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Convolution L -series

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1. Introduction

In the series of papers [4]–[7] we have developed several techniques for estimating the coefficients of L -functions which satisfy standard functional equations. Inspired by these works we now begin to examine convolution series formed by multiplying the coefficients.

Suppose we have two Dirichlet series

$$\mathcal{A}(s) = \sum_1^{\infty} a_n n^{-s}$$

$$\mathcal{B}(s) = \sum_1^{\infty} b_n n^{-s}$$

which converge absolutely in the half-plane $\operatorname{Re} s > 1$, which have analytic continuation to entire functions and which satisfy functional equations of the type

$$\mathcal{A}(1-s) = \Phi(s)\mathcal{A}(s)$$

$$\mathcal{B}(1-s) = \Psi(s)\mathcal{B}(s).$$

Here $\Phi(s)$ and $\Psi(s)$ are certain holomorphic functions in $\operatorname{Re} s > 0$ which have at most a polynomial growth on vertical lines. Furthermore we assume that $\mathcal{A}(s)$, $\mathcal{B}(s)$ have Euler products of degree k , l , i.e.,

$$\mathcal{A}(s) = \prod_p \mathcal{A}_p(p^{-s})^{-1}$$

$$\mathcal{B}(s) = \prod_p \mathcal{B}_p(p^{-s})^{-1}$$

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If $\operatorname{Re} s > 1$, where $\mathcal{A}_p(T), \mathcal{B}_p(T)$ are polynomials in T of degree k, l respectively with the constant term $\mathcal{A}_p(0) = \mathcal{B}_p(0) = 1$. We shall study the convolution series

$$C(s) = \sum_1^{\infty} a_n b_n n^{-s}.$$

Clearly, this convolution series converges absolutely in $\operatorname{Re} s > 2$. Our objective will be to prove the absolute convergence in $\operatorname{Re} s > 1$ and to establish the analytic continuation up to $\operatorname{Re} s > 1/2$ subject to some further natural conditions.

Motivating examples are the symmetric power L -functions attached to an automorphic form. These have been intensively studied along the lines of Langlands program (see the survey articles by F. Shahidi [8] and D. Bump [2]). In this context our approach to analytic continuation is more elementary but the results are not quite complete since we cannot reach the critical line $\operatorname{Re} s = 1/2$ and prove a functional equation for the convolution series $C(s)$. Nevertheless our applications illustrate what can be concluded directly from the existence of functional equations for Dirichlet series without appealing to automorphic theory.

As in [4]–[7] our approach requires suitable functional equations for the twisted series

$$\mathcal{A}(s, \chi) = \sum_1^{\infty} a_n \chi(n) n^{-s}$$

$$\mathcal{B}(s, \chi) = \sum_1^{\infty} b_n \chi(n) n^{-s}.$$

We assume that for any primitive character $\chi \pmod{q}$ the twisted series are entire functions of finite order and that they satisfy the compatible functional equations

$$\mathcal{A}(1-s, \chi) = \alpha_{\chi} q^{k(s-1/2)} \Phi_{\nu}(s) \mathcal{A}(s, \bar{\chi}) \quad (1)$$

$$\mathcal{B}(1-s, \chi) = \beta_{\chi} q^{l(s-1/2)} \Psi_{\nu}(s) \mathcal{B}(s, \bar{\chi}) \quad (2)$$

with $|\alpha_{\chi}| = |\beta_{\chi}| = 1$. Here we allow the factors $\Phi_{\nu}(s), \Psi_{\nu}(s)$ to depend on the parity index $\nu = \chi(-1) = \pm 1$ but not on χ itself. It is assumed that $\Phi_{\nu}(s), \Psi_{\nu}(s)$ are holomorphic in $\operatorname{Re} s > 0$ where they have a polynomial growth on vertical lines, i.e.,

$$\Phi_{\nu}(s), \Psi_{\nu}(s) \ll |s|^B \quad \text{if } \operatorname{Re} s = \sigma > 0, \quad (3)$$

where $B > 0$ and the implied constant depend on σ . Usually the factors $\Phi_v(s)$, $\Psi_v(s)$ of functional equations are products of suitable gamma functions but we do not need to assume this property because the s -aspect plays no role in the method.

However the signs α_χ , β_χ of functional equations play the key part. Usually they are expressible in terms of the Gauss sum

$$\tau(\chi) = \sum_{x \pmod{q}} \chi(x) e_q(x)$$

where

$$e_q(x) = e\left(\frac{x}{q}\right) = e^{2\pi i x/q}$$

denotes the additive character. Let $\varepsilon_\chi = \tau(\chi)q^{-1/2}$ be the sign of Gauss sum. It satisfies $\bar{\varepsilon}_\chi = v\varepsilon_\chi = \varepsilon_\chi^{-1}$ for any primitive character.

Let us give two examples. If $\mathcal{A}(s)$ is the $k-1$ th symmetric power L -function attached to a cusp form for the modular group then one expects (in accordance with the Langlands program) the functional equation for $\mathcal{A}(s, \chi)$ to have the sign $\alpha_\chi = \varepsilon_\chi^k$. Another interesting example is the shifted Riemann zeta-function $\mathcal{A}(s) = \zeta(ks - (k-1)/2)$ which is useful for generating k th powers. In this case we have the functional equation for $\mathcal{A}(s, \chi) = L(ks - (k-1)/2, \chi^k)$ with the sign $\alpha_\chi = \varepsilon_{\chi^k}$ provided χ^k is primitive.

What truly matters in our argument is the Fourier transform

$$K_q(c) = \sum_{\chi \pmod{q}}^* \chi(c) \alpha_\chi \bar{\beta}_\chi \quad (4)$$

where the star restricts the summation to primitive characters. We can compute $K_q(c)$ in two cases:

Case 1. Suppose $\alpha_\chi = \beta_\chi$. Then if $(c, q) = 1$ we have

$$K_q(c) = \sum_{\chi \pmod{q}}^* \chi(c) = \sum_{w|(q, c-1)} \varphi(w) \mu(q/w). \quad (5)$$

For $c = 1$ this yields the number of primitive characters to modulus q ,

$$K_q(1) = \sum_{sw=q} \mu(s) \varphi(w) = q \prod_{p|q} \left(1 - \frac{2}{p}\right) \prod_{p^2|q} \left(1 - \frac{1}{p}\right)^2.$$

If $c \neq 1$ then $K_q(c)$ is bounded on average in q . Moreover one senses a

“reciprocity law” for $K_q(c)$ as w is switched into the complementary divisor of $|c - 1|$ in (5).

Case 2. Suppose $\alpha_\chi = v^{-h} \varepsilon_\chi^k$ and $\beta_\chi = \varepsilon_\chi^l$ with $h = l - k > 0$. Then $2k$ Gauss sums out of $l + k$ annihilate themselves leaving $\alpha_\chi \bar{\beta}_\chi = \varepsilon_\chi^h$. Using (5) we infer that

$$\begin{aligned} q^{h/2} K_q(c) &= \sum_{\chi(\bmod q)}^* \bar{\chi}(c) \left(\sum_{x(\bmod q)} \chi(x) e_q(x) \right)^h \\ &= \sum_{sw=q} \mu(s) \varphi(w) \sum_{\substack{x_1, \dots, x_h(\bmod q) \\ x_1 \cdots x_h \equiv c(\bmod w)}}^* e_q(x_1 + \cdots + x_h) \\ &= \sum_{\substack{sw=q \\ (s, w)=1}} \mu(s)^{h+1} \varphi(w) \sum_{\substack{x_1, \dots, x_h(\bmod w) \\ x_1 \cdots x_h \equiv c(\bmod w)}}^* e_w((x_1 + \cdots + x_h) \bar{s}) \end{aligned}$$

where $\bar{s}$ denotes the multiplicative inverse of s modulo w . Observe that the innermost sum is the generalized Kloosterman sum for which P. Deligne [3] has established the bound $\tau_h(w) w^{(h-1)/2}$ (for prime modulus only but the extension to all moduli is straightforward). Employing Deligne’s bound we get

$$|K_q(c)| \leq \tau_h(q) q^{1/2}.$$

Of particular interest is the case of $h = 1$ because the sum

$$q^{1/2} K_q(c) = \sum_{\substack{sw=q \\ (s, w)=1}} \mu^2(s) \varphi(w) e_w(c \bar{s}) \quad \text{if } (c, q) = 1 \quad (6)$$

can be transformed by means of the following ‘reciprocity’ formula

$$e_w(c \bar{s}) e_s(c \bar{w}) = e_q(c). \quad (7)$$

2. Statement of results

In this paper we explore Case 2 for $k = 2$ and $l = 3$. It has been shown in [5] that both series formed by squaring the coefficients of $\mathcal{A}(s)$ and $\mathcal{B}(s)$ converge absolutely in $\operatorname{Re} s > 1$. Hence by Cauchy’s inequality the convolution series $\mathcal{C}(s)$ also converges absolutely in $\operatorname{Re} s > 1$. Now we look deeper into the critical strip to prove analytic continuation in $\operatorname{Re} s > 1/2$. For simplicity we shall assume more than the absolute convergence in $\operatorname{Re} s > 1$, namely that the local polynomials $\mathcal{A}_p(T)$ and $\mathcal{B}_p(T)$ have bounded coefficients. This is indeed the case for L -functions attached to holomorphic cusp forms (the Ramanujan

conjecture proved by P. Deligne [3]). In general, one can probably avoid this condition by using the bounds on average.

THEOREM 1. *Suppose $\mathcal{A}(s)$, $\mathcal{B}(s)$ are Euler products of degree two and three with bounded coefficients such that $\mathcal{A}(s, \chi)$, $\mathcal{B}(s, \chi)$ are entire functions of finite order which satisfy the functional equations with signs $\alpha_\chi = \varepsilon_\chi^2$ and $\beta_\chi = \varepsilon_\chi^3$ respectively for all primitive characters. Then the convolution series $\mathcal{C}(s)$ has analytic continuation without poles to the region $\operatorname{Re} s > 1/2$.*

If we take for $\mathcal{A}(s)$ the Hecke L -function attached to a cusp form for the modular group

$$L_1(s) = \prod_p (1 - \alpha_p p^{-s})^{-1} (1 - \beta_p p^{-s})^{-1}$$

and for $\mathcal{B}(s)$ we take the Shimura symmetric square L -function

$$L_2(s) = \prod_p (1 - \alpha_p^2 p^{-s})^{-1} (1 - \alpha_p \beta_p p^{-s})^{-1} (1 - \beta_p^2 p^{-s})^{-1}$$

then the convolution series $\mathcal{C}(s)$ becomes $L_1(s)L_3(s)P(s)$ where

$$L_3(s) = \prod_p (1 - \alpha_p^3 p^{-s})^{-1} (1 - \alpha_p^2 \beta_p p^{-s})^{-1} (1 - \alpha_p \beta_p^2 p^{-s})^{-1} (1 - \beta_p^3 p^{-s})^{-1}$$

is the symmetric cube L -function and $P(s)$ is given by the product

$$P(s) = \prod_p (1 - \alpha_p p^{-s})(1 - \beta_p p^{-s})(1 - (\alpha_p + \beta_p)p^{-s})^{-1}$$

which converges absolutely in $\operatorname{Re} s > 1/2$. By Theorem 1 we infer

COROLLARY. *The symmetric cube L -function $L_3(s)$ attached to a Hecke eigencusp-form for the modular group has meromorphic continuation to the region $\operatorname{Re} s > 1/2$ whereas $L_1(s)L_3(s)$ is holomorphic.*

REMARKS. The corollary is not new, it was proved in greater generality by F. Shahidi and others (see [8]) using quite different methods.

3. Applying the δ -symbol

Our approach to analytic continuation of $\mathcal{C}(s)$ features estimates for finite sums of the type

$$\mathcal{D}(X) = \sum_n a_n b_n n^2 \left(\frac{n}{X} \right)$$

where η is any smooth function supported in the interval $[1/2, 1]$. We shall prove that

$$\mathcal{D}(X) \ll X^{1/2+\varepsilon} \quad (8)$$

and this shows through the Mellin transform that $\mathcal{C}(s)$ is holomorphic in $\operatorname{Re} s > 1/2$.

We begin by writing

$$\mathcal{D}(X) = \sum_m \sum_n a_m b_n \eta\left(\frac{m}{X}\right) \eta\left(\frac{n}{X}\right) \delta_{mn}$$

where δ_{mn} is the diagonal symbol of Kronecker. Then, as in [4], we use the formula

$$Y\delta_{mn} = \sum_{q|(m-n)} \left(\omega(q) - \omega\left(\frac{|m-n|}{q}\right) \right)$$

where ω is any smooth function, compactly supported in $\mathbb{R}^+$ and $Y = \sum \omega(q)$. We choose $\omega(z) = \eta(z/\sqrt{X})$, so $Y \asymp \sqrt{X}$ and

$$Y\mathcal{D}(X) = \sum_q \sum_{m \equiv n \pmod{q}} \sum_n a_m b_n f\left(\frac{m}{X}, \frac{n}{X}, \frac{q}{\sqrt{X}}\right)$$

where

$$f(x, y, z) = \eta(x)\eta(y) \left(\eta(z) - \eta\left(\frac{|x-y|}{z}\right) \right).$$

Notice that $f(x, y, z)$ is smooth and supported in the box $[1/2, 1] \times [1/2, 1] \times [0, 1]$.

Next we write by means of multiplicative characters

$$Y\mathcal{D}(X) = \sum_{q|t} \varphi(qt)^{-1} \sum_{\chi \pmod{q}}^* \sum_{(mn, t)=1} \chi(m) \bar{\chi}(n) a_{rm} b_{rn} f\left(\frac{rm}{X}, \frac{rn}{X}, \frac{qrt}{\sqrt{X}}\right).$$

Further transformation of $\mathcal{D}(X)$ requires factoring the coefficients a_{rm} , b_{rn} as well as relaxing the condition $(mn, t) = 1$. To this end we exploit the Euler products for $\mathcal{A}(s)$ and $\mathcal{B}(s)$. There are numbers $a_r(a) \ll r^\varepsilon$ defined for $a|r$ such that for all m it holds that

$$a_{rm} = \sum_{am' = m} a_r(a) a_{m'}.$$

Also there are numbers $b_r(b) \ll r^\varepsilon$ defined for $b|r^2$ such that for all n it holds that

$$b_{rn} = \sum_{bn'=n} b_r(b)b_{n'}.$$

The above factorizations yield

$$\begin{aligned} Y\mathcal{D}(X) &= \sum_{qrt} \varphi(qt)^{-1} \sum_{(ab,qt)=1} a_r(a)b_r(b) \\ &\quad \times \sum_{\chi(\bmod q)}^* \sum_{(mn,t)=1} \chi(am)\bar{\chi}(bn)a_m b_n f\left(\frac{arm}{X}, \frac{brn}{X}, \frac{qrt}{\sqrt{X}}\right). \end{aligned}$$

To relax the co-primality condition we again appeal to the Euler products. One can define numbers $c_t(c) \ll t^\varepsilon$ for $c|t^2$ with the property that

$$a_m = \sum_{cm'=m} c_t(c)a_{m'}.$$

if $(m, t) = 1$, or else the sum vanishes. Also one can define numbers $d_t(d) \ll t^\varepsilon$ for $d|t^3$ with the property that

$$b_n = \sum_{dn'=n} d_t(d)b_{n'}.$$

if $(n, t) = 1$, or else the sum vanishes. The above relations yield

$$\begin{aligned} Y\mathcal{D}(X) &= \sum_{qrt} \varphi(qt)^{-1} \sum_{(ab,qt)=1} a_r(a)b_r(b) \sum_{(cd,q)=1} c_t(c)d_t(d) \\ &\quad \times \sum_{\chi(\bmod q)}^* \sum_m \sum_n \chi(acm)\bar{\chi}(bdn)a_m b_n f\left(\frac{acrm}{X}, \frac{bdrn}{X}, \frac{qrt}{\sqrt{X}}\right). \end{aligned} \quad (9)$$

4. Applying the functional equations

Now we are ready to execute the summation in m and n . Let us first consider a general character sum of the type

$$\Delta(\chi) = \sum_m \sum_n \chi(m)\bar{\chi}(n)a_m b_n g(m, n)$$

where g is a smooth function, compactly supported in $\mathbb{R}^+ \times \mathbb{R}^+$. Employing the functional equations for $\mathcal{A}(s, \chi)$ and $\mathcal{B}(s, \chi)$ by way of Mellin's transform

we infer

$$\Delta(\chi) = \alpha_\chi \bar{\beta}_\chi \sum_m \sum_n \bar{\chi}(m) \chi(n) a_m b_n g_v(mq^{-2}, nq^{-3}) q^{-5/2}$$

where g_v is an integral transform of g given by

$$g_v(x, y) = \frac{-1}{4\pi^2} \iint_{(\sigma_1, \sigma_2)} \check{g}(s_1, s_2) \Phi_v(s_1) \Psi_v(s_2) x^{-s_1} y^{-s_2} ds_1 ds_2$$

with $\sigma_1, \sigma_2 > 0$ and

$$\check{g}(s_1, s_2) = \iint g(u, v) u^{-s_1} v^{-s_2} du dv.$$

Note that g_v depends only on the parity index $v = \chi(-1) = \pm 1$ but not on the character itself. We put $g^+ = g_1 + g_{-1}$ and $g^- = g_1 - g_{-1}$ so $2g_v = g^+ + vg^-$. Summing over the primitive characters we evaluate the Fourier transform of $\Delta(\chi)$ as follows

$$\sum_{\chi(\bmod q)}^* \chi(e) \Delta(\chi) = \frac{1}{2} \sum_{(mn, q)=1} a_m b_n K_q(\pm e \bar{m} n) g^\pm(mq^{-2}, nq^{-3}) q^{-5/2}$$

for any $(e, q) = 1$. In particular this together with (9) gives

$$2Y\mathcal{D}(X) = X^2 \sum_{rt < \sqrt{X}} \sum_{\substack{a|r \\ (ab, t)=1}} \sum_{\substack{b|r^2 \\ (ab, t)=1}} \sum_{\substack{c|t^2 \\ d|t^3}} a_r(a) b_r(b) c_t(c) d_t(d) (abcd)^{-1} \mathcal{E} \quad (10)$$

where we have put

$$\mathcal{E} = \sum_{\substack{m \\ (abcdmn, q)=1}} \sum_{\substack{n \\ (abcdmn, q)=1}} \sum_{\substack{q \\ (abcdmn, q)=1}} q^{-5/2} \varphi(qt)^{-1} a_m b_n K_q(\pm acn \overline{b d m}) F^\pm \left(\frac{mX}{acrq^2}, \frac{nX}{bdrq^3}, \frac{qrt}{\sqrt{X}} \right)$$

and $F^\pm = F_1 \pm F_{-1}$, where $F(u, v, z)$ are the integral transforms of $f(x, y, z)$ given by

$$F_v(u, v, z) = \frac{-1}{4\pi^2} \iint_{(\sigma_1, \sigma_2)} \check{f}(s_1, s_2, z) \Phi_v(s_1) \Psi_v(s_2) u^{-s_1} v^{-s_2} ds_1 ds_2 \quad (11)$$

with

$$\check{f}(s_1, s_2, z) = \iint f(x, y, z) x^{-s_1} y^{-s_2} dx dy$$

on the lines $\operatorname{Re} s_1 = \sigma_1 > 0$ and $\operatorname{Re} s_2 = \sigma_2 > 0$.

5. Applying the reciprocity transformation

Let $(uv, q) = 1$. By (6) and (7) we get

$$q^{1/2} K_q(u\bar{v}) = e\left(\frac{u}{vq}\right) \sum_{\substack{sw=q \\ (s,w)=1}} \mu^2(s) \varphi(w) e\left(\frac{-u\bar{w}}{vs}\right).$$

Since $\varphi(qt)^{-1} = \varphi(st)^{-1} \varphi(w)^{-1} \sigma((t, w))$ with $\sigma(h) = \varphi(h)h^{-1}$ we obtain

$$\mathcal{E} = \sum_m \sum_n a_m b_n \sum_{\substack{(s,w)=1 \\ (sw, abcdmn)=1}} \mu^2(s) \frac{\sigma((t, w))}{\varphi(st)} e\left(\pm \frac{acn\bar{w}}{bdms}\right) G(m, n, sw)$$

where for notational simplicity we have put

$$G(m, n, q) = q^{-3} e\left(\frac{\pm acn}{bdmq}\right) F^{\pm}\left(\frac{mX}{acrq^2}, \frac{nX}{bdrq^3}, \frac{qrt}{\sqrt{X}}\right). \quad (12)$$

In the sequel we shall denote $u = \pm acn\delta^{-1}$, $v = bdm\delta^{-1}$ where $\delta = (acn, bdm)$. Furthermore we split $\mathcal{E} = \mathcal{E}_0 + \mathcal{E}_1$, where

$$\begin{aligned} \mathcal{E}_0 &= \sum_m \sum_n a_m b_n \sum_{\substack{(s,w)=1 \\ (sw, \delta uv)=1}} \frac{\mu(vs) \sigma((t, w))}{\varphi(st) \varphi(sv)} G(m, n, sw), \\ \mathcal{E}_1 &= \sum_m \sum_n a_m b_n \sum_{\substack{(s,w)=1 \\ (sw, \delta uv)=1}} \mu^2(s) \frac{\sigma((t, w))}{\varphi(st)} \left(e\left(\frac{u\bar{w}}{vs}\right) - \frac{\mu(vs)}{\varphi(vs)} \right) G(m, n, sw). \end{aligned}$$

6. Estimating $G(m, n, q)$

First let us estimate the transform (11). The partial derivatives of $f(x, y, z)$ are bounded by

$$f^{(ijk)}(x, y, z) \ll z^{-i-j-k}$$

with the implied constant depending on i, j, k only. Hence, by a repeated partial integration the Mellin transform satisfies

$$\frac{z^k \partial^k}{\partial z^k} \check{f}(s_1, s_2, z) \ll (1 + |s_1|z)^{-A} (1 + |s_2|z)^{-A}$$

for s_1, s_2 on the vertical lines $\operatorname{Re} s_1 = \sigma_1 > 0$, $\operatorname{Re} s_2 = \sigma_2 > 0$, where A is an arbitrary positive number, the implied constant depending on σ_1, σ_2, A and k only. Since $\Phi_v(s_1)$ and $\Psi_v(s_2)$ have at most a polynomial growth we obtain

$$u^i v^j z^k F_v^{(ijk)}(u, v, z) \ll u^{-\sigma_1} v^{-\sigma_2} z^{-2B}$$

for any $\sigma_1, \sigma_2 > 0$ and some constant $B > 0$. This yields

$$u^i v^j z^k F_v^{(ijk)}(u, v, z) \ll (1+u)^{-A} (1+v)^{-A} (uz^2)^{-B} (uv)^{-\varepsilon}$$

for any $\varepsilon, A > 0$ and some constant $B > 0$. Finally, by a change of variables we conclude from the above and (12) that

$$m^i n^j q^k G^{(ijk)}(m, n, q) \ll q^{-3} \left(1 + \frac{mX}{acrq^2}\right)^{-A} \left(1 + \frac{nX}{bdrq^3}\right)^{-A} X^\varepsilon \quad (13)$$

for $m, n, q \geq 1$, where $\varepsilon, A > 0$ are arbitrary and the implied constant depends on ε, A, i, j, k only.

Recall that $G(m, n, q)$ vanishes if $qrt > \sqrt{X}$. Therefore, by (13) all terms in $\mathcal{E}_0$ and $\mathcal{E}_1$ are very small except for

$$\begin{aligned} 1 \leq m \leq M & \quad \text{with } M = acr^{-1}t^{-2}X^\varepsilon \\ 1 \leq n \leq N & \quad \text{with } N = bdr^{-2}t^{-3}X^{1/2+\varepsilon} \\ QX^{-\varepsilon} < q < Q & \quad \text{with } Q = (rt)^{-1}X^{1/2}. \end{aligned} \quad (14)$$

Notice that the above conditions imply $M \leq X^\varepsilon$, $N \leq X^{1/2+\varepsilon}$, and $Q < X^{1/2}$.

7. Estimating $\mathcal{E}_0$

First we execute the summation over n in $\mathcal{E}_0$ by an appeal to the following general result:

LEMMA 1. Let $G(n)$ be a smooth function on $\mathbb{R}^+$ whose derivatives satisfy

$$n^j G^{(j)}(n) \ll \left(1 + \frac{n}{N}\right)^{-A}$$

for some $N \geq 1$ and any $A > 0$. Then we have

$$\sum_{(n,t)=1} b_{rn} G(n) \ll (rtN)^\varepsilon. \quad (15)$$

Proof. Using the functional equation for $\mathcal{B}(s)$ by contour integration we infer that

$$\sum_n b_n G(n) \ll N^\varepsilon.$$

Hence

$$\sum_{(n,t)=1} b_{rn} G(n) = \sum_{\substack{b|r^2 \\ (b,t)=1}} b_r(b) \sum_{d|t^3} d_t(d) \sum_n b_n G(bdn) \ll (rtN)^\varepsilon.$$

The sum over n in $\mathcal{E}_0$ is of the type (15). More precisely we have

$$\mathcal{E}_0 = \sum_m a_m \sum_{\delta|bdm} \sum_{\substack{(s,w)=1 \\ (sw,abcdmn)=1}} \sum_{\substack{(n,sw)=1 \\ (acn,bdm)=\delta}} \frac{\mu(vs)\sigma((t,w))}{\varphi(st)\varphi(sv)} b_n G(m, n, sw).$$

Therefore, by Lemma 1 we obtain

$$\begin{aligned} \mathcal{E}_0 &\ll \sum_m |a_m| \sum_{q>Q} t^{-1} q^{-3} \left(1 + \frac{mX}{acr q^2}\right)^{-A} X^\varepsilon \\ &\ll t^{-1} M Q^{-2} X^\varepsilon \ll acrt^{-1} X^{\varepsilon-1}. \end{aligned} \quad (16)$$

8. Estimating $\mathcal{E}_1$

We replace $\sigma((t, w))$ by

$$\sigma((t, w)) = \sum_{\tau|(t,w)} \tau^{-1} \mu(\tau)$$

and relax the condition $(w, \delta u) = 1$ using Möbius inversion to get

$$\begin{aligned} \mathcal{E}_1 &= \sum_m \sum_n a_m b_n \sum_{\substack{\tau|t \quad v|\delta u \\ (\tau, \delta uv) = (v, v) = 1}} \tau^{-1} \mu(\tau) \mu(v) \sum_{(s, \tau \delta uv) = 1} \mu^2(s) \varphi(st)^{-1} \\ &\quad \times \sum_{(w, sv) = 1} \left(e\left(\frac{u\tau vw}{vs}\right) - \frac{\mu(vs)}{\varphi(vs)} \right) G(m, n, \tau vsw). \end{aligned}$$

First, we shall show that small s contribute very little. To this end we establish the following general result:

LEMMA 2. *Let $G(w)$ be a smooth function supported in $[0, W]$ whose derivatives*

satisfy $G^{(j)} \ll W^{-j}$. Let $g < W^{1-\varepsilon}$. Then

$$\sum_{w \equiv a \pmod{g}} G(w) = \frac{1}{g} \sum_w G(w) + O(W^{-A}).$$

Proof. The Poisson summation gives

$$\frac{1}{g} \sum_h e\left(\frac{-ah}{g}\right) \hat{G}\left(\frac{h}{g}\right)$$

where $\hat{G}$ is the Fourier transform of G . Integrating by parts one proves that

$$\hat{G}(y) \ll (1 + |y|W)^{-A}.$$

Hence our sum is equal to $g^{-1}\hat{G}(0) + O(W^{-A})$. This shows that the sum does not depend on $a \pmod{g}$ up to the error term $O(W^{-A})$, giving the result.

COROLLARY. *If $(a, g) = 1$ and $g < W^{1-\varepsilon}$ then*

$$\sum_{(w, g)=1} \left(e\left(\frac{a\bar{w}}{g}\right) - \frac{\mu(g)}{\varphi(g)} \right) G(w) \ll W^{-A}. \quad (17)$$

The hypothesis of the above corollary to Lemma 2 is satisfied for $G(w) = G(m, n, \tau vsw)$ in the range $g = vs \leq (\tau vs)^{-1} QX^{-2\varepsilon}$ by virtue of (13), where $Q = (rt)^{-1} X^{1/2}$. We put

$$S = (\tau v)^{-1/2} Q^{1/2} X^{-\varepsilon},$$

so (17) can be used for all $s \leq S$. Therefore, the contribution of this range to $\mathcal{E}_1$ is

$$\mathcal{E}_{11} \ll X^{-A}. \quad (18)$$

In the remaining range of $s > S$ we estimate the contribution of the part $-\mu(vs)\varphi(vs)^{-1}$ trivially as follows:

$$\begin{aligned} \mathcal{E}_{12} &\ll \sum_m \sum_n |a_m b_n| \sum_{\tau|t} \tau^{-1} \sum_{v|\delta u} \varphi(t)^{-1} \varphi(v)^{-1} S^{-2} \sum_q |G(m, n, \tau vq)| \tau(q) \\ &\ll t^{-1} M N Q^{-3} X^\varepsilon \ll abcd t^{-3} X^{\varepsilon-1}. \end{aligned} \quad (19)$$

Now we are left with

$$\begin{aligned} \mathcal{E}_{13} &= \sum_m \sum_n a_m b_n \sum_{\substack{\tau|t \\ (\tau, \delta uv) = (v, v) = 1}} \tau^{-1} \mu(\tau) \mu(v) \\ &\quad \times \sum_{\substack{(s, \tau \delta uv) = 1 \\ s > S}} \mu^2(s) \varphi(st)^{-1} \sum_{(w, sv) = 1} e\left(\frac{u\tau vw}{vs}\right) G(m, n, \tau vsw). \end{aligned}$$

After changing the order of summation we estimate as follows:

$$\mathcal{E}_{13} \ll \varphi(t)^{-1} \sum_m |a_m| \sum_{\tau|t} \tau^{-1} \sum_{(v, v) = 1} \sum_{\delta | bdm} \mathcal{H} \quad (20)$$

where $\mathcal{H}$ is a sum in s, w, n given by

$$\mathcal{H} = \sum_{\substack{(s, \tau v) = 1 \\ s > S}} \varphi(s)^{-1} \sum_{\substack{(w, sv) = 1 \\ \tau vsw < Q}} \left| \sum_{\substack{(n, \tau s) = 1, v | acn \\ (acn, bdm) = \delta}} b_n e\left(\frac{u\tau vw}{vs}\right) G(m, n, \tau vsw) \right|.$$

By virtue of (13) we can restrict the summation to the range (14) up to a small error term $O(X^{-4})$. Moreover we can separate n from the other variables of $G(m, n, q)$ by any standard technique at no cost. In fact the integral representation (11) yields the desired separation without effort. We obtain

$$\mathcal{H} \ll Q^{-3} X^\varepsilon \sum_{\substack{(s, \tau v) = 1 \\ s > S}} s^{-1} \sum_{\substack{(w, sv) = 1 \\ \tau vsw < Q}} \left| \sum_{\substack{n > N, v | acn \\ (n, s) = 1}} \beta_n e\left(\frac{u\tau vw}{vs}\right) \right| + X^{-4}$$

for some $\beta_n \ll b_n$. For estimating this sum we shall use the large sieve inequality (see [1]).

LEMMA 3. *For any complex numbers β_n it holds that*

$$\sum_{\substack{s < T \\ (s, v) = 1}} \sum_{x \pmod{vs}}^* \left| \sum_{\substack{n < N \\ (n, s) = 1}} \beta_n e\left(\frac{nx}{vs}\right) \right|^2 \ll (vT^2 + N) \sum_n |\beta_n|^2.$$

From Lemma 3 by Cauchy's inequality we deduce the following:

COROLLARY. *If $(ab, v) = 1$ it holds that*

$$\sum_{\substack{sw < Q \\ (vs, abw) = 1 \\ s > S}} s^{-1} \left| \sum_{\substack{n < N \\ (n, s) = 1}} \beta_n e\left(an \frac{bw}{vs}\right) \right| \ll (vQ)^{1/2} \left(1 + \frac{Q}{vs}\right)^{1/2} \left(1 + \frac{N}{vs}\right)^{1/2} \left(\sum |\beta_n|^2\right)^{1/2}.$$

The corollary provides an estimate for a sum of the type we have in $\mathcal{H}$. It gives

$$\begin{aligned}\mathcal{H} &\ll Q^{-3} X^\varepsilon \left(\frac{vQ}{\tau v} \right)^{1/2} \left(1 + \frac{Q}{\tau v S^2} \right)^{1/2} \left(1 + \frac{(v, ac)N}{v S^2} \right)^{1/2} \left(\frac{(v, ac)}{v} N \right)^{1/2} \\ &\ll v^{-1}(v, ac) v^{1/2} Q^{-5/2} (1 + NQ^{-1})^{1/2} N^{1/2} X^\varepsilon \\ &\ll v^{-1}(v, ac) bdr^{3/2} t (1 + bdr^{-1} t^{-2})^{1/2} X^{\varepsilon-1} \ll v^{-1}(v, ac) bdr^2 t^{3/2} X^{\varepsilon-1}\end{aligned}$$

because $b|r^2$, $d|t^3$, so $bd \leq r^2 t^3$. Hence by (20) we derive

$$\mathcal{E}_{13} \ll bdr^2 t^{1/2} M X^{\varepsilon-1} \ll abcdrt^{-3/2} X^{\varepsilon-1}. \quad (21)$$

Gathering together (18), (19), and (21) we conclude that

$$\mathcal{E}_1 \ll abcdrt^{-3/2} X^{\varepsilon-1}. \quad (22)$$

From (16) and (22) we obtain

$$\mathcal{E} \ll abcdrt^{-1} X^{\varepsilon-1}. \quad (23)$$

Finally by (10) and (23) we conclude (8). This completes the proof of Theorem 1.

References

1. E. Bombieri: Le grand crible dans la théorie analytique des nombres, *Astérisque* 18 (1973).
2. D. Bump: The Rankin-Selberg method. In: *Number Theory, Trace Formulas and Discrete Groups*. Oslo, Norway (1987), Academic Press (1989).
3. P. Deligne: Le conjecture de Weil I, *Publ. Math. I.H.E.S.* 43 (1974), 273–307.
4. W. Duke and H. Iwaniec: Estimates for coefficients of L -functions. I. In: *Automorphic Forms and Analytic Number Theory*. CRM Publications, Montreal (1990) pp. 43–47.
5. W. Duke and H. Iwaniec: Estimates for coefficients of L -functions. II. In: *Analytic Number Theory*. Proceedings of the Amalfi Conference (1989), Università di Salerno (1992) pp. 71–82.
6. W. Duke and H. Iwaniec, Estimates for coefficients of L -functions. III. In: *The Séminaire de Théorie des Nombres*. Paris (1989–90) pp. 113–120.
7. W. Duke and H. Iwaniec: Estimates for coefficients of L -functions. IV. *Amer. J. Math.* 116 (1993) 1–11.
8. F. Shahidi: Automorphic L -functions. In: *Automorphic Forms, Shimura Varieties, and L -Functions*. Proceedings of the conference in Ann Arbor (1988), Academic Press, Boston (1990) pp. 415–437.