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An application of Kloosterman sums

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To the memory of Professor A. V. Malyshev.

Let \( n \) be a positive integer

\[
A_n = \{a : 1 \leq a \leq n, \ (a, n) = 1\},
\]

and for \( a \in A_n \) let \( \bar{a} \) denote the unique element of \( A_n \) satisfying \( a\bar{a} \equiv 1 \pmod{n} \).

For \( n \) odd, \( \varepsilon = 0 \) or \( 1 \), \( \delta = \pm1 \) put

\[
L^\varepsilon_n = \{a \in A_n : a - \bar{a} \equiv \varepsilon \pmod{2} \},
\]

\[
L^\varepsilon,\delta_n = \left\{a \in A_n : a - \bar{a} \equiv \varepsilon \pmod{2}, \ (a/n) = \delta \right\}.
\]

Zhang Wenpeng [5] recently conjectured that for every odd \( n \) and \( \eta > 0 \)

\[
\#L^1_n = \frac{1}{2} \phi(n) + O(n^{1/2+\eta})
\]

and proved it, even in a somewhat stronger form for \( n \) being a prime power or a product of two primes.

On the other hand, G. Terjanian [4] conjectured that \( L^\varepsilon,\delta_p \neq \emptyset \) for every prime \( p > 29 \) and every choice of \( \varepsilon \) and \( \delta \). This conjecture has been proved by Chaładus [1] by applying Nagell’s bound for the least quadratic nonresidue modulo \( p \).

We prove the following theorem, which confirms Zhang’s conjecture, improves his error term for \( n \) being a prime power, and improves Chaładus’s theorem except for finitely many primes.

THEOREM 1. For every choice of \( \varepsilon = 0, 1 \) and \( \delta = \pm1 \) we have

\[
\#L^\varepsilon,\delta_n = \frac{\phi(n)}{4} c_{n,\delta} + O(2^{\nu(n)}\sqrt{n}(\log n)^2),
\]

(1)

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where
\[
c_{n,\delta} = \begin{cases} 
1 + \delta & \text{if } n \text{ is a perfect square}, \\
1 & \text{otherwise}.
\end{cases}
\]

and \(v(n)\) is the number of distinct prime factors of \(n\).

Consider a positive integer \(n\), not necessarily odd, a positive integer \(m\) coprime to \(n\), \(0 \leq j, k < m\), an odd divisor \(r\) of \(n\), \(\delta = \pm 1\) and put
\[
S_n^m(j, k, r, \delta) = \# \left\{ a \in A_n : a \equiv j \pmod{m}, \bar{a} \equiv k \pmod{m}, \left( \frac{a}{r} \right) = \delta \right\}.
\]

We shall deduce Theorem 1 from the following estimate of this quantity.

**THEOREM 2.** For any choice of \(m < n\), coprime to \(n\), \(0 \leq j, k < m\), odd \(r\) dividing \(n\) and \(\delta = \pm 1\) we have
\[
S_n^m(j, k, r, \delta) = \frac{\phi(n)}{2m^2} c_{r,\delta} + O(2^{v(n)} \sqrt{n} (\log n)^2),
\]
where the constant in the \(O\) symbol is absolute and effective.

To obtain Theorem 1 from Theorem 2 we only need to observe that
\[
\# L^0_n,\delta = S_n^2(0, 0, n, \delta) + S_n^2(1, 1, n, \delta),
\]
\[
\# L^1_n,\delta = S_n^2(0, 1, n, \delta) + S_n^2(1, 0, n, \delta).
\]
The proof of Theorem 2 is based on four lemmas.

**LEMMA 1.** If \(r\) is an odd divisor of \(n\), we have
\[
\left| \sum_{u \in A_n} \left( \frac{u}{r} \right) e \left( \frac{uv}{n} \right) \right| \leq \sqrt{n(v, n)},
\]
where \(e(t) = \exp(2\pi it)\).

**Proof.** \((\frac{u}{r})\) is a character mod \(n\), whose conductor \(f\) is equal to the squarefree kernel of \(r\). Hence by a known formula (see [2], Chapter IV, Sect. 20, assertion IV)
\[
\left| \sum_{u \in A_n} \left( \frac{u}{r} \right) e \left( \frac{uv}{n} \right) \right| = \begin{cases} 
\phi(n) \left( \frac{n}{(v, n)} \right) \vert \mu \left( \frac{n}{(v, n)} \right) \vert \sqrt{f} & \text{if } f \left| \frac{n}{(v, n)} \right|, \\
0 & \text{otherwise}.
\end{cases}
\]

Since
\[
\phi \left( \frac{n}{(v, n)} \right) \geq \frac{\phi(n)}{(v, n)},
\]
we obtain
\[
\left| \sum_{u \in A_n} \left( \frac{u}{r} \right) e \left( \frac{uv}{n} \right) \right| \leq \begin{cases} \sqrt{f(v, n)} & \text{if } f(v, n)|n, \\ 0 & \text{otherwise}, \end{cases}
\]
which gives the lemma.

**Lemma 2.** For all integers \(v, w\) and an odd integer \(r\) dividing \(n\)

\[
\left| \sum_{u \in A_n} \left( \frac{u}{r} \right) e \left( \frac{uv + \bar{w}u}{n} \right) \right| \leq \sqrt{2n}2^v \sqrt{v, w, n),}
\]

where \(v\) is the number of distinct prime factors of \(n\).

**Proof.** This is a slight improvement of a result of Malyshev [3], where instead of the last factor \(\min\{\sqrt{v}, n\}, \sqrt{w}, n\}\) is obtained. We indicate only the necessary changes to Malyshev’s proof to obtain (2).

We use \(K_r(v, w, n)\) to denote the sum in the left side of (2). For prime-powers Malyshev shows

\[
|K_r(v, w, p^t)| \leq C_p p^{t/2}(v, p^t)^{1/2},
\]
where \(C_p = 2\) for odd primes and \(C_2 = 2\sqrt{2}\). By symmetry we also have

\[
|K_r(v, w, p^t)| \leq C_p p^{t/2}(w, p^t)^{1/2},
\]
and, taking into account that

\[
\min\{(v, p^t), (w, p^t)\} = (v, w, p^t),
\]
we conclude that

\[
|K_r(v, w, p^t)| \leq C_p p^{t/2}(v, w, p^t)^{1/2}. \tag{3}
\]

To treat the case of composite numbers Malyshev establishes the composition rule

\[
K_r(v, w, n) = \pm \prod K_{p_i^{t_i}}(v, w_i, p_i^{s_i}), \tag{4}
\]
where

\[n = \prod p_i^{t_i}, \quad r = \prod p_i^{s_i}\]
and the numbers \(w_i\) satisfy

\[w \equiv \sum w_i(n/p_i^{t_i})^2 \pmod{n}.\]
This implies that $(v, w_i, p_i^{t_i}) = (v, w, p_i^{t_i})$ and hence

$$\prod (v, w_i, p_i^{t_i}) = (v, w, n).$$

Consequently on substituting (3) into (4) we obtain

$$|K_r(v, w, n)| \leq \sqrt{n} \sqrt{v, w, n} \prod_{p \mid n} C_p,$$

and (2) follows by noting that $\Pi C_p \leq \sqrt{2} \cdot 2^\nu$.  

LEMMA 3. For any integer $n \geq 2$ we have

$$\sum_{v=1}^{n-1} \frac{\sqrt{(v, n)}}{v} \ll 2^\nu \log n,$$

where $\nu$ is the number of distinct prime divisors of $n$.

Proof.

$$\sum_{v=1}^{n-1} \frac{\sqrt{(v, n)}}{v} \leq \sum_{d \mid n} \sum_{j=1}^{[n/d]} \frac{\sqrt{d}}{d^j} \leq \sum_{d \mid n} d^{-1/2} \sum_{j=1}^{\infty} 1/j.$$  

Here the second sum is $O(\log n)$. We estimate the first sum as follows:

$$\sum_{d \mid n} d^{-1/2} \leq \prod_{p \mid n} \sum_{i=0}^{\infty} p^{-i/2} = \prod_{p \mid n} \frac{1}{1 - p^{-1/2}} \ll 2^\nu,$$

since each term is at most 2, except possibly those corresponding to $p = 2$ and $p = 3$.  

LEMMA 4. For any integer $n \geq 2$ we have

$$\sum_{v=1}^{n-1} \sum_{w=1}^{n-1} \frac{\sqrt{(v, w, n)}}{v w} \ll (\log n)^2.$$

Proof.

$$\sum_{v=1}^{n-1} \sum_{w=1}^{n-1} \frac{\sqrt{(v, w, n)}}{v w} \leq \sum_{d \mid n} \sum_{i=1}^{[n/d]} \sum_{j=1}^{[n/d]} \frac{\sqrt{d}}{(di)(dj)} \leq \sum_{d \mid n} d^{-3/2} \sum_{i=1}^{n} 1/i \sum_{j=1}^{n} 1/j.$$
Here the first sum is bounded from above by the convergent sum $\sum_{k=1}^{\infty} k^{-3/2}$, and the second and third sum is $O(\log n)$.

**Proof of Theorem 2.** For $0 \leq j < m$, $0 \leq u < n$ we define $\phi_j(u)$ as

$$
\phi_j(u) = \begin{cases} 
1 & \text{if } u \equiv j \pmod{m}, \\
0 & \text{otherwise}
\end{cases}
$$

and extend it periodically with period $n$. Clearly we have

$$
S = S^m_n(j, k, r, \delta) = \frac{1}{2} \sum_{u \in A_n} \phi_j(u) \phi_k(\bar{u}) \left(1 + \delta \left(\frac{u}{r}\right)\right).
$$

(5)

We develop $\phi_j$ into a trigonometric series:

$$
\phi_j(u) = \sum_{v=0}^{n-1} \alpha_{jv} e\left(\frac{uv}{n}\right).
$$

(6)

A substitution of expansion (6) into (5) yields

$$
S = \frac{1}{2} \sum_{v,w=0}^{n-1} \alpha_{jv} \alpha_{kw} \sum_{u \in A_n} e\left(\frac{uv + \bar{uw}}{n}\right) \left(1 + \delta \left(\frac{u}{r}\right)\right)
$$

(7)

$$
= \frac{1}{2} \sum_{v,w=0}^{n-1} \alpha_{jv} \alpha_{kw} T_{vw}.
$$

To estimate $T_{vw}$ we distinguish four cases.

(i) If $v = w = 0$, then clearly $T_{vw} = \phi(n) c_{r,\delta}$.

(ii) If $v \neq 0$, $w = 0$, then we have

$$
T_{vw} = \sum_{u=1}^{n-1} e\left(\frac{uv}{n}\right) \left(1 + \delta \left(\frac{u}{r}\right)\right).
$$

Applying Lemma 1 twice we obtain

$$
|T_{vw}| \leq 2\sqrt{n(v, n)}.
$$

(8)

(iii) If $v = 0$, $w \neq 0$, then by symmetry

$$
|T_{vw}| \leq 2\sqrt{n(w, n)}.
$$

(9)
(iv) If $v \neq 0$ and $w \neq 0$, then by Lemma 2

$$|T_{vw}| \leq 2^v (2n(v, w, n))^{1/2}.$$  

Substituting these estimates into (7) we obtain

$$S = \frac{\phi(n)}{2} c_{r, \delta} \alpha_j \alpha_{k_0} + R,$$  

where

$$|R| \leq 2\sqrt{n} \left( \sum_{v=1}^{n-1} |\alpha_j \alpha_{k_0}| \sqrt{v, n} \right) + \sum_{w=1}^{n-1} |\alpha_{j_0} \alpha_{k_0}| \sqrt{w, n} 
+ 2^{2^v} \sum_{v=1}^{n-1} \sum_{w=1}^{n-1} |\alpha_j \alpha_{k_0}| \sqrt{v, w, n}.$$  

The coefficients can be determined from an inversion formula:

$$\alpha_{j,v} = \frac{1}{n} \sum_{u=0}^{n-1} \phi_j(u) e \left( -\frac{vu}{n} \right)$$

$$= \frac{1}{n} \sum_{l=0}^{L} e \left( -\frac{v}{n} (j + lm) \right), \quad L = \left\lfloor \frac{n - 1 - j}{m} \right\rfloor.$$  

In particular,

$$\alpha_{j,0} = \frac{1}{n} \left[ \frac{n - 1 - j}{m} + 1 \right], \quad \frac{1}{m} - \frac{1}{n} \leq \alpha_{j,0} \leq \frac{1}{m} + \frac{1}{n}.$$  

Hence the main term of (11) satisfies

$$\frac{\phi(n)}{2} c_{r, \delta} \alpha_{j,0} \alpha_{k_0} = \frac{\phi(n)}{2m^2} c_{r, \delta} + R_1, \quad |R_1| \leq \frac{3}{m}.$$  

On the other hand, the geometric series in (13) can be easily summed. With $z = e(\nu m / n)$ we have

$$\alpha_{j,v} = \frac{1}{n} e \left( -\frac{v}{n} j \right) \frac{1 - z^{L+1}}{1 - z},$$

thus

$$|\alpha_{j,v}| = \frac{1}{n} \frac{|1 - z^{L+1}|}{|1 - z|} \leq \frac{1}{n} \frac{2}{|1 - z|} = \frac{1}{n |\sin \pi \nu m / n|}.$$
(we used the fact that \(|1 - e(t)| = 2|\sin \pi t|\)). As \(v\) runs from 1 to \(n - 1\), the residue of \(vm\) modulo \(n\) assumes the values 1 to \(n - 1\), since \((m, n) = 1\), and we have \((vm, n) = (v, n)\). Hence

\[
\sum_{v=1}^{n-1} \alpha_{jv} |\sqrt{(v, n)}| \leq \frac{1}{n} \sum_{v=1}^{n-1} \frac{\sqrt{(v, n)}}{|\sin \pi vm/n|} = \frac{1}{n} \sum_{v=1}^{n-1} \frac{\sqrt{(v, n)}}{\sin \pi v/n} \leq 2 \left(\frac{n/2}{n} \frac{\sqrt{(v, n)}}{\sin \pi v/n}\right).
\]

Since \(\sin t \geq \frac{2}{\pi} t\) on \([0, \pi/2]\), this sum is

\[
\leq \sum_{v=1}^{[n/2]} \frac{\sqrt{(v, n)}}{v} \ll 2^\nu \log n
\]

by Lemma 3. Thus the first sum in estimate (12) of \(R\) is \(O(2^\nu \log n)\), and by symmetry so is the second.

By the same arguments, the third sum can be estimated as follows:

\[
\sum_{v=1}^{n-1} \sum_{w=1}^{n-1} \alpha_{jv} \alpha_{kw} |\sqrt{(v, w, n)}| \leq \frac{1}{n^2} \sum_{v=1}^{n-1} \sum_{w=1}^{n-1} \frac{\sqrt{(v, w, n)}}{|\sin \pi vm/n \sin \pi wm/n|} = \frac{1}{n^2} \sum_{v=1}^{n-1} \sum_{w=1}^{n-1} \frac{\sqrt{(v, w, n)}}{\sin \pi v/n \sin \pi w/n} \leq 4 \left(\frac{n/2}{n} \frac{\sqrt{(v, w, n)}}{\sin \pi v/n \sin \pi w/n}\right) \leq \sum_{v=1}^{[n/2]} \sum_{w=1}^{[n/2]} \frac{\sqrt{(v, w, n)}}{vw} \ll (\log n)^2
\]

by Lemma 4.

Substituting these estimates into (12) we obtain

\[|R| \ll 2^\nu \sqrt{n(\log n)^2}.\]
Theorem 2 follows from (12), (14) and (15).

References