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Generic simple coverings of the affine plane

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0. Introduction

In [1, 2, 4, 5, 6] ideas of Deligne are used to prove the factoriality of the surface \( \mathbb{Z}^p = f(X, Y) \) for a generic choice of polynomial \( f(X, Y) \) of arbitrary degree \( \geq 4 \) (with \( p \geq 3 \)). In this paper we study the class group of surface \( \mathbb{Z}^n = f(X, Y) \) for arbitrary positive integer \( n \).

The above mentioned calculation leads us naturally to conjecture that the class group of \( \mathbb{Z}^n = f(X, Y) \) is factorial for a generic choice of \( f \). To be more precise, let \( f = \sum T_{ij}X^iY^j \) be a generic polynomial with indeterminate coefficients and let \( A_n = K\langle X, Y, Z \rangle / (Z^n - f) \) where \( K \) is the algebraic closure of \( \mathbb{F}_p(T_{ij}) \) with \( \mathbb{F}_p \) the prime field of \( p \) elements (\( p \geq 3 \)). Assume the degree of \( f \) is at least 4. Then we conjecture

0.1. For all \( n \in \mathbb{Z}^+ \), \( A_n \) is factorial.

In this paper we prove that (0.1) reduces to the case \( \gcd(p, n) = 1 \). We feel that this latter case can be approached by adapting a theorem of Steenbrink [9] from characteristic 0 to characteristic \( p \) by systematically replacing singular cohomology by étale cohomology; a project we are currently working on.

In Sections 1 and 2, descent techniques are used to study the class group of arbitrary surfaces \( \mathbb{Z}^n = f \). Two main results proved are (2.16), which reduces (0.1) to the case \( n = pm \) where \( \gcd(p, m) = 1 \), and (2.5), which shows that if (0.1) is true for some \( n \), then it is true for all divisors of \( n \).

In Section 3 the reduction of (0.1) to the case \( \gcd(p, n) = 1 \) is accomplished by analyzing the action of \( G = \text{Gal}(K, \mathbb{F}_p(T_{ij})) \) on the divisor class group of \( \mathbb{Z}^{pm} = f \) (3.8).
1. Galois descent

1.1. NOTATION. If \( R \) is a commutative ring with unity and \( P \) is a prime ideal of \( R \), denote the residue field of \( R \) at \( P \) by \( k(P) = R_P/PR_P \).

If \( R \) is a Krull domain, let \( \text{Cl}(R) \) denote the divisor class group of \( R \) as defined in P. Samuel’s Tata notes [7] (also see [3]).

1.2. DISCUSSION. This section makes use of Galois descent techniques and the next section employs radical descent methods. Suppose \( G \) is a finite group of automorphisms acting on a Krull domain \( B \) and \( A \) is the fixed subring of \( B \). Denote the multiplicative set of units in \( B \) and \( A \) by \( B^* \) and \( A^* \), respectively. Since \( G \) is a finite group, the ring \( B \) is integral over \( A \). The inclusion \( A \to B \) induces a homomorphism \( \varphi: \text{Cl}(A) \to \text{Cl}(B) \) by the following theorem.

1.3. THEOREM. Let \( A \subseteq B \) be Krull rings with \( B \) integral over \( A \) or with \( B \) flat as an \( A \)-module. Then there is a well defined group homomorphism \( \varphi: \text{Cl}(A) \to \text{Cl}(B) \) such that for each height one prime \( P \) of \( A \)

\[
\varphi(P) = \sum_{P'} e(P', P)P'
\]

where the \( P' \) are the prime ideals of \( B \) lying over \( P \) and \( e(P', P) \) is the ramification index of \( P' \) over \( P \) ([7], pp. 19–20).

1.4. THEOREM. Let \( A \) and \( B \) be as in (1.2). Then \( \varphi \) induces an injection \( 0: \ker \varphi \to H^1(G, B^*) \). If every prime divisorial ideal of \( B \) is unramified over \( A \), then \( 0 \) is a bijection ([7], p. 55).

1.5. REMARK. If \( G \) in (1.2) is a finite cyclic group generated by an element \( \pi \), then \( H^1(G, B^*) \) is the homology of the complex \( B^* \xrightarrow{h} B^* \xrightarrow{N} A^* \) where \( h(x) = \pi(x)/x \) for \( x \in B^* \) and \( N \) is the norm on \( B^* \) ([7], p. 57).

1.6. LEMMA. Assume in (1.2) that \( G \) is cyclic of order \( n \) and \( B \) is a unique factorization domain. Assume that for each prime element \( b \in B \) either

(i) \( \pi^s(b)B \neq \pi^t(b)B \) whenever \( s \neq t \) (mod \( n \)), or

(ii) \( b \in A \).

Then \( H^1(G, B^*) = 0 \).

Proof. By (1.5) \( H^1(G, B^*) \) is the homology of the complex \( B^* \xrightarrow{h} B^* \xrightarrow{N} A^* \). Assume \( u \) is a unit in \( B \) and \( N(u) = 1 \). Let \( L \) denote the field of fractions of \( B \). Each element of \( L^* \) can be written as a fraction \( b/a \) where \( b \in B, a \in A \). Then by Hilbert’s Theorem 90 there exists \( x \in B \) such that \( h(x) = u. x \) can be written as a product \( x = wb_1^{e_1} \cdots b_r^{e_r} \) where \( w \in B^* \), the \( b_i \) are prime elements in \( B \) and \( e_i \in \mathbb{Z}^+ \), \( 1 \leq i \leq r \).
Note that since $\pi(x) = ux$, if $\pi(b)B = b_jB$, then $\pi(b_j)$ multiplied by a unit must appear in the prime factorization of $x$ in $B$ with the same exponent as $b_j$. Therefore, in order to show that $u \in h(B^*)$ we may reduce to the case $x = wb\pi(b) \cdots \pi^{m-1}(b)$ where $m$ is the smallest positive integer such that $\pi^m(b)B = bB$. By hypothesis either $b \in A$ and $m = 1$, or $m = n$, in which case $x = wN(b)$. In either case $u = \pi(x)/x = \pi(w)/w$, so that $u$ is a boundary.

1.7. LEMMA. Assume in (1.2) that $G$ is cyclic of order $n$ and $B$ is a unique factorization domain. Assume for each prime element $b \in B$ either $[k(bB) : k(bB \cap A)] = 1$ or $b \in A$. Assume also that $B$ is unramified over $A$. Then $H^1(G, B^*) = 0$.

Proof. Let $b$ be a prime element of $B$ and $b \not\in A$. Then by hypothesis there are exactly $n$ height one primes of $B$ lying over $bB \cap A$ and each of them is generated by a conjugate of $b$. Thus $b$ satisfies condition (i) of (1.6).

1.8. NOTATION. If $E$ is a field, $A = E[X_1, \ldots, X_n]$ is the polynomial ring in $s$ variables over $E$ and $h \neq 0$ is an element of $A$, let $\deg(h)$ denote the degree of $h$ and $h^+$ the highest degree form of $h$. If $g \neq 0$ also belongs to $A$ define $\deg(h/g) = \deg(h) - \deg(g)$.

1.9. ASSUMPTIONS. Throughout $K$ will be an algebraically closed field of characteristic $p \geq 3$. Assume $f \in K[X, Y]$ is an irreducible polynomial in two variables $X, Y$ of degree at least 4. We will assume that $\partial f/\partial X$ and $\partial f/\partial Y$ meet transversally and in the maximum possible number of points of $K^2$. This number is $(\deg f - 1)^2$ if $\deg f \not\equiv 0 \pmod{p}$ and $(\deg f)^2 - 3 \deg f + 3$ otherwise (see [5, pp. 287–288]). Implicit in these assumptions is the fact that $f^+ \notin K[X^p, Y^p]$. We remark that a generic $f$ of degree at least 4 satisfies the conditions stated above.

For each $n \in \mathbb{Z}^+$, let $A_n = K[X, Y, Z]/(Z^n - f)$ and $E_n$ denote the field of fractions of $A_n$. Let $x, y, z$ denote the images of $X, Y, Z$ in $A_n$. Then the subring of $K[x, y]$ of $A_n$ is isomorphic to $K[X, Y]$.

Let $W_n = \text{Spec}(A_n)$. Since $W_n$ has only finitely many singular points, $A_n$ is noetherian integrally closed and hence a Krull ring.

1.10. LEMMA. Assume $n \in \mathbb{Z}^+$ and $\text{Cl}(A_n) = 0$. Then $\text{Cl}(A_m) = 0$ for all $m \in \mathbb{Z}^+$ such that $m$ divides $n$ and $\gcd(p, n/m) = 1$.

Proof. It’s enough to prove the case $n = mq$ where $q$ is a prime number. Let $c \in K$ be a primitive $q$-th root of unity and let $\pi$ be the $K(X, Y)$-automorphism on $K(X, Y, Z)$ defined by $\pi(Z) = cZ$. Then $\pi$ induces an automorphism on $A_n$. Let $G$ be the cyclic group generated by $\pi$ and $A$ be the fixed subring of $A_n$. Then $A = K[x, y, z^n] \cong A_m$.

Let $b$ a prime element of $A_n$. Then $b$ can be written $b = \sum_{i=0}^{q-1} a_i z^i$ for unique $a_i \in A$. Since $[E_n : E_m] = q$, $[k(bA_n) : k(bA_n \cap A)] = 1$ unless $a_i = 0$ for
Since $f$ is irreducible in $K[X, Y]$, $z$ is a prime element in $A_n$. Since $A_n\left[\frac{1}{z}\right]$ is unramified over $A_n\left[\frac{1}{z}z^q\right]$, we obtain by (1.7) that $H^1(G, A_n\left[\frac{1}{z}\right]^*) = 0$. By (1.3) and (1.4) it follows that $\text{Cl}(A_m\left[\frac{1}{z}\right]) = 0$, which by Nagata's lemma implies $\text{Cl}(A_m) = 0$.

2. Radical descent

2.1. DISCUSSION. Let $B$ be a Krull ring of characteristic $p \neq 0$, and let $L$ be its quotient field. Let $\Delta$ be a derivation of $L$ such that $\Delta(B) \subset B$. Let $L' = \ker(\Delta)$ and $A = L' \cap B$. Then $A$ is a Krull ring and $B$ is integral over $A$ since $B^p \subset A \subset B$. By (1.3) there is a well defined group homomorphism $\varphi: \text{Cl}(A) \to \text{Cl}(B)$.

Set $L' = \{t^{-1} \Delta t: t^{-1} \Delta t \in B, t \in L^*\}$ and $L^* = \{u^{-1} \Delta u: u \in B^*\}$. Then $L'$ is an additive subgroup of $B$ and $L'^*$ is a subgroup of $L^*$.

2.2. THEOREM. (a) There exists a canonical monomorphism $0: \ker\varphi \to L'/L'^*$. (b) If $[L: L'] = p$ and if $\Delta(B)$ is not contained in any height one prime of $B$, then $0$ is an isomorphism ([7], p. 62).

2.3. PROPOSITION. If $[L: L'] = p$ in (2.1) then there exists $a \in A$ such that $\Delta^p = a \Delta$ ([7], p. 63).

2.4. PROPOSITION. If $[L: L'] = p$ in (2.1), then an element $x \in L$ is logarithmic derivative (i.e., $x = t^{-1} \Delta t$ for some $t \in L$) if and only if $\Delta^{p-1} x = ax + x^p = 0$, where $\Delta^p = a \Delta$ ([7], p. 64).

2.5. PROPOSITION. Assume $n \in \mathbb{Z}^+$ and $\text{Cl}(A_n) = 0$. Then $\text{Cl}(A_m) = 0$ for all positive divisors $m$ of $n$.

Proof. It's enough to prove the case $n = mq$ where $q$ is a prime number. The case $\gcd(p, q) = 1$ is (1.10). Thus we are left with the case $n = mp$.

The derivation $d = \partial/\partial Z$ defines a derivation on $A_n$ with kernel $K[x, y, z^p] \cong A_m$. By (2.2) $\text{Cl}(A_m) \cong L'/L''$, where $L = \{u^{-1} du: u \in E_n$ and $u^{-1} du \in A_n\}$ and $L'' = \{u^{-1} du: u \in A_n^*\}$. Let $t \in L\setminus\{0\}$. We have $t = \sum_{i=0}^{n-1} t_i z^i$ for unique $t_i \in k[x, y]$. By (2.4) $d^{p-1} t = -t^p$. If we compare coefficients of $z^{(r-1)p}$ on both sides of this equality, we obtain for each $r = 1, 2, \ldots, m$,

$$t_{r, p-1} = \sum_{j=0}^{p-1} t_{r, p-1-jm} z^{nj}.$$ (2.5.1)

Since $z^n = f$, we have for each $r = 1, 2, \ldots, m$,
Choose s such that \(\deg(t_{sp-1}) \geq \deg(t_{rp-1})\) for each \(r\). \(t_{sp-1}\) appears on the right side of one of the equations in (2.5.2). Let \(t_{up-1}\) be the element on the left side of this equation. Since \(1, f^+, \ldots, (f^+)^{p-1}\) are independent over \(K(X^p, Y^p)\), \(\deg t_{sp-1} \geq \deg(t_{up-1}) \geq \deg(t_{sp-1}) > p \deg(t_{sp-1})\), which is impossible. Therefore \(\mathcal{L} = 0\). \(\square\)

The next proposition follows easily by (2.2), (2.3) and (2.4). Details are provided in [5]. Also see the proof of (2.13).

2.6. PROPOSITION. Let \(D\) be the derivation on \(K(X, Y)\) defined by

\[
D = \frac{\partial f}{\partial Y} \frac{\partial}{\partial X} - \frac{\partial f}{\partial X} \frac{\partial}{\partial Y}.
\]

(a) \(\ker D \cap K[X, Y] = K[X^p, Y^p, f]\);

(b) \(A_p\) is isomorphic to \(K[X^p, Y^p, f]\);

(c) \(\text{Cl}(A_p)\) is isomorphic to \(\mathcal{L}_0 = \{u^{-1}Du : u \in K(X, Y)\}\) and \(u^{-1}Du \in K[X, Y]\);

(d) There exists \(a_0 \in K[X^p, Y^p, f]\) such that \(D^p = a_0 D\) and \(\deg(a_0) \leq (p - 1)(\deg(f) - 2)\) ([5], pp. 616–622).

2.7. THEOREM. Let \(\Phi\) be an algebraically closed field of characteristic \(p \neq 0\). Let \(g \in \Phi[X, Y]\), \(D = g_x \frac{\partial}{\partial Y} - g_y \frac{\partial}{\partial X}\) and \(a\) be such that \(D^p = aD\). Let \(Q \in \Phi^2\) be such that \(g_x(Q) = g_y(Q) = 0\) and \(\sqrt{H(Q)}\) a root of \(T^2 = H(Q)\), where \(H = g_{xx}g_{yy} - g_{xy}^2\). Then \(a(Q) = (\sqrt{H(Q)})^{p-1}\) (see [4, Theorem 1.5]).

2.8. NOTATION. Let \(S = \{Q \in K^2 : f_x(Q) = f_y(Q) = 0\}\).

2.9. LEMMA. If \(t \in K[X, Y]\), then \(\{Q \in S : t(Q) = 0\}\) has less than or equal to \(\deg(t) \cdot (\deg(f) - 1)\) elements.

Proof. Let \(t = t_1^{i_1} \cdots t_s^{i_s}\) be the prime factorization of \(t\) in \(K[X, Y]\). Since \(f_x\) and \(f_y\) have no common factors, \(t_i\) is relatively prime to either \(f_x\) or \(f_y\), \(1 \leq i \leq s\). By Bezout's Theorem [8] the number of points \(Q \in S\) such that \(t_i(Q) = 0\) is at most \((\deg t_i)(\deg f - 1)\). It then follows that the number of \(Q \in S\) such that \(t(Q) = 0\) is at most \((\sum \deg t_i) \cdot (\deg f - 1) \leq \deg(t)(\deg f - 1)\). \(\square\)

2.10. LEMMA. If \(t \in K[X, Y]\) and \(t(Q) = 0\) for each \(Q \in S\), then either \(t = 0\) or \(\deg t > \deg f - 2\).

Proof. Assume \(t \neq 0\) and \(\deg t \leq \deg f - 2\). By (2.9), the number of points \(Q \in S\) such that \(t(Q) = 0\) is at most \((\deg f - 2)(\deg f - 1)\). By (1.9),
there is at least one point \( Q \in S \) such that \( t(Q) \neq 0 \). \( \square \)

2.11. LEMMA. Assume \( a_0 \in K[X^p, Y^p, f] \) is such that \( D^p = a_0 D \). If \( t \in K[X, Y] \), \( \deg t \leq \deg f - 2 \) and \( D^{p-1} t - a_0 t = 0 \), then \( t = 0 \).

Proof. Given \( Q \in S \), \( (D^{p-1} t)(Q) = 0 \) and \( a_0(Q) \neq 0 \) by (1.9) and (2.7) (recall that \( \partial f / \partial X \) and \( \partial f / \partial Y \) meet transversally at \( Q \)). Therefore \( t(Q) = 0 \).

By (2.10) we obtain \( t = 0 \). \( \square \)

2.12. NOTATION. The derivation \( D \) on \( K(X, Y) \) extends to a derivation on \( K(X, Y, Z) \) with \( Z^n - f \) in its kernel. Thus \( D \) induces a derivation on \( E_n \) which we denote by \( D_n \). \( L_n \) will denote the additive group of logarithmic derivatives of \( D_n \) in \( A_n \), \( L'_n = \{ u^{-1} D_n u : u \in E_n \text{ and } u^{-1} D_n u \in A_n \} \). \( L_n \) will denote the subgroup of \( L_n \) of logarithmic derivatives of units in \( A_n \).

2.13. PROPOSITION. (a) \( A_{np} \) is isomorphic to \( \ker D_n \cap A_n \); (b) there is a well defined group homomorphism \( \varphi_n : Cl(A_{np}) \to Cl(A_n) \) with \( \ker \varphi_n = L_n / L'_n \).

Proof. \( \ker D_n \cap A_n \cong K[x^p, y^p, z] \), the latter is clearly isomorphic to \( K[X, Y, Z]/(Z^{np} - f^{(p)}) \), where \( f^{(p)} \) is obtained from \( f \) by raising each coefficient of \( f \) to the \( p \)-th power. Since \( K \) is perfect, the automorphism \( \alpha \to \alpha^p \) of \( K \) induces an isomorphism \( A_{np} \to K[x^p, y^p, z] \). It follows that \( K[x^p, y^p, z] \) is integrally closed. Since \( [E_n : K(x^p, y^p, z)] = p \), \( \ker D_n \cap A_n \) and \( K[x^p, y^p, z] \) have the same field of fractions. Since \( \ker D_n \cap A_n \) is integral over \( K[x^p, y^p, z] \), we obtain (a). (b) is an immediate consequence of (a) and (2.2). \( \square \)

2.14. PROPOSITION. Let \( t = \sum_{i=0}^{n-1} t_i z^i \in A_n \), where \( t_i \in K[x, y] \), \( 0 \leq i < n \). For each \( i = 0, 1, \ldots, n - 1 \), let \( J(i) = \{ j : 0 \leq j < n \text{ and } p j \equiv i \pmod{n} \} \). Then \( t \in L_n \) if and only if for each \( i = 0, 1, \ldots, n - 1 \),

\[
D^{p-1} t_i - a_0 t_i = - \sum_{j \in J(i)} t_j^p f^{(p j - i)/n},
\]

where \( a_0 \) is such that \( D^p = a_0 D \).

Proof. By (2.4), \( t \in L_n \) if and only if \( D^{p-1} t - a_0 t = - t^p \); which holds if and only if \( \sum (D^{p-1} t_i - a_0 t_i) z^i = - \sum t_j^p z^{ip} \). Since \( 1, z, \ldots, z^{n-1} \), is a basis for \( E_n \) over \( K(x, y) \) and since \( Z^n = f \) we obtain the desired result by comparing powers of \( z \) on both sides of the above equation.

2.15. LEMMA. Let \( t = \sum_{i=0}^{n-1} t_i z^i \in A_n \), where \( t_i \in K[x, y] \), \( 0 \leq i < n \). If \( t \in L_n \), then \( \deg t_i \leq \deg f - 2 \) for each \( i \).

Proof. Let \( r \) be such that \( \deg t_r \geq \deg t_i \) for each \( i \). We consider two cases.

Case 1. \( \gcd(p, n) = 1 \).

We have \( pr = nq + s \) for \( q, s \in \mathbb{Z} \) with \( q \geq 0 \), \( 0 \leq s < n \). By (2.14),
\(D_{p-1}^n t_s - a_0 t_s = -t_p f^a\). By (2.6), \(\deg a_0 \leq (\deg f - 2)(p - 1)\). A simple induction shows that \(\deg(D_{p-1}^n t_s) \leq \deg t_s + (\deg f - 2)(p - 1)\). Thus \(p \deg t_r \leq \deg(D_{p-1}^n t_s - a_0 t_s) \leq \deg t_s + (\deg f - 2)(p - 1) \leq \deg t_r + (\deg f - 2)(p - 1)\). Hence \(\deg t_r \leq \deg f - 2\).

**Case 2.** \(p | n\).

Again \(p \tau = nq + s\) as in Case 1. By (2.14),

\[
D_{p-1}^n t_s - a_0 t_s = -\sum_{j \in J(s)} t_j^p f^{(pj - s)/n}\]

Since \(p\) divides \(n\) and each \(j \in J(s)\) is less than \(n\), the integers \((pj - s)/n\) are distinct modulo \(n\). Since \(f^+ \notin K(x^n, y^n)\) by (1.9) and since \(r \in J(s)\) it follows

\[
\deg t_r = \deg(t_r^p) \leq \deg(t_r^p f^a) \leq \deg(\sum t_j^p f^{(pj - s)/n})
\]

\[
= \deg(D_{p-1}^n t_s - a_0 t_s) \leq \deg t_s + (\deg f - 2)(p - 1)
\]

\[
\leq \deg t_r + (\deg f - 2)(p - 1).
\]

Hence \(\deg t_r \leq \deg f - 2\). □

2.16. **THEOREM.** Let \(m \in \mathbb{Z}^+\) such that \(\gcd(p, m) = 1\). If \(\Cl(A_{pm}) = 0\) then \(\Cl(A_{pm}) = 0\) for all \(r \geq 0\).

**Proof.** The case \(r = 0\) follows by (2.5). The case \(r = 1\) is by hypothesis. To prove the remaining cases we need to establish the below claim.

**CLAIM.** If \(p\) divides \(n\), then the composition \(A_{n/p} \rightarrow K[x, y, z^n] \hookrightarrow A_n\) maps \(\mathcal{L}_{n/p}\) isomorphically onto \(\mathcal{L}_n\).

**Proof of Claim.** Let \(t = \sum_{i=0}^{n-1} t_i z^i \in \mathcal{L}_n\), where \(t_i \in K[x, y]\) and \(n = p^n m\). Since \(s \geq 1\), we have that if \(\gcd(i, p) = 1\), then by (2.14), \(D_{p-1}^n t_i - a_0 t_i = 0\); which by (2.11) and (2.15) implies \(t_i = 0\). Thus \(t \in K[x, y, z^n] \cong A_{n/p}\). Therefore the isomorphism that maps \(A_{n/p}\) onto \(K[x, y, z^n]\) maps \(\mathcal{L}_{n/p}\) onto \(\mathcal{L}_n\).

Now \(\Cl(A_{pm}) = 0\) and (2.13) imply \(\mathcal{L}_m/\mathcal{L}_m' = 0\). Then the claim shows that \(\mathcal{L}_{p^rm}/\mathcal{L}_{p^rm} = 0\) for all \(r \geq 1\). The remaining cases of the theorem follow by (2.13) and a simple induction. □

2.17. **PROPOSITION.** The kernel of \(\phi_n: \Cl(A_{np}) \rightarrow \Cl(A_n)\) is finite \(p\)-group of type \((p, \ldots, p)\) of order \(p^M\), where \(M \leq \deg f(\deg f - 1)/2\).

**Proof.** By (2.13) we need only show that \(\mathcal{L}_n\) has the stated properties. By the claim in the proof of (2.16) we may reduce to the case \(\gcd(p, n) = 1\).

Let \(t = \sum_{i=0}^{n-1} t_i z^i \in \mathcal{L}_n\), where \(t_i \in K[x, y]\), \(0 \leq i < n\). By (2.15), each \(t_i = \sum \alpha^{(i)}_{rs} x^r y^s\) where each \(\alpha^{(i)}_{rs} \in K\) and \(\deg t_i \leq \deg f - 2\). \(pi = nq + j\) for
$q, j \in \mathbb{Z}$, $q \geq 0$, $0 \leq j < n$. $\gcd(p, n) = 1$ implies $J(i) = \{i\}$; which by (2.14) yields

$$D^{p-1} t_j - a_0 t_j = -t^p f^q.$$  

(2.17.1)

Comparing the coefficients of $x^e y^a z^b$ on both sides of (2.17.1) we see that for each triple of nonnegative integers $(e, a, b)$ with $e < n$ and $a + b \leq \deg f - 2$, $\alpha_{ab}^i$ must satisfy an equation of the form

$$L_{(e, a, b)} = (\alpha_{ab}^i)^p.$$  

(2.17.2)

where $L_{ab}$ is a linear expression in the $\alpha_{ab}^i$ with coefficients in $K$. There are a total of $n \deg f (\deg f - 1)/2$ such equations. The ring $R = K[\ldots, \alpha_{rs}^i, \ldots]$ with these relations is a finite dimensional $K$-vector space spanned by all monomials in the $\alpha_{rs}^i$ of degree $(p - 1)n \deg f (\deg f - 1)/2$. This shows $R$ is Artinian and has a finite number of maximal ideals. Thus the equations in (2.17.2) have only a finite number of solutions in $K$, which by Bezout's theorem [8, p. 198] is at most $p^n \deg f (\deg f - 1)/2$.

Since $\mathcal{L}_n \subset K[x, y, z]$, each element of $\mathcal{L}_n$ has $p$-torsion.

2.18. REMARK. Our main objective is to reduce conjecture (0.1) to the case $\gcd(p, n) = 1$. Theorem (2.16) allows us to reduce to the case $n = pm$ where $\gcd(p, m) = 1$. In the next section we use results concerning $\text{Gal}(K(T_{ij})/K(T_{ij}))$ to complete the project. Proposition (2.5) gives us some flexibility when attempting (0.1). For example, we may reduce (0.1) to the case $n \equiv 1 \pmod{p}$.

3. The action of the Galois group

3.1. NOTATION. In this section $\mathbb{F}_p$ is the prime field of characteristic $p > 3$, $T_{ij}$ are indeterminates algebraically independent over $\mathbb{F}_p$ where $0 \leq i + j \leq M$ with $M$ a positive integer greater than or equal to 4. We denote the following:

$$f = \Sigma T_{ij} X^i Y^j$$

$$H = f_{xx}f_{yy} - f_{xy}^2$$

the hessian of $f$,

$$K = \overline{\mathbb{F}_p}(T_{ij}),$$

the algebraic closure of $\mathbb{F}_p(T_{ij})$,

$$\mathcal{G} = \text{Gal}(K, \mathbb{F}_p(T_{ij})),$$

$$S = \{Q \in K^2: f_X(Q) = f_Y(Q) = 0\},$$
For $n \in \mathbb{Z}^+$, let $\mathcal{S}_n = \{ (\alpha, \beta, \gamma) \in K^3 : (\alpha, \beta) \in S \text{ and } \gamma^n = f(\alpha, \beta) \}$.

In [1, 4] it is shown that $\mathcal{S}$ has the maximum possible number of elements as described in (1.9). Let $Q_1, \ldots, Q_I$ be a listing of the elements of $\mathcal{S}$. Then we can list the elements of $\mathcal{S}_n$ as $Q_{ij}$, where if $Q_{ij} = (\alpha, \beta, \gamma)$, then $(\alpha, \beta) = Q_i$. Finally, for each $i$, let $\sqrt{H(Q_i)}$ denote a fixed root of the equation $T^2 = H(Q_i)$.

The next two theorems are proved in [2] and [4].

3.2. **THEOREM.** $\mathcal{G}$ acts on $\mathcal{S}$ as the full symmetric group (see [4, p. 353] and [2, p. 296]).

3.3. **THEOREM.** For every pair $Q_i \neq Q_j \in \mathcal{S}$, there exists $\sigma \in \mathcal{G}$ such that $\sigma$ acts as the identity on $\mathcal{S}$, and

$$
\sigma(\sqrt{H(Q_e)}) = \begin{cases} 
-\sqrt{H(Q_e)}, & \text{if } e = i, j \\
\sqrt{H(Q_e)}, & \text{otherwise}.
\end{cases}
$$

([4, p. 354] and [2, p. 297]).

3.4. **REMARK.** Assume $n \in \mathbb{Z}^+$ such that $\gcd(p, n) = 1$. Let $c \in K$ be a primitive $n$-th root of unity. Let $\pi$ be the $K(X, Y)$-automorphism on $K(X, Y, Z)$ defined by $\pi(Z) = cZ$. Then $\pi$ induces an automorphism on $A_n$ and let $T : A_n \to K[x, y]$ denote the trace map.

Since the points $Q_{ij} \in \mathcal{S}_n$ lie on the surface $z^n = f$, we may define $t(Q_{ij})$ for $t \in A_n$ by evaluating any preimage of $t$ in $K[X, Y, Z]$ at $Q_{ij}$. Observe that if for a fixed $i$, $t(Q_{ij}) = 0$ for all $j$, then for each $j$, $T(t)(Q_{ij}) = 0$, which yields $T(t)(Q_i) = 0$.

3.5. **LEMMA.** Assume $\gcd(p, n) = 1$ and $t = \sum_{r=0}^{n-1} t_r z^r \in A_n$. If for a fixed $i$, $t(Q_{ij}) = 0$ for each $j$, then $t_r(Q_i) = 0$ for each $r = 0, 1, \ldots, n - 1$.

Proof. It is well known that $f(Q_i) \neq 0$ for each $i$ (it also follows by (3.2)). Let $s$ be a nonnegative integer less than $n$. Then $t(Q_{ij}) = 0$ for each $j$ implies $z^{n-s} t(Q_{ij}) = 0$ for each $j$. As we saw in (3.4) we obtain $T(z^{n-s} t)(Q_i) = n z^n t_s(Q_i) = n f(Q_i) t_s(Q_i) = 0$; hence $t_s(Q_i) = 0$.

3.6. **LEMMA.** Assume $\gcd(p, n) = 1$. For each $t \in \mathcal{L}_n$ and $Q_i \in \mathcal{S}$, there is an $r_{ij} \in \mathbb{F}_p$ such that $t(Q_{ij}) = r_{ij} \sqrt{H(Q_i)}$. Furthermore, the map

$$
\Phi : \mathcal{L} \to \bigoplus_{i,j} \mathbb{F}_p : \sqrt{H(Q_{ij})}
$$

defined by $\Phi(t) = (t(Q_{ij}))$ is an injection of groups.

Proof. Given $t \in \mathcal{L}_n$, $D_n^{-1} t - a_0 t = -t^p$ where $a_0 \in K[x^n, y^p, f]$ such that $D^p = a_0 D$ by (2.4). Evaluate both sides of this equality at $Q_{ij}$ to obtain
Now use (2.7) to obtain the first statement of the lemma.

Write \( t = \sum_s t_s z^s \cdot \Phi(t) = 0 \) implies \( t_s(Q_i) = 0 \) for each \( i \) by (3.5). By (2.10) and (2.15), each \( t_s = 0 \).

3.7. THEOREM. Assume \( \gcd(p, n) = 1 \). Then the map \( \text{Cl}(A_{np}) \to \text{Cl}(A_n) \) is an injection.

Proof. By (2.13) it’s enough to show \( L_n = 0 \). Let \( t \in L_n \) and suppose \( t \neq 0 \). Assume \( \Phi(t) = (r_{ij} \sqrt{H(Q_i)}) \). If \( \sigma \in \mathcal{G} \) then \( \sigma(t) \in L_n \) and the action of \( \sigma \) on \( t \) is compatible with the action of \( \sigma \) on \( \Phi(t) \). By (3.2) we may assume that \( r_{11} \neq 0 \). By (3.3), there is \( \sigma', \sigma'' \in \mathcal{G} \) such that

\[
\sigma'\left(\sqrt{H(Q_i)}\right) = \begin{cases} -\sqrt{H(Q_i)}, & i = 1, 2 \\ \sqrt{H(Q_i)}, & \text{otherwise} \end{cases} \\
\sigma''\left(\sqrt{H(Q_i)}\right) = \begin{cases} -\sqrt{H(Q_i)}, & i = 1, 3 \\ \sqrt{H(Q_i)}, & \text{otherwise} \end{cases}
\]

Then \( \hat{t} = t - \sigma'(t) - \sigma''(t) + \sigma''\sigma'(t) \in L_n \) and has the property that \( \hat{t}(Q_{ij}) = 0 \) for all \( i \geq 2, 0 \leq j < n \), and \( \hat{t} \neq 0 \) since the first coordinate of \( \Phi(\hat{t}) \) is \( 4r_{11}\sqrt{H(Q_1)} \neq 0 \).

We have \( \hat{t} = \sum_{s=0}^{n-1} t_s z^s \), where \( t_s \in K[x, y] \), \( 0 \leq s < n \). By (3.5) \( t_s(Q_i) = 0 \) for each \( s \) and each \( i \geq 2 \). We now show that this implies each \( t_s = 0 \); thus obtaining a contradiction.

If \( \deg f \equiv 0 \pmod{p} \), then \( S \) has \( (\deg f - 1)^2 \) distinct points. By (2.15), \( \deg t_s \leq \deg f - 2 \). If \( t_s \neq 0 \) then \( t_s(Q) = 0 \) at most \( (\deg f - 2)(\deg f - 2) \) points \( Q \in S \) by (2.9). Hence \( t_s = 0 \).

The case \( \deg f \equiv 0 \pmod{p} \) requires a bit more effort. For each \( s = 1, \ldots, n - 1 \), let \( m(s) \) be the smallest positive integer \( m \) such that \( p^m s > n \). We proceed by induction to show that \( t_s = 0 \).

If \( m = 1 \), then \( ps = nq + r \) where \( q, r \in \mathbb{Z}^+, r < n \). By (2.14) \( D^{p-1} t_r - a_0 t_r = -t_r f^q \). The degree of the left side of the equality is at most \( p(\deg f - 2) \) by (2.6) and (2.15). Since \( q \geq 1 \), we obtain \( \deg t_s \leq \deg f - 3 \). By (2.9) and the fact that \( S \) has \( (\deg f)^2 - 3 \deg f + 3 \) points, we have \( t_s = 0 \).

Assume that \( t_s = 0 \) whenever \( m(s) < d \) and \( 1 \leq s_0 < n \) with \( m(s_0) = d \geq 2 \). By (2.14), \( D^{p-1} t_{ps_0} - a_0 t_{ps_0} = -t_{ps_0} \). Since \( m(ps_0) = m(s_0) - 1 \), \( t_{ps_0} = 0 \); hence \( t_{s_0} = 0 \). From this it follows that \( \hat{t} = t_0 \in K[x, y] \). In the introduction we mentioned that \( \text{Cl}(A_p) = 0 \) for a generic \( g \) of degree \( \geq 4 \), which shows \( t_0 = 0 \) by (2.13).

\[ \square \]
3.8. THEOREM. For a generic $f$ of degree at least 4 the following two statements are equivalent:

1. $\text{Cl}(A_n) = 0$ for all $n \in \mathbb{Z}^+$;
2. $\text{Cl}(A_n) = 0$ for all $n \in \mathbb{Z}^+$ where $\gcd(p, n) = 1$.

Proof. By (2.16) and (3.7).

4. References

5. Lang, J., The divisor class group of the surface $x^p = G(x, y)$ over fields of characteristic $p > 0$, J. Alg., 84 (1983).