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Successive minima on arithmetic varieties

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Mumford's theory of stability, when applied to varieties over number fields, has interesting consequences, as was shown in recent years by several authors [12], [5], [13], [3], [16], [20]. In this paper, we use it to get informations on the successive minima of the lattice of sections of bundles on arithmetic varieties.

More precisely, let E be a projective module of rank N over the ring of integers in a number field K , and $E_K^\vee = \text{Hom}(E, K)$. Consider a closed subvariety $X_K \subset \mathbb{P}(E_K^\vee)$ in the projective space of lines in E_K^\vee . Fix a hermitian metric on $E \otimes_{\mathbb{Z}} \mathbb{C}$. Bost proved in [3] that Chow semi-stability of X_K in $\mathbb{P}(E_K^\vee)$ implies a lower bound for the height of X_K (see 3.1 below). By a different method we show that the proof that X_K is semi-stable gives, in some cases, a stronger inequality (see however the remark in 3.1.2) which involves the successive minima of E . Our general result, Theorem 1, can be applied to surfaces of general type, Theorem 3, using the work of Gieseker [7], and to line bundles on smooth curves, Theorem 4, using the work of Morrison [14]. A variant of Theorem 1 gives results for rank two stable bundles on curves, Theorem 5, by using the work of Gieseker and Morrison [8]. Finally, we derive another inequality for successive minima on arithmetic surfaces, Theorem 6, from the vanishing theorem proved in [16].

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1. Preliminaries

1.1. Let M be a free \mathbb{Z} -module of finite rank and $\|\cdot\|$ a norm on the complex vector space $M \otimes_{\mathbb{Z}} \mathbb{C}$. We equip $M \otimes_{\mathbb{Z}} \mathbb{R}$ with the Haar measure for which the unit ball has volume equal to the volume of the standard euclidean ball of the same dimension and we let $\text{covol}(M \otimes_{\mathbb{Z}} \mathbb{R}/M)$ be the covolume of M in $M \otimes_{\mathbb{Z}} \mathbb{R}$ for that measure. We then define the Euler characteristic of $(M, \|\cdot\|)$ to be the real number $\chi(M, \|\cdot\|) = -\log \text{covol}(M \otimes_{\mathbb{Z}} \mathbb{R}/M)$.

Clearly, if $\|\cdot\|'$ is another norm on $M \otimes_{\mathbb{Z}} \mathbb{C}$ such that $\|x\| \leq \|x\|'$ for all $x \in M \otimes_{\mathbb{Z}} \mathbb{C}$, we have

$$\chi(M, \|\cdot\|') \leq \chi(M, \|\cdot\|).$$

1.2. Let K be a number field, of degree $[K : \mathbb{Q}]$, let \mathcal{O}_K be its ring of integers, let $S = \text{Spec}(\mathcal{O}_K)$ be the associated scheme, let Σ be the set of complex embeddings

of K and let D_K be the discriminant of K over \mathbb{Q} . These notations will be valid throughout this paper.

If M is a torsion free \mathcal{O}_K -module of finite rank such that, for all $\sigma \in \Sigma$, the corresponding complex vector space $M_\sigma = M \otimes_{\mathcal{O}_K} \mathbb{C}$ is equipped with a norm $|\cdot|_\sigma$, we may think of M as a free \mathbb{Z} -module equipped with the norm $|\cdot|$ on $M \otimes_{\mathbb{Z}} \mathbb{C} = \bigoplus_{\sigma} M_\sigma$ defined by

$$\left| \sum_{\sigma} x_{\sigma} \right| = \sup_{\sigma} |x_{\sigma}|_{\sigma}.$$

In particular, consider an hermitian vector bundle (E, h) over S , in the sense of [9]. In other words, E is a torsion free \mathcal{O}_K -module of finite rank and, for all $\sigma \in \Sigma$, E_σ is equipped with an hermitian scalar product h , compatible with the isomorphism $E_\sigma \simeq E_{\bar{\sigma}}$ induced by complex conjugation. We will then denote by $\|\cdot\|_{\sigma}$ the associated norm on E_{σ} and $\|\cdot\|$ the norm on $E \otimes_{\mathbb{Z}} \mathbb{C}$ defined as above. Also we let $\widehat{\deg}(E, h) \in \mathbb{R}$ be the *arithmetic degree* of (E, h) , which can be computed as follows. Let N be the rank of E and $\Lambda^N E$ its top exterior power. We equip $\Lambda^N E$ with the metric induced by h : if $v_1, \dots, v_N, w_1, \dots, w_N$ lie in E_{σ}

$$\Lambda^N h(v_1 \wedge \dots \wedge v_N, w_1 \wedge \dots \wedge w_N) = \det(h(v_i, w_j)), \quad 1 \leq i, j \leq N).$$

We have then

$$\widehat{\deg}(E, h) = \widehat{\deg}(\Lambda^N E, \Lambda^N h).$$

On the other hand, if (L, h) is an hermitian line bundle on S , and if $s \in L$ is any nontrivial section, denote by $[L : \mathcal{O}_K s]$ the index in L of the submodule generated by s . We have

$$\widehat{\deg}(L, h) = \log[L : \mathcal{O}_K s] - \sum_{\sigma \in \Sigma} \log \|s\|_{\sigma}.$$

1.3.

LEMMA 1. *Let $\varphi : E \rightarrow M$ be a morphism of torsion free \mathcal{O}_K -modules of finite type, h a hermitian metric on E , with associated norm $\|\cdot\|_{\sigma}$ on E_{σ} , and $|\cdot|_{\sigma}$ a norm on each M_{σ} , $\sigma \in \Sigma$. We assume that, for all $x \in E_{\sigma}$, $|\varphi(x)|_{\sigma} \leq \|x\|_{\sigma}$. If x_1, \dots, x_N in E are such that $\varphi(x_1), \dots, \varphi(x_N)$ is a basis of $M \otimes_{\mathcal{O}_K} K$, the following inequality holds:*

$$\chi(M, |\cdot|) \geq -[K : \mathbb{Q}] \sum_{i=1}^N \log \|x_i\| - \frac{N}{2} \log |D_K|.$$

Proof. Let $\|\cdot\|'_\sigma$ be the norm induced from $\|\cdot\|_\sigma$ by the projection map $E_\sigma \rightarrow M_\sigma$. Since $|\cdot|_\sigma \leq \|\cdot\|'_\sigma$ we deduce from 1.1 that $\chi(M, |\cdot|) \geq \chi(M, \|\cdot\|'_\sigma)$, so we may assume that $|\cdot|_\sigma = \|\cdot\|'_\sigma$. Furthermore, $\|\cdot\|'_\sigma$ is the norm coming from the hermitian metric h' induced by E_σ on M_σ , and $\|\varphi(x_i)\|' \leq \|x_i\|$, so we may assume that $(M, \|\cdot\|') = (E, \|\cdot\|)$.

We know that

$$\widehat{\text{deg}}(E, h) = \chi(E, \|\cdot\|) + \frac{N}{2} \log |D_K|$$

(e.g. [4], (2.1.13)). The element $s = x_1 \wedge \cdots \wedge x_N$ of $\Lambda^N E$ is nonzero, so we get, by Hadamard inequality,

$$\begin{aligned} \widehat{\text{deg}}(E, h) &= \log[\Lambda^N E : \mathcal{O}_K s] - \sum_{\sigma \in \Sigma} \log \|x_1 \wedge \cdots \wedge x_N\|_\sigma \geq \\ &= -[K : \mathbb{Q}] \sum_{i=1}^N \log \|x_i\|. \end{aligned}$$

The Lemma 1 follows from this.

1.4. Let (E, h) be a hermitian vector bundle of rank N over S . For any integer $i \leq N$ we let λ_i be the infimum of the set of real numbers λ such that there exist v_1, \dots, v_i in E , linearly independent over K and such that $\|v_\alpha\| \leq \lambda$, $1 \leq \alpha \leq i$. These are the *successive minima* of (E, h) . We can choose $x_1, \dots, x_N \in E$, linearly independent over K , such that $\|x_i\| = \lambda_{N-i+1}$.

If we let

$$\mu_i = \log \lambda_i, \quad 1 \leq i \leq N,$$

and

$$\mu = \frac{1}{N} \sum_{i=1}^N \mu_i,$$

it follows from Bombieri–Vaaler’s version of Minkowski’s theorem on successive minima that

$$-N\mu[K : \mathbb{Q}] \leq \widehat{\text{deg}}(E, h) \leq C(N, K) - N\mu[K : \mathbb{Q}], \tag{1}$$

where $C(N, K)$ is the following constant

$$\begin{aligned} C(N, K) &= N(r_1 + r_2) \log(2) \\ &\quad + N(\log |D_K|)/2 - r_1 \log V_N - r_2 \log V_{2N}, \end{aligned}$$

where r_1 and r_2 are the number of real and complex places of K , and V_n is the standard euclidean volume of the unit ball in \mathbb{R}^n (see [4], 5.2.3).

1.5. The \mathcal{O}_K -module $\omega_S = \text{Hom}_{\mathbb{Z}}(\mathcal{O}_K, \mathbb{Z})$ is locally free of rank one. We fix an hermitian metric on ω_S by deciding that the trace morphism $\text{Tr} \in \omega_S$ has norm $|\text{Tr}|_{\sigma} = 1$ (resp. $|\text{Tr}|_{\sigma} = 2$), if $\sigma = \bar{\sigma}$ (resp. $\sigma \neq \bar{\sigma}$).

Let (E, h) be a hermitian vector bundle of rank N on S . We denote by $E^{\vee} = \text{Hom}(E, \mathcal{O}_S)$ its dual and we equip $E' = E^{\vee} \otimes_{\mathcal{O}_S} \omega_S$ with the tensor product of the metric dual to h on E^{\vee} with the chosen metric on ω_S . If $x = \sum_{\sigma \in \Sigma} x_{\sigma}$ lies in $E' \otimes_{\mathbb{Z}} \mathbb{C} = \bigoplus_{\sigma} E'_{\sigma}$ we let

$$\|x\|' = \sum_{\sigma \in \Sigma} \|x_{\sigma}\|.$$

The \mathbb{Z} -modules underlying E and E' , equipped with the norms $\|\cdot\|$ and $\|\cdot\|'$, are then dual to each other (see e.g. [10], 2.4.2). Let λ_i be the successive minima of $(E, \|\cdot\|)$ and λ'_i those of $(E', \|\cdot\|')$, $1 \leq i \leq n = \text{rk}_{\mathbb{Z}}(E) = [K : \mathbb{Q}]N$. In other words, λ_i is the infimum of the real numbers $\lambda \geq 0$ such that there exist $v_1, \dots, v_i \in E$, linearly independent over \mathbb{Z} , with $\|v_{\alpha}\| \leq \lambda$ for all $\alpha \leq i$.

From [1] Theorem 2.1 and John theorem, as in op. cit. Section 3, we get the inequalities

$$\lambda_i \lambda'_{n+1-i} \leq n^{3/2}, \quad i = 1, \dots, n. \quad (2)$$

2. The main result

2.1. Let (E, h) and x_1, \dots, x_N be as in 1.4 above, let $E^{\vee} = \text{Hom}(E, \mathcal{O}_S)$ be the dual of E , and let $\mathbb{P}(E^{\vee})$ be the associated projective space (representing lines in E^{\vee}).

Consider a closed subvariety $X_K \subset \mathbb{P}(E^{\vee}_K)$ of dimension d over K . We let $\text{deg}(X_K) \in \mathbb{N}$ be its (algebraic) degree and $h(X_K) \in \mathbb{R}$ its Faltings height, denoted $h_F(X_K)$ in [4], (3.1.1) and (3.1.5). If $\overline{\mathcal{O}(1)}$ is the canonical line bundle on $\mathbb{P}(E^{\vee})$ equipped with the metric induced from h , and if X is the Zariski closure of X_K in $\mathbb{P}(E^{\vee})$, we have from [4], loc. cit.,

$$h(X_K) = \widehat{\text{deg}} \left(\hat{c}_1(\overline{\mathcal{O}(1)})^{d+1} | X \right) \in \mathbb{R}.$$

When m is large enough, $m \geq m_0$ say, the cup-product map

$$\varphi : E_K^{\otimes m} \rightarrow H^0(X_K, \mathcal{O}(m))$$

is surjective, so that $H^0(X_K, \mathcal{O}(m))$ is generated by the monomials

$$x_1^{\alpha_1} \cdots x_N^{\alpha_N} = \varphi(x_1^{\otimes \alpha_1} \otimes \cdots \otimes x_N^{\otimes \alpha_N}),$$

$\alpha_1 + \dots + \alpha_N = m$. A *special basis* is a basis of $H^0(X_K, \mathcal{O}(m))$ made of such elements.

Assume N real numbers r_1, \dots, r_N are given, and let $\mathbf{r} = (r_1, \dots, r_N)$. We define the weight of x_i to be r_i , $1 \leq i \leq N$, the weight of a monomial in $E_K^{\otimes m}$ to be the sum of the weight of the x_i 's occurring in it, and the weight of a monomial $u \in H^0(X_K, \mathcal{O}(m))$ to be the minimum $\text{wt}_{\mathbf{r}}(u)$ of the weights of the monomials in the x_i 's mapping to u by φ . The weight $\text{wt}_{\mathbf{r}}(\mathcal{B})$ of a special basis \mathcal{B} is the sum of the weights of its elements, and $w_{\mathbf{r}}(m)$ is the minimum weight of a special basis of $H^0(X_K, \mathcal{O}(m))$.

When $r_1, \dots, r_N \in \mathbb{N}$, there is a natural integer $e_{\mathbf{r}}$ such that, as m goes to infinity,

$$w_{\mathbf{r}}(m) = e_{\mathbf{r}} \frac{m^{d+1}}{(d+1)!} + O(m^d)$$

([14], Corollary 3.3).

THEOREM 1. *Assume there exists a continuous function $\psi : \mathbb{R}^N \rightarrow \mathbb{R}$ such that $\psi(tx) = t\psi(x)$ for all $t \in \mathbb{R}$ and $x \in \mathbb{R}^N$, and such that $e_{\mathbf{r}} \leq \psi(\mathbf{r})$ when $r_1 \geq r_2 \geq \dots \geq r_N = 0$ are integers. Then the following inequality holds:*

$$\begin{aligned} \frac{h(X_K)}{[K : \mathbb{Q}]} + (d+1) \deg(X_K) \mu_1 \\ + \psi(\mu_N - \mu_1, \mu_{N-1} - \mu_1, \dots, \mu_2 - \mu_1, 0) \geq 0. \end{aligned} \tag{3}$$

2.2. Our first step to prove Theorem 1 is the following. Fix real numbers $\varepsilon > 0$ and $r_1 \geq r_2 \geq \dots \geq r_N = 0$. Then there exists a constant C such that, for any positive integer $m \geq m_0$,

$$w_{\mathbf{r}}(m) \leq (\psi(\mathbf{r}) + \varepsilon) \frac{m^{d+1}}{(d+1)!} + Cm^d. \tag{4}$$

Indeed we may choose a positive real number $\eta > 0$ and rational numbers $s_i = p_i/q$ with $p_1 \geq p_2 \geq \dots \geq p_N = 0$, $|s_i - r_i| < \eta$, and $\psi(\mathbf{s}) \leq \psi(\mathbf{r}) + \varepsilon/2$.

If $m \geq m_0$ and if \mathcal{B} is any special basis of $H^0(X_K, \mathcal{O}(m))$ we have

$$\sum_{u \in \mathcal{B}} \text{wt}_{\mathbf{r}}(u) \leq \sum_{u \in \mathcal{B}} \text{wt}_{\mathbf{s}}(u) + m\eta \text{card}(\mathcal{B}).$$

By the usual theory of Hilbert polynomials,

$$h^0(X_K, \mathcal{O}(m)) = \deg(X_K) \frac{m^d}{d!} + O(m^{d-1}). \tag{5}$$

Therefore

$$w_{\mathbf{r}}(m) \leq w_{\mathbf{s}}(m) + \deg(X_K) \frac{m^{d+1}}{d!} \eta + O(m^d).$$

Since $w_{\mathbf{s}}(m) = w_{\mathbf{p}}(m)/q$ and $\psi(\mathbf{s}) = \psi(\mathbf{p})/q$, we get from our hypothesis on ψ the inequality

$$w_{\mathbf{s}}(m) \leq \psi(\mathbf{s}) \frac{m^{d+1}}{(d+1)!} + O(m^d),$$

hence

$$w_{\mathbf{r}}(m) \leq \left(\psi(\mathbf{r}) + \frac{\varepsilon}{2} + (d+1) \deg(X_K) \eta \right) \frac{m^{d+1}}{(d+1)!} + O(m^d).$$

If η is small enough this means that

$$w_{\mathbf{r}}(m) \leq (\psi(\mathbf{r}) + \varepsilon) \frac{m^{d+1}}{(d+1)!} + O(m^d),$$

i.e. (4) holds.

2.3. Given $m \geq m_0$ we let $M = H^0(X, \mathcal{O}(m))$. If $\sigma \in \Sigma$, denote by X_σ the corresponding set of complex points of X_K . We equip $M_\sigma = H^0(X_\sigma, \mathcal{O}(m))$ with the sup norm on X_σ :

$$|s|_\sigma = \sup_{x \in X_\sigma} \|s(x)\|_\sigma,$$

where $\|\cdot\|_\sigma$ is the norm on $\mathcal{O}(1)^{\otimes m}$ induced by E . The morphism

$$\varphi : E^{\otimes m} \rightarrow M$$

is then norm decreasing. If $u = \varphi(x_1^{\otimes \alpha_1} \otimes \cdots \otimes x_N^{\otimes \alpha_N})$ is a monomial, we have

$$|u| \leq \|x_1^{\otimes \alpha_1} \otimes \cdots \otimes x_N^{\otimes \alpha_N}\| \leq \prod_{i=1}^N \|x_i\|^{\alpha_i}.$$

Let

$$r_i = \mu_{N-i+1} - \mu_1, \quad 1 \leq i \leq N.$$

Then $r_1 \geq r_2 \geq \cdots \geq r_N = 0$ and the previous inequalities imply

$$\log |u| \leq \log \|x\| \leq \sum_{i=1}^N \alpha_i \log \|x_i\| = \sum_{i=1}^N \alpha_i r_i + m \mu_1,$$

where $x = x_1^{\otimes \alpha_1} \otimes \dots \otimes x_N^{\otimes \alpha_N} \in E^{\otimes m}$.

By definition of $\text{wt}_{\mathbf{r}}(u)$, for any $\varepsilon' > 0$ we may find x with $\varphi(x) = u$ and

$$\log \|x\| \leq \text{wt}_{\mathbf{r}}(u) + \varepsilon' + m\mu_1.$$

Applying Lemma 1 we conclude from this that for any special basis \mathcal{B} of M

$$\begin{aligned} \chi(M, |\cdot|) &\geq -[K : \mathbb{Q}] \sum_{u \in \mathcal{B}} (\text{wt}_{\mathbf{r}}(u) + \varepsilon' + m\mu_1) \\ &\quad - h^0(X_K, \mathcal{O}(m)) \frac{\log |D_K|}{2}, \end{aligned}$$

hence

$$\chi(M, |\cdot|) \geq -[K : \mathbb{Q}] (w_{\mathbf{r}}(m) + mh^0(X_K, \mathcal{O}(m))\mu_1) + O(m^d).$$

Using (4) and (5) we deduce that

$$\chi(M, |\cdot|) \geq -[K : \mathbb{Q}] (\psi(\mathbf{r}) + \varepsilon + (d+1) \deg(X_K)\mu_1) \frac{m^{d+1}}{(d+1)!} + O(m^d). \quad (6)$$

On the other hand, by a result of Zhang, [19] Theorem 1.4, we have

$$\chi(M, |\cdot|) = h(X_K) \frac{m^{d+1}}{(d+1)!} + o(m^{d+1}).$$

Comparing with (6) for all $\varepsilon > 0$, we get the inequality (3).

3. Applications

3.1. CHOW SEMI-STABILITY

3.1.1. We keep the notations of Section 2.1 and denote by $X_{\bar{K}} = X_K \otimes_K \bar{K}$ the projective variety obtained from X_K by extending scalars from K to an algebraic closure \bar{K} . Let $E_{\bar{K}}^{\vee} = E^{\vee} \otimes_{\mathcal{O}_K} \bar{K}$.

THEOREM 2. *Assume that the projective variety $X_{\bar{K}} \subset \mathbb{P}(E_{\bar{K}}^{\vee})$ is Chow semi-stable. Then*

$$\frac{h(X_K)}{[K : \mathbb{Q}]} + (d+1) \deg(X_K)\mu \geq 0. \quad (7)$$

Proof. Let V_i be the subspace of $E_{\bar{K}}$ generated by x_1, \dots, x_i , $1 \leq i \leq N$. If $r_1 \geq r_2 \geq \dots \geq r_N = 0$ are integers, it follows from Mumford's criterion for

semi-stability, [15] Theorem 2.9 applied to the weighted flag (V_i, r_i) and from [14] Corollary 3.3 that

$$e_{\mathbf{r}} \leq (d+1) \deg(X_K) \left(\sum_{i=1}^N r_i/N \right).$$

Therefore we may apply Theorem 1 with

$$\psi(\mathbf{r}) = (d+1) \deg(X_K) \left(\sum_{i=1}^N r_i/N \right).$$

We get

$$\frac{h(X_K)}{[K : \mathbb{Q}]} + (d+1) \deg(X_K) \left(\left(\sum_{i=1}^N (\mu_i - \mu_1)/N \right) + \mu_1 \right) \geq 0,$$

i.e. (7) holds.

3.1.2. Using (1) we deduce from Theorem 2 the following

COROLLARY. Under the assumptions of Theorem 2,

$$\begin{aligned} h(X_K) - (d+1) \deg(X_K) \widehat{\deg}(E, h)/N \\ \geq -(d+1) \deg(X_K) C(N, K)/N. \end{aligned}$$

This inequality is Bost's Theorem 1 in [3], except that the constant on the right hand side of this inequality is different from the one in loc. cit. (which is a constant multiple of $[K : \mathbb{Q}]$). In order to get a constant multiple of $[K : \mathbb{Q}]$ one could try to replace the successive minima μ_i , $1 \leq i \leq N$, by the slopes of the canonical polygon of Stuhler [17] and Grayson [11]. It is mentioned in [3] 4.3 that the inequality of loc. cit. can be applied to stable bundles on curves, surfaces of general type and abelian varieties.

3.2. SURFACES OF GENERAL TYPE

Let Y be a smooth surface of general type defined over K and $n \geq 5$ a fixed integer. The n th power L of the canonical line bundle on Y has then no base point [2]. With the notations of 2.1, we assume that $E_K = H^0(Y, L)$ and that X_K is the image of the morphism $Y \rightarrow \mathbb{P}(H^0(Y, L)^\vee)$.

THEOREM 3. *If n is big enough, the following inequality holds:*

$$\frac{h(X_K)}{[K : \mathbb{Q}]} + 3 \deg(X_K) \mu \geq \deg(X_K) (\mu_N - \mu_1)/N.$$

Proof. This result follows from Theorem 1 and Gieseker’s work [7]. Indeed, let $r(1) \leq r(2) \leq \dots \leq r(N)$ be relative integers such that $\sum_{i=1}^N r(i) = 0$. Denote by $c_1(Y)^2$ the (algebraic) self intersection of the canonical line bundle on Y . Let $p \geq 1$ and $M \gg 0$ be integers and $m = M(p + 1)$. Then, according to [7], Lemma 6.6, Lemma 5.15, Definition 5.3 and Section 2, the vector space $H^0(X_K, L^{\otimes m})$ has a distinguished basis of weight at most

$$\begin{aligned} & \frac{M^3 p^3}{2} \left(r(N)c_1(Y)^2 n^2 - (Nr(N) + \frac{1}{3}(r(N) - r(1))) \frac{n^2}{N} c_1(Y)^2 \right) \\ & + o(M^3 p^3) \\ & = -\frac{M^3 p^3}{6} \deg(X_K)(r(N) - r(1))/N + o(M^3 p^3) \end{aligned}$$

with respect to $(r(1), \dots, r(N))$, as M goes to infinity. If $r_1 \geq r_2 \geq \dots \geq r_N = 0$ are integers, we let $r(i) = r_{N-i+1} - (\sum_{i=1}^N r_i/N)$. We get $e_{\mathbf{r}} \leq \psi(\mathbf{r})$ with

$$\psi(\mathbf{r}) = \left(-r_1 + 3 \sum_{i=1}^N r_i \right) \deg(X_K)/N,$$

hence Theorem 3 follows from Theorem 1.

4. Smooth curves

4.1. We keep the notations of Section 2.1.

THEOREM 4. *Assume that $X_K \subset \mathbb{P}(E_K^{\vee})$ is a smooth geometrically irreducible curve of genus g and degree $d_0 = \deg(X_K) \geq 2g + 1$. Then the following inequality holds when $E_K = H^0(X_K, \mathcal{O}(1))$:*

$$\frac{h(X_K)}{[K : \mathbb{Q}]} + 2d_0\mu \geq \frac{2d_0g(d_0 - 2g)}{d_0^2 + d_0 - 2g^2}(\mu - \mu_1). \tag{8}$$

Proof. By a result of Morrison, [14] Theorem 4.4, the hypotheses of Theorem 1 are satisfied with

$$\psi(\mathbf{r}) = \frac{2d_0^2}{d_0^2 + d_0 - 2g^2} \left(\sum_{i=1}^N r_i \right)$$

(as noticed by the referee, the computation in [14], loc. cit., is not correct; the constant above is what comes out instead). Since $N = h^0(X_K, \mathcal{O}(1)) = d_0 + 1 - g$, we get from (3) the inequality

$$\frac{h(X_K)}{[K : \mathbb{Q}]} + 2d_0\mu_1 + \frac{2d_0^2(d_0 + 1 - g)}{d_0^2 + d_0 - 2g^2}(\mu - \mu_1) \geq 0,$$

i.e. (8) holds.

REMARK. Another way to prove (8), which does not use Zhang's result [19] Theorem 1.4, consists in comparing the height of X_K with the height of its projections to $\mathbb{P}(V_i^\vee)$, where $V_i \subset E_K$ is the subspace generated by x_1, \dots, x_i , $1 \leq i \leq N$. One may then combine [4] 3.3.2 with Morrison's combinatorial results, [14] Corollary 4.3 and Theorem 4.4, to obtain the inequality (8).

4.2. We shall now consider vector bundles of rank two on curves. Let X_K be a smooth geometrically irreducible curve of genus $g \geq 2$ over K , let F be a rank two vector bundle on X_K of degree d_0 (big enough with respect to g), let $L = \Lambda^2 F$ be the second exterior power of F , and let $F_{\bar{K}}$ be its restriction to $X_{\bar{K}}$. Assume (E, h) is an hermitian vector bundle on S such that $E_K = H^0(X_K, F)$. According to [8], Lemma 3.2, the map

$$\psi : \Lambda^2 H^0(X_K, F) \rightarrow H^0(X_K, L)$$

is surjective. Therefore the lattice $E' = \psi(\Lambda^2 E)$ is such that $E'_K = H^0(X_K, L)$, and we let h' be the metric induced by h on E' . We let $h(X_K)$ be the height of X_K for the projective embedding $X_K \subset \mathbb{P}(H^0(X_K, L)^\vee)$, with respect to (E', h') . Denote by $\lambda_1, \dots, \lambda_N$ the successive minima of (E, h) , $N = h^0(X_K, F) = d_0 + 2 - 2g$, $\mu_i = \log \lambda_i$, $1 \leq i \leq N$, and

$$\mu = \frac{1}{N} \sum_{i=1}^N \mu_i.$$

THEOREM 5. *There exists a positive constant $a(g, d_0)$ and an integer D such that if $d_0 > D$ and the bundle $F_{\bar{K}}$ is stable the following inequality holds*

$$\frac{h(X_K)}{[K : \mathbb{Q}]} + 4d_0\mu \geq a(g, d_0)(\mu - \mu_1);$$

furthermore, if $d_0 > D$ and $F_{\bar{K}}$ is semi-stable, then

$$\frac{h(X_K)}{[K : \mathbb{Q}]} + 4d_0\mu \geq 0.$$

Proof. Theorem 5 follows from [8] by a method similar to Theorem 1. Choose $x_1, \dots, x_N \in E$, linearly independent over K , such that $\|x_i\| = \lambda_{N-i+1}$. Consider the morphism

$$\varphi : (\Lambda^2 E)^{\otimes m} \rightarrow M = H^0(X, \mathcal{O}(m)),$$

where X is the Zariski closure of X_K in $\mathbb{P}(E^\vee)$, obtained by cup-product from the canonical morphism

$$\Lambda^2 E \rightarrow E' \rightarrow H^0(X, \mathcal{O}(1)).$$

When m is big enough, the image of φ has maximal rank over K . Given a set of N real numbers $\mathbf{r} = (r_1, \dots, r_N)$, we define the weight of $y_{ij} = x_i \wedge x_j \in \Lambda^2 E$ to be $r_i + r_j$, $1 \leq i \neq j \leq N$. The weight of a monomial $y_{i_1 j_1} \otimes y_{i_2 j_2} \otimes \dots \otimes y_{i_m j_m} \in (\Lambda^2 E)^{\otimes m}$ is the sum of the weights of its factors, a special basis \mathcal{B} of $H^0(X_K, \mathcal{O}(m))$ is a basis made of the images by φ of some of these monomials. We define its weight $\text{wt}_{\mathbf{r}}(\mathcal{B})$ as in 2.1, and $w_{\mathbf{r}}(m)$ is the minimum weight of a special basis of $H^0(X_K, \mathcal{O}(m))$. When m goes to infinity

$$w_{\mathbf{r}}(m) = e_{\mathbf{r}} \frac{m^2}{2} + O(m). \tag{9}$$

From the proof of [8], Theorem 5.1, it follows that, if $r_1 \geq r_2 \geq \dots \geq r_N = 0$ are rational numbers such that $r_1 + r_2 + \dots + r_N = 1$ and if $F_{\bar{K}}$ is stable (resp. semi-stable) and d_0 is big enough, we have

$$e_{\mathbf{r}} \leq (4d_0 - a(g, d_0))/N$$

(resp. $e_{\mathbf{r}} \leq 4d_0/N$) for some positive constant $a(g, d_0)$. As in 2.2, we deduce from this that if $r_1 \geq r_2 \geq \dots \geq r_N = 0$ are real numbers, then

$$w_{\mathbf{r}}(m) \leq (\psi(\mathbf{r}) + \varepsilon) \frac{m^2}{2} + Cm,$$

with

$$\psi(\mathbf{r}) = \frac{4d_0 - a(g, d_0)}{N} \left(\sum_{i=1}^N r_i \right),$$

(resp.

$$\psi(\mathbf{r}) = \frac{4d_0}{N} \left(\sum_{i=1}^N r_i \right)).$$

If we equip $M = H^0(X, \mathcal{O}(m))$ with the sup-norm coming from the metric induced by E' on L , and if $u = \varphi(y_{i_1 j_1} \otimes y_{i_2 j_2} \otimes \dots \otimes y_{i_m j_m})$ is a decomposable element, we have

$$\begin{aligned} |u| &\leq \|y_{i_1 j_1}\| \|y_{i_2 j_2}\| \dots \|y_{i_m j_m}\| \\ &\leq \|x_{i_1}\| \|x_{j_1}\| \|x_{i_2}\| \dots \|x_{i_m}\| \|x_{j_m}\|. \end{aligned}$$

If we let $r_i = \mu_{N-i+1} - \mu_1$, $1 \leq i \leq N$, it follows that

$$\log|u| \leq \text{wt}_{\mathbf{r}}(u) + 2m\mu_1.$$

Therefore, using Lemma 1 as in 2.3, we get

$$\chi(M, |\cdot|) \geq -[K : \mathbb{Q}](w_{\mathbf{r}}(m) + 2m h^0(X_K, \mathcal{O}(m))\mu_1) + O(m). \quad (10)$$

Since

$$h^0(X_K, \mathcal{O}(m)) = d_0 m + O(1),$$

it follows from (9), (10) and [19] Theorem 1.4 as in 2.3, that

$$\frac{h(X_K)}{[K : \mathbb{Q}]} + 4d_0\mu_1 + (4d_0 - a(g, d_0))(\mu - \mu_1) \geq 0$$

if $F_{\bar{K}}$ is stable, and

$$\frac{h(X_K)}{[K : \mathbb{Q}]} + 4d_0\mu \geq 0$$

if $F_{\bar{K}}$ is semi-stable. This proves Theorem 5.

REMARK. From the proof of [8] Theorem 5.1, one can derive the following estimate:

$$a(g, d_0) \geq 0.8.$$

4.3. The vanishing theorem of [16] provides more information on the successive minima of sections of line bundles on curves. Namely, let $f : X \rightarrow S$ be a semi-stable curve over S , with geometrically irreducible generic fiber X_K . Consider a line bundle L on X of degree $m \geq 2$ on X_K . Choose an hermitian metric h on L with positive first Chern form $c_1(L, h)$.

We assume that the arithmetic degree of $\bar{L} = (L, h)$ on any irreducible divisor of X is nonnegative, and we let $\bar{L}^2 \in \mathbb{R}$ be the arithmetic self-intersection $\hat{c}_1(\bar{L})^2$ of the first Chern class of \bar{L} .

We equip the tangent space of $X(\mathbb{C})$ with the metric whose associated (normalized) Kähler form is $c_1(L, h)/m$, and the relative dualizing sheaf $\omega_{X/S}$ with the dual metric.

The \mathcal{O}_K -module $E' = H^0(X, L \otimes \omega_{X/S})$ is then equipped with the L^2 -metric. If $x = \sum_{\sigma \in \Sigma} x_{\sigma}$ lies in $E' \otimes_{\mathbb{Z}} \mathbb{C}$ we let $\|x\|' = \sum_{\sigma \in \Sigma} \|x_{\sigma}\|_{L^2}$. Let $n = [K : \mathbb{Q}]h^0(X_K, L \otimes \omega_{X/S})$ be the rank of E' over \mathbb{Z} , λ'_n the top successive minimum of $(E', \|\cdot\|')$ and $\mu'_n = \log \lambda'_n$.

THEOREM 5.

(a) *Under the above assumptions, the following inequality holds*

$$\mu'_n \leq -\frac{\bar{L}^2}{m^2[K : \mathbb{Q}]} + \frac{\log|D_K|}{[K : \mathbb{Q}]} + 1 + \frac{3}{2} \log(n). \quad (11)$$

(b) Assume furthermore that X_K has genus $g \geq 2$, that $\omega_{X/S}$ is equipped with the Arakelov metric, and that \bar{L} is the k -th power of $\bar{\omega}_{X/S}$, $k \geq 1$. Then

$$\mu'_n \leq -\frac{(k+1)\bar{\omega}_{X/S}^2}{4g(g-1)[K:\mathbb{Q}]} + \frac{\log|D_K|}{[K:\mathbb{Q}]} + 1 + \frac{3}{2}\log(n). \quad (12)$$

Proof. By Serre duality, if we let L^{-1} be the dual of L , the quotient of the \mathcal{O}_K -module $H^1(X, L^{-1})$ by its torsion subgroup, when equipped with the L^2 -metric, is the dual of $H^0(X, L \otimes \omega_{X/S})$ over S . Let ω_S be as in 1.5 above, let λ_1 be the smallest norm $\|v\| = \text{Sup}_\sigma \|v_\sigma\|_{L^2}$ of nonzero vectors v in

$$E = (H^1(X, L^{-1}) \otimes \omega_S) / \text{torsion} = H^1(X, L^{-1} \otimes f^*\omega_S) / \text{torsion},$$

and let $\mu_1 = \log \lambda_1$. From (2) we know that

$$\mu'_n \leq -\mu_1 + \frac{3}{2}\log(n). \quad (13)$$

Let $M = L \otimes f^*\omega_S^{-1}$ be equipped with the tensor product of the chosen metrics. Using [10], p. 355, we compute

$$\begin{aligned} \hat{c}_1(\bar{M})^2 &= \hat{c}_1(\bar{L})^2 - 2\hat{c}_1(\bar{L})\hat{c}_1(f^*\bar{\omega}_S) \\ &= \bar{L}^2 - 2m \widehat{\text{deg}}(\bar{\omega}_S), \end{aligned}$$

where

$$\widehat{\text{deg}}(\bar{\omega}_S) = \log|D_K| - 2r_2 \log(2) \leq \log|D_K|$$

is the arithmetic degree of $\bar{\omega}_S$.

Similarly, let $P \in X(\bar{K})$ be an algebraic point on X_K , defined on a finite extension K' of K , and $u: \text{Spec}(\mathcal{O}_{K'}) \rightarrow X$ the morphism defined by P . The normalized height of P with respect to \bar{M} is then

$$\frac{\widehat{\text{deg}}(u^*\bar{M})}{[K':K]} = \frac{\widehat{\text{deg}}(u^*\bar{L})}{[K':K]} - \widehat{\text{deg}}(\bar{\omega}_S).$$

From our hypotheses on \bar{L} we get

$$\frac{\widehat{\text{deg}}(u^*\bar{M})}{[K':K]} \geq -\widehat{\text{deg}}(\bar{\omega}_S).$$

In case (a), we may then apply [16] Theorem 2 to get

$$\begin{aligned} [K:\mathbb{Q}]m^2(\mu_1 + 1) &\geq \hat{c}_1(\bar{M})^2 + (m^2 - 2m)e(\bar{M}) \\ &\geq \bar{L}^2 - m^2 \widehat{\text{deg}}(\bar{\omega}_S). \end{aligned} \quad (14)$$

The inequality (11) follows from (13) and (14). Similarly, in case (b), we get as in [16] Theorem 3 that

$$[K : \mathbb{Q}](\mu_1 + 1) \geq \frac{(k+1)\bar{\omega}_{X/S}^2}{4g(g-1)} - \widehat{\deg}(\bar{\omega}_S), \quad (15)$$

and (12) follows from (13) and (15).

REMARK. Since n is an affine function of k , Theorem 5(b) implies that λ'_n goes to zero as k goes to infinity. As was noticed by Ullmo, this proves that, if $k \geq k_0$, the lattice $H^0(X, \omega_{X/S}^{\otimes k+1})$ contains a set of sections of L^2 -norm less than one which has maximal rank. This also follows from Zhang's result [18] Theorem 1.5, but this proof is effective in the sense that k_0 can be evaluated from (12).

References

1. W. Banaszczyk: New bounds in some transference theorems in the geometry of numbers, *Math. Annalen*, 296 (1993), 625–635.
2. E. Bombieri: Canonical models of surfaces of general type, *Publ. Math. IHES*, 42 (1973), 171–219.
3. J.-B. Bost: Semi-stability and heights of cycles, *Inventiones Math.*, 118 (1994), 223–253.
4. J.-B. Bost, H. Gillet, C. Soulé: Heights of projective varieties and positive Green forms, *Journal of the AMS*, 7 (1994), 903–1027.
5. J.-F. Burnol: Remarques sur la stabilité en arithmétique, International Mathematics Research Notices, *Duke Math. J.*, 6 (1992), 117–127.
6. G. Faltings: Calculus on arithmetic surfaces, *Annals of Math.*, 119 (1984), 387–424.
7. D. Gieseker: Global moduli for surfaces of general type, *Inventiones Math.*, 43 (1977), 233–282.
8. D. Gieseker and I. Morrison: Hilbert stability of rank-two bundles on curves, *J. Differential geometry*, 19 (1984), 1–29.
9. H. Gillet and C. Soulé: Characteristic classes for algebraic vector bundles with Hermitian metrics, I, II, *Annals of Math.*, 131 (1990), 163–203 and 205–238.
10. H. Gillet and C. Soulé: On the number of lattice points in convex symmetric bodies and their duals, *Israel Journal of Math.*, 74, no 2–3 (1991), 347–357.
11. D. Grayson: Reduction theory using semistability, *Comment. Math. Helvetici*, 59 (1984), 600–634.
12. Y. Miyaoka: Bogomolov inequality on arithmetic surfaces, talk at the Oberwolfach conference on “Arithmetical Algebraic Geometry”, G. Harder and N. Katz org. (1988).
13. A. Moriwaki: Inequality of Bogomolov–Gieseker’s type on arithmetic surfaces, *Duke Math. Journal*, 74 (1994), 713–761.
14. I. Morrison: Projective stability of ruled surfaces, *Inventiones Math.*, 56 (1980), 269–304.
15. D. Mumford: Stability of projective varieties, *l’Ens. Math.*, 23 (1977), 39–110.
16. C. Soulé: A vanishing theorem on arithmetic surfaces, *Inventiones Math.* 116 (1994), 577–599.
17. U. Stuhler: Eine Bemerkung zur Reduktionstheorie Quadratischen Formen, *Archiv. der Math.*, 27 (1976), 604–610.
18. S. Zhang: Positive line bundles on arithmetic surfaces, *Annals of Math.*, 136 (1992), 569–587.
19. S. Zhang: Positive line bundles on arithmetic varieties, preprint Princeton University (1992).
20. S. Zhang: Geometry reductivity at archimedean places, preprint Princeton University (1993).