COMPOSITIO MATHEMATICA

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Compositio Mathematica, tome 91, nº 3 (1994), p. 305-319
<http://www.numdam.org/item?id=CM_1994__91_3_305_0>

Compositio Mathematica **91**: 305–319, 1994.

On the Betti numbers of perfect ideals

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Received 22 October 1992; accepted in final form 14 May 1993

Let V be a non degenerate arithmetically Cohen-Macaulay projective variety of codimension n and degree e in $\mathbf{P}^r$. Let $S := k[X_0, \dots, X_r]$, I the defining ideal of V in S and $A := S/I$ the homogeneous coordinate ring of V.

The S-module A has a minimal graded free resolution

$$0 \to \bigoplus_{j=1}^{b_n} S(-d_{nj}) \to \cdots \to \bigoplus_{j=1}^{b_1} S(-d_{1j}) \to S \to A \to 0.$$

The integers b_i are called the *Betti numbers* of V, often but improperly the Betti numbers of I.

These integers are very important numerical invariants of the embedding of V in $\mathbf{P}^r$; thus for example b_1 is just the minimal number of generators of I and b_n is the so-called Cohen-Macaulay type of V, a number which measures how far is V from being arithmetically Gorenstein.

In this paper we find upper bounds for the b_i's when: (i) V ranges over the class of arithmetically Cohen-Macaulay non degenerate projective varieties of a given codimension and degree and when: (ii) V ranges over the class of arithmetically Cohen-Macaulay non degenerate projective varieties of a given codimension, degree and initial degree.

All the bounds we obtain in this paper are sharp since we show them to be attained by a set of distinct points in $\mathbf{P}^n$ which have maximal Hilbert function with respect to the given data.

The results of this paper are the completion of the work started in [ERV] and taken up in [EGV] where the problem was solved for b_1 and b_n respectively.

For some time now these results were purely conjecture. Now the crucial work done by Eliahou and Kervaire in [EK] and the upper bounds for the Betti numbers in the resolution of a standard graded algebra having a given Hilbert function, proved independently by Bigatti and Hulett in [B] and [H], gave us the possibility to prove them.

The crucial part of the proof is a formula giving the Betti numbers of a zero dimensional lex-segment ideal in terms of the multiplicities of certain hyper-

The author was partially supported by M.P.I. (Italy).

plane sections. For this we need notations and results from the theory of lex-segments ideals which can be found in [ERV], [EK] and [B].

Let $R = k[X_1, \dots, X_n]$ be a polynomial ring over the field k. For every homogeneous ideal I of R, the Hilbert function of the standard graded algebra $A := R/I = \oplus_{t \geqslant 0} A_t$, is the numerical function defined by

$$H_A(t) := \dim_k(A_t).$$

A classical result of Macaulay (see [M]) describes among all the numerical functions, those which are the Hilbert function of a standard k-algebra. To explain this we recall that if p and q are positive integers, then p can be uniquely written as

$$p = \binom{p(q)}{q} + \binom{p(q-1)}{q-1} + \cdots + \binom{p(j)}{j}$$

where

$$p(q) > p(q-1) > \cdots > p(j) \geqslant j \geqslant 1.$$

This is called the *q-binomial expansion of p*.

We define for every positive integers p and q:

$$p^{\langle q \rangle} := \binom{p(q)+1}{q+1} + \binom{p(q-1)+1}{q} + \cdots + \binom{p(j)+1}{j+1}$$

and

$$p_{\langle q \rangle} := \binom{p(q)-1}{q} + \binom{p(q-1)-1}{q-1} + \cdots + \binom{p(j)-1}{j}.$$

Notice that the first of these formulas is the $(q+1)$-expansion of $p^{\langle q \rangle}$ while the second is the q expansion of $p_{\langle q \rangle}$ if and only if $p(j) > j$.

Macaulay proved that given a numerical function

$$H: \mathbf{N} \to \mathbf{N},$$

such that $H(0) = 1$, this is the Hilbert function of a standard k-algebra A if and only if

$$H(t+1) \leqslant H(t)^{\langle t \rangle}$$

for every $t \geqslant 1$.

Macaulay gave also a way to produce an algebra A with a given permissible numerical function H. For this we need the notion of *lex-segment ideal.*

Let T_R be the monoid of terms in $X_1, \ldots, X_n$; we consider in T_R the total ordering defined as follows and called the degree-lexicographic order. We say that

$$X_1^{a_1} \cdots X_n^{a_n} > X_1^{b_1} \cdots X_n^{b_n}$$

if the first non zero coordinate of the vector $(\Sigma_i(a_i - b_i), a_1 - b_1, \ldots, a_n - b_n)$ is positive.

For instance if $n = 3$ the terms of degree 3 of R are ordered as follows:

$$X_1^3 > X_1^2X_2 > X_1^2X_3 > X_1X_2^2 > X_1X_2X_3 > X_1X_3^2 > X_2^3 \\ > X_2^2X_3 > X_2X_3^2 > X_3^3$$

and any term of degree three is smaller than any term of degree $\geqslant 4$.

It makes sense, therefore, to talk about a lex-segment as a sequence of terms of the same degree, which, along with a term u, contains any term v such that $v \geqslant u$.

For example

$$X_1^3, X_1^2X_2, X_1^2X_3, X_1X_2^2, X_1X_2X_3$$

is a lex segment in R_3.

DEFINITION 1. Let $I = \Sigma_{t \geqslant 0} I_t$ be a graded ideal of R. We say that I is a lex-segment ideal if for every $t \geqslant 0$, I_t is generated, as a k-vector space, by a lex-segment.

Given a numerical function

$$H: \mathbf{N} \to \mathbf{N}$$

such that $H(0) = 1$ and $H(t + 1) \leqslant H(t)^{\langle t \rangle}$ for every $t \geqslant 1$, if for every $d \geqslant 0$ we delete the smallest $H(d)$ terms in R_d, the remaining monomials form a k-vector base for a monomial ideal which is a lex-segment ideal (see [M] and [S]). This ideal is uniquely determined by the given numerical function H and is called the lex-segment ideal associated to H.

Now, for any monomial ideal I in R, we can consider its canonical minimal system of generators. It is the set of all monomials in I which are not proper multiples of any monomial in I. We shall denote this generator system $M(I)$. Of course $M(I)$ is a finite set.

For any monomial u in R, we denote by $\max(u)$ the largest index of the variables actually occurring in u. Thus if $u = X_1^{a_1} \cdots X_n^{a_n}$, then

$$\max(u) = \max\{i \mid a_i > 0\}.$$

As in [EK] we say that a monomial ideal is *stable* if for every monomial $w \in I$ and for every positive integer $i < m = \max(w)$, we have $X_i w / X_m \in I$.

It is clear that if $i < m$,

$$X_i \frac{w}{X_m} \geqslant w,$$

so that every lex-segment ideal is stable.

For stable monomial ideals Eliahou and Kervaire found a minimal free resolution, from which one easily computes the following Betti numbers (see [EK], p. 16):

$$b_i = \sum_{u \in M(I)} \binom{\max(u) - 1}{i - 1}.$$

Now for every zero dimensional lex-segment ideal I in R, and for every integer $j = 0, \ldots, n-1$ we denote by I_j the image of I in the polynomial ring

$$R_j := k[X_1, \ldots, X_j]$$

under the canonical projection. Hence

$$I_j = (I + (X_{j+1}, \ldots, X_n))/(X_{j+1}, \ldots, X_n).$$

It is clear that $I_0 = (0)$; further a minimal system of generators of I_j is obtained by considering the monomials $u \in M(I)$ such that $\max(u) \leqslant j$.

Also it is not difficult to see that I_j is still a lex-segment ideal. The relationship between the numerical invariants of I and those of I_j has been studied deeply in [ERV]. Here we collect what we need in the sequel. If A is either a local ring or a graded k-algebra, we write $e(A)$ for the multiplicity of A. If I is an ideal of A, we write $v(I)$ for the minimal number of generators of I.

1. For every lex-segment ideal I of R we have

$$H_{R_{n-1}/I_{n-1}}(t) = (H_{R/I}(t))_{\langle t \rangle}$$

for every $t \geqslant 1$ (see [ERV], Corollary 2.8).

2. For every non degenerate lex-segment ideal I of R we have:

$$v(I) = v(I_{n-1}) + e(R_{n-1}/I_{n-1})$$

(see [ERV], Theorem 2.9).

With these notations and remarks we can prove now the following crucial formula where for simplicity we write $e(j)$ instead of $e(R_j/I_j)$.

PROPOSITION 2. *Let I be a zero dimensional non degenerate lex-segment ideal of R. Then, for every $i = 1,\dots,n$, we have*:

$$b_i(R/I) = \sum_{j=i-1}^{n-1} \binom{j}{i-1} e(j).$$

Proof. We have for every $i = 1,\dots,n$

$$b_i = \sum_{u\in M(I)} \binom{\max(u) - 1}{i-1} = \sum_{j=1}^{n} \binom{j-1}{i-1} \#\{u \in M(I) \mid \max(u) = j\},$$

where for a finite set X we denote by $\#(X)$ the number of elements of X. Since, as we remarked above, for every $j = 0,\dots,n-1$ we have:

$$v(I_j) = \#\{u \in M(I) \mid \max(u) \leqslant j\},$$

we get

$$\#\{u \in M(I) \mid \max(u) = j\} = v(I_j) - v(I_{j-1}).$$

Since for every $j \geqslant 1$, I_j is a lex-segment ideal with the same initial degree as I and $I_{j-1} = (I_j)_{j-1}$, we can apply the equality in 2, to get

$$\#\{u \in M(I) \mid \max(u) = j\} = v(I_j) - v(I_{j-1}) = e(j-1).$$

This implies

$$b_i = \sum_{j=i}^{n} \binom{j-1}{i-1} e(j-1) = \sum_{j=i-1}^{n-1} \binom{j}{i-1} e(j).$$

We remark that if we apply the above formula for $i = 1$ and $i = n$ we get

$$b_1 = \sum_{j=0}^{n-1} e(j)$$

$$b_n = e(n-1),$$

as proved in [ERV] and [EGV] respectively.

EXAMPLE. Let $n = 4$ and

$$\begin{aligned}I = (&X_1^3, X_1^2X_2, X_1^2X_3, X_1^2X_4, X_1X_2^2, X_1X_2X_3^2, X_1X_2X_3X_4, X_1X_2X_4^2, X_1X_3^3,\\ &X_1X_3^2X_4, X_1X_3X_4^2, X_1X_4^3, X_2^4, X_2^3X_3, X_2^3X_4, X_2^2X_3^2, X_2^2X_3X_4, X_2^2X_4^2,\\ &X_2X_3^3, X_2X_3^2X_4, X_2X_3X_4^2, X_2X_4^3, X_3^4, X_3^3X_4, X_3^2X_4^2, X_3X_4^4, X_4^6)\end{aligned}$$

be the lex-segment ideal corresponding to the zero dimensional Hilbert function

$$\{1, 4, 10, 15, 2, 1, 0, \dots\}.$$

Then we have $I_0 = (0)$, $I_1 = (X_1^3)$, $I_2 = (X_1^3, X_1^2X_2, X_1X_2^2, X_2^4)$, and finally

$$I_3 = (X_1^3, X_1^2X_2, X_1^2X_3, X_1X_2^2, X_1X_2X_3^2, X_1X_3^3, X_2^4, X_2^3X_3, X_2^2X_3^2, X_2X_3^3, X_3^4).$$

We easily get $e(0) = 1$, $e(1) = 3$, $e(2) = 7$ and $e(3) = 16$ so that $b_1 = 27$, $b_2 = 65$, $b_3 = 55$ and $b_4 = 16$.

We need now to recall the main result proved in [ERV]. Given the polynomial ring $R = k[X_1, \dots, X_n]$, if we fix a number $e \geqslant n + 1$, we can consider the zero dimensional Hilbert function of multiplicity e, maximal with respect to the rule given by Macaulay's criterion. If $t = t(e)$ is defined as the unique integer such that

$$\binom{n+t-1}{t-1} \leqslant e < \binom{n+t}{t},$$

the function is defined by the formula

$$H(p) = \begin{cases} \dbinom{n+p-1}{p} & \text{for } p \leqslant t - 1 \\[2ex] e - \dbinom{n+t-1}{t-1} & \text{for } p = t \\[2ex] 0 & \text{for } p > t. \end{cases}$$

With these notations we let $J(e)$ be the lex-segment ideal corresponding to this maximal Hilbert function.

In the same manner, given the integer e and defined as before the integer $t(e)$ which is certainly $\geqslant 2$, let i be any integer such that $2 \leqslant i \leqslant t(e)$.

Let us consider the maximal zero dimensional Hilbert function corresponding to the multiplicity e and the initial degree i. If $s = s(e, i)$ is defined as the unique integer such that

$$\binom{n+s-1}{s-1} - \binom{n+s-i-1}{s-i-1} \leqslant e < \binom{n+s}{s} - \binom{n+s-i}{s-i},$$

the function is defined by the formula

$$H(p) = \begin{cases} \binom{n+p-1}{p} & \text{for } 0 \leqslant p \leqslant i-1 \\ \binom{n+p-2}{p} + \binom{n+p-3}{p-1} + \cdots + \binom{n+p-i-1}{p-i+1} & \text{for } p = i, \ldots, s-1 \\ e - \binom{n+s-1}{s-1} + \binom{n+s-i-1}{s-i-1} & \text{for } p = s \\ 0 & \text{for } p > s. \end{cases}$$

It is easy to see that we have

$$H(j+1) = H(j)^{\langle j \rangle}$$

for every $j \leqslant s-2, j \neq i-1$. Moreover the multiplicity of this Hilbert function is exactly e.

With these notations we let $J(e, i)$ be the lex-segment ideal corresponding to this maximal Hilbert function.

The following result is the main tool for our paper. One can get it easily by connecting several statements in [ERV] (see Lemma 3.9, Lemma 4.1, Lemma 4.3 and the proof of Theorem 3.10).

THEOREM 3

1. *Given an integer $e \geqslant n+1$, for any zero dimensional lex-segment ideal I with $e(R/I) = e$, we have for every $j = 0, \ldots, n-1$:*

$$e(R_j/I_j) \leqslant e(R_j/J(e)_j).$$

2. *Given the integers e, i with $e \geqslant n+1, 2 \leqslant i \leqslant t(e)$, for any zero dimensional lex-segment ideal I with initial degree i and $e(R/I) = e$, we have for every $j = 0, \ldots, n-1$:*

$$e(R_j/I_j) \leqslant e(R_j/J(e, i)_j).$$

We can prove now the main result of the paper. We formulate it in the local version which is more general, not more difficult.

THEOREM 4

1. *Let α be a codimension n perfect ideal of the regular local ring (B, m) such that $\alpha \subseteq m^2$ and $e = e(B/\alpha)$. Then, for every $j = 1, \ldots, n$, we have*

$$b_j(B/\alpha) \leqslant b_j(R/J(e)).$$

2. *Let α be a codimension n perfect ideal of the regular local ring (B, m) such that $\alpha \subseteq m^2$ and $e = e(B/\alpha)$. If $i \geqslant 2$ is the initial degree of α, then, for every $j = 1, \ldots, n$,*

$$b_j(B/\alpha) \leqslant b_j(R/J(e, i)).$$

Proof. Let $d = \dim(B/\alpha)$. We know that there exists a minimal reduction $\underline{l} := l_1, \ldots, l_d$ of m modulo α, such that the initial degree of α is the same as the initial degree of $\bar{\alpha} := \alpha + (\underline{l})/(\underline{l})$ (see for example [EI]). The local ring $(\bar{B} := B/(\underline{l}), \bar{m} := m/(\underline{l}))$ is a regular local ring of dimension n. The associated graded ring of the artinian ring $\bar{B}/\bar{\alpha}$ is the artinian graded ring

$$gr_{\bar{m}/\bar{\alpha}}(\bar{B}/\bar{\alpha}) = k[X_1, \ldots, X_n]/I,$$

where I is the zero dimensional homogeneous ideal of $R := k[X_1, \ldots, X_n] = gr_{\bar{m}}(\bar{B})$, whose elements are the $\bar{m}$-initial forms of the elements of $\bar{\alpha}$.

It is clear that I, $\bar{\alpha}$ and α have the same initial degree. On the other hand since $\underline{l}$ is a regular sequence modulo α, we also have

$$e = e(B/\alpha) = e(\bar{B}/\bar{\alpha})$$

and

$$b_j(B/\alpha) = b_j(\bar{B}/\bar{\alpha}).$$

Further, by passing to the associated graded ring, the multiplicity does not change, while the Betti numbers can only increase (see for example [HRV], Corollary 3.2). Hence we get

$$e = e(B/\alpha) = e(R/I)$$

and for every $j = 1, \ldots, n$

$$b_j(B/\alpha) \leqslant b_j(R/I).$$

In particular, since $I \subseteq (X_1, \ldots, X_n)^2$, we have

$$e = \dim_k(R/I) \geqslant n + 1.$$

We use now the result of Bigatti-Hulett which says that all the Betti numbers of a given ideal are bounded above by the Betti numbers of the lex-segment ideal with the same Hilbert function (see [B] and [H]). Hence, if we denote by $L(I)$ the lex-segment ideal with the same Hilbert function as I, we get for every $j = 1, \ldots, n$

$$b_j(B/\alpha) \leqslant b_j(R/I) \leqslant b_j(R/L(I)).$$

Now, for short, let $J = J(e)$ in case (1) and $J = J(e, i)$ in case (2). Since the initial degree of J is certainly bigger or equal than the initial degree of $L(I)$ which is the same as that of α, we may apply Proposition 2 to the zero dimensional lex-segment ideals $L(I)$ and J. This and Theorem 3 gives, for every $j = 1, \ldots, n$:

$$\begin{aligned} b_j(B/\alpha) \leqslant b_j(R/L(I)) &= \sum_{k=j-1}^{n-1} \binom{k}{j-1} e(R_k/L(I)_k) \\ &\leqslant \sum_{k=j-1}^{n-1} \binom{k}{j-1} e(R_k/J_k) = b_j(R/J). \end{aligned}$$

This proves the theorem.

We can explicitly compute the Betti numbers of the ideal $J(e)$ and $J(e, i)$. We define $r_{\langle t \rangle(0)} := r$ and inductively

$$r_{\langle t \rangle(k)} = (r_{\langle t \rangle(k-1)})_{\langle t \rangle}.$$

In particular we have $r_{\langle t \rangle(1)} = r_{\langle t \rangle}$.

PROPOSITION 5

1. *Let e be an integer $e \geqslant n + 1$. Let $t = t(e)$ be the unique integer such that*

$$\binom{n+t-1}{t-1} \leqslant e < \binom{n+t}{t},$$

and let

$$r := e - \binom{n+t-1}{t-1}.$$

Then, for every $j = 1,\dots,n$, *we have*

$$b_j(R/J(e)) = \binom{t+j-2}{t-1}\binom{t+n-1}{j+t-1} + \sum_{k=j-1}^{n-1} \binom{k}{j-1} r_{\langle t\rangle(n-k)}.$$

2. *Let* e, i *be integers such that* $e \geqslant n+1$ *and* $2 \leqslant i \leqslant t(e)$. *Let* $s = s(e, i)$ *be the unique integer such that*

$$\binom{n+s-1}{s-1} - \binom{n+s-i-1}{s-i-1} \leqslant e < \binom{n+s}{s} - \binom{n+s-i}{s-i}$$

and let

$$r := e - \binom{n+s-1}{s-1} + \binom{n+s-i-1}{s-i-1}.$$

Then

$$b_1(R/J(e, i)) = 1 + \binom{n+s-1}{s} - \binom{n+s-i-1}{s-i} + r^{\langle s\rangle} - r$$

while for every $j = 2,\dots,n$, *we have*:

$$b_j(R/J(e, i)) = \binom{s+j-2}{s-1}\binom{s+n-1}{s+j-1} - \binom{s+j-i-2}{j-1}\binom{s+n-i-1}{s+j-i-1} + \sum_{k=j-1}^{n-1} \binom{k}{j-1} r_{\langle s\rangle(n-k)}.$$

Proof. We write J instead of $J(e)$. By using the definition of the Hilbert function of R/J and the remark before Proposition 2, we get for every $k = 0,\dots,n-1$,

$$e(R_k/J_k) = \sum_{p\geqslant 0} H_{R_k/J_k}(p) = 1 + \sum_{p=1}^{t-1} \binom{n+p-n+k-1}{p} + r_{\langle t\rangle(n-k)}$$
$$= 1 + \sum_{p=1}^{t-1} \binom{p+k-1}{p} + r_{\langle t\rangle(n-k)} = \binom{t-1+k}{k} + r_{\langle t\rangle(n-k)}.$$

Using Proposition 2 we get:

$$b_j(R/J) = \sum_{k=j-1}^{n-1} \binom{k}{j-1}\binom{t-1+k}{k} + \sum_{k=j-1}^{n-1} \binom{k}{j-1} r_{\langle t\rangle(n-k)}$$
$$= \binom{t+j-2}{t-1}\binom{n+t-1}{j+t-1} + \sum_{k=j-1}^{n-1} \binom{k}{j-1} r_{\langle t\rangle(n-k)},$$

where the last equality follows from the easy verified identity:

$$\sum_{a=0}^{b} \binom{a}{v}\binom{w+a-1}{a} = \binom{w+v-1}{v}\binom{w+b}{w+v}. \tag{$*$}$$

This gives the conclusion.

In case (2), let us write again J instead of $J(e, i)$. As before we have

$$e(R_0/J_0) = 1.$$

and for every $k = 1, \ldots, n-1$:

$$e(R_k/J_k) = \sum_{p \geq 0} H_{R_k/J_k}(p)$$
$$= 1 + \sum_{p=1}^{i-1} \binom{n+p-n+k-1}{p} + \sum_{p=i}^{s-1} \left(\sum_{m=p-i+1}^{p} \binom{n+m-n+k-2}{m} \right)$$
$$+ r_{\langle s\rangle(n-k)}$$
$$= \sum_{p=0}^{i-1} \binom{p+k-1}{p} + \sum_{p=i}^{s-1} \left(\sum_{m=p-i+1}^{p} \binom{m+k-2}{m} \right) + r_{\langle s\rangle(n-k)}$$
$$= \sum_{p=0}^{i-1} \binom{p+k-1}{p} + \sum_{p=i}^{s-1} \binom{p+k-1}{p} - \sum_{p=i}^{s-1} \binom{p-i+k-1}{p-i} + r_{\langle s\rangle(n-k)}$$
$$= \sum_{p=0}^{s-1} \binom{p+k-1}{p} - \sum_{h=0}^{s-1-i} \binom{h+k-1}{h} + r_{\langle s\rangle(n-k)}$$
$$= \binom{s-1+k}{s-1} - \binom{s-1-i+k}{s-i-1} + r_{\langle s\rangle(n-k)}$$
$$= \binom{s-1+k}{s-1} - \binom{s+k-i-1}{s-i-1} + r_{\langle s\rangle(n-k)}$$

Using Proposition 2 we get:

$$\begin{aligned}
b_1(R/J) &= \sum_{k=0}^{n-1} e(R_k/J_k) \\
&= 1 + \sum_{k=1}^{n-1} \binom{s+k-1}{s-1} - \sum_{k=1}^{n-1} \binom{s+k-i-1}{s-i-1} + \sum_{k=1}^{n-1} r_{\langle s\rangle(n-k)} \\
&= 1 + \binom{n-1+s}{s} - 1 - \binom{n-1+s-i}{s-i} + 1 + \sum_{k=1}^{n-1} r_{\langle s\rangle(n-k)} \\
&= 1 + \binom{n-1+s}{s} - \binom{n-1+s-i}{s-i} + r^{\langle s\rangle} - r
\end{aligned}$$

where we used the equality:

$$r_{\langle s\rangle} + r_{\langle s\rangle(2)} + \cdots + r_{\langle s\rangle(n-1)} = r^{\langle s\rangle} - r$$

proved in [ERV], Property 1.12.

If $j \geqslant 2$, we get:

$$\begin{aligned}
b_j(R/J) &= \sum_{k=j-1}^{n-1} \binom{k}{j-1}\binom{s+k-1}{k} - \sum_{k=j-1}^{n-1} \binom{k}{j-1}\binom{s+k-i-1}{k} \\
&\quad + \sum_{k=j-1}^{n-1} \binom{k}{j-1} r_{\langle s\rangle(n-k)} \\
&= \binom{s+j-2}{s-1}\binom{s+n-1}{s+j-1} - \binom{s+j-i-2}{j-1}\binom{s+n-i-1}{s+j-i-1} \\
&\quad + \sum_{k=j-1}^{n-1} \binom{k}{j-1} r_{\langle s\rangle(n-k)}
\end{aligned}$$

where we used twice the formula $(*)$. This gives the conclusion.

We remark that if we apply the above formulas for $j = n$ we get

$$b_n(R/J(e)) = \binom{t+n-2}{t-1} + r_{\langle t\rangle}$$

and

$$b_n(R/J(e, i)) = \binom{s+n-2}{s-1} - \binom{s+n-i-2}{n-1} + r_{\langle s\rangle}$$

which are the formulas proved in [EGV].

If we apply the above formulas for $j = 1$, we get, using again Property 1.12 in [ERV],

$$b_1(R/J(e)) = v(J(e)) = \binom{n+t-1}{t} + r^{\langle t\rangle} - r,$$

so that we recover the formulas proved in [ERV], Propositions 4.2 and 4.4.

We can compute for example the Betti numbers of $J(33)$ for $n = 4$ and we can compare them with the Betti numbers of the ideal given in the Example following Proposition 2.

We get $t = 3$ and $r = 18$. Hence

$$b_1 = \binom{4+3-1}{3} + 18^{\langle 3\rangle} - 18 = 30,$$

$$b_2 = \binom{3}{2}\binom{6}{4} + 18_{\langle 3\rangle(3)} + 2\cdot 18_{\langle 3\rangle(2)} + 3\cdot 18_{\langle 3\rangle} = 73,$$

$$b_3 = \binom{4}{2}\binom{6}{5} + 18_{\langle 3\rangle(2)} + 3\cdot 18_{\langle 3\rangle} = 62,$$

$$b_4 = \binom{5}{2}\binom{6}{6} + 18_{\langle 3\rangle} = 18.$$

These are upper bounds for the Betti numbers of every perfect non degenerate codimension four ideal with multiplicity 33.

REMARK 1. The bounds we found in this paper are sharp since they are attained by a monomial ideal. We remark that we can reach these bounds with radical ideals, in fact with ideals which define a zero dimensional reduced scheme in $\mathbf{P}^n$.

This can be seen by considering a result of Hartshorne which says that monomial ideals in $k[X_1,\ldots,X_n]$ can be lifted to ideals of distinct points in $\mathbf{P}^n$ with the same Betti numbers, the same multiplicity and the same initial degree (see for example [GGR]).

The way to lift a monomial ideal is very easy and can be described as follows. For every monomial, say

$$u = X_1^{a_1}X_2^{a_2}\cdots X_n^{a_n}$$

in $R = k[X_1,\ldots,X_n]$, we consider the monomial

$$l(u) = \prod_{j=1}^{n} X_j(X_j - X_0)(X_j - 2X_0)\cdots(X_j - (a_j - 1)X_0)$$

in $S = k[X_0,\ldots,X_n]$.

A lifting for the ideal I is the ideal of S generated by $l(u)$, u running among a set of minimal generators of I.

It is easy to see that this ideal is a radical ideal, thus defining a set of $e(R/I)$ distinct points in $\mathbf{P}^n$. For example, if we want to find 11 distinct points in $\mathbf{P}^3$ lying on a quadric, with the highest possible Betti numbers according to the main theorem, we can consider the maximal zero-dimensional Hilbert function in $R = k[X_1, X_2, X_3]$ with multiplicity 11 and initial degree 2. This is $\{1, 3, 5, 2, 0, \ldots\}$. The lex-segment ideal with this Hilbert function is

$$J = J(11, 2) = (X_1^2, X_1X_2^2, X_1X_2X_3, X_1X_3^2, X_2^3, X_2^2X_3, X_2X_3^2, X_3^4).$$

A lifting for J in $S = k[X_0, X_1, X_2, X_3]$ is the radical ideal

$$\begin{aligned} I = (&X_1(X_1 - X_0), X_1X_2(X_2 - X_0), X_1X_2X_3, X_1X_3(X_3 - X_0),\\ &X_2(X_2 - X_0)(X_3 - X_0), X_2(X_2 - X_0)X_3, X_2X_3(X_3 - X_0),\\ &X_3(X_3 - X_0)(X_3 - 2X_0)(X_3 - 3X_0)). \end{aligned}$$

This is the defining ideal of the following 11 points in $\mathbf{P}^3$:

$$\begin{aligned} &P_1 = (1, 0, 0, 0),\ P_2 = (1, 0, 0, 1),\ P_3 = (1, 0, 0, 2),\ P_4 = (1, 0, 0, 3),\\ &P_5 = (1, 0, 1, 0),\ P_6 = (1, 0, 1, 1),\ P_7 = (1, 0, 1, 2),\\ &P_8 = (1, 0, 2, 0),\ P_9 = (1, 1, 0, 0),\ P_{10} = (1, 1, 0, 1),\ P_{11} = (1, 1, 1, 0) \end{aligned}$$

This set of points has Betti numbers $b_1 = 8$, $b_2 = 12$ and $b_3 = 5$, the highest possible Betti numbers of an arithmetically Cohen-Macaulay non degenerate projective variety of codimension 3 and degree 11.

REMARK 2. The above theorem does not extend to ideals which are non perfect.

Let $R = k[X, Y]$ and $I = (X) \cap (X, Y)^n$. Then

$$b_1(R/I) = n,$$

and

$$b_2(R/I) = n - 1$$

while $e(R/I) = 1$ does not depend on n.

REMARK 3. In order to extend the results of this paper, we can ask for example the following question.

Can we find upper bounds for the Betti numbers of non degenerate perfect codimension three ideals of multiplicity 17 containing exactly two elements of degree two?

If we try to answer this question along the ideas of this paper, we should consider the zero-dimensional Hilbert function in $R = k[X_1, X_2, X_3]$ maximal with respect to the given data. Since $4 = \binom{3}{2} + \binom{1}{1}$, we have $4^{\langle 2 \rangle} = 5$, hence this function is $\{1, 3, 4, 5, 4, 0, \ldots\}$.

The corresponding lex-segment ideal is

$$J = (X_1^2, X_1X_2, X_1X_3^3, X_2^4, X_2^3X_3^2, X_2^2X_3^3, X_2X_3^4, X_3^5).$$

This ideal has Betti numbers $b_1 = 8$, $b_2 = 12$ and $b_3 = 5$. But the lex-segment ideal corresponding to the Hilbert function $\{1, 3, 4, 4, 5, 0, \ldots\}$ is the ideal

$$J' = (X_1^2, X_1X_2, X_1X_3^2, X_2^5, X_2^4X_3, X_2^3X_3^2, X_2^2X_3^3, X_2X_3^4, X_3^5).$$

This ideal has Betti numbers $b_1 = 9$, $b_2 = 14$ and $b_3 = 6$.

This proves that our result cannot be freely extended to a more general situation. One needs some technical assumption which we do not want to discuss here (see [ERV]).

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