COMPOSITIO MATHEMATICA

ARTHUR BARAGAR

Asymptotic growth of Markoff-Hurwitz numbers

Compositio Mathematica, tome 94, n° 1 (1994), p. 1-18
<http://www.numdam.org/item?id=CM_1994__94_1_1_0>

Compositio Mathematica **94**: 1-18, 1994.

Asymptotic growth of Markoff-Hurwitz numbers

ARTHUR BARAGAR
Department of Mathematics, University of Texas, Austin, TX 78712

Received 12 November 1991; accepted in final form 8 April 1993

Introduction

In 1879, Markoff [Ma] made the equation

$$x^2 + y^2 + z^2 = 3xyz \tag{0.1}$$

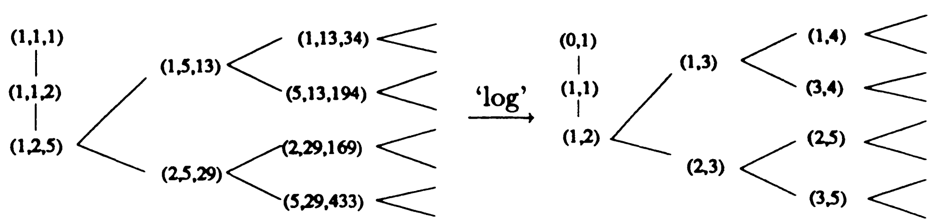
famous when he noted the connection between its integral solutions, classes of quadratic forms, and diophantine approximation. Using a descent argument, he showed all the integral solutions (except $(0, 0, 0)$) can be generated by the fundamental solution $(1, 1, 1)$ and a group of automorphisms $\mathcal{G}$ of the surface (0.1). The set of positive ordered integral solutions to (0.1) has a natural tree structure (Fig. 1, left side).

Hurwitz [Hu] noted that Markoff's descent technique can be applied to the equation

$$M_{a,n}\colon x_1^2 + \cdots + x_n^2 = ax_1 \cdots x_n. \tag{0.2}$$

Like the Markoff equation, the set of positive ordered integral solutions to (0.2) can be separated into a finite number of trees [Hu].

Zagier [Z] did some beautiful work when he counted the Markoff numbers – those positive numbers that appear in integral solutions to the Markoff equation. He showed that the number of Markoff numbers less than H grows



The Markoff tree | The Euclid tree

Fig. 1.

asymptotically like $c(\log H)^2$ (modulo the unicity conjecture), and gave a rapidly converging series for $c \approx 0.1807$. His technique involves an idea (due to Cohn [Co]) of comparing the Markoff tree with the Euclid tree (Fig. 1).

Suppose (p, q, r) with $0 < p \leqslant q \leqslant r$ is a solution of the Markoff equation. Then

$$p^2 + q^2 + r^2 = 3pqr$$

$$r \leqslant 3pq \leqslant 3r$$

and if p, q and r are large, then

$$\log r \approx \log p + \log q.$$

Hence the branching operations of the Markoff tree

$$(p, q, r) \begin{array}{l} \nearrow (p, r, 3pr - q) \\ \searrow (q, r, 3qr - p) \end{array}$$

can be compared with the branching operations

$$(a, b, a + b) \begin{array}{l} \nearrow (a, a + b, 2a + b) \\ \searrow (b, a + b, a + 2b) \end{array}$$

or more simply

$$(a, b) \begin{array}{l} \nearrow (a, a + b) \\ \searrow (b, a + b). \end{array}$$

The tree thus generated is called a Euclid tree since going left down the tree is the Euclidean algorithm. In Part 1 we make a similar 'logarithmic' comparison between Hurwitz trees and an $(n - 1)$-branch generalization of the Euclid tree.

Part 2 is devoted to counting $\gamma_{\mathbf{a}}(x)$ – the number of nodes with height less than x in a k-branch Euclid tree rooted at $\mathbf{a} \in (\mathbb{R}^+)^k$. The classical 2-branch Euclid tree is relatively easy to count, and Zagier shows

$$\gamma_{(a_1, a_2)}(x) = \frac{3}{\pi^2 a_1 a_2} x^2 + O\left(\frac{x}{a_1}\right) + O\left(\frac{x}{a_2} \log\left(\frac{x}{a_2}\right)\right).$$

The k-branch Euclid tree rooted as $\mathbf{1} = (1, 1, \ldots, 1)$ sits naturally in $(\mathbb{Z}^+)^k$. Hence

$$\gamma_1(x) = O(x^k),$$

and it is tempting to conjecture $\gamma_{\mathbf{a}}(x)$ is of order x^k. We show that for any $\varepsilon > 0$,

$$\gamma_{\mathbf{a}}(x) \leqslant x^{\alpha(k)+\varepsilon}$$

for all sufficiently large x, and

$$\gamma_{\mathbf{a}}(x) \geqslant x^{\alpha(k)-\varepsilon}$$

for an unbounded set of x's. We also give bounds on $\alpha(k)$ and show the surprising result that for large k, $\alpha(k)$ grows like $\log k$. For some small values of k,

$$2.248 < \alpha(3) < 2.637$$

$$2.524 < \alpha(4) < 3.172$$

$$2.767 < \alpha(5) < 3.611.$$

From this we conclude results about the growth of Hurwitz numbers.

This paper is in part derived from results in the author's thesis written under the supervision of Joseph H. Silverman at Brown University [B1].

1. A comparison of trees

Define a height

$$h(\mathbf{x}) = \max\{|x_1|, \ldots, |x_n|\}.$$

It is clear that the number of integral solutions of $M_{a,n}$ with bounded height H is finite. Hence it makes sense to define

$$g_{a,n}(H) = \#\{\mathbf{x} \in M_{a,n}(\mathbb{Z}) : h(\mathbf{x}) \leqslant H\}.$$

Zagier actually dealt with the quantity $g_{3,3}(H)$:

THEOREM 1.1 (Zagier.) *Without assuming the unicity conjecture,*

$$g_{3,3}(H) = c(\log H)^2 + O(\log H \log\log H)$$

where $c \approx 4.337$.

Modulo the unicity conjecture and a factor of 24, $g_{3,3}(H)$ counts the number of Markoff numbers less than H.

The aim of this paper is to approximate $g_{a,n}(H)$ for the Hurwitz equations $M_{a,n}$ which have non-trivial integral solutions. Deciding which pairs (a, n) yield Hurwitz equations $M_{a,n}$ that have non-trivial integral solutions is an interesting problem partially addressed by Hurwitz [Hu] and again by Herzberg [He].

The Hurwitz equations $M_{a,n}$ admit large groups of automorphisms $\mathscr{G}_{a,n}$ generated by the deck transformation

$$\phi:(x_1,\ldots,x_n)\mapsto(x_1,\ldots,x_{n-1}, ax_1\cdots x_{n-1} - x_n),$$

the sign change

$$\sigma:(x_1,\ldots,x_n)\mapsto(-x_1, -x_2, x_3,\ldots,x_n),$$

and the group S_n of permutations on the variables $\{x_1,\ldots,x_n\}$. Note that the set of integral solutions $M_{a,n}(\mathbb{Z})$ of $M_{a,n}$ is closed under the action of $\mathscr{G}_{a,n}$. Hence if we have a non-trivial integral solution $\mathbf{s}$ of $M_{a,n}$, then we can obtain many more integral solutions by considering the $\mathscr{G}_{a,n}$-orbit of $\mathbf{s}$: $\mathscr{G}_{a,n}\{\mathbf{s}\}$. In each orbit $\mathscr{G}_{a,n}\{\mathbf{s}\}$ there exists a positive ordered element $\mathbf{r}$ with minimal height. We say $\mathbf{r}$ is the *fundamental solution* or *root* of $\mathscr{G}_{a,n}\{\mathbf{s}\}$. Hurwitz showed that $\mathbf{r}$ is unique for each orbit $\mathscr{G}_{a,n}\{\mathbf{s}\}$, and the set of roots $\mathscr{F}_{a,n}$ in $M_{a,n}$ is finite. Investigating the possible size of $\mathscr{F}_{a,n}$ is another interesting question studied in [B2].

Let $\mathfrak{M}_{a,n}\{\mathbf{r}\}$ be the Hurwitz tree for an equation $M_{a,n}$ rooted at $\mathbf{r} = (r_1,\ldots,r_n)$ with $0 < r_1 \leqslant \cdots \leqslant r_n$, and generated by the branching operations

$$\phi_j:(q_1,\ldots,q_n)\mapsto\left(q_1,\ldots,\widehat{q_{n-j}},\ldots,q_n, a\left(\prod_{i\neq j} q_i\right) - q_j\right)$$

for $j = 1, 2,\ldots,n-1$ (for a positive ordered solution $\mathbf{s}$, ϕ_0 is the descending branch when $\mathbf{s}$ is not a root.) The hat ^ indicates that that element is omitted. This definition does not demand that $\mathbf{r}$ be a root, or even that $\mathbf{r}\in M_{a,n}(\mathbb{Z})$. However, if $\mathbf{r}$ is a root, then $\mathfrak{M}_{a,n}\{\mathbf{r}\}\cup\mathfrak{M}_{a,n}\{\phi_0(\mathbf{r})\}$ is the subset of positive ordered integral solutions in $\mathscr{G}_{a,n}\{\mathbf{r}\}$.

We compare the Hurwitz tree with $\mathfrak{E}_{n-1}\{\mathbf{b}\}$, the Euclid-like tree rooted at $\mathbf{b}\in(\mathbb{R}^+)^{n-1}$ and defined by the branching operations

$$E_j:(c_1,\ldots,c_{n-1})\mapsto(c_1,\ldots,\widehat{c_{n-j}},\ldots,c_{n-1}, c_1 + \cdots + c_{n-1})$$

for $j = 1,\ldots,n-1$. We use the subscript $n-1$ to emphasize the number of branches. Both $\mathfrak{M}_{a,n}\{\mathbf{r}\}$ and $\mathfrak{E}_{n-1}\{\mathbf{b}\}$ are $(n-1)$-branch trees.

Conceptually, these trees are easy to understand, but the notation we need to describe them is cumbersome. It would be convenient if we could describe each node by the vector at that place, but since we have allowed the possibility of duplication, we cannot.

For a set S of k letters, let us denote by $\langle S\rangle$ the free monoid on the k letters in S. So $\langle S\rangle$ is the set of formal words written with letters in S and can be thought of as a formal k-branch tree, rooted at the empty word (the word with no letters.) For the Hurwitz trees and Euclid-like trees, the elements $\{\phi_1,\ldots,\phi_{n-1}\}$ and $\{E_1,\ldots,E_{n-1}\}$ are free as elements of a monoid (but not as elements of a group), so the trees can be represented by $\langle\phi_1,\ldots,\phi_{n-1}\rangle$ and $\langle E_1,\ldots,E_{n-1}\rangle$. However, since we are primarily interested in the action of the elements in each monoid on the root solutions $\mathbf{r}$ and $\mathbf{b}$, let us denote these trees by

$$\mathfrak{M}_{a,n}\{\mathbf{r}\} = \{(\omega, \omega\mathbf{r}) : \omega\in\langle\phi_1,\ldots,\phi_{n-1}\rangle\},$$

$$\mathfrak{E}_{n-1}\{\mathbf{b}\} = \{(W, W\mathbf{b}) : W\in\langle E_1,\ldots,E_{n-1}\rangle\},$$

We then have the obvious map between these trees

$$\Theta : \mathfrak{M}_{a,n}\{\mathbf{r}\} \to \mathfrak{E}_{n-1}\{\mathbf{b}\}$$

$$\left(\bigcirc_{i=1}^{m} \phi_{j_i}, \left(\bigcirc_{i=1}^{m} \phi_{j_i}\right)(\mathbf{r})\right) \mapsto \left(\bigcirc_{i=1}^{m} E_{j_i}, \left(\bigcirc_{i=1}^{m} E_{j_i}\right)(\mathbf{b})\right)$$

that pairs matching nodes in each tree. Sometimes we will abuse our notation, and write $\Theta(\mathbf{q}) = \mathbf{c}$ where $\mathbf{q} = \omega\mathbf{r}$ and $\Theta(\omega, \omega\mathbf{r}) = (W, \mathbf{c})$.

LEMMA 1.2. *Suppose* $(\omega, \mathbf{q})\in\mathfrak{M}_{a,n}\{\mathbf{r}\}$, $\Theta(\omega, \mathbf{q}) = (W, \mathbf{c})\in\mathfrak{E}_{n-1}\{\mathbf{b}\}$, *and*

$$\log(a^{1/(n-2)}r_i) \leqslant b_i$$

for $i = 1,\ldots,n-1$. *Then*

$$\log(a^{1/(n-2)}q_i) \leqslant c_i.$$

Proof. (By induction on m, the number of letters in ω.)
We need only check that if

$$\log(a^{1/(n-2)}q_i) \leqslant c_i$$

for some $\mathbf{q}\in\mathfrak{M}_{a,n}\{\mathbf{r}\}$ and $\mathbf{c} = \Theta(\mathbf{q})$, then

$$\log(a^{1/(n-2)}q_n) \leqslant c_1 + \cdots + c_{n-1}.$$

But $q_n + q_n' = aq_1 \cdots q_{n-1}$ and $q_n' \geqslant 1$. Hence

$$q_n \leqslant aq_1 \cdots q_{n-1}$$

$$a^{1/(n-2)}q_n \leqslant a^{n-1/(n-2)}q_1 \cdots q_{n-1} = \prod_{i=1}^{n-1} (a^{1/(n-2)}q_i)$$

$$\log(a^{1/(n-2)}q_n) \leqslant \sum_{i=1}^{n-1} \log(a^{1/(n-2)}q_i) \leqslant c_1 + \cdots + c_{n-1}. \tag{1.1}$$

LEMMA 1.3. *Suppose*

(i) $q_n \geqslant kq_1 \cdots q_{n-1}$ *for all* $(\omega, \mathbf{q}) \in \mathfrak{M}_{a,n}\{\mathbf{r}\}$,
(ii) $\log(k^{1/(n-2)}r_i) \geqslant b_i$ *for all* $i = 1, \ldots, n-1$,

and let $(W, \mathbf{c}) = \Theta(\omega, \mathbf{q}) \in \mathfrak{E}_{n-1}\{\mathbf{b}\}$. *Then*

$$\log(k^{1/(n-2)}q_i) \geqslant c_i$$

for $i = 1, \ldots, n-1$.

The proof is identical to the proof of Lemma 1.2, except we use

$$q_n \geqslant kq_1 \cdots q_{n-1}$$

at (1.1).

For a Hurwitz tree $\mathfrak{M}_{a,n}\{\mathbf{r}\}$, define

$$g_{\mathbf{r}}(H) = \#\{(\omega, \mathbf{q}) \in \mathfrak{M}_{a,n}\{\mathbf{r}\} : q_n \leqslant H\}.$$

$g_{\mathbf{r}}(H)$ is not quite the same as $g_{a,n}(H)$, but it is close. We have omitted the solutions of $M_{a,n}$ obtained from $\mathbf{q}$ by sign changes and permutations of the variables. This accounts for a factor of roughly $n!2^{n-1}$. We have also allowed the possibility that $(\omega, \mathbf{p}) \neq (\omega', \mathbf{q})$ yet $\mathbf{p} = \mathbf{q}$, but this is only possible if $\mathbf{r}$ has components r_i and r_j such that $r_i = r_j$. The number of duplications that can occur is easy to determine, and is bounded for every $\mathbf{q}$. Another difference is that we have not chosen $\mathbf{r}$ to be a fundamental solution of $M_{a,n}(\mathbb{Z})$. A further complication is that $M_{a,n}(\mathbb{Z})$ may have more than one fundamental solution; however, it is important that we know $M_{a,n}(\mathbb{Z})$ has a finite number of roots.

For a Euclid tree $\mathfrak{E}_{n-1}\{\mathbf{b}\}$, define the height

$$h(\mathbf{c}) = c_1 + \cdots + c_{n-1}$$

and the counting function

$$\gamma_{\mathbf{b}}(x) = \#\{(W, \mathbf{c}) \in \mathfrak{C}_{n-1}\{\mathbf{b}\} : h(\mathbf{c}) \leqslant x\}.$$

THEOREM 1.4. *Suppose*

(i) $q_n \geqslant kq_1 \cdots q_{n-1}$ *for all* $\mathbf{q} \in \mathfrak{M}_{a,n}\{\mathbf{r}\}$,
(ii) $b_i \geqslant \log(a^{1/(n-2)}r_i)$ *for all* i,
(iii) $c_i \leqslant \log(k^{1/(n-2)}r_i)$ *for all* i.

Then

$$\gamma_{\mathbf{b}}(\log(a^{1/(n-2)}H)) \leqslant g_{\mathbf{r}}(H) \leqslant \gamma_{\mathbf{c}}(\log(k^{1/(n-2)}H)).$$

The proof is straightforward from Lemma 1.2 and Lemma 1.3.

We still need a few more results before Theorem 1.4 can be useful. For instance, we need to know such a k exists. If we choose $\mathbf{q} \in \mathfrak{M}_{a,n}\{\mathbf{r}\}$ and $\mathbf{q} \neq \mathbf{r}$, then we can descend from $\mathbf{q}$. Hence

$$2q_n \geqslant aq_1 \cdots q_{n-1},$$

and $k = a/2$ works. We can do better:

LEMMA 1.5. *Suppose* $\mathbf{q} \in \mathfrak{M}_{a,n}\{\mathbf{r}\}$. *Then*

$$q_n \geqslant kq_1 \cdots q_{n-1}$$

for $k = a - 1/r_1 \cdots r_{n-2}$.

Proof. We have

$$(a-k)(r_1 \cdots r_{n-2}) = 1.$$

Since $0 < r_i \leqslant q_i$ for all i,

$$(a-k)(q_1 \cdots q_{n-2}) \geqslant 1.$$

The claim is clear for $\mathbf{q} = \mathbf{r}$, so assume $\mathbf{q} \neq \mathbf{r}$. Then we can descend from $\mathbf{q}$, and $q_n' \leqslant q_{n-1} \leqslant q_n$ where $q_n' = a(q_1 \cdots q_{n-1}) - q_n$ (see Cassels [Ca], p. 27, for a clever proof of this, or [B2]). Hence

$$(a-k)(q_1 \cdots q_{n-1}) \geqslant a(q_1 \cdots q_{n-1}) - q_n$$
$$q_n \geqslant kq_1 \cdots q_{n-1}.$$

We also need to know there exist vectors $\mathbf{b}$ and $\mathbf{c}$ that satisfy (ii) and (iii) and have positive components. If $\mathbf{b}$ or $\mathbf{c}$ has a non-positive component, then $\gamma(x)$ may not be finite. We can guarantee $\mathbf{b}$ and $\mathbf{c}$ have positive components if $a \geqslant 3$. A more delicate argument is needed for $a = 1$ and 2, but we leave it to the reader.

Lastly, we need to know what $\gamma_{\mathbf{b}}(x)$ looks like. If we know this, then we may be able to get a good approximation for $g_{\mathbf{r}}(H)$.

By Lemma 1.5, k is near a if $r_1 \cdots r_{n-2}$ is large. In that case, choose $b_i = \log(a^{1/(n-2)} r_i)$ and $\mathbf{c} = \varepsilon \mathbf{b}$ so that $\mathbf{c}$ satisfies (iii). Since k is near a, we can choose ε near 1. And since

$$\gamma_{\varepsilon \mathbf{b}}(x) = \gamma_{\mathbf{b}}\left(\frac{x}{\varepsilon}\right),$$

Theorem 1.4 must give a good approximation to $g_{\mathbf{r}}(H)$. So to count a Hurwitz tree $\mathfrak{M}_{a,n}\{\mathbf{r}\}$, first take a subtree whose end points are a number of solutions $\{\mathbf{r}_k\}_{k=1}^{m}$ with the property that $r_{k,1} \cdots r_{k,n-2}$ is large for each k. Then approximate the counting of each tree rooted at $\mathbf{r}_k$ using Theorem 1.4, and sum up.

Zagier works out the details for $\mathfrak{M}_{3,3}\{(1, 1, 1)\}$ in [Z]. The first detail is to approximate $\gamma_{\mathbf{b}}(x)$, which Zagier does rather successfully with Theorem 1.1. Approximating $\gamma_{\mathbf{b}}(x)$ for $n \geqslant 4$ is the subject of the following section.

2. The *k*-branch Euclid tree

In this section, we show:

THEOREM 2.1. *For a k-branch Euclid-like tree rooted at* $\mathbf{b} = (b_1, \ldots, b_k) \in (\mathbb{R}^+)^k$ *with* $k \geqslant 3$, *the quantity*

$$\limsup_{x \to \infty} \frac{\log(\gamma_{\mathbf{b}}(x))}{\log x} = \alpha$$

is finite. Furthermore, $\alpha = \alpha(k)$ *depends only on k, and satisfies*

$$\frac{\log k}{\log 2} + o(1) < \alpha(k) < \frac{3 \log k}{2 \log 2} + o(1).$$

For small k,

$$2.248 < \alpha(3) < 2.637$$

$$2.524 < \alpha(4) < 3.172$$

$$2.767 < \alpha(5) < 3.611.$$

In other words, for any $\varepsilon > 0$,

$$\gamma_{\mathbf{b}}(x) = \begin{cases} O(x^{\alpha(k)+\varepsilon}) \\ \Omega(x^{\alpha(k)-\varepsilon}), \end{cases}$$

where $\alpha(k)$ grows like $\log k$, and the 'big omega' notation means $\gamma_{\mathbf{b}}(x) > x^{\alpha(k)-\varepsilon}$ for all x in some unbounded subset of the positive reals.

Zagier was able to directly compute an asymptotic formula for $\gamma_{(a,b)}(x)$. For the k-branch Euclid trees with $k > 2$, our approach is much more convoluted.

We begin by defining another Euclid-like tree $\mathfrak{E}'_k\{\mathbf{r}\}$ rooted at $\mathbf{r}$ and generated by the branching operations

$$T_j:(a_1,\dots,a_k)\mapsto((a_j,\, a_1 + a_j,\dots,\widehat{a_j + a_j},\dots, a_k + a_j).$$

$\mathfrak{E}'_k\{\mathbf{1}\}$ and $\mathfrak{E}_k\{\mathbf{1}\}$ are related by the linear map

$$L:\mathfrak{E}'_k\{\mathbf{1}\} \to \mathfrak{E}_k\{\mathbf{1}\}$$
$$(L\mathbf{a})_j = (a_1 + \cdots + a_k) - (k-1)a_{k+1-j},$$

and in general, $\mathfrak{E}'_k\{\mathbf{r}\}$ is mapped by L to $\mathfrak{E}_k\{L(\mathbf{r})\}$. Note that

$$\begin{aligned}(a_1 + \cdots + a_k) &= \sum_{j=1}^{k} ((a_1 + \cdots + a_k) - (k-1)a_{k+1-j}) \\ &= A_1 + \cdots + A_k\end{aligned}$$

where $\mathbf{A} = L\mathbf{a}$. Hence counting the elements in $\mathfrak{E}'_k\{\mathbf{r}\}$ with height less than x is the same as counting the elements in $\mathfrak{E}_k\{L(\mathbf{r})\}$ with height less than x. The tree $\mathfrak{E}'_k\{\mathbf{r}\}$ is easier to deal with because it has the classical two branch Euclid tree naturally imbedded in it.

We now define a different sort of counting function:

$$f_{\mathbf{r}}(t) = \sum_{\mathbf{a}\in\mathfrak{E}'_k\{\mathbf{r}\}} \frac{1}{(h(\mathbf{a}))^t} = \sum_{\mathbf{a}\in\mathfrak{E}_k(L(\mathbf{r}))} \frac{1}{(h(\mathbf{a}))^t}.$$

Our goal is to show that $f_{\mathbf{r}}(t)$ converges for all $t>\alpha$ and diverges for all $t<\alpha$. The following result will help us determine when f converges:

LEMMA 2.2. *If $f_{\mathbf{r}}(t)$ converges at t, then $f_{c\mathbf{r}}(t)$ converges at t and*

$$f_{c\mathbf{r}}(t) = \frac{1}{c^t} f_{\mathbf{r}}(t)$$

for all $c>0$. If $r_i \geqslant s_i$ for all i, and $f_{\mathbf{s}}(t)$ converges at t, then $f_{\mathbf{r}}(t)$ converges at t and

$$f_{\mathbf{r}}(t) \leqslant f_{\mathbf{s}}(t).$$

The proofs just require node by node comparisons of the trees in question.

COROLLARY 2.3. *For every $\mathbf{r} \in (\mathbb{R}^+)^k$, $f_{\mathbf{r}}(t)$ converges at t if and only if $f_{\mathbf{1}}(t)$ converges at t.*

Proof. Without loss of generality, we may set $0 < r_1 \leqslant \cdots \leqslant r_k$. Then

$$f_{r_k \mathbf{1}}(t) \leqslant f_{\mathbf{r}}(t) \leqslant f_{r_1 \mathbf{1}}(t)$$
$$\frac{1}{r_k^t} f_{\mathbf{1}}(t) \leqslant f_{\mathbf{r}}(t) \leqslant \frac{1}{r_1^t} f_{\mathbf{1}}(t),$$

from which the result follows.

Consequently, we may regard $\alpha = \alpha(k)$ as depending only on the dimension of $\mathbf{r}$. We may also fix $\mathbf{r}$, and choose $\mathbf{r} = (1, 1, \ldots, 1) = \mathbf{1}$. As is implied by our notation, $\alpha(k)$ is in fact the limsup defined in Theorem 2.1. This result will be shown in Theorems 2.8 and 2.9. Let us first find bounds for $\alpha(k)$:

THEOREM 2.4 *$f_{\mathbf{1}}(t)$ converges for all $t > \alpha(k)$ and diverges for all $t < \alpha(k)$ where $\alpha(k)$ satisfies*

$$\frac{1}{k(k-1)(k-2)} \leqslant Z(\alpha(k)) \leqslant \frac{2^{\alpha(k)}}{k(k-1)(k-2)}$$
$$Z(\alpha) = \frac{1}{2} \sum_{m=5}^{\infty} \frac{\phi(m) - 2}{(m-1)^{\alpha}},$$

and ϕ is the Euler phi function.

As mentioned before, the property that makes $\mathfrak{E}_k'\{\mathbf{1}\}$ nice to work with is that it has the 2-branch Euclid tree (which, with a little abuse of notation, we also call $\mathfrak{E}_2$) naturally imbedded in it:

$$\mathfrak{E}_2 = \{(W, W(1, 2, \ldots, 2)) : W \in \langle T_1, T_2 \rangle\}$$

and we have a very good description of the elements $W(1, 2, \ldots, 2) \in \mathfrak{E}_2$. Since we start at $(1, 2, \ldots, 2)$, $\mathfrak{E}_2$ has the property that if $W(1, 2, \ldots, 2) = W'(1, 2, \ldots, 2)$, then $W = W'$. So each node $(W, W(1, 2, \ldots, 2))$ is completely characterized by $W(1, 2, \ldots, 2)$. Also, we know:

LEMMA 2.5. *$\mathbf{a} = W(1, 2, \ldots, 2)$ for some $W \in \langle T_1, T_2 \rangle$ if and only if a_1 and a_2 are relatively prime, $a_1 < a_2$, and*

$$a_j = a_1 + a_2 - 1$$

for all $j \geq 3$.

Proof. Only the last needs some justification. We can show $a_j = a_1 + a_2 - 1$ using induction, or by noting that $a_3 = \cdots = a_k$ and

$$\begin{aligned} L\mathbf{a} &= (LWL^{-1})L\mathbf{1} \\ &= W'\mathbf{1} \end{aligned}$$

where $W' \in \langle LT_1L^{-1}, LT_2L^{-1} \rangle = \langle E_1, E_2 \rangle$. But E_1 and E_2 leave the first $k-2$ components of $\mathbf{c}$ fixed for every $\mathbf{c}$. Hence

$$\begin{aligned} (L\mathbf{a})_j &= L(\mathbf{1})_j \\ (a_1 + \cdots + a_k) - (k-1)a_j &= k - (k-1) \\ a_1 + a_2 + (k-2)a_j - (k-1)a_j &= 1 \\ a_j &= a_1 + a_2 - 1, \end{aligned}$$

for $j \geq 3$.

Hence we can write

$$\mathfrak{C}_2 = \{(a, b, c, \ldots, c) : (a, b) = 1, a < b, c = a + b - 1\}.$$

Also imbedded in $\mathfrak{C}_2$ is the trivial one branch tree

$$\mathfrak{C}_1 = \{(1, b, b, \ldots, b) : b \in \mathbb{Z}\}.$$

Note that $T_j(1, 1, \ldots, 1) = (1, 2, \ldots, 2)$ for all j; for any $\mathbf{a} \in \mathfrak{C}_1$, $T_j\mathbf{a} = T_2\mathbf{a} \in \mathfrak{C}_2$ for all $j \geq 2$; and for any $\mathbf{a} \in \mathfrak{C}_2$, $T_j\mathbf{a} = T_3\mathbf{a}$ for all $j \geq 3$. Thus, we can write

$$f_1(t) = k(k-1)(k-2) \sum_{\mathbf{a} \in \mathfrak{C}_2 \setminus \mathfrak{C}_1} f_{T_3\mathbf{a}}(t) + k(k-1) \sum_{\mathbf{a} \in \mathfrak{C}_2 \setminus \mathfrak{C}_1} (h(\mathbf{a}))^{-t} + k \sum_{\mathbf{a} \in \mathfrak{C}_1}$$

Note also that if $\mathbf{a} = (a, b, c, \ldots, c) \in \mathfrak{C}_2$, then

$$T_3\mathbf{a} = (c, a + c, b + c, 2c, \ldots, 2c)$$

so

$$f_{T_3\mathbf{a}}(t) \geqslant f_{2c1}(t) = (2(a+b-1))^{-t} f_1(t).$$

Thus,

$$f_1(t) \geqslant k(k-1)(k-2) \sum_{\substack{(a,b)=1 \\ 1 \neq a < b}} \frac{1}{(2(a+b-1))^t} f_1(t)$$
$$\geqslant k(k-1)(k-2) \sum_{n=5}^{\infty} \frac{\phi(n)-2}{2^{t+1}(n-1)^t} f_1(t).$$

So, if $t > \alpha(k)$, then the series converges and we must have

$$\frac{1}{k(k-1)(k-2)} \geqslant \sum_{n=5}^{\infty} \frac{\phi(n)-2}{2^{t+1}(n-1)^t} = 2^{-t} Z(t)$$

and with a little help from a computer, this gives the lower bounds on $\alpha(k)$. Note that this also shows $\alpha(k) > 2$ for all $k \geqslant 3$—a result we will need later.

We find the upper bounds in a similar fashion, but unfortunately must contend with an 'error' term. To deal with this term, we want $f_1(t)$ to be large, so we exploit the divergence for $f_1(t)$ with $t < \alpha(k)$, and instead consider the partial sums

$$f_{\mathbf{r}}(t, y) = \sum_{\substack{\mathbf{a} \in \mathfrak{C}_{\mathbf{r}} \\ h(\mathbf{a}) < y}} h(\mathbf{a})^{-t}.$$

These partial sums have properties similar to those outlined in Lemma 2.2:

LEMMA 2.6. *For $c > 0$,*

$$f_{c\mathbf{r}}(t, y) = \frac{1}{c^t} f_{\mathbf{r}}(t, y/c)$$

and if $r_i \geqslant s_i$ for all i, then

$$f_{\mathbf{r}}(t, y) \leqslant f_{\mathbf{s}}(t, y).$$

We can decompose the partial sum $f_1(t, y)$ the same way we decomposed the convergent infinite sum $f_1(t)$:

$$f_1(t, y) = k(k-1)(k-2) \sum_{\mathbf{a} \in \mathfrak{C}_2 \setminus \mathfrak{C}_1} f_{T_3\mathbf{a}}(t, y) + k(k-1) \sum_{\substack{\mathbf{a} \in \mathfrak{C}_2 \setminus \mathfrak{C}_1 \\ h(\mathbf{a}) \leqslant y}} (h(\mathbf{a}))^{-t}$$
$$+ k \sum_{\substack{\mathbf{a} \in \mathfrak{C}_1 \\ h(\mathbf{a}) \leqslant y}} (h(\mathbf{a}))^{-t}.$$

This time we note

$$f_{T_3\mathbf{a}}(t, y) \leqslant f_{c1}(t, y) = c^{-t} f_1(t, y/c) \leqslant c^{-t} f_1(t, y)$$

so

$$f_1(t, y) \leqslant k(k-1)(k-2) \sum_{\mathbf{a} \in \mathfrak{C}_2 \setminus \mathfrak{C}_1} c^{-t} f_1(t, y) + k(k-1) \sum_{\mathbf{a} \in \mathfrak{C}_2} (h(\mathbf{a}))^{-t}.$$

The second term on the right-hand side is what we have referred to as the 'error' term. We note that

$$\begin{aligned}\sum_{\mathbf{a} \in \mathfrak{C}_2} (h(\mathbf{a}))^{-t} &= \sum_{\substack{(a,b)=1,\, a<b \\ c=a+b-1}} ((k-1)c+1)^{-t} \\ &= \sum_{n=1}^{\infty} \frac{\phi(n)}{((k-1)(n-1)+1)^t} \\ &\leqslant \sum_{n=1}^{\infty} \frac{\phi(n)}{n^t} = \frac{\zeta(t-1)}{\zeta(t)}\end{aligned}$$

where $\zeta(t)$ is the Reimann Zeta function. Thus, this term converges for all $t > 2$. Since $\alpha(k) > 2$ for all $k \geqslant 3$, we may choose t so that $2 < t < \alpha(k)$. Then, after dividing through by $f_1(t, y)$ and letting y go to infinity, we get

$$1 \leqslant k(k-1)(k-2)Z(t),$$

which gives us our upper bounds for $\alpha(k)$.

We now show $\alpha(k)$ grows like $\log k$:

THEOREM 2.7.

$$\frac{\log k}{\log 2} + O(k^{-1}) < \alpha(k) < \frac{3 \log k}{2 \log 2} + O(k^{-.3}).$$

Proof.

$$Z(t) = \frac{1}{2} \sum_{n=5}^{\infty} \frac{\phi(n) - 2}{(n-1)^t}$$

$$> \frac{1}{4^t}.$$

Hence

$$\frac{1}{4^{\alpha(k)}} < Z(\alpha(k)) < \frac{2^{\alpha(k)}}{k(k-1)(k-2)}$$

$$\alpha(k) \log 8 > 3 \log k + \log\left(1 - \frac{3}{k} + \frac{2}{k^2}\right)$$

$$\alpha(k) > \frac{\log k}{\log 2} + O\left(\frac{1}{k}\right).$$

Since $\phi(n) \leqslant n - 1$, we have for $t > 2$

$$Z(t) < \frac{1}{4^t} + \frac{1}{2} \sum_{n=7}^{\infty} \frac{1}{(n-1)^{t-1}}$$

$$< \frac{1}{4^t} + \frac{1}{2} \int_5^{\infty} \frac{dx}{x^{t-1}}$$

$$< \frac{1}{4^t} + O\left(\frac{1}{t5^t}\right)$$

and

$$\frac{1}{k(k-1)(k-2)} < Z(\alpha(k)) < \frac{1}{4^{\alpha(k)}} + O\left(\frac{1}{5^{\alpha(k)}}\right)$$

$$3 \log k > \alpha(k) \log 4 - \log\left(1 + O\left(\left(\tfrac{4}{5}\right)^{\alpha(k)}\right)\right)$$

and using the inequality established above,

$$\alpha(k) < \frac{3 \log k}{\log 4} + O\left(\left(\tfrac{4}{5}\right)^{2 \log k/\log 4}\right)$$

$$< \frac{3 \log k}{2 \log 2} + O\left(k^{2(1 - \log 5/\log 4)}\right).$$

We are now ready to relate these results to the function $\gamma_{\mathbf{r}}(x)$:

THEOREM 2.8. *Suppose* $\mathbf{r} \in (\mathbb{R}^+)^k$. *Then for every* $\varepsilon > 0$, *there exists a* $C(\varepsilon)$ *such that*

$$\gamma_{\mathbf{r}}(x) < x^{\alpha(k)+\varepsilon}$$

for every $x > C(\varepsilon)$.

Proof. Suppose not. Then there exists a sequence $\{x_n\}_{n=1}^{\infty}$ that goes to infinity, and has the property that

$$\gamma_{\mathbf{r}}(x_n) > x_n^{\alpha(k)+\varepsilon}$$

for every n. But then

$$\begin{aligned} f_{L^{-1}(\mathbf{r})}(t) &= \sum_{\mathbf{a} \in \mathfrak{C}_{\mathbf{r}}} \frac{1}{(h(\mathbf{a}))^t} \\ &> \frac{x_n^{\alpha(k)+\varepsilon}}{x_n^t}, \end{aligned}$$

for every n. So, choose $\alpha(k) < t < \alpha(k) + \varepsilon$. Then $f_{L^{-1}(\mathbf{r})}(t)$ converges, but the right-hand side diverges as x_n approaches infinity – a contradiction. Thus, if x is sufficiently large, then

$$\gamma_{\mathbf{r}}(x) < x^{\alpha(k)+\varepsilon}.$$

THEOREM 2.9. *Suppose* $\mathbf{r} \in (\mathbb{R}^+)^k$. *Then for every* $\varepsilon > 0$, *there exists an unbounded set* X *such that*

$$\gamma_{\mathbf{r}}(x) > x^{\alpha(k)-\varepsilon}$$

for all x *in* X.

Proof. Suppose not. Then there exists a sequence $\{x_n\}_{n=1}^{\infty}$ and a constant $c > 0$ such that $0 < (x_{n+1} - x_n) < c$; x_n approaches infinity as n goes to infinity; and

$$\gamma_{\mathbf{r}}(x_n) < x_n^{\alpha(k)-\varepsilon}$$

for every n. So set $\alpha(k) - \varepsilon < t < \alpha(k)$ and consider $f_{L^{-1}(\mathbf{r})}(t)$, which we know diverges at t. But

$$
\begin{aligned}
f_{L^{-1}(\mathbf{r})}(t) &= \sum_{\mathbf{a}\in\mathfrak{E}_1} \frac{1}{(h(\mathbf{a}))^t} \\
&< \frac{\gamma_{\mathbf{r}}(x_1)}{x_0^t} + \sum_{n=1}^{\infty} \frac{\gamma_{\mathbf{r}}(x_{n+1}) - \gamma_{\mathbf{r}}(x_n)}{x_n^t} \\
&< \sum_{n=1}^{\infty} \gamma_{\mathbf{r}}(x_n)\left(\frac{1}{x_{n-1}^t} - \frac{1}{x_n^t}\right),
\end{aligned} \tag{2.1}
$$

where $x_0 = h(\mathbf{r})$. We can justify the reordering in the infinite sum, since the difference of the nth partial sums is

$$
\begin{aligned}
\frac{\gamma_{\mathbf{r}}(x_{n+1})}{x_n^t} &< \frac{(x_n + c)^{\alpha(k)-\varepsilon}}{x_n^t} \\
&< x_n^{\alpha(k)-\varepsilon-t}\left(1 + \frac{c}{x_n}\right)^{\alpha(k)-\varepsilon},
\end{aligned}
$$

which goes to zero as x_n goes to infinity for $t > \alpha(k) - \varepsilon$.

Continuing with (2.1):

$$
\begin{aligned}
f_{L^{-1}(\mathbf{r})}(t) &< g(t) + \sum_{n=2}^{\infty} x_n^{\alpha(k)-\varepsilon}\left(\frac{x_n^t - x_{n-1}^t}{x_n^t x_n^t\left(1 - \frac{c}{x_n}\right)^t}\right) \\
&< g(t) + \sum_{n=2}^{\infty} x_n^{(\alpha(k)-\varepsilon-2t)}\left(\frac{(x_n - x_{n-1})t x_n^{t-1}}{\left(1 - \frac{c}{x_n}\right)^t}\right),
\end{aligned}
$$

where $g(t)$ is the first term, and hence converges for all t. Since $x_2 > c$, we know $(1 - c/x_n)$ is bounded away from zero, and since $t < \alpha(k)$, we can replace it with a constant, so there exists a constant C such that

$$
f_{L^{-1}(\mathbf{r})}(t) < g(t) + C\sum_{n=2}^{\infty} x_n^{(\alpha(k)-\varepsilon-t-1)}
$$

which converges if $t > \alpha(k) - \varepsilon$, a contradiction. Hence, we could not have had these 'fence posts' of inequalities, and there exists an unbounded set X such that

$$
\gamma_{\mathbf{r}}(x) > x^{\alpha(k)-\varepsilon}
$$

for all x in X.

COROLLARY 2.10 (*of Theorems* 2.8 *and* 2.9).

$$\limsup_{x\to\infty} \frac{\log(\gamma_{\mathbf{b}}(x))}{\log x} = \alpha(k).$$

Gathering Theorems 2.4, 2.7 and Corollary 2.10 gives us Theorem 2.1. Using Theorem 1.4, the remarks following Theorems 1.3 and 1.5 and Theorem 2.1, our main result follows:

THEOREM 2.11. *If $M_{a,n}$ has a non-trivial integral solution, then for every $\varepsilon > 0$,*

$$g_{a,n}(H) = \begin{cases} O((\log H)^{\alpha(n-1)+\varepsilon}) \\ \Omega((\log H)^{\alpha(n-1)-\varepsilon}), \end{cases}$$

where the exponent $\alpha(n-1)$ depends only on n, and satisfies

$$\frac{\log(n)}{\log 2} + o(1) < \alpha(n-1) < \frac{3 \log(n)}{2 \log 2} + o(1),$$

and for small n.

$$2.248 < \alpha(3) < 2.637$$

$$2.524 < \alpha(4) < 3.172$$

$$2.767 < \alpha(5) < 3.611.$$

The constants and subset implied by the 'big oh' and 'big omega' depend on ε, a and n.

References

[B1] A. Baragar: *The Markoff Equation and Equations of Hurwitz*, Ph.D. Thesis, Brown University, 1991.

[B2] A. Baragar: Integral solutions of Markoff-Hurwitz equations, *J. Number Theory* (to appear).

[Ca] J. W. S. Cassels: *An Introduction to Diophantine Approximation*, Chap. II, Cambridge Univ. Press, Cambridge, 1957.

[Co] H. Cohn: Growth types of Fibonacci and Markoff, *Fibonacci Quart.* 17(2) (1979) 178–183.

[He] N. P. Herzberg: On a problem of Hurwitz, *Pac. J. Math.* 50 (1974) 485–493.

[Hu] A. Hurwitz: Über eine aufgabe der unmbestimmten analysis, *Archiv. Math. Phys.* 3 (1907) 185–196; *Mathematisch Werke*, Vol. 2, Chap. LXX (1933 and 1962) 410–421.

[Ma] A. A. Markoff: Sur les formes binaires indéfinies, *Math. Ann.* 17 (1880) 379–399.

[Mo1] L. J. Mordell: On the integer solutions of the equation $x^2+y^2+z^2+2xyz=n$, *J. London Math. Soc.* 28 (1953) 500–510.

[Mo2] L. J. Mordell: *Diophantine Equations*, Chap. 13.4. Academic Press, New York, London, 1969, pp. 106–110.

[S] J. H. Silverman: The Markoff equation $x^2+y^2+z^2=axyz$ over quadratic imaginary fields, *J. Number Theory* 35(1) (1990) 72–104.

[Z] D. Zagier: On the number of Markoff numbers below a given bound, *Math. Comp.* 39 (1982) 709–723.