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A problem of D. H. Lehmer and its generalization (II)*

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Abstract. Let $q > 2$ be an odd number. For each integer x with $0 < x < q$ and $(q, x) = 1$, we define $\bar{x}$ by $x\bar{x} \equiv 1 \pmod q$ and $0 < \bar{x} < q$. Let $r(q)$ be the number of integers x with $0 < x < q$ for which x and $\bar{x}$ are of opposite parity. The main purpose of this paper is to give a sharper asymptotic formula for $r(q)$ for all odd numbers q.

1. Introduction

Let q be an odd integer >2. For each integer x with $0 < x < q$ and $(q, x) = 1$, we know that there exists one and only one $\bar{x}$ with $0 < \bar{x} < q$ such that $x\bar{x} \equiv 1 \pmod q$. Let $r(q)$ be the number of cases in which x and $\bar{x}$ are of opposite parity. For example, $r(3) = 0$, $r(5) = 2$, $r(7) = 0$, $r(13) = 6$. For $q = p$ a prime, D. H. Lehmer [1] asks us to find $r(p)$ or at least to say something nontrivial about it. It is known that $r(p) \equiv 2$ or $0 \pmod 4$ according to $p \equiv \pm 1 \pmod 4$. About this problem, the author [2] obtained an asymptotic formula for $r(p^\alpha)$ and $r(p_1 p_2)$, where p, p_1 and p_2 are primes. In this paper, as an improvement of [2], we shall give an asymptotic formula for $r(q)$ for all odd numbers q. The constants implied by the O-symbols and the symbols $\ll$, $\gg$ used in this paper do not depend on any parameter, unless otherwise indicated. By using estimates for character sum and Kloosterman sums, and the properties of Dirichlet L-functions, we prove the following two theorems:

THEOREM 1. *For every prime $p > 2$ we have the asymptotic formula*

$$\begin{aligned} r(p) = \frac{1}{2}p + \frac{1}{\pi^2}\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\frac{1}{ab}\{&S(a, b; p) - S(a, -b; p) \\ &+ 4S(\bar{4}a, b; p) - 4S(\bar{4}a, -b; p) - 4S(\bar{2}a, b; p) \\ &+ 4S(\bar{2}a, -b; p)\} + O(\ln^3 p) \end{aligned}$$

where $d\bar{d} \equiv 1 \pmod p$, $S(m, n; p) = \sum'_{d \pmod p} e\left(m\frac{d}{p} + n\frac{\bar{d}}{p}\right)$ is the Kloosterman sum, and $e(y) = e^{2\pi i y}$.

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THEOREM 2. *For every odd integer $q > 2$ we have*

$$r(q) = \tfrac{1}{2}\phi(q) + O(q^{1/2}d^2(q)\ln^2 q)$$

where $\phi(q)$ is the Euler function and $d(q)$ is the divisor function.

From theorem 1 we can see that if we could get a nontrivial upper bound estimate for the mean value $\sum_{q=1}^{p-1}\sum_{b=1}^{p-1}\frac{1}{ab}S(ma, b; p)$, then we may obtain a more accurate asymptotic formula. From theorem 2 we can also deduce the following:

COROLLARY. *For every prime $p > 2$ we have the asymptotic formula*

$$r(p) = \tfrac{1}{2}p + O(p^{1/2}\ln^2 p)$$

2. Some lemmas

To complete the proofs of the theorems, we need some lemmas. First we have:

LEMMA 1. *Let $q > 2$ be an odd number. Then we have*

$$r(q) = \frac{1}{2}\phi(q) - \frac{2}{\phi(q)}\sum_{\chi(-1)=-1}\chi(4)\left(\sum_{a=1}^{(q-1)/2}\chi(a)\right)^2$$

where the summation is over all odd characters mod q.

Proof. From the definition of $r(q)$ and the orthogonality of characters we get

$$\begin{aligned} r(q) &= \frac{1}{2}\sum_{\substack{a=1\\ ab\equiv 1(q)}}^{q-1}\sum_{b=1}^{q-1}\{1-(-1)^{a+b}\} \\ &= \frac{1}{2}\phi(q) - \frac{1}{2}\sum_{\substack{a=1\\ ab\equiv 1(q)}}^{q-1}\sum_{b=1}^{q-1}(-1)^{a+b} \\ &= \frac{1}{2}\phi(q) - \frac{1}{2\phi(q)}\sum_{\chi \bmod q}\left(\sum_{a=1}^{q-1}(-1)^a\chi(a)\right)^2 \\ &= \frac{1}{2}\phi(q) - \frac{1}{2\phi(q)}\sum_{\chi\neq\chi^0}\left(\sum_{a=1}^{q-1}(-1)^a\chi(a)\right)^2 \qquad (1) \end{aligned}$$

where $\sum_{\chi\neq\chi^0}$ denotes the summation over all nonprincipal characters mod q.

Now if $\chi(-1) = 1$ and $\chi \neq \chi^0$, then we have

$$\sum_{a=1}^{q-1} (-1)^a \chi(a) = 0 \tag{2}$$

while if $\chi(-1) = -1$, then

$$\sum_{a=1}^{q-1} (-1)^a \chi(a) = 2 \sum_{a=1}^{(q-1)/2} \chi(2a) \tag{3}$$

Combining (1), (2) and (3) we may immediately deduce lemma 1. □

LEMMA 2. *Let $q > 1$ be any odd number and let χ be any Dirichlet character modulo q, not necessarily primitive. Then*

$$(1 - 2\chi(2)) \sum_{\gamma=1}^{q-1} \gamma\chi(\gamma) = \chi(2) q \sum_{\gamma=1}^{(q-1)/2} \chi(\gamma)$$

Proof. (See reference [3]). □

LEMMA 3. *Let m, n, q be integers with $q > 1$. Then*

$$S(m, n; q) = {\sum_{d=1}^{q-1}}' e\left(m\frac{d}{q} + n\frac{\bar{d}}{q}\right) \ll (m, n, q)^{1/2} q^{1/2} d(q)$$

where $d\bar{d} \equiv 1 \pmod q$, $d(q)$ is the divisor function, and (m, n, q) denotes the greatest common factor of m, n and q. ${\sum_a}'$ denotes the summation over a such that $(a, q) = 1$.

Proof. (See reference [4]). □

LEMMA 4. *Let q be an odd integer >2. Then for any integer b we have the estimate*

$$\sum_{\chi(-1)=-1} \tau^2(\chi)\chi(b) L^2(1, \bar{\chi}) \ll \phi(q) q^{1/2} d(q) \ln^2 q$$

where $L(s, \chi)$ is the Dirichlet L-function and $\tau(\chi)$ is the Gauss sum corresponding to χ.

Proof. First for any integer r with $(r, q) = 1$ we have

$$\sum_{\chi(-1)=-1} \chi(r) = \begin{cases} \frac{1}{2}\phi(q), & \text{if } r \equiv 1 \pmod q, \\ -\frac{1}{2}\phi(q), & \text{if } r \equiv -1 \pmod q, \\ 0, & \text{otherwise.} \end{cases} \tag{4}$$

Now for any $N > 1$, from the well-known Pólya-Vinogradov inequality (See Theorem 13.15 of [5]) we get for $\chi \neq \chi^0$

$$L(1, \bar{\chi}) = \sum_{n \leqslant N} \frac{\bar{\chi}(n)}{n} + O\left(\frac{\sqrt{q}\ln q}{N}\right) \tag{5}$$

If $(b, q) > 1$, then $\chi(b) = 0$ and in this case it is clear that lemma 4 holds. Now we suppose that $(b, q) = 1$. On noticing that $|\tau(\chi)| \ll q^{1/2}$ and $\sum_{n \leqslant N} \frac{\bar{\chi}(n)}{n} \ll \ln N$, for $N \geqslant q$, we obtain from (4) and (5)

$$\begin{aligned}
&\sum_{\chi(-1)=-1} \tau^2(\chi)\chi(b)L^2(1, \bar{\chi}) \\
&= \sum_{\chi(-1)=-1} \tau^2(\chi)\chi(b)\left(\sum_{n \leqslant N} \frac{\bar{\chi}(n)}{n} + O\left(\frac{\sqrt{q}\ln q}{N}\right)\right)^2 \\
&= \sum_{a=1}^{q-1} \sum_{c=1}^{q-1} \sum_{1 \leqslant m,n \leqslant N} \frac{1}{mn} \sum_{\chi(-1)=-1} \chi(a)\chi(b)\chi(c)\bar{\chi}(mn) e\left(\frac{a+c}{q}\right) \\
&\quad + O\left(\frac{\phi(q)q^{3/2}\ln q \ln N}{N}\right) \\
&= \sum_{a=1}^{q-1} \sum_{c=1}^{q-1} \sum_{1 \leqslant m,n \leqslant N}{}' \frac{1}{mn} \sum_{\chi(-1)=-1} \chi(abc\bar{m}\bar{n}) e\left(\frac{a+c}{q}\right) + O\left(\frac{\phi(q)q^{3/2}\ln q \ln N}{N}\right) \\
&= \frac{1}{2}\phi(q) \sum_{1 \leqslant m,n \leqslant N}{}' \frac{1}{mn} \sideset{}{'}\sum_{a=1}^{q-1} \sideset{}{'}\sum_{\substack{c=1 \\ abc \equiv mn(q)}}^{q-1} e\left(\frac{a+c}{q}\right) \\
&\quad - \frac{1}{2}\phi(q) \sum_{1 \leqslant m,n \leqslant N}{}' \frac{1}{mn} \sideset{}{'}\sum_{a=1}^{q-1} \sideset{}{'}\sum_{\substack{c=1 \\ abc \equiv -mn(q)}}^{q-1} e\left(\frac{a+c}{q}\right) + O\left(\frac{\phi(q)q^{3/2}\ln q \ln N}{N}\right) \\
&= \frac{1}{2}\phi(q) \sum_{1 \leqslant m,n \leqslant N}{}' \frac{1}{mn} \{S(1, \bar{b}mn; q) - S(1, -\bar{b}mn; q)\} + O\left(\frac{\phi(q)q^{3/2}\ln q \ln N}{N}\right)
\end{aligned} \tag{6}$$

Taking $N = q$, applying lemma 3 we may immediately get

$$\sum_{\chi(-1)=-1} \tau^2(\chi)\chi(b)L^2(1, \bar{\chi}) \ll \phi(q)q^{1/2}d(q)\ln^2 q.$$

This completes the proof of the lemma 4. □

LEMMA 5. *Let p be a prime* >3. *Then for any integer b with* $(b, p) = 1$, *we have*

$$\frac{2}{p-1}\sum_{\chi_p(-1)=-1}\tau^2(\chi)\chi(b)L^2(1,\bar{\chi})=\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\frac{1}{ac}S(\bar{b},ac;p)$$
$$-\sum_{a=1}^{p-1}\sum_{c=1}^{p-1}\frac{1}{ac}S\,(\bar{b},\,-ac;p)+O(\ln^3 p)$$

Proof. Let $K(r,q)=\sum_{q=1}^{p-1}a^q e\left(\frac{ra}{p}\right)$, $p\nmid r$, then we have the estimate

$$K(r,q)\ll\frac{p^q}{\left|\sin\left(\frac{\pi r}{p}\right)\right|} \tag{7}$$

Namely we have

$$K(\gamma,q)\left(1-e\left(\frac{\gamma}{p}\right)\right)=\sum_{a=1}^{p-1}a^q\left(e\left(\frac{\gamma a}{p}\right)-e\left(\frac{(a+1)\gamma}{p}\right)\right)$$
$$=\sum_{a=1}^{p-2}((a+1)^q-a^q)e\left(\frac{(a+1)\gamma}{p}\right)+e\left(\frac{\gamma}{p}\right)-(p-1)^q e\left(\frac{p\gamma}{p}\right)$$
$$\ll p^q+\sum_{a=1}^{p-2}((a+1)^q-a^q)\ll p^q.$$

Thus

$$|K(\gamma,q)|\ll\frac{p^q}{\left|1-e\left(\frac{\gamma}{p}\right)\right|}\ll\frac{p^q}{\left|\sin\left(\frac{\pi\gamma}{p}\right)\right|}.$$

This completes the proof of (7). □

By applying (6) with $q=p$, $N=p^2+p$ we get

$$\frac{2}{p-1}\sum_{\chi_p(-1)=-1}\tau^2(\chi)\chi(b)L^2(1,\bar{\chi})$$
$$=\sum_{1\leqslant m,n\leqslant p^2+p}{}'\frac{1}{mn}\{S(1,\bar{b}mn;p)-S(1,-\bar{b}mn;p)\}+O\left(\frac{\ln^2 p}{\sqrt{p}}\right)$$
$$=\sum_{a=1}^{p-1}\sum_{n=0}^{p}\sum_{c=1}^{p-1}\sum_{m=0}^{p}\frac{1}{(np+a)(mp+a)}\{S(1,\bar{b}ac;p)-S(1,-\bar{b}ac;p)\}$$
$$+O\left(\frac{\ln^2 p}{\sqrt{p}}\right)$$

$$= \sum_{a=1}^{p-1} \sum_{c=1}^{p-1} \frac{1}{ac} \{S(\bar{b}, ac; p) - S(\bar{b}, -ac; p)\} + O\left(\frac{\ln^2 p}{\sqrt{p}}\right)$$

$$+ \sum_{a=1}^{p-1} \sum_{c=1}^{p-1} \sum_{m=1}^{p} \frac{1}{a(mp+c)} \{S(1, \bar{b}ac; p) - S(1, -\bar{b}ac; p)\}$$

$$+ \sum_{a=1}^{p-1} \sum_{c=1}^{p-1} \sum_{n=1}^{p} \frac{1}{c(np+a)} \{S(1, \bar{b}ac; p) - S(1, -\bar{b}ac; p)\}$$

$$+ \sum_{a=1}^{p-1} \sum_{c=1}^{p-1} \sum_{m=1}^{p} \sum_{n=1}^{p} \frac{1}{(np+a)(mp+c)} \{S(1, \bar{b}ac; p) - S(1, -\bar{b}ac; p)\}$$

$$\equiv M_1 + O\left(\frac{\ln^2 p}{\sqrt{p}}\right) + M_2 + M_3 + M_4 \tag{8}$$

It is clear that M_1 is the main term of lemma 5. Now we shall estimate the other three terms M_2, M_3 and M_4. Using the power series expansion of $(1-x)^{-1}$ we get

$$\sum_{a=1}^{p-1} \sum_{c=1}^{p-1} \sum_{m=1}^{p} \frac{1}{a(mp+c)} S(1, \bar{b}ac; p)$$

$$= \sum_{a=1}^{p-1} \frac{1}{a} \sum_{c=1}^{p-1} \sum_{m=1}^{p} \frac{1}{(m+1)p - (p-c)} S(1, \bar{b}ac; p)$$

$$= \sum_{a=1}^{p-1} \frac{1}{a} \sum_{k=0}^{\infty} \frac{1}{p^{k+1}} \sum_{m=1}^{p} \frac{1}{(m+1)^{k+1}} \sum_{c=1}^{p-1} (p-c)^k S(1, \bar{b}ac; p)$$

$$= \sum_{a=1}^{p-1} \frac{1}{a} \sum_{k=0}^{\infty} \frac{1}{p^{k+1}} \sum_{m=1}^{p} \frac{1}{(m+1)^{k+1}} \sum_{c=1}^{p-1} c^k S(1, -\bar{b}ac; p) \tag{9}$$

Applying (7) we get the estimate

$$\sum_{c=1}^{p-1} c^k S(1, -\bar{b}ac; p) = \sum_{d=1}^{p-1} e\left(\frac{d}{p}\right) \sum_{c=1}^{p-1} c^k e\left(\frac{-\bar{b}ac\bar{d}}{p}\right)$$

$$\ll \sum_{d=1}^{p-1} \frac{p^k}{\left|\sin\left(\frac{\pi \bar{b}a\bar{d}}{p}\right)\right|} \ll p^{k+1} \ln p \tag{10}$$

From (9) and (10) we immediately deduce the estimate

$$\sum_{a=1}^{p-1} \sum_{c=1}^{p-1} \sum_{m=1}^{p} \frac{1}{a(mp+c)} S(1, \bar{b}ac; p)$$

$$\ll \sum_{a=1}^{p-1} \frac{1}{a} \sum_{m=1}^{p} \sum_{k=0}^{\infty} \frac{1}{(m+1)^{k+1} p^{k+1}} \cdot p^{k+1} \ln p$$

$$\ll \sum_{a=1}^{p-1} \frac{1}{a} \sum_{m=1}^{p} \frac{1}{m+1} \ln p \ll \ln^3 p \tag{11}$$

Similarly we can deduce that

$$\sum_{a=1}^{p-1} \sum_{c=1}^{p-1} \sum_{m=1}^{p} \frac{1}{a(mp+c)} S(1, -\bar{b}ac; p) \ll \ln^3 p \tag{12}$$

Combining (11) and (12) we get

$$M_2 \ll \ln^3 p \tag{13}$$

In the same way we get the estimates

$$M_3 \ll \ln^3 p \tag{14}$$

$$M_4 \ll \ln^3 p \tag{15}$$

Now lemma 5 follows at once from (8), (13), (14) and (15). □

LEMMA 6. *Suppose* χ *is an odd character* mod q, *generated by the primitive character* χ_m mod m. *Then we have*

$$\sum_{a=1}^{q} a\chi(a) = \frac{q}{m} \Bigg(\prod_{\substack{p \mid q \\ p \nmid m}} (1 - \chi_m(p)) \Bigg) \Bigg(\sum_{a=1}^{m} a\chi_m(a) \Bigg)$$

Proof. Let l be the largest divisor of q that is coprime with m. Then we have

$$\sum_{a=1}^{q} a\chi(a) = \sum_{i=0}^{(q/ml)-1} \sum_{j=1}^{ml} (iml + j)\chi(iml + j)$$

$$= \sum_{i=0}^{(q/ml)-1} \sum_{j=1}^{ml} (iml + j)\chi(j) = \frac{q}{ml} \sum_{j=1}^{lm} j\chi(j)$$

$$= \frac{q}{lm} \sum_{a=1}^{lm} a\chi_m(a) \sum_{\substack{d \mid a \\ d \mid l}} \mu(d)$$

$$= \frac{q}{lm} \sum_{d \mid l} \mu(d) \sum_{\substack{a=1 \\ d \mid a}}^{lm} a\chi_m(a)$$

$$= \frac{q}{lm} \sum_{d \mid l} \mu(d) d \chi_m(d) \sum_{b=1}^{lm/d} b \chi_m(b)$$

$$= \frac{q}{lm} \sum_{d \mid l} \mu(d) d \chi_m(d) \sum_{i=0}^{(l/d)-1} \sum_{j=1}^{m} (im+j) \chi_m(im+j)$$

$$= \frac{q}{lm} \sum_{d \mid l} \mu(d) d \chi_m(d) \cdot \frac{l}{d} \sum_{a=1}^{m} a \chi_m(a)$$

$$= \frac{q}{m} \Bigg(\prod_{\substack{p \mid q \\ p \nmid m}} (1 - \chi_m(p)) \Bigg) \Bigg(\sum_{a=1}^{m} a \chi_m(a) \Bigg)$$

This proves lemma 6. □

LEMMA 7. *Let* χ *be a primitive character* mod m with $\chi(-1) = -1$. *Then we have*

$$\frac{1}{m} \sum_{\gamma=1}^{m} \gamma \chi(\gamma) = -\frac{i}{\pi} \tau(\chi) L(1, \bar{\chi})$$

Proof. (See Theorem 12.11 and Theorem 12.20 of [5]). □

3. Proof of the theorems

In this section, we shall complete the proofs of the theorems. First we prove theorem 2. From lemmas 2, 4, 6 and lemma 7 we know that for every odd integer $q > 2$,

$$\sum_{\chi(-1)=-1} \chi(4) \Bigg(\sum_{a=1}^{(q-1)/2} \chi(a) \Bigg)^2$$

$$= \frac{1}{q^2} \sum_{\chi(-1)=-1} (1 - 2\chi(2))^2 \Bigg(\sum_{\gamma=1}^{q-1} \gamma \chi(\gamma) \Bigg)^2$$

$$= \sum_{m \mid q} \sideset{}{^*}\sum_{\substack{\chi \bmod m \\ \chi(-1)=-1}} \frac{1}{m^2} (1 - 2\chi(2))^2 \Bigg(\prod_{\substack{p \mid q \\ p \nmid m}} (1 - \chi_m(p)) \Bigg)^2 \Bigg(\sum_{a=1}^{m} a \chi_m(a) \Bigg)^2$$

$$= -\frac{1}{\pi^2} \sum_{m \mid q} \sideset{}{^*}\sum_{\substack{\chi \bmod m \\ \chi(-1)=-1}} (1 - 2\chi_m(2))^2 \Bigg(\prod_{\substack{p \mid q \\ p \nmid m}} (1 - \chi_m(p)) \Bigg)^2 \tau^2(\chi_m) L^2(1, \bar{\chi}_m)$$

$$= -\frac{1}{\pi^2} \sum_{m \mid q} \sum_{d \mid m} \mu\left(\frac{m}{d}\right) \sum_{\substack{\chi \bmod m \\ \chi(-1)=-1}} (1 - 2\chi(2))^2 \Bigg(\prod_{\substack{p \mid q \\ p \nmid m}} (1 - \chi(p)) \Bigg)^2 \tau^2(\chi) L^2(1, \bar{\chi})$$

$$\ll \sum_{m|q} \sum_{k|m} 2^{\omega(q)-\omega(m)} k^{1/2} d(k)\phi(k) \ln^2 k$$

$$\ll \phi(q) q^{1/2} d^2(q) \ln^2 q \tag{16}$$

where $\omega(n)$ denotes the number of all distinct prime divisor of n. Now lemma 1 and (16) imply that

$$r(q) = \tfrac{1}{2}\phi(q) + O\left(q^{1/2} d^2(q) \ln^2 q\right)$$

This completes the proof of theorem 2. □

Now we prove theorem 1. Let p be a prime >2. Note that every odd character $\chi \bmod p$ is a primitive character $\bmod p$. Thus, from lemma 2 and lemma 7 we derive

$$\sum_{\chi_p(-1)=-1} \chi(4) \left(\sum_{a=1}^{(p-1)/2} \chi(a) \right)^2$$

$$= -\frac{1}{\pi^2} \sum_{\chi_p(-1)=-1} (1 - 2\chi(2))^2 \tau^2(\chi) L^2(1, \bar{\chi})$$

$$= -\frac{1}{\pi^2} \sum_{\chi_p(-1)=-1} (1 - 4\chi(2) + 4\chi(4)) \tau^2(\chi) L^2(1, \bar{\chi}) \tag{17}$$

From (17), lemma 1 and lemma 5 we deduce that

$$r(p) = \frac{1}{2} p + \frac{1}{\pi^2} \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \frac{1}{ab} \{S(1, ab; p) - S(1, -ab; p)$$

$$+ 4S(\bar{4}, ab; p) - 4S(\bar{4}, -ab; p) - 4S(\bar{2}, ab; p)$$

$$+ 4S(\bar{2}, -ab; p)\} + O(\ln^3 p) \tag{18}$$

Notice that

$$\sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \frac{1}{ab} S(m, ab; p) = \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \frac{1}{ab} S(ma, b; p).$$

Together with (18) this implies theorem 1.

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