

$$\log Y_L = 2(\rho-2)^{-1} \log C \ll (\rho-2)^{-1}(r \log M(F) + r \log r + \log h).$$

But $M(F) \leqslant (r+1)H$ according to Mahler [5], so that we obtain

$$\log Y_L \ll r(\rho-2)^{-1}(\log H + \log r + \log h^{1/r}).$$

The same upper bound holds for $\log x_\nu$. Comparison with (4.16) yields

$$\nu \ll r^2 s \rho^{-1}(\rho-2)^{-1}(1 + r^{-1} \log h^{1/r}).$$

Taking account of the summation over $\alpha^* \in S^*$ we obtain Proposition 2.

5. Small Solutions

LEMMA 6. *Let $p(y) = A_s y^{r_s} + \cdots + A_1 y^{r_1} + A_0$ be a polynomial with real coefficients with $r = r_s > \cdots > r_1 > r_0 = 0$ and with $|A_s| \geqslant 1$. Let $h > 0$ and $0 \leqslant \gamma < r - s$, $\rho = r - \gamma$ as in (1.4). Then the real numbers y with*

$$|p(y)| \leqslant h|y|^\gamma \tag{5.1}$$

make up a set of measure

$$\mu < 18(rs^2(\gamma+1))^{s/\rho} h^{1/\rho}. \tag{5.2}$$

Given $x \geqslant 1$, the numbers y with (5.1) and $|y| > x$ make up a set of measure

$$\mu < 18(\gamma+1) rs^2 h^{1/s} x^{1-(\rho/s)}. \tag{5.3}$$

This lemma corresponds to Lemma 19 of [6], but the role of the variables has been interchanged.

Proof. Define

$$p_i(y) = \sum_{j=i}^{s} (r_j - r_0)(r_j - r_1)\dots(r_j - r_{i-1}) A_j y^{r_j - r_i}, \quad (0 \leqslant i \leqslant s).$$

Then $p_0(y) = p(y)$,

$$p_i'(y) = p_{i+1}(y) y^{r_{i+1} - r_i - 1}, \quad (0 \leqslant i < s). \tag{5.4}$$

We now introduce a new parameter Z. We will initially concentrate on numbers y with

$$|y| \geqslant Z. \tag{5.5}$$

Set

$$g(y) = p'(y) - hs^2|y|^\gamma/Z. \tag{5.6}$$

Then $g(y)$ restricted to $y > 0$ (or to $y < 0$) is a polynomial of degree $r-1$ if $\gamma \in \mathbb{Z}$, and in general is a linear combination of powers of y (or of $-y$), the highest power of y occurring being y^{r-1}. In fact it is a sum of at most $s+1$ powers, therefore has at most $2s+1$ real zeros. The same is true if the $-$ sign in (5.6) is replaced by $+$. Therefore there are at most $4s+2$ real numbers y with $|p'(y)| = hs^2|y|^\gamma/Z$. The real numbers y with (5.5) and $|p'(y)| > hs^2|y|^\gamma/Z$ make up at most $4s+4 \leqslant 8s$ intervals and half-lines. If $y_1,\ y_2$ with (5.1) lie in such an interval, we have on the one hand

$$|p(y_2) - p(y_1)| < h(|y_1|^\gamma + |y_2|^\gamma).$$

On the other hand, if, say, $0 < y_1 < y_2$, then

$$\begin{aligned} |p(y_2) - p(y_1)| &= \left|\int_{y_1}^{y_2} p'(y)\,\mathrm{d}y\right| > hs^2 Z^{-1}\int_{y_1}^{y_2} y^\gamma\,\mathrm{d}y \\ &= hs^2(\gamma+1)^{-1}Z^{-1}(y_2^{\gamma+1} - y_1^{\gamma+1}) \\ &> hs^2(\gamma+1)^{-1}Z^{-1}(y_2 - y_1)y_2^\gamma. \end{aligned}$$

Therefore $y_2 - y_1 < 2(\gamma+1)s^{-2}Z$, so that our interval is of length $< 2(\gamma+1)s^{-2}Z$. Thus if we neglect a set of measure $< 16(\gamma+1)s^{-1}Z$, we may concentrate on numbers y with $|p'(y)| \leqslant hs^2|y|^\gamma/Z$, i.e., with

$$|p_1(y)y^{r_1-1}| \leqslant hs^2|y|^\gamma/Z. \tag{5.7}$$

We now repeat the argument with hs^2/Z in place of h and $q(y) = p_1(y)y^{r_1-1}$ in place of $p(y)$. If we neglect a further set of measure $< 16(\gamma+1)s^{-1}Z$, we may suppose that $|q'(y)| \leqslant hs^4|y|^\gamma/Z^2$. Now $q'(y) = p_1'(y)y^{r_1-1} + (r_1-1)p_1(y)y^{r_1-2}$. The second summand here is of modulus $\leqslant (r_1-1)hs^2|y|^\gamma/Z^2$ by (5.5), (5.7), so that we obtain $|p_1'(y)y^{r_1-1}| \leqslant r_1hs^4|y|^\gamma/Z^2$, whence by (5.4),

$$|p_2(y)y^{r_2-2}| \leqslant r_1s^4h|y|^\gamma/Z^2.$$

We now deal with this in a manner analogous to (5.7). We have to replace $p_1,\ r_1,\ h$ by $p_2,\ r_2-1,\ r_1s^2h/Z$. So if we neglect a further set of measure $< 16(\gamma+1)s^{-1}Z$, we may suppose that

$$|p_3(y)y^{r_3-3}| \leqslant r_1(r_2-1)s^6h|y|^\gamma/Z^3.$$

And so on. The conclusion is that except for a set of measure $\leqslant s \cdot 16(\gamma+1)s^{-1}Z = 16(\gamma+1)Z$, the numbers y with (5.1), (5.5) have

$$|p_i(y)y^{r_i-i}| \leqslant r_1 \dots r_{i-1} s^{2i} h|y|^\gamma / Z^i, \quad (i = 1, \dots, s) \tag{5.8}$$

(where $r_1 \dots r_{i-1} = 1$ when $i = 1$). Incidentally, here we have used the fact that $p_i(y)y^{r_i-i}$ has degree $r-i$, so that its derivative is of degree $r-i-1 \geqslant r-s > \gamma$ for $i < s$, and therefore the analogue of the function in (5.6) at the ith step of the argument is not zero.

We now apply (5.8) with $i = s$ and note that $p_s(y)$ is a constant of modulus $\geqslant r$. We get

$$Z^s|y|^{\rho-s} \leqslant r^{s-2}s^{2s}h. \tag{5.9}$$

If we also neglect y with $|y| \leqslant (\gamma+1)Z$, then altogether we are neglecting a set of measure $< 18(\gamma+1)Z$. Now (5.5) holds, and $|y| > (\gamma+1)Z$ in conjunction with (5.9) yields $Z^\rho(\gamma+1)^{\rho-s} < r^{s-2}s^{2s}h$. This is impossible if we choose $Z = Z_0 = (r^s s^{2s} h)^{1/\rho}(\gamma+1)^{(s/\rho)-1}$. Therefore the numbers y with (5.1) constitute a set of measure $< 18(\gamma+1)Z_0$, giving (5.2).

On the other hand when $|y| \geqslant x$, then (5.9) gives $Z^s < r^s s^{2s} h x^{s-\rho}$. This is impossible if we choose $Z = Z_1 = rs^2h^{1/s}x^{1-(\rho/s)}$. Therefore the numbers y with (5.1), (5.5) and $|y| \geqslant x$ constitute a set of measure $< 16(\gamma+1)Z_1$. The interval $|y| < Z_1$ (the complement of (5.5)) has measure $2Z_1$, so that we get altogether $< 18(\gamma+1)Z_1$, i.e., (5.3).

We now turn to the proof of Proposition 3. The problem is to estimate the number of solutions with $\langle \underline{x} \rangle < Y_S$. We may supose that $|x| \leqslant |y|$ and $|x| < Y_S$. This number is $\Sigma z(x)$ over x with $|x| < Y_S$, where $z(x)$ is the number of integers y with (1.2) and $|x| \leqslant |y|$. Given x, the number of *real* y with $F(x, y) = \pm|y|^\gamma$ or with $y = \pm x$ is $\leqslant 4s + 7$ (e.g. $\leqslant s + 1$ solutions of $F(x, y) = |y|^\gamma$ with $y > 0$, and the same for $y < 0$ or $F(x, y) = -|y|^\gamma$, plus 3 solutions with $y = \pm x$ or $y = 0$). Thus the real numbers y with $|x| \leqslant |y|$, $|F(x, y)| \leqslant h|y|^\gamma$ make up at most $2s + 4$ intervals. The number $z(x)$ then is $\leqslant \mu(x) + 2s + 4$, where $\mu(x)$ is the total measure of these intervals. The number of small solutions then is

$$\leqslant (2s+4) \sum_{\substack{x \\ |x|<Y_s}} 1 + \sum_x \mu(x). \tag{5.10}$$

The first summand here is

$$\ll sY_S \ll c_9(r, s, \rho)h^{1/(\rho-2s)},$$

with

$$c_9(r, s, \rho) = s((e^6 s)^r R^{2s})^{1/(\rho-2s)}$$

by (2.4), (2.5). In the case when $\rho \geqslant 4s$ we obtain

$$\ll c_9(r,\ s,\ \rho)h^{2/\rho},$$

with

$$c_9(r,\ s,\ \rho) = s((e^6 s)^r R^{2s})^{1/(\rho-2s)} \ll s^a e^{12r/\rho} R^{4s/\rho},$$

where

$$a = 1 + r(\rho - 2s)^{-1} = 1 + r\rho^{-1} + 2rs/\rho(\rho - 2s) \leqslant 1 + r\rho^{-1} + 4rs\rho^{-2}.$$

Thus

$$\begin{aligned} c_9(r,\ s,\ \rho) &\ll s^{1+(r/\rho)} \exp(\rho^{-1}(12r + 3200s\ \log^3 r + 4rs\rho^{-1}\ \log\ s)) \\ &= c_3(r,\ s,\ \rho). \end{aligned}$$

The second summand in (5.10) is $\Sigma_1 + \Sigma_2$, with $\Sigma_1,\ \Sigma_2$ respectively a sum over $|x| \leqslant c_{10}h^{1/\rho},\ |x| > c_{10}h^{1/\rho}$ where

$$c_{10} = ((\gamma + 1)rs^2)^{s/\rho}.$$

By (5.2) of Lemma 6

$$\sum\nolimits_1 \ll c_{10}((\gamma + 1)rs^2)^{s/\rho}h^{2/\rho} = ((\gamma + 1)rs^2)^{2s/\rho}h^{2/\rho}.$$

By (5.3) of Lemma 6

$$\sum\nolimits_2 \ll (\gamma + 1)rs^2 h^{1/s} \sum_{x > c_{10}h^{1/\rho}} x^{1-(\rho/s)}.$$

The sum on the right is

$$\begin{aligned} &\ll (c_{10}h^{1/\rho})^{1-(\rho/s)} + (\rho s^{-1} - 2)^{-1}(c_{10}h^{1/\rho})^{2-(\rho/s)} \\ &\ll \rho(\rho - 2s)^{-1}c_{10}^{2-(\rho/s)}h^{(2/\rho)-(1/s)}, \end{aligned}$$

so that

$$\sum\nolimits_2 \ll (\gamma + 1)(\rho - 2s)^{-1}\rho rs^2 c_{10}^{2-(\rho/s)}h^{2/\rho} = c_{11}(r,\ s,\ \rho)h^{2/\rho},$$

with

$$c_{11}(r,\ s,\ \rho) = \rho(\rho - 2s)^{-1}(rs^2(\gamma + 1))^{2s/\rho}.$$

When $\rho \geqslant 4s$ we have

$$c_{11}(r, s, \rho) \ll (rs)^{4s/\rho} \leqslant r^{8s/\rho} = \exp(\rho^{-1} \cdot 8s \log r) \leqslant c_3(r, s, \rho).$$

Combining our results we see that the total number of small solutions to (1.2) is $\ll c_6(r, s, \rho)h^{\kappa}$ with $\kappa = \max(2/\rho, 1/(\rho - 2s))$, and a certain constant c_6, and it is $\ll c_3(r, s, \rho)h^{2/\rho}$ when $\rho \geqslant 4s$.

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