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The homeomorphism group of the hairy arc

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Abstract. In many pictures of planar Julia sets similar subsets appear which can be loosely described as hairy objects, i.e., objects which contain Cantor sets of arcs such that each arc is a limit from both sides of other arcs. The Julia set of the exponential family $\lambda \exp z$, $0 < \lambda < (1/e)$, is a prototype of such objects. In [AO] it was shown that these objects are topologically unique and admit essentially only one embedding in the plane; the unique object is called the hairy arc. Since the hairy arc arises in many non-conjugate dynamical systems, it is natural to study the homeomorphisms they admit. In this paper we show that the group of homeomorphisms $\mathcal{H}$ of the hairy arc $\mathbb{H}$ is one-dimensional and totally disconnected. We will also show that $\mathcal{H}$ is dynamically much richer than the corresponding group of homeomorphisms of the interval.

1. Introduction

In [AO] we defined the hairy arc $\mathbb{H}$ and showed that it appears as the Julia set in the exponential family [D2]. The hairy arc is topologically unique. In this paper we shall study the homeomorphism group $\mathcal{H}$ of $\mathbb{H}$. In this section we shall present some extensions and refinements of the results of [AO]. In the next section we show that $\mathcal{H}$ is one-dimensional and in the last section we study the conjugacy classes of $\mathcal{H}$.

A special representation of the hairy arc is the sosha. First we shall give the definition of a *straight one-sided hairy arc* (abbreviated *sosha*) in the plane. A *hairy arc* is then defined as a topological image of a sosha. The unit interval [0, 1] is denoted by $\mathbb{I}$.

1.1. DEFINITIONS. A sosha X is a compact subset of the unit square $\mathbb{I}^2$ of the following form. There exists a function $l\colon \mathbb{I} \to \mathbb{I}$, called the length function, such that

(1) For all (ξ, η) in $\mathbb{I}^2$ we have $(\xi, \eta) \in X$ if and only if $0 \leqslant \eta \leqslant l(\xi)$,
(2) The sets $\{\xi \in \mathbb{I} \mid l(\xi) > 0\}$ and $\{\xi \in \mathbb{I} \mid l(\xi) = 0\}$ are both dense in $\mathbb{I}$ and $l(0) = l(1) = 0$.
(3) For each ξ in $\mathbb{I}$ with $l(\xi) > 0$ there exist sequences $\langle\alpha_n\rangle_n$ and $\langle\beta_n\rangle_n$ in $\mathbb{I}$ such that $\alpha_n \uparrow \xi$, $\beta_n \downarrow \xi$ and $\lim l(\alpha_n) = \lim l(\beta_n) = l(\xi)$.

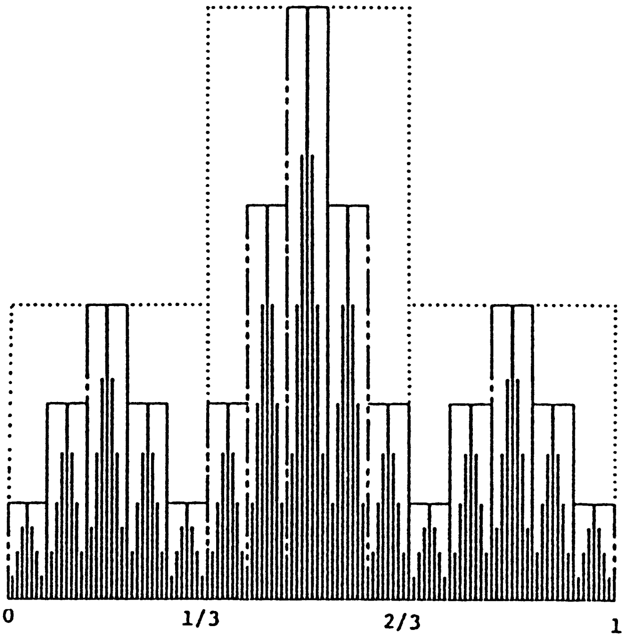

Fig. 1. A prototype of the hairy arc $\mathbb{H}$

The set $\mathbb{I} \times \{0\}$ is called the *base* of the sosha. Throughout the base of the sosha is denoted by B. If $x = (\xi, 0)$, $\xi \in (0, 1)$ and $l(\xi) = 0$, then we say that x is a *hairless* base point. The points (0, 0) and (1, 0) are called end points of the base. Note that the end points are not included in the set of hairless points. For each $x = (\xi, 0)$ in B with $l(\xi) > 0$, the set $\{(\xi, \eta) \mid 0 \leqslant \eta \leqslant l(\xi)\}$ is called the *hair at* x and denoted by h_x; the point x is called the *base point* of the hair h_x and $e_x = (\xi, l(\xi))$ is called the *endpoint* of the hair. By abuse of language, sometimes we write $l(x)$ in stead of $l(\xi)$ where $x = (\xi, 0)$. There is a natural order on the base $B : (\alpha, 0) < (\beta, 0)$ if and only if $\alpha < \beta$.

The main result of [AO] is that any two soshas are homeomorphic. It follows that any two hairy arcs are homeomorphic. Moreover under mild conditions all soshas are equivalently embedded in the plane. The topologically unique hairy arc is denoted by $\mathbb{H}$.

The main object of this paper is the study of the homeomorphism group $\mathcal{H}$ of $\mathbb{H}$. Besides the uniqueness of the hairy arc the following facts from [AO] are relevant for the discussion. The length function l of a sosha is upper semi-continuous. Consequently l attains a maximum on each closed subinterval $[\alpha, \beta]$ of $\mathbb{I}$ and assumes on $[\alpha, \beta]$ all values between 0 and the maximum. The set of endpoints E of the hairy arc $\mathbb{H}$ is dense in $\mathbb{H}$ and $E \cup B$ is a connected set. Without loss of generality we shall assume that a sosha X has a unique longest hair M and that $M = \{(\frac{1}{2}, t) | 0 \leqslant t \leqslant 1\}$. If X and Y are soshas, they have the hair M in common and there is a homeomorphism of X to Y that is the identity on M. We write b and e instead of $(\frac{1}{2}, 0)$ and $(\frac{1}{2}, 1)$ respectively.

The first lemma states that the action of a homeomorphism of the hairy arc is determined by its action on the base. The restriction of the homeomorphism F to the subset A of the domain of F is denoted by $f \restriction A$.

1.2. RIGIDITY LEMMA. *Suppose $F, G \in \mathcal{H}$. If $F \restriction B = G \restriction B$, then $F = G$.*

Proof. It is easily seen that for every H in $\mathcal{H}$ and every x in B we have $l(x) > 0$ if and only if $l(H(x)) > 0$. From this it follows that $H(e_x) = e_{H(x)}$ for every x with $l(x) > 0$. Consequently, if $F \restriction B = G \restriction B$, then $F(y) = G(y)$ for every endpoint y of $\mathbb{H}$. As the set of endpoints is dense, we have $F = G$.

1.3. COROLLARY. *$\mathcal{H}$ is totally disconnected.*

Proof. If $F, G \in \mathcal{H}$ and $F \neq G$, then by the Rigidity Lemma $F(x) \neq G(x)$ for some x in B. Without loss of generality we may assume that $F(x) < G(x)$. As the set of hairless base points is dense, we may also assume that x is a hairless base point. Let c be a base point between $F(x)$ and $G(x)$ with $l(c) > 0$. Then $\{H \in \mathcal{H} \mid H(x) < c\}$ and $\{H \in \mathcal{H} \mid H(x) > c\}$ are disjoint neighborhoods of F and G respectively whose union is $\mathcal{H}$.

The reader may wonder whether Lemma 1.2 can be improved. In particular the Rigidity Lemma suggests that $F(b) = b$ (and hence $F(M) = M$) forces F to be the identity on the hair M. The following lemma shows that this is false and highlights the fact that the structure of $\mathcal{H}$ is more complicated than the subgroup of homeomorphisms of $\mathbb{I}$ which keeps the set of hairless base points invariant.

1.4. EXTENSION LEMMA. *Suppose $f\colon M \to M$ is a homeomorphism such that $f(b) = b$. Then there exists an F in $\mathcal{H}$ such that $F \restriction M = f$.*

Proof. Let X be a sosha. Define $G\colon \mathbb{I}^2 \to \mathbb{I}^2$ by $G(x,t) = (x, f(t))$. From the definition of a sosha it easily follows that $G(X)$ is a sosha. By the remark preceding Lemma 1.2 there exists a homeomorphism H of $G(X)$ to X which is the identity on M. Then $F = H \circ G$ satisfies the required conditions.

1.5 COROLLARY. *There are five homeomorphism types of points in $\mathbb{H}$, namely*

(1) *end points of the base,*
(2) *end points of hairs,*
(3) *hairless base points,*
(4) *base points with hairs attached,*
(5) *interior points of hairs.*

Proof. It is easily seen that points in distinct sets of the list are topologically different. It remains to show that for any two points listed under the same number there is a homeomorphism in $\mathcal{H}$ that sends the one point to the other. This follows easily from the uniqueness of $\mathbb{H}$ for the points listed under (1) and (3). For the points listed under (2) and (4) one argues as follows. Given a hair in a sosha there is a simple homeomorphism which transforms this hair into the longest hair of another sosha. And we know already that any two soshas are homeomorphic under a homeomorphism that sends the longest hair to the longest hair. For the points listed under (5) this follows from the Extension Lemma and (2).

2. Dimension of $\mathcal{H}$

In this section we show that $\mathcal{H}$ is one-dimensional. We shall use the following notation. The sum metric on $\mathbb{I}^2$ is denoted by d. The supremum metric on function spaces is denoted by ρ. The function spaces we consider are $\mathcal{H}$ and the space of all homeomorphisms of the longest hair M to itself both endowed with the supremum metric. For a sosha X we define projections $\pi_1 : X \to B$ and $\pi_2 : X \to M$ by $\pi_1(\xi, \eta) = (\xi, 0)$ and $\pi_2(\xi, \eta) = (\frac{1}{2}, \eta)$. The mapping π_1 is a monotone retraction. For $a = (\alpha, 0)$, $b = (\beta, 0)$ in $\mathbb{I}$ with $\alpha < \beta$ we define

$$a \mathbin{\triangle} b = \pi_1^{-1}([0, \alpha] \times \{0\}) \cup \pi_1^{-1}([\beta, 1] \times \{0\}),$$

$$a \mathbin{\square} b = \pi_1^{-1}([\alpha, \beta] \times \{0\}).$$

The following lemmas are refinements of the Extension Lemma. The main feature is the control of the change in the second coordinate.

2.1. LEMMA. *Suppose that X and Y are soshas with common unique longest hair M. Suppose a, c are hairless base points of X such that $a < b < c$ and p, r are hairless base points of Y such that $p < b < r$. Then for each $\varepsilon > 0$ there exists a homeomorphism $F\colon X \to Y$ such that*

(1) $F \restriction M = \mathrm{id}_M$,
(2) *$F(a) = p$ and $F(c) = r$,*
(3) *$|\pi_2(x) - \pi_2 \circ F(x)| < \varepsilon$ for all x in $a \mathbin{\square} c$.*

Proof. The proof follows from the proof of the uniqueness of the hairy arc in [AO, Sect. 3]. The homeomorphism between two soshas is obtained by constructing a sequence of homeomorphisms of the square $\mathbb{I}^2$. The first homeomorphism of the sequence does not affect the second coordinate and can easily be made in such a way that (2) is satisfied. For any other homeomorphism of the sequence there is control over the change in the second coordinate; the change can be made arbitrarily small.

2.2. LEMMA. *Suppose that X and Y are soshas. Let $f : M \to M$ be a homeomorphism with $f(b) = b$. Then for each $\varepsilon > 0$ there exists a homeomorphism $F\colon X \to Y$ such that $F \restriction M = f$ and $|f \circ \pi_2(x) - \pi_2 \circ F(x)| < \varepsilon$ for all x.*

Proof. The homeomorphism F is obtained as the composition $H \circ G$; G is the restriction of the homeomorphism $\mathrm{id} \times f\colon \mathbb{I}^2 \to \mathbb{I}^2$ to X and the homeomorphism $H : G(X) \to Y$ is obtained by applying Lemma 2.1 (with the same ε), where $a = 0 = p$ and $c = 1 = r$.

2.3. LEMMA. *Suppose that G is a homeomorphism of the sosha X to the sosha Y such that $G(b) = b$. Then for every $\varepsilon > 0$ there exist hairless base points a and c of X with $a < b < c$ such that for each homeomorphism $f\colon M \to M$ with $f(b) = b$ there exists a homeomorphism $F\colon X \to Y$ such that the following conditions are satisfied*

(1) $d(a, c) < \varepsilon$ *and* $d(G(a), G(c)) < \varepsilon$,
(2) $F \restriction M = f$,
(3) $F \restriction (a \triangle c) = G \restriction (a \triangle c)$,
(4) $|f \circ \pi_2(x) - \pi_2 \circ F(x)| < \varepsilon$ *for all x in $a \square c$*,
(5) $\rho(F, G) \leqslant \rho(f, G \restriction M) + 3\varepsilon$.

Proof. Using the continuity of G and π_2, choose hairless base points a and c such that $a < b < c$, $d(a, c) < \varepsilon$, $d(G(a), G(c)) < \varepsilon$, and $|G \circ \pi_2(x) - \pi_2 \circ G(x)| < \varepsilon$ for each x in $a \square c$.

Define $F \restriction (a \triangle c) = G \restriction (a \triangle c)$. Using Lemma 2.2, define an extension F of f on $a \square c$ such that $F(a) = G(a)$, $F(c) = G(c)$ and (2) as well as (4) are satisfied. Then, for each x in $a \square c$, we have

$$
\begin{aligned}
d(F(x), G(x)) &= |\pi_2 \circ F(x) - \pi_2 \circ G(x)| + |\pi_1 \circ F(x) - \pi_1 \circ G(x)| \\
&\leqslant |\pi_2 \circ F(x) - f \circ \pi_2(x)| + |f \circ \pi_2(x) - G \circ \pi_2(x)| \\
&\quad + |G \circ \pi_2(x) - \pi_2 \circ G(x)| + \varepsilon \\
&< \varepsilon + \rho(f, G \restriction M) + \varepsilon + \varepsilon.
\end{aligned}
$$

Since (3) holds, $\rho(F, G) \leqslant \rho(f, G \restriction M) + 3\varepsilon$. This completes the proof of the lemma.

2.4. THEOREM. *Suppose that X is a sosha and $\mathcal{H}$ is the group of homeomorphisms of X. For each open-and-closed subset U of $\mathcal{H}$ with* id $\in U$ *and for each y in $M \setminus \{b, e\}$ we have*

$$\inf\{\pi_2 \circ G(y) \mid G \in U \text{ and } G(b) = b\} = 0.$$

Before proving the theorem we first discuss an important corollary.

2.5. COROLLARY. *For each open-and-closed subset U of $\mathcal{H}$,* $\operatorname{diam}(U) \geqslant 1$. *Hence,* $\operatorname{diam} \mathcal{H} \geqslant 1$.

Proof. If id $\in U$, the result easily follows from the preceding theorem. Let U be an open-and-closed subset of $\mathcal{H}$ and suppose $F \in U$. The right translation $G \to G \circ F^{-1}$ is an isometry of $\mathcal{H}$ onto itself sending U to an open-and-closed neighborhood of id.

Proof of Theorem 2.4. We may assume that $\mathcal{H}$ is the group of homeomorphisms of the sosha X. The proof is by way of contradiction. Assume that there is an open-and-closed neighborhood of id and a $y = (\frac{1}{2}, \eta)$ in $M\backslash\{b, e\}$ such that

$$\inf\{\pi_2 \circ G(y) \mid G \in U \text{ and } G(b) = b\} > 0.$$

Let $g_t : M \to M$ be a piecewise linear homeomorphism such that $g_t(b) = b$, $g_t(e) = e$ and $g_t(y) = (\frac{1}{2}, t)$, for t in $(0, \eta]$. Then there exists $\delta > 0$ such that for each extension G_t of g_t to a homeomorphism $G_t\colon X \to X$ we have $G_t \notin U$ for all t in $(0, \delta)$.

We will construct a Cauchy sequence $\langle F_n \rangle_n$ in $\mathcal{H}$ such that $F_\infty = \lim F_n$ belongs to the boundary of U, which would contradict the assumption that U is open-and-closed.

Put $F_0 = \text{id}$. Since $F_0 \in U$ there exists $\delta_0 > 0$ such that $B_{\delta_0}(\text{id}) \subset U$.

To construct F_1 we apply Lemma 2.3 with $G = F_0$ and $\varepsilon = \varepsilon_1 < \min\{\frac{\delta_0}{4}, \frac{1}{8}\}$ to find a_1 and c_1 such that for each t in $(0, \eta]$ there exists a homeomorphism G_t^1 satisfying

(1) $d(a_1, c_1) < \varepsilon_1$ and $d(F_0(a_1), F_0(c_1)) < \varepsilon_1$,
(2) $G_t^1 \restriction M = g_t$,
(3) $G_t^1 \restriction (a_1 \triangle c_1) = F_0 \restriction (a_1 \triangle c_1)$,
(4) $|g_t \circ \pi_2(x) - \pi_2 \circ G_t^1(x)| < \varepsilon_1$ for all x in $a_1 \square c_1$,
(5) $\rho(G_t^1, F_0) \leqslant \rho(g_t, \text{id} \restriction M) + 3\varepsilon_1$.

Since $g_\eta = \text{id} \restriction M$, (5) implies that $\rho(F_0, G_\eta^1) \leqslant 3\varepsilon_1 < \delta_0$, whence $G_\eta^1 \in U$. Let $t_1' = \sup\{t \in (0, \eta) \mid G_t^1 \notin U\}$. Then $t_1' < \eta$. Choose t_1 in (t_1', η) and s_1 in $(0, t_1')$ such that $\rho(g_{t_1}, g_{s_1}) < \frac{1}{8}$ and $G_{s_1}^1 \notin U$. Put $F_1 = G_{t_1}^1$.

By (5) we have $\rho(F_0, F_1) \leqslant \rho(g_\eta, g_{t_1}) + 3 \cdot \frac{1}{8}$ and by an argument similar to that in the proof of Lemma 2.3 for each x in X we get

$$\begin{aligned} d(G_{t_1}^1(x), G_{s_1}^1(x)) &\leqslant |\pi_2 \circ G_{t_1}^1(x) - g_{t_1} \circ \pi_2(x)| + \rho(g_{t_1}, g_{s_1}) \\ &\quad + |g_{t_1} \circ \pi_2(x) - \pi_2 \circ G_{s_1}(x)| + |c_1 - a_1| \\ &< 4 \cdot \tfrac{1}{8} = \tfrac{1}{2}. \end{aligned}$$

Hence $\rho(F_1, \mathcal{H}\backslash U) \leqslant \rho(G_{t_1}^1, G_{s_1}^1) \leqslant \frac{1}{2}$. Since $G_{t_1}^1 = F_1 \in U$, there is a $\delta_1 > 0$ such that $B_{\delta_1}(F_1) \subset U$.

Inductively for $n > 0$, construct base points a_n and c_n such that $a_n < b < c_n$, $[a_{n+1}, c_{n+1}] \subset (a_n, c_n)$ and $\lim(c_n - a_n) = 0$ and homeomorphisms $F_n = G_{t_n}^n$ in U with $t_1 > t_2 > \cdots > \delta$ such that

$$\rho(F_n, F_{n+1}) < \rho(g_{t_n}, g_{t_{n+1}}) + \frac{1}{2^{n-1}}, \tag{*}$$

$$\rho(F_n, \mathcal{H}\backslash U) < \frac{1}{2^n}, \tag{**}$$

$$F_n \restriction (a_n \bigtriangleup c_n) = F_m \restriction (a_n \bigtriangleup c_n) \quad \text{for all} \quad m \geqslant n. \tag{***}$$

Since $\lim t_n = t_\infty \geqslant \delta > 0$, $\lim g_{t_n} = g_\infty$ is a piecewise linear homeomorphism and the convergence is uniform. Hence $\langle g_{t_n} \rangle_n$ is a Cauchy sequence and, by $(*)$, $\langle F_n \rangle_n$ is a Cauchy sequence. The sequence $\langle F_n \rangle_n$ converges to a function F_∞ in the space of all continuous functions of X to itself (with the supremum norm). By $(***)$, $F_\infty \restriction X \backslash M$ is one-to-one. Moreover, since $F_\infty \restriction M = g_\infty$, it follows that F_∞ is one-to-one and hence a homeomorphism. By $(**)$, F_∞ belongs to the boundary of U, as required.

To prove that $\dim \mathcal{H} \leqslant 1$ we use some results of [OT]. A topological space X is called *almost zero-dimensional* provided there exists a basis $\mathcal{B}$ for the open sets such that $X \backslash \mathrm{cl}\, B$ is the union of open-and-closed sets for every B in $\mathcal{B}$. There is the following sufficient condition for a separable metric space to be almost zero-dimensional. Denote the closed ball of radius ε around x by $C_\varepsilon(x)$. From Lemma 3 in [OT] it follows that a separable metric space X is almost zero-dimensional whenever $X \backslash C_\varepsilon(x)$ is the union of open-and-closed sets for each x in X and every $\varepsilon > 0$. A uniquely and locally arcwise connected separable metric space is called a $\mathbb{R}$*-tree*. It is known that a non-degenerate $\mathbb{R}$-tree is one-dimensional. The following theorem is Theorem 2 of [OT].

2.6. THEOREM. *Let X be an almost zero-dimensional space. Then X embeds in the set of endpoints of an $\mathbb{R}$-tree. In particular, X is at most one-dimensional.*

We use this result to show that $\mathcal{H}$ is at most one-dimensional.

2.7. THEOREM. *$\mathcal{H}$ is almost zero-dimensional.*

2.8. COROLLARY. *$\mathcal{H}$ is one-dimensional.*

Proof of Theorem 2.7. We may assume that $\mathcal{H}$ is the homeomorphism group of the sosha X. It suffices to show that for all F, G in $\mathcal{H}$ with $\rho(F, G) > \varepsilon$ there exists a clopen set U in $\mathcal{H}$ such that $G \in U$ and $U \cap C_\varepsilon(F) = \emptyset$.

Since the set of endpoints E of X is dense, there exists a hair h_x with endpoint e_x and an $\eta > 0$ such that $d(F(e_x), G(e_x)) = \varepsilon + \eta$. Note that $F(x) \neq G(x)$ for the base point x of h_x.

Suppose, for a moment, that $\pi_2(F(e_x)) \geqslant \pi_2(G(e_x))$. It is easily seen that $C_{\varepsilon+(\eta/2)}(F(e_x)) \cap h_{G(x)} = \emptyset$. Hence it is possible to choose a and c such that $B_{\varepsilon+(\eta/2)}(F(e_x)) \cap G(a \,\square\, c) = \emptyset$. In this way it follows that there exist a and c with $a < x < c$ such that $F(a \,\square\, c) \cap G(a \,\square\, c) = \emptyset$ and

$$\text{either } B_{\varepsilon+(\eta/2)}(F(e_x)) \cap G(a \,\square\, c) = \emptyset \text{ or } B_{\varepsilon+(\eta/2)}(G(e_x)) \cap F(a \,\square\, c) = \emptyset.$$

Notice that in the first case $d(F(e_x), G(y)) \geqslant \varepsilon + \frac{\eta}{2}$ for all y in $a \,\square\, c$, while in the second case $d(F(y), G(e_x)) \geqslant \varepsilon + \frac{\eta}{2}$ for all y in $a \,\square\, c$.

In the final part of the proof we make use of the linear order of B, in particular the usual interval notations. Choose points p and q in B and open intervals P and Q in B such that

(1) $a < p < x < q < c$,
(2) $G(p) \in P$ and $G(q) \in Q$,
(3) $l(p) > 0$ and $l(q) > 0$,
(4) $\mathrm{cl}\, P \subset G((a, x))$ and $\mathrm{cl}\, Q \subset G((x, b))$,
(5) the endpoints of the intervals P and Q are hairless base points.

Define $U = \{H \in \mathcal{H} \mid H(p) \in P \text{ and } H(q) \in Q\}$. Then U is an open-and-closed subset of $\mathcal{H}$ and $G \in U$. If $H \in U$, then there are r and s in (p, q) such that $H(e_r) = G(e_x)$ and $H(e_x) = G(e_s)$. We have

$$\begin{aligned}\rho(H, F) &\geqslant \max\{d(H(e_r), F(e_r)), d(H(e_x), F(e_x))\} \\ &= \max\{d(G(e_x), F(e_r)), d(G(e_s), F(e_x))\} \geqslant \varepsilon + \frac{\eta}{2}.\end{aligned}$$

Hence $\rho(F, H) > \varepsilon$ and $C_\varepsilon(F) \cap U = \emptyset$.

In [KOT] it is shown that the set of end points of the hairy arc is homeomorphic to the set $\mathcal{E}$ of points in Hilbert space all of whose coordinates are irrational. It is not known whether any two homogeneous almost zero-dimensional, topologically complete, one-dimensional sets are homeomorphic. In particular, the following problem is open.

2.9. PROBLEM. *Is $\mathcal{H}$ homeomorphic to the set of end points of $\mathbb{H}$?*

3. Conjugacy classes of $\mathcal{H}$

Let X be a sosha and let N be the subset of B of hairless base points. Let $\mathcal{I}$ be the subgroup of the group of homeomorphisms of the interval B that keeps N invariant. It is easy to see that $\mathcal{I}$ is zero-dimensional. By the Rigidity Lemma the continuous function $\Psi\colon \mathcal{H} \to \mathcal{I}$, defined by $\Psi(H) = H \restriction B$ for H in $\mathcal{H}$, is one-to-one. Since $\mathcal{H}$ and $\mathcal{I}$ have different dimension, Ψ is not a topological embedding.

It is well known that any two homeomorphisms in $\mathcal{I}$ with exactly two fixed points are topologically conjugate. For each n, there are only finitely many topological conjugacy classes of homeomorphisms with n fixed points. We shall establish a similar result for homeomorphisms in $\mathcal{H}$ with exactly two fixed points, but in all other cases the analogy disappears. That the homeomorphisms F and G are topologically conjugate is denoted by $F \simeq G$.

3.1. THEOREM. *Suppose X is a sosha and $F, G\colon X \to X$ are homeomorphisms with exactly two fixed points. Suppose that $F \restriction B$ and $G \restriction B$ are order preserving. Then $F \simeq G$.*

Proof. The proof is similar to that for the interval (Note that a homeomorphism of the interval with exactly two fixed points is automatically order preserving). Without loss of generality we may assume that both $F \restriction B$ and $G \restriction B$ are increasing. We define a conjugacy H of F to G in the following way. $H(0, 0) = (0, 0)$ and $H(1, 0) = (1, 0)$. Let $x_0 = y_0$ be a hairless base point. For n in $\mathbb{Z}$ we define let $x_n = F^n(x_0)$ and $y_n = G^n(y_0)$. The hairy arcs $x_0 \square x_1$ and $y_0 \square y_1$ are the fundamental domains (cf. [D1]). Let H^* be an order preserving homeomorphism of $x_0 \square x_1$ to $y_0 \square y_1$. The definition of H is completed by defining H from $x_n \square x_{n+1}$ to $y_n \square y_{n+1}$ by $H = G^n \circ H^* \circ F^{-n}$, for $n \in \mathbb{Z}$. It is easily seen that H is conjugacy of F to G.

Now we discuss a situation of exactly four fixed points. The result shows that the structure of $\mathcal{H}$ is much richer that of $\mathbb{I}$. The set of all countable ordinals is denoted by Ω.

3.2. THEOREM. *Let X be a sosha and let $f\colon M \to M$ be a homeomorphism with exactly two fixed points. Then there exist extensions $F_\alpha\colon X \to X$ of f, $0 < \alpha < \Omega$, such that for distinct α and β we have $(F_\alpha \restriction B) \simeq (F_\beta \restriction B)$, but $F_\alpha \not\simeq F_\beta$.*

Proof. After replacing f by f^{-1} if necessary, we may assume $\pi_2(f(\frac{1}{2},\, t)) < t$. We construct for each α in $\Omega\backslash\{0\}$ a sosha X_α and a homeomorphism $G_\alpha : X_\alpha \to X_\alpha$ which extends f. Since all soshas are homeomorphic, there exists for each α a homeomorphism $\varphi_\alpha : X_\alpha \to X$ such that $\varphi \restriction M = \mathrm{id}$. By choosing dynamically distinct homeomorphisms G_α on X_α, the induced homeomorphisms $F_\alpha = \varphi_\alpha \circ G_\alpha \circ \varphi_\alpha^{-1}$ on X are not conjugate.

By symmetry it suffices to construct X_α only in $[0,\, \frac{1}{2}] \times [0,\, 1]$. For each α in Ω there exists a compact zero-dimensional subset E_α of $(\frac{1}{8},\, \frac{1}{4}) \times \{1\}$ such that α is the smallest ordinal with the property that the derived set of order α is empty (See [K]). It is not difficult to construct for each α a sosha Y^*_α such that $E_\alpha \cup \{(\frac{1}{2},\, 1)\}$ is exactly the set of endpoints of hairs of length 1 and $(\frac{1}{8},\, 0)$ and $(\frac{1}{4},\, 0)$ are hairless base points. Let Y^0_α be the subset $(\frac{1}{8},\, 0) \square (\frac{1}{4},\, 0)$ of Y^*_α. Define $g(x) = \frac{2}{3}x + \frac{1}{6}$. Notice that $g(\frac{1}{8}) = \frac{1}{4}$ and $\lim_{n\to\infty} g^n(x) = \frac{1}{2}$. Put

$$Y^n_\alpha = \begin{cases} \{(2^n x,\, 2^n y) \mid (x,\, y) \in Y^0_\alpha\}, & \text{for } n < 0, \\ \{(g^n(x),\, f^n(y)) \mid (x,\, y) \in Y^0_\alpha\}, & \text{for } n > 0. \end{cases}$$

Then $L_\alpha = M \cup \{(0,\, 0)\} \cup \bigcup\{Y^n_\alpha \mid n \in \mathbb{Z}\}$ is the required left half of the sosha X_α. Define $G_\alpha\colon L_\alpha \to L_\alpha$ by

$$G_\alpha(x,\, y) = \begin{cases} (2x,\, 2y) & \text{if } (x,\, y) \in Y^n_\alpha \text{ and } n < 0, \\ (g(x),\, f(y)) & \text{if } (x,\, y) \in Y^n_\alpha \text{ and } n \geqslant 0, \\ (0,\, 0) & \text{if } (x,\, y) = (0,\, 0), \\ (\frac{1}{2},\, f(y)) & \text{if } x = \frac{1}{2}. \end{cases}$$

Clearly $(G_\alpha \restriction B) \simeq (G_\beta \restriction B)$. Notice that for all $(x,\, y)$ with $(x,\, y) \neq (\frac{1}{2},\, 1)$ we have $\lim_{n\to\infty} \pi_2 \circ G^n_\alpha(x,\, y) = b$ if and only if $G^n_\alpha(x,\, y) \cap E_\alpha = \emptyset$ for all n in $\mathbb{Z}$. It follows that $\lim_{n\to\infty} G^n_\alpha(x,\, 1) = e$ for all $(x,\, 1)$ in E_α. From these considerations we conclude that the set $Z_\alpha = \{e\} \cup \bigcup\{G_\alpha^n(E_\alpha) \mid n \in \mathbb{Z}\}$ can be characterized dynamically as the set of points z such that $\lim_{n\to\infty} G_\alpha^n(z) = e$. As α is the smallest number such that the derived set of order $\alpha+1$ of Z_α is empty, it follows that no F_α is conjugate to any F_β when $\alpha \neq \beta$.

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References

[AO] Aarts, J. M. and Oversteegen, L. G.: The geometry of Julia sets, *Trans. Amer. Math. Soc.* 338 (1993) 897–918.

[D1] Devaney, R. L.: An introduction to chaotic dynamical systems, Benjamin, Menlo Park, Ca 94025.

[D2] Devaney, R. L.: e^z, Dynamics and bifurcations, *International Journal of Bifurcation and Chaos* 1 (1991) 287–308.

[K] Kuratowski, C.: *Topologie I et II*, PWN, Warszawa.

[KOT] Kawamura, K., Oversteegen, L. G. and Tymchatyn, E. D.: On homogeneous totally disconnected 1-dimensional spaces (Preprint).

[OT] Oversteegen, L. G. and Tymchatyn, E. D.: On the dimension of certain totally disconnected spaces (Preprint).