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Moduli for stable actions of an abelian group on trees of projective lines

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Introduction

The general frame into which this article fits is the following:

Trees of projective lines (henceforth called TPL's) arise in several natural ways in the study of the boundary of the Deligne-Mumford compactification $\bar{\mathcal{M}}_g$ of the moduli space of curves, i.e. in relation with stable curves: Recall that a stable curve (over a field k) is a connected projective curve C/k with at most ordinary double points as singularities; moreover any irreducible component of C that is isomorphic to $\mathbb{P}_k^1$ has to meet the rest of the curve in at least three points.

A stable curve is called *totally degenerate* if all its irreducible components are rational. Blowing up as many of the singular points of a totally degenerate curve as possible without disconnecting it yields our first example of a TPL: a connected projective curve with all components isomorphic to $\mathbb{P}_k^1$ such that the intersection graph is a (finite) tree (the vertices of the intersection graph are the components, the edges correspond to intersection points). Moreover the TPL comes along with a *marking:* Each of the blown-up nodes determines a pair of (distinct!) points on the TPL. In this way we get a finite map from the moduli space B_{2g} of stable $2g$-pointed TPL's to the subspace $D_g \subset \bar{\mathcal{M}}_g$ of totally degenerate stable curves of (arithmetic) genus g. The B_{2g} are smooth projective rational varieties studied systematically in [GHP].

Infinite TPL's arise when the uniformization theory of Riemann surfaces is extended to stable curves (working over $k = \mathbb{C}$, now):

For example the universal covering of a totally degenerate curve C of genus g is an infinite TPL on which the fundamental group of C (isomorphic to the free group F_g of rank g) acts in a natural way.

Smooth Riemann surfaces of genus g also admit coverings with group of decktransformations isomorphic to F_g: Schottky uniformization; here the covering space is an open dense subset of the Riemann sphere $\hat{\mathbb{C}} = \mathbb{P}^1_{\mathbb{C}}$ which we view as a trivial TPL (but with nontrivial action of F_g!). It is shown in [GH] that also all other stable curves of genus g may be represented as a quotient of an open dense subset of a TPL by a discontinuous group isomorphic to F_g (the construction is, roughly speaking, a mixture of the two "extreme cases" described above). Moreover it is shown in [GH] that both the TPL and the F_g-action can be recovered from the relative position (i.e. the generalized cross ratios) of the fixed points of all (primitive) elements of F_g. Thus using the fixed points as markings and extending the construction of B_n in [GHP] to infinite dimensions one obtains a (huge) moduli space containing the "extended Schottky space", i.e. the space of Schottky uniformizations of stable Riemann surfaces.

The goal of this paper is now to embed this construction in a more general theory in the following way: We fix a (finitely generated) group Γ and ask for the moduli space that classifies all possible Γ-actions on TPL's that are stable in some suitable sense. If such a moduli space exists (as a complex provariety, say) we may consider the subspace of discontinuous actions with compact quotient; it will classify the uniformizations of stable complex projective curves by a group isomorphic to Γ. To classify Γ-actions on TPL's we associate with such an action a marking of the TPL on which the action takes place, i.e. a set (in general infinite) of points on the TPL determined in some way by the action. In that way we relate the classification of Γ-actions on TPL's to that of stable marked TPL's which was carried out in [H].

In this approach several technical difficulties arise that necessitate careful generalizations of the familiar notions: first, the fixed points of transformations that act by translating the components are "end points" of the TPL. Hence we have to work with TPL's that are compactified by adding the end points. One reason for the at first sight perhaps somewhat strange looking definition of a TPL in Section 2 is to include such points.

Secondly, an infinite TPL is not a projective variety and, due to the end points and also to the fact that a component may intersect infinitely many others, not even a scheme. A fortiori, our moduli space cannot be expected to be a scheme. But all occuring spaces turn out to be projective limits of projective varieties (so-called provarieties). Therefore we shall work with notions of TPL (and of intersection tree) that are stable under taking projective limits.

Unfortunately the method sketched above of associating with a Γ-action on a TPL the fixed points of the primitive elements only "works" for groups that are (algebraically) close to free groups (the precise condition is that the centralizer of any element different from the identity be cyclic). By this we mean that only for these groups we obtain in this way a bijection between stable actions and stable markings. Even more unfortunate, we do not know of any way of attaching stable markings to stable actions that would yield a bijection for all finitely generated groups (and we doubt that such a method exists at all). So we contend ourselves with the study of another natural assignment of (stable) markings to stable Γ-actions: the marking of an (arbitrarily chosen) Γ-orbit with trivial stabilizer. As we will show in Section 3 this yields a bijection (even an equivalence of categories) for abelian groups Γ.

The most important step in the proof of this theorem is the following result which is of some independent interest: If an abelian group acts on a compact TPL then there is a common fixed point for the whole group (see Thm. 2.23 for precise statement and exceptions).

For the proof of this result we pass from a TPL to its intersection graph, which is a tree. Since our TPL's are projective limits of projective curves, the intersection tree will also be a projective limit of usual trees. Therefore we first introduce in Section 1 a category of trees that contains the usual trees of graph theory, their projective limits, and also the Λ-trees investigated e.g. in [M]. The fixed point theorem is first proved for these trees.

The fundamentals on trees and TPL's were laid in [H]. We take here the opportunity to supplement the definition of a TPL in [H] in order to assure certain global topological properties that were tacitly assumed in [H] but turned out not to be deducible from the axioms there.

In the last two sections we describe in some detail the moduli spaces for the simplest cases, namely finite abelian groups, and the integers. Of course, a finite group has only finitely many different stable actions on TPL's; more interesting than the number of actions is the scheme structure on this moduli space (determined explicitly in Section 5). The moduli space of stable $\mathbb{Z}$-actions on TPL's is in Section 4 itself shown to be an (infinite and locally infinite) compact TPL. The subset $\bar{S}_1$ of discontinuous actions with compact quotient consists of the uniformizations of tori by an infinite cyclic group, i.e. in the form $\mathbb{C}^*/q^{\mathbb{Z}}$ for some q with $|q| \neq 1$.

The next group to be investigated would be $\mathbb{Z}^2$; here the corresponding moduli space, which is a two-dimensional provariety, contains the usual uniformizations of elliptic curves as quotients of the complex plane by a lattice (all elements of $\mathbb{Z}^2$ have to act parabolically, i.e. by translations). The irreducible components of this provariety are not all isomorphic to each other; in particular the base component (corresponding to actions on a single $\mathbb{P}^1$) is itself only a provariety, not a scheme (more precisely it is a $\mathbb{P}^1 \times \mathbb{P}^1$ blown up in infinitely many points). Our results concerning this moduli space are very incomplete, so we do not give details here.

1. Trees and their projective limits

As mentioned in the introduction we need a notion of "tree" that allows us to treat projective limits of usual (finite) trees. One obvious phenomenon is that between two vertices a and b in a projective limit of finite trees there may be infinitely many edges. This suggests to replace the path between a and b by the segment $S(a, b)$, i.e. the set of all vertices and edges between a and b, and to formulate axioms for the segments, which now are just subsets of the set V of all vertices and edges. This idea is also used in the concept of a Λ-tree, see [M], where a set of axioms different from ours is used (see Remark 1.2 for the comparison).

Once working with segments it is possible and even natural to drop the distinction between vertices and edges, see [M] and [H]. That we keep here this distinction is done with regard to intersection graphs of TPL's, to be introduced in Section 2 (Def. 2.13): they in a natural way have vertices (components) and edges (intersections), and to have them separated is very helpful in showing that a TPL in our definition is indeed the projective limit of its finite sub-TPL's. On the other hand, vertices and edges are not elements of V of different nature: we only impose (see 1.1(a)(vi) below) the completely symmetric condition that between any two vertices there is at least one edge, and conversely, between any two edges there is at least one vertex. In particular the "end point" of a segment may well be an edge.

1.1. DEFINITION.
(a) A *tree* is a tripel (V_c, V_s, S) where:
V_c is a set (the set of vertices)
V_s is a set (the set of edges)
$V := V_s \cup V_c$ is nonempty
$S: V \times V \to \mathscr{P}(V)$ is a map with the following properties:
(i) $S(a, a) = \{a\}$
(ii) $a, b \in S(a, b)$
(iii) $S(a, b) = S(b, a)$
(iv) $c \in S(a, b)$ implies $S(a, b) = S(a, c) \cup S(c, b)$
(v) For all $a, b, c \in V$ there is a unique $\mu = \mu(a, b, c) \in V$ such that

$$\{\mu\} = S(a, b) \cap S(a, c) \cap S(b, c)$$

μ is called the *median* of a, b and c
(vi) If a, b are different elements of V_s, then $S(a, b) \cap V_c \neq \emptyset$
If a, b are different elements of V_c, then $S(a, b) \cap V_s \neq \emptyset$
$S(a, b)$ is called the *segment* between a and b.

(b) (U_c, U_s, S') is a *subtree* of (V_c, V_s, S), if it is a tree, $U_s \subset V_s$, $U_c \subset V_c$ and $\forall a, b \in U$ one has $S'(a, b) = S(a, b) \cap U$

(c) (V_c, V_s, S) is called *finite*, if V is a finite set

(d) A *morphism* $\varphi: (V_c, V_s, S) \to (U_c, U_s, S')$ is a map $\varphi: V \to U$ with $\varphi(S(a, b)) = S'(\varphi(a), \varphi(b))$ for all $a, b \in V$

(e) φ is an *isomorphism* if $\varphi(V_s) = U_s$, $\varphi(V_c) = U_c$ and $\varphi: V \to U$ is bijective.

(One could give a definition of morphism such that a bijective morphism fulfills the condition of (e), but it would be rather complicated and unnecessary for our purpose.)

We will usually denote a tree (V_s, V_c, S) just as the set V.

1.2. REMARK. We want to show that the category of trees just defined contains the Λ-trees in the sense of [M] (for a totally ordered abelian group Λ) and hence in particular the graph theoretical trees (for $\Lambda = \mathbb{Z}$); this last statement is already shown in [H], Lemma 2.1.

PROPOSITION. *Let V be a set and $S: V \times V \mapsto \mathcal{P}(V)$ a map satisfying* (i), (ii) *and* (iii).

(a) *then* (iv) *and* (v) *implies*

(iv′) *for any a, b, c in V, $S(a, b) \cap S(a, c) = S(a, d)$ for some $d \in V$*

(v′) *for any a, b, c in V with $S(a, b) \cap S(a, c) = \{a\}$, $S(a, b) \cup S(a, c) = S(b, c)$*

(b) *if moreover $S(a, b)$ is a Λ-segment for any $a, b \in V$, then* (iv′) *and* (v′) *implies* (iv) *and* (v).

REMARK. It is possible (though may be somewhat artificial) to endow any Λ-tree T with a structure $T = T_s \cup T_c$ satisfying (vi).

Proof. (a) to show (iv′) let $d = \mu(a, b, c)$. Then $S(a, d) \subset S(a, b) \cap S(a, c)$; conversely let $x \in S(a, b) \cap S(a, c)$. If $x \in S(b, c)$, then $x = d$ by (v), so let $x \notin S(b, c)$. Assume $x \notin S(a, d)$ then by (iv) $x \in S(d, b) \cap S(d, c) \subset S(b, c)$, a contradiction.

To show (v′), note that by (v) $a = \mu(a, b, c) \in S(b, c)$, whence by (iv) $S(a, b) \cup S(a, c) = S(b, c)$.

(b)(iv) is a defining property of a Λ-segment. For (v) let $\mu \in V$ such that $S(a, b) \cap S(a, c) = S(a, \mu)$, and similarly $S(a, b) \cap S(b, c) = S(\mu', b)$; $S(a, c) \cap S(b, c) = S(a, \mu'')$.

Now $S(\mu, \mu') \cap S(\mu, \mu'') \subset S(a, b) \cap S(a, c) = S(a, \mu)$; but the intersection is also contained in $S(\mu, b)$, hence is equal to $\{\mu\}$. Then it follows from (v′) that $\mu \in S(\mu', \mu'')$. By symmetry we also have $\mu' \in S(\mu, \mu'')$ and $\mu'' \in S(\mu, \mu')$, hence all three coincide, as all segments are Λ-segments.

1.3. PROPOSITION. *For a finite subtree U of a tree V there exists a (canonical) morphism $\pi_U^V: V \to U$ such that $\pi|_U = \mathrm{id}_U$. π_U^V is called the projection from V onto U.*

Proof. For $x \in V$, let $U_x := \{y \in U | S(x, y) \cap U = \{y\}\}$, then $U_x \neq \emptyset$ as U is finite.

(1) $U_x = \{y\}$: Set $\pi_U^V(x) := y$

(2) $U_x = \{y_1, y_2\} \Rightarrow S(y_1, y_2) \cap U = S'(y_1, y_2\} = \{y_1, y_2\}$
$\Rightarrow$ w.l.o.g. $y_1 \in U_c, y_2 \in U_s$ (condition (vi)), let $\pi_U^V(x) := y_2$

(3) If $y_1, y_2, y_3 \in U_x$ are all distinct then $x = \mu(y_1, y_2, y_3) \Rightarrow x \in U \Rightarrow U_x = \{x\}$.
Then π_U^V is a morphism of trees.

1.4. PROPOSITION. *$U \subset U' \subset V$ finite subtrees* $\Rightarrow \pi_U^V = \pi_U^{U'} \circ \pi_{U'}^V$.
Proof. Easy computation.

1.5. DEFINITION. V a tree, $\mathfrak{A}_V :=$ system of finite subtrees. Then $\mathfrak{A}_V$ is a projective system (Prop. 1.4). Let every finite subtree have the discrete topology, then $\bar{V} := \varprojlim \mathfrak{A}_V$ is a topological space.

For $x = (x_U)_{U \in \mathfrak{A}_V}, y = (y_U)_{U \in \mathfrak{A}_V}$ let:

$$\bar{S}(x, y) = \{(a_U)_{U \in \mathfrak{A}_V} \in \bar{V} | a_U \in S(x_U, y_U) \, \forall U \in \mathfrak{A}_V\}$$

Let $\bar{V}_c = V_c, \bar{V}_s = \bar{V} \setminus V_c$.

1.6. PROPOSITION. *With the notation of Definition* 1.5 *one has:*

(i) $(\bar{V}_c, \bar{V}_s, \bar{S})$ *is a tree*

(ii) $\bar{V}$ *is a compact topological space*

(iii) *There is a canonical injective map* $i: V \to \bar{V}$ *with dense image; this induces a topology on* V.

(iv) *Every morphism of trees is continuous.*

(v) $\mu: \bar{V} \times \bar{V} \times \bar{V} \to \bar{V}$ *is continuous.*

(vi) *For* $U \in \mathfrak{A}_V$ *let* $\pi_U: \bar{V} \to U$ *be the projection; then* $\pi_U = \pi_U^{\bar{V}}$ *as defined in Proposition* 1.3.

(vii) $\bar{S}(a, b)$ *is closed for all* $a, b \in \bar{V}$.

Proof.

(i) + (vi) + (vii) are easy to prove, one only needs the fact, that every finite subset of V is contained in a finite subtree.

(ii) + (iii) hold by definition.

(iv) Let $\varphi: \bar{V} \to \bar{V}'$ be a morphism of trees. If $U' \subset \bar{V}'$ is a finite subtree, then one can find a finite subtree $U \subset \bar{V}$ and a morphism $\tilde{\varphi}: U \to U'$ such that

$$\tilde{\varphi} \circ \pi_U^{\bar{V}} = \pi_{U'}^{\bar{V}'} \circ \varphi$$

As every morphism of finite trees is continuous this shows that φ is continuous.

(v) As μ is continuous for every finite tree, the argument is the same as in (iv).

1.7. DEFINITION. A tree V is called *compact* if $V = \bar{V}$.

We now want to proof that every automorphism of a compact tree has a fixed point. For this we need some technical results, which are obvious for finite trees.

1.8. PROPOSITION. *V a tree;* $a, b, c, d \in V, b \neq c,\ c \in S(a, b),\ b \in S(c, d) \Rightarrow b, c \in S(a, d)$.

1.9. COROLLARY.

$$S(a, d) = S(a, c) \cup S(c, b) \cup S(b, d)$$

$$S(a, c) \cap S(c, b) = \{c\},\ S(c, b) \cap S(b, d) = \{b\},\ S(a, c) \cap S(b, d) = \emptyset.$$

Proof. (Prop. 1.8). $\mu := \mu(a, c, d)$, if $c \notin S(a, d)$ then $\mu \neq c, \mu \in S(c, d) = S(c, b) \cup S(b, d)$, as $b \in S(c, d)$.

(1) $\mu \in S(b, c)$
$c \in S(a, b) \Rightarrow c = \mu(a, b, c)$
$\mu \in S(a, c)$ by definition, $S(a, b) = S(a, c) \cup S(c, b)$, as $c \in S(a, b) \Rightarrow \mu \in S(a, b) \Rightarrow \mu \in S(a, b) \cap S(a, c) \cap S(b, c) = \{\mu(a, b, c)\} = \{c\} \Rightarrow \mu = c$

(2) $\mu \in S(b, d) \Rightarrow \mu = b$ as above. Now $c \in S(a, b) = S(a, \mu)$ and $\mu \in S(a, c) \Rightarrow c, \mu \in S(a, c) \cap S(a, \mu) \cap S(c, \mu) \Rightarrow c = \mu$.

1.10. COROLLARY. $S(a, b) = S(c, d) \Leftrightarrow \{a, b\} = \{c, d\}$.

1.11. PROPOSITION.

$$\mu_1 = \mu(a, b, c),\ \mu_2 = \mu(b, c, d),\ \mu_3 = \mu(c, d, e)$$

$$\Rightarrow \mu(\mu_1, \mu_2, \mu_3) \in \{\mu_1, \mu_2, \mu_3\}$$

Proof. Let

$$S := S(b, c) \cup S(c, d) \cup S(d, b) = S(b, \mu_2) \cup S(c, \mu_2) \cup S(d, \mu_2) = S_1 \cup S_2 \cup S_3,$$

as $\mu_2 = \mu(b, c, d)$, then $\mu_1, \mu_3 \in S$. As

$$S_1 \cap S_2 = S_2 \cap S_3 = S_1 \cap S_3 = \{\mu_2\},$$

$\mu = \mu_2$ unless all μ_i are elements of the same S_j.

It follows easily from Corollary 1.9 that the median of such "collinear" points is one of the points.

1.12. THEOREM. *Let V be a compact tree,* $\varphi: V \to V$ *a bijective morphism,* $\emptyset \neq F \subset V$ *closed such that* $\varphi(F) = F$ *and F is closed under taking medians. Then there exists* $p \in F$ *such that* $\varphi^2(p) = p$ *and* $S(p, \varphi(p)) \cap F = \{p, \varphi(p)\}$.

1.13. LEMMA. *Theorem* 1.12 *is true for* $V = S(a, b)$, $a, b \in F$.

Proof. $S(\varphi(a), \varphi(b)) = \varphi(V) = S(a, b) \Rightarrow$ (Corollary 1.10) $\{\varphi(a), \varphi(b)\} = \{a, b\}$. If $\varphi(a) = a$ take $p = a$. If $\varphi(a) = b$, $\varphi(b) = a$ let $Z = \{z \in F \mid \mu(a, z, \varphi(z)) = z\}$. Z is closed because φ and μ are continuous and V is a Hausdorff space.

CLAIM 1. $x, y \in Z \Rightarrow F \cap S(x, y) \subset Z$.

Proof. $z \in S(x, y) \cap F$, w.l.o.g. $\mu(a, x, y) = x$, i.e. $x \in S(a, y)$, then $\varphi(y) \in S(y, b)$, because $y \in S(a, \varphi(y))$ by definition of Z. $z \in S(x, y) \subset S(a, y)$ because $x \in S(a, y) \Rightarrow \varphi(z) \in S(\varphi(y), b) \subset S(y, b)$. So $z \in S(a, y)$, $\varphi(z) \in S(y, b) \Rightarrow \mu(a, z, \varphi(z)) = z$.

For $z \in Z$ one has $\varphi^2(z) \in Z$ and for $z \notin Z$ one has $\varphi(z) \in Z$, therefore $F = Z \cup \varphi(Z)$. For $z \in Z$ let $M_z = S(z, \varphi(z)) \cap F$, then M_z is closed. Let $M := \bigcap_{z \in Z} M_z$.

CLAIM 2. $z, z' \in Z$, $z' \in M_z \Rightarrow M_{z'} \subset M_z$. If $z' \neq z, \varphi(z)$ one has $M_{z'} \neq M_z$.

Proof. $S(a, b) = S(a, z) \cup S(z, \varphi(z)) \cup S(\varphi(z), b) =: S_1 \cup S_2 \cup S_3$ and $S_1 \cap S_2 = \{z\}$, $S_2 \cap S_3 = \{\varphi(z)\}$, $S_1 \cap S_3 = \varnothing$ if $z \neq \varphi(z)$. $z' \in S_2 \Rightarrow z' \notin S_1 \Rightarrow \varphi(z') \notin S_3$. If $\varphi(z') \notin S_2 \Rightarrow \varphi(z') \in S_1 \Rightarrow \mu(a, z', \varphi(z')) = \varphi(z') \Rightarrow z' \notin Z$ (as $z' \neq \varphi(z')$). $M_{z'} \neq M_z$ because otherwise $S(z, \varphi(z)) = S(z', \varphi(z')) \Rightarrow \{z, \varphi(z)\} = \{z', \varphi(z')\}$ (Cor. 1.10). Therefore $M_z \subset M_{z'}$ or $M_{z'} \subset M_z \forall z, z' \in Z$.

CLAIM 3. $M \cap Z \neq \varnothing$.

Proof. If $M \cap Z = \varnothing$ one has $z_1, \ldots, z_n \in Z$ such that $\bigcap_{i=1}^n M_{z_i} \cap Z = \varnothing$, as $S(a, b)$ is compact and M_{z_i}, Z are closed $\Rightarrow M_{z_{i_0}} \cap Z = \varnothing$ for some i_0, as $M_z \subset M_{z'}$ or $M_{z'} \subset M_z \forall z, z' \in Z$, but $z_{i_0} \in M_{z_{i_0}} \cap Z$.

For $z_0 \in M \cap Z$ one has

$$z_0 \in M_z \forall z \in Z \Rightarrow M_{z_0} \subset M_z \forall z \Rightarrow M_{z_0} \subset M \subset M_{z_0} \Rightarrow M = M_{z_0}.$$

CLAIM 4. $\varphi^2(z_0) = z_0$.

Proof. $\varphi(z_0) \in S(z_0, b)$, as $z_0 \in Z \Rightarrow \varphi^2(z_0) \in S(\varphi(z_0), a) = S(a, z_0) \cup S(z_0, \varphi(z_0))$.

(1) $\varphi^2(z_0) \in S(z_0, \varphi(z_0)) = M_{z_0}$

$$\varphi^2(z_0) \in Z \Rightarrow M_{\varphi^2(z_0)} \subset M_{z_0} \Rightarrow M_{\varphi^2(z_0)} = M_{z_0} = M$$
$$\Rightarrow \text{(Claim 2) } \varphi^2(z_0) \in \{z_0, \varphi(z_0)\}.$$

(2) $\varphi^2(z_0) \in S(a, z_0)$

$$\varphi(z_0) \in S(z_0, b) \Rightarrow z_0 \in S(\varphi^2(z_0), \varphi(z_0))$$
$$\Rightarrow \varphi^{-1}(z_0) \in S(z_0, \varphi(z_0)) \subset S(\varphi(z_0), \varphi^2(z_0))$$
$$\Rightarrow \varphi^{-2}(z_0) \in S(z_0, \varphi(z_0))$$
$$\Rightarrow \text{(as in 1) } \varphi^{-2}(z_0) = z_0$$

CLAIM 5. $M_{z_0} = \{z_0, \varphi(z_0)\}$.

Proof. Let $z' \in S(z_0, \varphi(z_0)) \cap F \Rightarrow \varphi(z') \in S(\varphi(z_0), \varphi^2(z_0)) = S(z_0, \varphi(z_0))$. Now $z' \in Z$ or $\varphi(z') \in Z$ so $M_{z'} = M_{z_0}$ or $M_{\varphi(z')} = M_{z_0}$. Then $z' = z_0$ or $\varphi(z') = z_0$. In the latter case $z' = \varphi^{-1}(z_0) = \varphi(z_0)$. □

Proof (Theorem 1.12): Let $x \in F$. If $x = \varphi(x)$ take $p = x$. If $x = \varphi^2(x)$ use Lemma 1.13 for $V = S(x, \varphi(x))$. Otherwise let $y := \mu(x, \varphi(x), \varphi^2(x))$, then $\mu = \mu(y, \varphi(y), \varphi^2(y)) \in \{y, \varphi(y), \varphi^2(y)\}$ (Prop. 1.11).

(1) $\mu = \varphi(y)$

$$\Rightarrow S(y, \varphi^2(y)) = S(y, \varphi(y)) \cup S(\varphi(y), \varphi^2(y))$$

$$\mu(\varphi(y), \varphi^2(y), \varphi^3(y)) = \varphi(\mu) = \varphi^2(y)$$

$$\Rightarrow S(\varphi(y), \varphi^3(y)) = S(\varphi(y), \varphi^2(y)) \cup S(\varphi^2(y), \varphi^3(y))$$

$$\Rightarrow \text{(Prop. 1.11) } S(y, \varphi^3(y)) = S(y, \varphi(y)) \cup S(\varphi(y), \varphi^2(y)) \cup S(\varphi^2(y), \varphi^3(y))$$

$$\Rightarrow \text{(induction) } S(y, \varphi^n(y)) = S(y, \varphi(y)) \cup \cdots \cup S(\varphi^{n-1}(y), \varphi^n(y)) \qquad (*)$$

CLAIM. $\varphi^n(y)$ converges to some $y_0 \in F$.

Proof. $(\varphi^n(y))_{n\in\mathbb{N}}$ has an accumulation point $y_0 \in F$, as F is compact. If there is another one $y_0' \neq y_0$ let $\mu_n := \mu(y_0, y_0', \varphi^n(y)) \in F$; then y_0, y_0' are accumulation points of $(\mu_n)_{n\in\mathbb{N}}$, as μ is continuous and $\mu(y_0, y_0', y_0) = y_0$, $\mu(y_0, y_0', y_0') = y_0'$. Now for $n < m$

$$S(\mu_n, \mu_m) = S(\mu_n, \mu_{n+1}) \cup \cdots \cup S(\mu_{m-1}, \mu_m) \qquad (**)$$

because $\{\mu(a, b, z) | z \in S(x, x')\} = S(\mu(a, b, x), \mu(a, b, x'))$. (Take $a = y_0$, $b = y_0'$, $x = \varphi^n(y)$, $x' = \varphi^m(y)$ and use $(*)$).

If $S(y_0, y_0') \neq \{y_0, y_0'\}$ take $a \in S(y_0, y_0')$, $a \neq y_0, y_0'$. Then $S(y_0, a)$, $S(a, y_0')$ are closed so one can choose open neighborhoods $U(y_0)$, $U'(y_0')$ such that $U(y_0) \cap S(a, y_0') = \emptyset$, $U'(y_0') \cap S(a, y_0) = \emptyset$. If $\mu_n, \mu_m \in U(y_0)$, then

$$\mu_n, \mu_m \in S(y_0, y_0') \backslash S(a, y_0'),$$

so

$$\begin{aligned} \mu_n, \mu_m \in S(a, y_0) &\Rightarrow S(\mu_n, \mu_m) \subset S(a, y_0) \\ &\Rightarrow (**) \mu_i \in S(a, y_0) \forall n \leqslant i \leqslant m \\ &\Rightarrow \mu_i \notin U'(y_0') \forall n \leqslant i \leqslant m \\ &\Rightarrow \mu_i \notin U'(y_0') \forall n_0 \leqslant i, \end{aligned}$$

as y_0 is an accumulation point of $(\mu_n)_{n\in\mathbb{N}}$ and hence there are infinitely many m with $\mu_m \in U(y_0)$.

If $S(y_0, y_0') = \{y_0, y_0'\}$ then with the same argument as above $\mu_m = y_0 \, \forall m \geqslant m_0$. So $(\varphi^n(y))_{n\in\mathbb{N}} \xrightarrow[n\to\infty]{} y_0$ and therefore $\varphi(y_0) = y_0$.

(2) $\mu = y$

$$\begin{aligned} y \in S(\varphi(y), \varphi^2(y)) &\Rightarrow \varphi^n(y) \in S(\varphi^{n+1}(y), \varphi^{n+2}(y)) \\ &\Rightarrow S(\varphi(y), \varphi^2(y)) = S(\varphi(y), y) \cup S(y, \varphi^2(y)) \\ &\Rightarrow S(\varphi^2(y), \varphi^3(y)) = S(\varphi^2(y), \varphi(y)) \cup S(\varphi(y), \varphi^3(y)) \\ &\Rightarrow S((\varphi^3(y), \varphi^4(y)) = S(\varphi^3(y), \varphi^2(y)) \cup S(\varphi^2(y), \varphi^4(y)) \end{aligned}$$

Using Proposition 1.8 one can deduce, that $\varphi^2(y) \in S(y, \varphi^4(y))$. Using Part 1 shows, that φ^2 has a fixed point a. Let $b = \varphi(a) \Rightarrow \varphi(S(a, b)) = S(\varphi(a), \varphi(b)) = S(b, a)$, so Lemma 1.13 shows the theorem in this case.

(3) $\mu = \varphi^2(y)$. Replace φ by φ^{-1} then

$$\mu(y, \varphi^{-1}(y), \varphi^{-2}(y)) = \varphi^{-2}(\mu(\varphi^2(y), \varphi(y), y) = \varphi^{-2}(\varphi^2(y)) = y$$

so one can use Part 2. □

1.14. THEOREM. *Every automorphism $\varphi: V \to V$ of a compact tree has a fixed point.*

Proof. Theorem 1.12 gives us a $p \in V$ such that $S(p, \varphi(p)) = \{p, \varphi(p)\}$. If $p \in V_s$ (resp. V_c) then $\varphi(p) \in V_s$ (resp. V_c) by definition of isomorphism and $S(p, \varphi(p)) \cap V_c$ (resp. $S(p, \varphi(p)) \cap V_s$) $= \varnothing$, so by condition (vi) of Definition 1.1 we get $p = \varphi(p)$.

1.15. THEOREM. *Let Γ be a commutative group acting on a compact tree V by bijective morphisms then there exist $p, p' \in V$ such that $S(p, p') = \{p, p'\}$ and $\gamma(\{p, p'\}) = \{p, p'\}$ for all $\gamma \in \Gamma$. If Γ acts by automorphisms we have $p = p'$.*

Proof. If there exist $\gamma_0 \in \Gamma, a, b \in V, a \neq b, \gamma_0(a) = b, \gamma_0(b) = a, S(a, b) = \{a, b\}$, then $\gamma(\{a, b\}) = \{a, b\} \, \forall \gamma \in \Gamma$, because:

Let $a' = \gamma(a), b' = \gamma(b)$ then $\gamma_0(a') = \gamma \circ \gamma_0 \circ \gamma^{-1}(a') = \gamma \circ \gamma_0(a) = \gamma(b) = b'$ and $\gamma_0(b') = a'$, so

$$\gamma_0(S(a', b')) = S(a', b') = \{a', b'\}.$$

Let $\mu = \mu(a, b, a') \in S(a, b) = \{a, b\}$. *w.l.o.g.* $\mu = a \Rightarrow a \in S(a', b) \Rightarrow \gamma_0(a) = b \in S(\gamma_0(a'), \gamma_0(b)) = S(b', a) \Rightarrow$ (Prop. 1.8) $a, b, \in S(a', b') = \{a', b'\}$.

If no such γ_0 exists, let $F_\gamma = \{p \in V | \gamma(p) = p\}$, then one can apply Theorem 1.12, so $F_\gamma \neq \varnothing$.

If $\bigcap_\gamma F_\gamma = \varnothing$ one can find $\gamma_1, \ldots, \gamma_n \in \Gamma$ such that $\bigcap_{i=1}^{n-1} F_{\gamma_i} = F \neq \varnothing$ and $F \cap F_{\gamma_n} = \varnothing$, as F_γ is compact. Now γ_n fixes F as Γ is commutative so Theorem 1.12 gives us a $p \in F$ such that

$$S(p, \gamma_n(p)) \cap F = \{p, \gamma_n(p)\}$$

and $\gamma_n^2(p) = p$.

If $p \neq \gamma_n(p)$ then by Lemma 1.13 one can find $q \in S(p, \gamma_n(p))$ such that $\gamma_n^2(q) = q$ and

$$S(q, \gamma_n(q)) = \{q, \gamma_n(q)\}.$$

The case $q \neq \gamma_n(q)$ is already done so let $q = \gamma_n(q)$. $\gamma_i(q) \in S(p, \gamma_n(p))$ so w.l.o.g.

$$\gamma_i(q) \in S(p, q) \Rightarrow \gamma_n \circ \gamma_i(q) \in S(\gamma_n(p), q).$$

But $\gamma_n \circ \gamma_i(q) = \gamma_i \circ \gamma_n(q) = \gamma_i(q)$ so $\gamma_i(q) \in S(p, q) \cap S(\gamma_n(p), q) = \{q\}$, so $q \in F$ and $F \cap F_{\gamma_n} \neq \varnothing$.

1.16. REMARK. Note that for the trees considered here the situation is more complicated than for Λ-trees because F_γ need not be "connected". In particular Lemma 12 of [M] does not hold for non commuting automorphisms.

Moreover in general F_γ is not a subtree of V, as condition (vi) of Definition 1.1 is not fulfilled by F_γ.

1.17. DEFINITION. For a, $b \in V$ let $\pi_{S(a,b)}: V \to S(a, b)$ be defined by $\pi_{S(a,b)}(r) := \mu(a, b, r)$.

The following result will be used later:

1.18. LEMMA.
(1) $\pi_{S(a,b)}$ is a projection onto $S(a, b) =: S$.
(2) π_S *is continuous*
(3) *Let* $\gamma, \delta \in \Gamma$, Γ *abelian and* $\gamma(a) = a$, $\gamma(b) = b$; $p \in \pi_S(\delta(S) \backslash S) \Rightarrow \gamma(p) = p$

Proof. (1) and (2) are trivial; to prove (3) let $a' = \pi_S(\delta(a))$, $b' = \pi_S(\delta(b))$; then $a' = \mu(a, b, \delta(a))$ and

$$\gamma(a') = \gamma(\mu(a, b, \delta(a)) = \mu(\gamma(a), \gamma(b), \gamma(\delta(a))) = \mu(a, b, \delta(a)) = a'$$

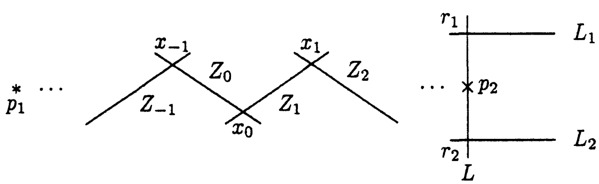
If $a' = b'$ then $\pi_S(\delta(S)) = S(a', b') = a'$.

If $a' \neq b'$ then $\delta(S) = S(\delta(a), a') \cup S(a', b') \cup S(b', \delta(b))$ and $\delta(S) \backslash S = S(\delta(a), \delta(b)) \backslash S(a', b')$, so $\pi_S(\delta(S) \backslash S) = \{a', b'\}$, as $\pi_S(S(\delta(a), a') = \{a'\}$, $\pi_S(S(b', \delta(b)) = \{b'\}$.

2. Trees of projective lines and intersection trees

To understand the definition of a TPL below recall that we want the category of TPL's to be closed under projective limits. Since a projective limit of projective curves is almost never a scheme we have to define TPL's as locally ringed spaces with certain properties. Later (Lemma 2.19) we shall show that any TPL is open and dense in a provariety, i.e. a projective limit of projective curves (see [H], Section 1 for some basic properties of provarieties).

2.1. EXAMPLE. Consider the following projective limit of finite TPL's:



This is to be understood as follows:

- the "end point" p_1 is an irreducible component; the local ring at p_1 consists only of the constant functions
- the middle part is a chain of $\mathbb{P}^1$-components.
- every neighbourhood of p_1 (resp. p_2) contains infinitely many $\mathbb{P}^1$-components of the middle part.
- nevertheless p_2 is a smooth point on L.

Due mainly to the presence of points like p_1 and p_2 in TPL's, the definition of the intersection graph of a TPL is much more subtle than in the finite case, and we avoid it in our definition of a TPL. Instead we characterize the tree-like nature of a TPL in terms of the *segments* $S(x, y)$ (see Definition 2.2(a)(iv) and (v) below); thus the following definition somewhat parallels that of a tree in 1.1:

2.2. DEFINITION.

(a) Let k be a field; a connected locally ringed space $(C, \mathcal{O})$ over k is called a *tree of projective lines* (*TPL*) over k if:

(i) for any closed point $x \in C$, the local ring $\mathcal{O}_{C,x}$ is isomorphic to $k, k[t]_{(t)}$ or $(k[s,t]_{/(s\cdot t)})_{(s,t)}$.

(ii) every irreducible component of C is isomorphic to $\mathbb{P}^1_k$ or Spec k. The union of the $\mathbb{P}^1$-components is dense in C.

(iii) For any $x \in C$, $C \backslash \{x\}$ has at most two connected components.

(iv) For $x, y \in C$ the intersection $S(x, y)$ of all closed connected subsets of C containing x and y is connected.

(v) $c \in S(a, b)$, $a \in S(b, c)$, $b \in S(a, c) \Rightarrow a, b, c \in L$, L component of C.

(vi) Let D be a connected component of $C \backslash S(x, y)$, then $\bar{D} \cap S(x, y)$ is a single point.

(vii) every connected component of an open set is open.

(b) A closed point $x \in C$ is called *special*, if $C \backslash \{x\}$ is not connected.
(c) A point $x \in C$ is called an *endpoint* of C, if $\mathcal{O}_{C,x} \cong k$ and $C \backslash \{x\}$ is connected.

2.3. REMARK. Properties (vi) + (vii) control the global topology of C. Without them there are some strange examples of locally ringed spaces C such that Lemma 2.20 (resp. Prop. 5 of [H]) is not true.

In Example 2.1, p_1 is an end point; the special points are p_2 and the intersection points of $\mathbb{P}^1$-components.

2.4. DEFINITION. For $X, Y \subset C$ let $S(X, Y) = S(X \cup Y)$ be the intersection of all closed connected subsets of C containing $X \cup Y$. $S(X, Y)$ is called the *segment* between X and Y. It is easy to see that one has:

2.5. LEMMA: *$S(X, Y)$ is connected.*

In the first place, a morphism of TPL's is simply a morphism of the locally ringed spaces; but as in [H], Section 3, we shall only consider contractions:

2.6. DEFINITION. A (contraction) morphism of TPL's C, C' is a morphism $\pi: C \to C'$ of locally ringed spaces satisfying

(i) for each $\mathbb{P}^1$-component L of C, $\pi|C$ is either constant or an isomorphism onto $\pi(L)$
(ii) for each $\mathbb{P}^1$-component L' of C' there is at most one $\mathbb{P}^1$-component L of C such that $\pi(L) = L'$.

We now want to define the intersection tree of a *TPL*.

For this we need some technical results, mainly to show that there are no topological pathologies around.

2.7. LEMMA. *For $a, b \in C$ there is a unique morphism $\pi_{S(a,b)}: C \to S(a, b)$ such that $\pi|_{S(a,b)} = \mathrm{id}_{S(a,b)}$. One has:*

$$\pi_{S(a,b)}(S(c, d)) = S(\pi_{S(a,b)}(c), \pi_{S(a,b)}(d))$$

Proof. Let $D \subset C \backslash S(a, b)$ a connected component, then by property (vi) $\bar{D} \cap S(a, b) = \{p_D\}$. Let $\pi|_D = p_D$. Let $\pi|_{S(a,b)} = \mathrm{id}_{S(a,b)}$, then π is defined everywhere and it is easy to see that π is continuous (using property (vii) of Definition 2.2). The formula holds by definition of $\pi_{S(a,b)}$ and S.

Uniqueness follows from the fact that by Definition 2.6 $\pi_{S(a,b)}$ must be constant on any $\mathbb{P}^1$-component of $C \backslash S(a, b)$.

2.8. LEMMA. $x \in S(a, b) \Rightarrow S(a, b) = S(a, x) \cup S(x, b)$. *Let $X = S(a, x) \cap S(x, b)$, then $X = \{x\}$ or $X = L$, where L is a component of C. In the latter case clearly $x \in L$.*

Proof. "$\subset$": $S(a, x) \cup S(x, b)$ is a closed connected set containing a and b.
"$\supset$": $S(a, b)$ is a closed connected set containing a, b and x.

Let $L \subset X$ be a component, then $S(a, L) \cap S(L, b) \subset X$.

Let $x \in S(a, L)$, then $S(a, x) \subset S(a, L) \subset S(a, x)$, so $S(a, x) = S(a, L)$. We first want to prove that $x \in L$. If $a \in L$, then $S(a, L) = L$. Suppose $a, x \notin L$. Let D be the connected component of $C \backslash S(L, L) = C \backslash L$ with $a \in D$, then $\bar{D} \backslash D \subset L$ and $\bar{D} \cap L \neq \emptyset$, so $D \cup L$ is a closed connected subset of C containing a and $L. \Rightarrow x \in S(a, L) \subset D \cup L \Rightarrow x \in D \Rightarrow S(a, x) \subset \bar{D}. \Rightarrow L \subset \bar{D}$ and $L \cap \bar{D} = \{\text{point}\}$, so L is a component isomorphic to Spec k.

Let $p = \pi_{S(x,L)}(a)$, then $p \in S(x, L)$, and

$$L \in S(a, x) \Rightarrow L = \pi_{S(L,x)}(L) \in S(\pi_{S(L,x)}(a), \pi_{S(L,x)}(x)) = S(p, x)$$

$$x \in S(a, L) \Rightarrow x = \pi_{S(L,x)}(x) \in S(\pi_{S(L,x)}(L), \pi_{S(L,x)}(a)) = S(L, p)$$

by Lemma 2.7. So by property (v) of Definition 2.2 we get: $x \in S(L, p)$, $L \in S(p, x)$, $p \in S(x, L) \Rightarrow p, x, L \in L', L'$ component of $C \Rightarrow L' = L$ and $x = p = L$. So for every component $L \subset X$ we know that $x \in L$.

Take $x' = (\text{generic point of } L) \in X. \Rightarrow S(a, x) = S(a, L) = S(a, x')$, $S(x, b) = S(x', b)$ and $S(a, x') \cap S(x', b) = X$. Using the first part for x' we get that $X = L$.

2.9. LEMMA. *$a_1, a_2, a_3 \in C, X = S(a_1, a_2) \cap S(a_2, a_3) \cap S(a_3, a_1). \Rightarrow X$ is a component of C or X is a single point.*

Proof. Let $S(a_i, a_j) = S_{ij}$. Assume $S_{12} \cap S_{13} \cap S_{23} = \emptyset$. Let $T_{12} = S_{12} \cap S_{23}$, $T_{13} = S_{13} \cap S_{23}$, then $T_{12} \cap T_{13} = \emptyset$. $T_{12} \cup T_{13} = (S_{12} \cup S_{13}) \cap S_{23}$. $S_{12} \cup S_{13}$ is a closed connected subset containing a_2 and $a_3 \Rightarrow S_{23} \subset S_{12} \cup S_{13}$. $\Rightarrow (S_{12} \cup S_{13}) \cap S_{23} = S_{23} = T_{12} \cup T_{13}$ is not connected.

Let $S = S_{12} \cap S_{13} \cap S_{23} \neq \emptyset$. It is easy to see that for $x, y \in S$ one has $S(x, y) \subset S$. If $x \neq y$ then $S(x, y)$ contains a component L of C. By Lemma 2.8 one has: $S_{ij} = S(a_i, L) \cup S(L, a_j)$ and $S(a_i, L) \cap S(L, a_j) = L$. Using Lemma 2.8 one can compute that

$$\begin{aligned} S_{12} \cap S_{13} \cap S_{23} = {} & [S(a_1, L) \cup S(a_2, L)] \cap [S(a_1, L) \cup S(L, a_3)] \\ & \cap [S(a_2, L) \cup S(L, a_3)] = L \end{aligned}$$

2.10. LEMMA. *$x \in S(a, b), x \neq a, b, S(a, b) \backslash \{x\}$ not connected $\Rightarrow x$ is special.*

Proof. $C \backslash S(a, b) = \bigcup_{i \in I} D_i$, where D_i are the connected components. $S(a, x) \cap S(x, b) = \{x\}$, as $S(a, b) \backslash \{x\}$ is not connected.

Let

$$I_a = \{i \in I \mid \pi_{S(a,b)}(D_i) \in S(a, x) \backslash \{x\}\}$$

$$I_b = \{i \in I \mid \pi_{S(a,b)}(D_i) \in S(x, b) \backslash \{x\}\}$$

$$I_x = \{i \in I \mid \pi_{S(a,b)}(D_i) = x\}$$

and

$$D_a = \bigcup_{i \in I_a} D_i,\ D_b = \bigcup_{i \in I_b} D_i,\ D_x = \bigcup_{i \in I_x} D_i$$

Then $C \backslash S(a, b)$ is the disjoint union of D_a, D_b and D_x. $S(a, x) \backslash \{x\}$ is a closed connected subset of $S(a, b) \backslash \{x\}$

$$\Rightarrow \tilde{D}_a = \pi^{-1}_{S(a,b)}(S(a, x) \backslash \{x\}) = D_a \cup S(a, x) \backslash \{x\}$$
$$\tilde{D}_b = \pi^{-1}_{S(a,b)}(S(x, b) \backslash \{x\}) = D_b \cup S(x, b) \backslash \{x\}$$

are closed connected subsets of $C \backslash \{x\}$.

It follows that $C \backslash \{x\} = \tilde{D}_a \cup \tilde{D}_b \cup \bigcup_{i \in I_x} D_i$ are the connected components of $C \backslash \{x\}$. Of course $\tilde{D}_a$ and $\tilde{D}_b$ are nonempty, hence x is special. (Property (iii) of Definition 2.2 shows moreover that $I_x = \emptyset$).

2.11. COROLLARY. *Let $x \in C$ with local ring k. Then: x is an endpoint of $C \Leftrightarrow (x \in S(a, b) \Rightarrow x \in \{a, b\})$.*

Proof. "$\Rightarrow$": Suppose $x \notin \{a, b\}$. $S(a, x) \cap S(x, b) = \{x\}$ by Lemma 2.8, so $S(a, b) \backslash \{x\}$ is not connected. By Lemma 2.10, $C \backslash \{x\}$ is not connected, so x is not an endpoint.

"$\Leftarrow$": Suppose $C \backslash \{x\} = C_1 \cup C_2$ are the connected components. Choose $a_i \in C_i$, then $x \notin S(a_1, a_2)$ as $x \notin \{a_1, a_2\}$. It follows from property (vi) of Definition 2.2 that $\bar{C}_i = C_i \cup \{x\}$, so $\bar{C}_i \cap S(a_1, a_2) = C_i \cap S(a_1, a_2)$. $\Rightarrow$ $S(a_1, a_2) = [C_1 \cap S(a_1, a_2)] \cup [C_2 \cap S(a_1, a_2)]$ is not connected.

2.12. COROLLARY. *$x \in C$ with local ring $(k[x, y]/(xy))_{(x,y)} \Rightarrow x$ is special.*

2.13. REMARK. The reverse statement of Lemma 2.10 is not true. In Example 2.1 the point p_2 is special, but $S(L_1, L_2) \backslash \{p_2\}$ is connected.

Now we can define the intersection tree of a *TPL*:

2.14. DEFINITION. Let C be a *TPL* over k. The *intersection tree*

$$T(C) = (V_c(C),\ V_s(C),\ S_C)$$

is defined in the following way:

$$V_c = \{\mathbb{P}^1\text{-components of } C\}$$

$$V_s = \{x \in C | x \text{ is special}\} \cup \{x \in C | x \text{ is an endpoint}\}$$

$$V(C) = V_c(C) \,\dot{\cup}\, V_s(C)$$

For X, $Y \in V(C)$ let

$$S_C(X, Y) = \{X, Y\} \cup \{L \in V_c(C) | L \subset S(X, Y)\} \cup A(X, Y)$$

where

$$A(X, Y) \subset V_s(C) \cap S(X, Y)$$

and $x \in A(X, Y) \Leftrightarrow S(X, Y) \setminus \{x\}$ is not connected.

2.15. EXAMPLE. Let C be the TPL of Example 2.1, then

$$V_c(C) = \{L, L_1, L_2, Z_i : i \in \mathbb{Z}\}$$
$$V_s(C) = \{p_2, r_1, r_2, x_i : i \in \mathbb{Z}\} \cup \{p_1\}$$
$$S_C(p_1, p_2) = \{p_1, p_2, x_i, Z_i\}$$
$$S_C(L_1, L_2) = \{L_1, r_1, L, r_2, L_2\} \not\ni p_2$$

2.16. PROPOSITION. *$T(C) = (V_c(C), V_s(C), S_C)$ is a tree.*

Proof. Properties (i)–(iii) of Definition 1.1 are trivial, properties (iv) and (v) are shown by Lemma 2.8 resp. Lemma 2.9.

To show (vi) let $a, b \in V_s(C)$: Let X be an irreducible component of $S(a, b)$. If $X \cong \mathbb{P}^1_k$, then X is a component of C, so $X \in S_C(a, b) \cap V_c(C)$. Suppose every irreducible component of $S(a, b)$ is a point. Suppose $X = \{x\}$ and $x \in L$, L a $\mathbb{P}^1$-component of C. $S(a, x) \cap S(x, b) = \{x\}$, so x is special by Lemma 2.10 and $C \setminus \{x\} = D_1 \cup D_2$, where D_i are the connected components. Lemma 2.10 also shows that w.l.o.g. $a \in D_1$, $b \in D_2$, also w.l.o.g. $L \setminus \{x\} \subset D_1$. Let $Z = \overline{D_1 \setminus L} \cap (L \setminus \{x\})$, then $Z = \{z\}$, as D_1 is connected and by property (vi) of 2.2. Now $L \not\subset S(a, z)$, as $S(a, z) \subset \overline{D_1 \setminus L}$, $L \not\subset S(a, x)$ but $S(a, x) \cup S(a, z)$ is a connected closed subset containing x and z, so $S(x, z) \subset [S(a, x) \cup S(a, z)] \cap L = \{x, z\}$ is not connected.

This shows that we can assume that every $x \in S(a, b)$, $x \neq a, b$ is a component of C. Let $\pi = \pi_{S(a,b)}$, then $\pi^{-1}(x) = \{x\}$, by property (iii) of 2.2. $S(a, b) \setminus \{a, b\}$ is open in $S(a, b)$, so $\pi^{-1}(S(a, b) \setminus \{a, b\}) = S(a, b) \setminus \{a, b\}$ is open in $C \Rightarrow C \setminus S(a, b) \cup \{a, b\}$ is closed in C. As this set is dense by property (ii) of 2.2 we know that $C \setminus S(a, b) \cup \{a, b\} = C$ or $S(a, b) = \{a, b\}$. Then $S(a, b)$ is not connected and the first part is shown.

Let $L_1, L_2 \in V_c(C)$, $L_1 \neq L_2$:

Let $D =$ connected component of $C \setminus L_1$ with $D \cap L_2 \neq \emptyset$, then $\bar{D} \cap L_1 = \{x\}$ and $x \in S(L_1, L_2)$, $S(L_1, x) \cap S(x, L_2) = S(L_1, x) \cap L_2 = \{x\}$, so x is special and $x \in S_C(L_1, L_2)$.

2.17. REMARK. The map $C \to T(C)$ is functorial. This is shown in [H], Proposition 4.

2.18. DEFINITION. Let $V = (V_s, V_c, S)$ a tree, $v \in V$, v is called an *endpoint* of V if $\forall a, b \in V\, v \in S(a, b) \Rightarrow v \in \{a, b\}$.

2.19. LEMMA. *Let $U \subset T(C)$ be a finite subtree of $T(C)$ such that every endpoint of U is in U_c. Then there is a finite TPL C_U with intersection tree U and a projection $\pi: C \to C_U$ such that $T(\pi): T(C) \to T(C_U) = U$ is equal to π_U as defined in* 1.3.

Proof. As U is finite and every endpoint of U is a vertex, U is an ordinary tree. So it is easy to find a finite $TPL\, C_U$ with intersection tree U. $U_c \subset V_c$ so one can identify the $\mathbb{P}^1$-components of C_U with $\mathbb{P}^1$-components of C.

To define $\pi: C \to C_U$ let L be a component of C. If $L \in U$, let π be given by the identification of the $\mathbb{P}^1$-components. If $L \notin U$, let $v = \pi_U(L) \in U$. If $v \in U_s$, then v corresponds to a special point in C_U, let $\pi(L) \equiv v$. If $v = L' \in U_c$, let $D \subset C$ be the connected component of $C \backslash L'$ with $L \cap D \neq \emptyset$, then $\bar{D} \cap L = \{p\}$. Let $\pi(L) = p$. It is easy to prove that this is the right π.

2.20. LEMMA. *Let $\mathfrak{W}_{T(C)}$ be the set of all finite subtrees U of $T(C)$ such that U has only endpoints in U_c. If C is a compact TPL then*

$$\varprojlim_{U \in \mathfrak{W}_{T(C)}} C_U = C$$

Proof. The proof is the same as that of Proposition 5 in [H].

2.21. PROPOSITION. *Let C be a compact TPL, then $T(C)$ is compact.*

Proof. Let us first assume that C has no endpoints. In this case

$$\overline{T(C)} = \varprojlim_{U \in \mathfrak{W}_{T(C)}} U$$

as any finite subtree U of $T(C)$ is contained in a finite subtree U' such that every endpoint of U' is in U'_c, as every endpoint of $T(C)$ is in $V_c(C)$. The morphisms $\pi_U: C \to C_U$ induce morphisms $T(\pi_U): T(C) \to T(C_U) = U$, so we get a morphism

$$\phi: T(C) \to \varprojlim_{U \in \mathfrak{W}_{T(C)}} U = \overline{T(C)}$$

By definition of $\overline{T(C)}$ we know that ϕ is bijective on $V_c(C)$. Proposition 1.6(iii) shows that ϕ is injective.

To show that ϕ is surjective, let $E = \{v \in T(C) | v \text{ is endpoint}\}$, then $E \subset V_c(C)$. As C is compact $C = \bigcup_{L_1, L_2 \in E} S(L_1, L_2)$. Let $T(E) = \bigcup_{v_1, v_2 \in E} \bar{S}_C(v_1, v_2) \subset \overline{T(C)}$.

As $E \subset V_c(C)$ it is easy to see that $T(E) = \overline{T(C)}$. Let $x = (x_U)_{U \in \mathfrak{W}_{T(C)}} \in \overline{T(C)} \setminus V_c(C)$, then there are $v_1, v_2 \in E$ such that $x \in \bar{S}_C(v_1, v_2)$. Let v_1, v_2 correspond to L_1, L_2 and let $C' = S(L_1, L_2)$ then $T(C') = \bar{S}_C(v_1, v_2)$ and

$$\varprojlim_{U \in \mathfrak{W}_{T(C')}} U = \bar{S}_C(v_1, v_2)$$

This shows that we can reduce the problem to chains C', where a chain is a TPL with components L_1, L_2 such that $S(L_1, L_2) = C'$. In this case let $\mathfrak{W}'_{T(C')} = \{U \in \mathfrak{W}_{T(C')} | L_1, L_2 \in U\}$, then

$$C' = \varprojlim_{U \in \mathfrak{W}'_{T(C')}} C_U$$

and

$$\overline{T(C')} = \varprojlim_{U \in \mathfrak{W}'_{T(C')}} U$$

Let $x = (x_U)_{U \in \mathfrak{W}'_{T(C')}} \in \overline{T(C')} \setminus V_c(C')$ then by construction $x_U \in U_s \forall U \in \mathfrak{W}'_{T(C')}$. Therefore x_U corresponds to a point $p_U \in C_U$, and clearly

$$p = (p_U)_{U \in \mathfrak{W}'_{T(C')}} \in \varprojlim_{U \in \mathfrak{W}'_{T(C')}} C_U = C'$$

If p is a special point of C' then p corresponds to x and $x \in T(C')$, so assume p is not special. In this case there is a $\mathbb{P}^1$-component $L \subset C'$ with $p \in L, p$ a smooth point on L. Let $U \subset T(C')$ a finite subtree such that $U_{C'} = \{L_1, L, L_2\}$, then $\pi^{C'}_{C_U}: C' \to C_U$ is an isomorphism on L, so $p_U = \pi^{C'}_{C_U}(p)$ is not a special point on C_U. This finishes the first case.

Now let C be an arbitrary compact TPL, and let $E = \{\text{endpoints of } C\}$. Define a new $TPL\, C'$ in the following way:

For every $e \in E$ take a $\mathbb{P}^1$-component L_e and a point $p_e \in L_e$ and replace e by the component L_e such that $\overline{C \setminus L_e} \cap L_e = \{p_e\}$. If C is compact then C' is a compact TPL without endpoints, so $T(C')$ is compact. $T(C) \subset T(C')$ and $T(C') \setminus T(C) = \{L_e | e \in E\}$. $\{L_e\} \subset T(C')$ is open, so $\{L_e | e \in E\} \subset T(C')$ is open. This shows that $T(C) = T(C') \setminus \{L_e | e \in E\}$ is closed in $T(C')$, so it is compact.

2.22. DEFINITION. For a group Γ acting on a $TPL\, C$ and a $\mathbb{P}^1$-component L of C let

$$\Gamma_L := \{\gamma \in \Gamma : \gamma(L) = L\}$$

$$\Gamma_L^0 := \{\gamma \in \Gamma_L : \gamma|_L = \mathrm{id}_L\}$$

$$\bar{\Gamma}_L := \Gamma_{L/\Gamma_L^0}$$

2.23. THEOREM. *Let Γ be an abelian group acting on a compact TPL C, then either all elements of Γ have a common fixpoint on C, or there is a $\mathbb{P}^1$-component $L \subset C$ such that $\Gamma_L = \Gamma$ and $\bar{\Gamma}_L \cong (\mathbb{Z}/2\mathbb{Z})^2$. In suitable coordinates $\bar{\Gamma}_L$ acts on L by $z \mapsto \pm z^{\pm 1}$.*

Proof. As C is compact $T(C)$ is compact and G acts on $T(C)$. By Theorem 1.15 the elements of Γ have a common fixpoint $p \in T(C)$. If p is a special point, then this is a fixpoint of Γ for the action on C. If p is a $\mathbb{P}^1$-component L of C then Γ/Γ_L is an abelian group acting effectively on a $\mathbb{P}^1$. Then either Γ/Γ_L has a common fixpoint, or Γ/Γ_L acts on $\mathbb{P}^1$ by $z \mapsto \pm z^{\pm 1}$.

2.24. THEOREM. *If L is a $\mathbb{P}^1$-component of C then one of the following cases occurs:*

(0) $\Gamma_L^0 = \Gamma$
(1) $\Gamma_L = \Gamma$, $\bar{\Gamma}_L = (\mathbb{Z}/2\mathbb{Z})^2$
(2) $\Gamma_L^0 \neq \Gamma$ and it exists $F \subset L$ consisting of one or two points such that

$$\pi_L(\gamma(L)) \in F \ \forall \gamma \in \Gamma \backslash \Gamma_L$$

Proof. Suppose $\Gamma_L^0 \neq \Gamma$, if $\bar{\Gamma}_L = (\mathbb{Z}/2\mathbb{Z})^2$ then $\bar{\Gamma}_L$ acts on L by $z \mapsto \pm z^{\pm 1}$, so there is no common fixpoint of $\bar{\Gamma}_L$ on L.

Suppose $\Gamma_L \neq \Gamma$. Let $\gamma \in \Gamma \backslash \Gamma_L$ and $p = \pi_L(\gamma(L))$. Choose $\varphi \in \Gamma_L$ such that $\varphi(p) \neq p$ then

$$\begin{aligned} p = \pi_L(\gamma(\varphi(L))) &= \pi_L(\varphi(\gamma(L))) = \varphi(\pi_{\varphi^{-1}(L)}(\gamma(L))) \\ &= \varphi(\pi_L(\gamma(L))) = \varphi(p) \neq p \end{aligned}$$

This shows that $\Gamma_L = \Gamma$.

So let $\bar{\Gamma}_L \neq (\mathbb{Z}/2\mathbb{Z})^2$. If $\bar{\Gamma}_L \neq \{1\}$ then there are one or two fixpoints of $\bar{\Gamma}_L$ on L as $\bar{\Gamma}_L$ is abelian. With the same argument as above we see that $\pi_L(\gamma(L))$ is a fixpoint of $\bar{\Gamma}_L$ for all $\gamma \in \Gamma \backslash \Gamma_L$.

If $\bar{\Gamma}_L = \{1\}$ let $\gamma \in \Gamma \backslash \Gamma_L$ and $F = \{\pi_L(\gamma(L)), \pi_L(\gamma^{-1}(L))\}$. Let $C \backslash L = \bigcup_{i \in I} D_i$, where D_i are the connected components. Let $\bar{D}_1 \cap L = \pi_L(\gamma(L))$, $\bar{D}_2 \cap L = \pi_L(\gamma^{-1}(L))$: Suppose there is a $\delta \in \Gamma \backslash \Gamma_L$ such that $\pi_L(\delta(L)) \notin F$; then $\delta(L) \cap \bar{D}_1 = \varnothing$, $\delta(L) \cap \bar{D}_2 = \varnothing$. Let D be the connected component of $C \backslash L$ such that $\delta(L) \cap D \neq \varnothing$; then $\gamma(\bar{D}) \subset D_1$ and $\delta(\bar{D}_1) \subset D$. But then $\gamma(\delta(L)) \subset \gamma(\bar{D}) \subset D_1$, $\delta(\gamma(L)) \subset \delta(\bar{D}_1) \subset D$, so $L \subset D_1 \cap D = \varnothing$, as Γ is abelian.

3. Stable group actions on TPL's

The aim of this section is to classify the possible actions of a given group Γ on TPL's. As usual in moduli problems this necessitates a suitable notion of stability. We propose the following:

3.1. DEFINITION. A Γ-action $\rho\colon \Gamma \to \mathrm{Aut}(C)$ on a TPL C over k is called *stable* if the following holds:

(i) there is $x \in C(k)$ with $\Gamma_x = \{1\}$, where $\Gamma_x := \{\gamma \in \Gamma\colon \rho(\gamma)(x) = x\}$ denotes the stabilizer of x.

(ii) if C' is a TPL over k, $\rho'\colon \Gamma \to \mathrm{Aut}(C')$ a Γ-action, $\varphi\colon C \to C'$ a surjective Γ-equivariant morphism, and $x' \in C'$ with $\Gamma_{x'} = \{1\}$, then φ is an isomorphism.

A point x as in (i) will be called *markable* if it is either non-special or an end point of C.

Very roughly speaking, the first condition ensures that the TPL is not too small for the given action, and the second that it is not too big.

To obtain a well defined moduli functor we extend this definition to families of TPL's and introduce the obvious equivalence relation:

3.2. DEFINITION. (a) Let S be a provariety and $\pi\colon C \to S$ a TPL over S. A homomorphism $\rho\colon \Gamma \to \mathrm{Aut}_S C/S$ $(:= \{\alpha \in \mathrm{Aut(C)}\colon \pi \circ \alpha = \pi\})$ is called a stable Γ-action on C if there is a section $x\colon S \to C$ such that for any $s \in S$ the restriction $\rho_s\colon \Gamma \to \mathrm{Aut}\, C_s$ $(C_s := \pi^{-1}(s))$ is a stable Γ-action on the TPL C_s over $k(s)$ with $x(s)$ as markable point.

(b) Two stable Γ-actions (C, ρ) and (C', ρ') on TPL's C, C' are called equivalent if there is an isomorphism $f\colon C \to C'$ such that $\rho' = f^* \circ \rho$, where $f^*(\alpha) := f \circ \alpha \circ f^{-1}$ for $\alpha \in \mathrm{Aut}(C)$.

This moduli functor is closely related to the functor of "(Γ, M)-TPL's" investigated in [H], Section 4: recall that for a set M with a given Γ-action, a (Γ, M)-TPL is a stable M-marked TPL (C, φ) together with a Γ-action $\rho\colon \Gamma \to \mathrm{Aut}(C)$ on C that is compatible with φ in the sense that $\rho(\gamma)\varphi(m) = \varphi(\gamma m)$ for all $\gamma \in \Gamma$, $m \in M$. Since for a given (Γ, M)-TPL (C, φ) the corresponding Γ-action ρ is unique (see the proof of [H], Prop. 8) we obtain a natural transformation of moduli functors "(Γ, M)-TPL's" $\to$ "stable Γ-actions on TPL's" which on the level of objects is given by $(C, \varphi, \rho) \mapsto (C, \rho)$.

We shall investigate this transformation in the special case $M = \Gamma$, the Γ-action being left multiplication. Our main result is:

3.3. THEOREM. *For an abelian group Γ, the map $(C, \varphi, \rho) \mapsto (C, \rho)$ induces an equivalence between the categories of isomorphy classes of stable Γ-equivariantly Γ-marked TPL's on the one hand and equivalence classes of stable Γ-actions on TPL's on the other hand.*

Together with [H], Proposition 9, which states that the functor "(Γ, M)-TPL's" is representable by a provariety B_M^Γ, this theorem means that for abelian Γ we have a fine moduli space for stable Γ-actions on TPL's.

We briefly recall the construction of B_M^Γ from [H], Section 4: it is based on the notion of the (generalized) cross ratio $\lambda_{x_1, x_2, x_3, x_4}$ of four mutually distinct points x_1, x_2, x_3, x_4 in a TPL C (over a field k): If L denotes the median component of x_1, x_2, x_3, in C, then $\lambda_{x_1,x_2,x_3,x_4}$ is defined to be the usual cross ratio of $\pi_L(x_1), \ldots, \pi_L(x_4)$ on the projective line L. These cross ratios satisfy the usual relations, see [GHP], (1.3)

$$\lambda_{x_1,x_2,x_3,x_4} = \lambda^{-1}_{x_2,x_1,x_3,x_4}$$

$$\lambda_{x_1,x_2,x_3,x_4} = 1 - \lambda_{x_2,x_3,x_4,x_1}$$

$$\lambda_{x_1,x_2,x_3,x_5} = \lambda_{x_1,x_2,x_3,x_4} \cdot \lambda_{x_1,x_2,x_4,x_5}$$

Now for a set M, a stable M-marked TPL (C, φ), and any four distinct elements $v_1, \ldots, v_4$ in M, we put

$$\lambda_{v_1,\ldots,v_4}(C, \varphi) := \lambda_{\varphi(v_1),\ldots,\varphi(v_4)}$$

In this way we get coordinates for M-TPL's: the subset of $(\mathbb{P}^1)^V$ ($V = \{(v_1, \ldots, v_4) \in M^4 | v_i \neq v_j, i \neq j\}$) defined by the above equations turns out to be a provariety B_M that classifies M-TPL's.

If a group Γ acts on M, then Γ-equivariant markings on TPL's with Γ-action are characterized by the additional equations

$$\lambda_{\gamma(v_1),\ldots,\gamma(v_4)} = \lambda_{v_1,\ldots,v_4}$$

for all $\gamma \in \Gamma$ and all $v_1, \ldots, v_4$. They define the sub-provariety B^Γ_M of B_M which classifies (Γ, M)-TPL's.

For the rest of this section, Γ will be an abelian group of at least three elements. The compatibility of our map $(C, \varphi, \rho) \mapsto (C, \rho)$ with morphisms of provarieties being obvious, the proof of the theorem is reduced to showing that for any provariety S the set $B^\Gamma_\Gamma(S)$ of isomorphism classes of (Γ, Γ)-TPL's over S is mapped bijectively onto the set $A_\Gamma(S)$ of equivalence classes of stable Γ-actions on TPL's over S. We shall treat first the case $S = \mathrm{Spec}(k)$ in some detail, leaving the easy extension of the arguments to the case of a family to the reader (see also remark 3.11).

So fix a field k. Our first aim is to show:

3.4 PROPOSITION. *For any $(C, \varphi, \rho) \in B^\Gamma_\Gamma(k)$, the corresponding Γ-action ρ on C is stable.*

The proof requires some preparation:

For $(C, \varphi, \rho) \in B^\Gamma_\Gamma(k)$, we know by theorem 2.23 that there is either a common fixed point $P \in C(k)$ of all $\gamma \in \Gamma$, or a $\mathbb{P}^1$-component L of C on which Γ acts through the Klein group V_4. In this case let P be an arbitrary non-special point of $L(k)$.

3.5 LEMMA. *Let $S := S(x, P)$ be the segment between $x := \varphi(1)$ and P. Then for any $\mathbb{P}^1$-component D of S with $D \cap \Gamma x = \emptyset$ there exists $\gamma \in \Gamma \backslash \{1\}$* such that $\gamma | D = \mathrm{id}_D$.

Proof. Since Γx makes C into a stable Γ-TPL, each $\mathbb{P}^1$-component of C is median component of three translates of x. Thus there are $\gamma_1, \gamma_2 \in \Gamma$ such that $D = \mu(x, \gamma_1(x), \gamma_2(x))$. We may assume that $\pi_D(\gamma_1(x))$ is different from $Q := \pi_D(x)$ and $Q' := \pi_D(P)$.

Case 1. $P \in D$.

If P is nonspecial, then $\gamma(D) = D$ for all $\gamma \in \Gamma$. Otherwise there is one further $\mathbb{P}^1$-component $\tilde{D}$ of C such that $\gamma(D) = D$ or $\tilde{D}$ for all $\gamma \in \Gamma$.

Since $\pi_D(\gamma_1(x)) = \pi_D(\gamma_1(Q)) \neq P$ by assumption, we must have $\gamma_1(D) = D$. Note that $\gamma_1(Q) = \pi_D(\gamma_1(x)) \neq Q$.

As $x \notin D$, there is a $\mathbb{P}^1$-component $D' \subset S$, $D' \neq D$, and there is $\gamma'_1 \in \Gamma$ such that $D' = \mu(x, \gamma'_1(x), \gamma_2(x))$. Then $\pi_D(\gamma'_1(x)) = \pi_D(x) \neq P$, thus $\gamma'_1(D) = D$, and moreover $\gamma'_1(Q) = Q$.

Now γ'_1 and γ_1 both act on D, but not with the same fixed points. Then they cannot commute unless $\gamma'_1 | D = \mathrm{id}_D$.

Case 2. $P \notin D$, $\gamma_1(D) = D$.

We may choose γ_2 in such a way that $\pi_D(\gamma_2(x)) = \pi_D(P) = Q'$. Then Q' is a fixed point of γ_1. Choose a $\mathbb{P}^1$-component D' of S with $D' \not\subset S(D, P)$, and choose γ'_1 such that $D' = \mu(x, \gamma'_1(x), \gamma_2(x))$. Then if $\gamma'_1(D) = D$, the same argument as in case 1 applies.

Next suppose that $\gamma'_1(D) \subset S(D, P)$, $\gamma'_1(D) \neq D$. Then $\gamma_1(\gamma'_1(D)) = \gamma'_1(D)$, and $\pi_{\gamma'_1(D)}(D)$, $\pi_{\gamma'_1(D)}(P)$ are fixed points of γ_1. Then we get a contradiction from

$$\gamma_1 \gamma'_1(Q) = \gamma_1(\pi_{\gamma'_1(D)}(D)) = \gamma'_1(Q) \neq \gamma'_1(\gamma_1(Q))$$

Finally suppose $D \subset S(\gamma'_1(D), P)$, $\gamma'_1(D) \neq D$. Again γ_1 acts on $\gamma'_1(D)$, implying this time that $Q = \pi_D(\gamma'_1(D))$ should be a fixed point of γ_1, which is false by assumption.

Case 3. $P \notin D$, $\gamma_1(D) \neq D$.

Changing the roles of $\gamma_1(x)$ and x if necessary we may assume that $D \subset S(\gamma_1(D), P)$ (otherwise $\pi_D(\gamma_1(x))$ had to be one of Q, Q'). We claim that the union $\tilde{S}$ of all $S(\gamma_1^n(D), P)$, $n \geqslant 1$ is a chain of projective lines on which γ_1 acts: $\tilde{S}$ is obviously γ_1-invariant. If a, b, c are in $\tilde{S}$ we find $n \geqslant 0$ such that a, b, c are all in $\gamma_1'^n(S(P, D))$ since $S(\gamma_1'^n(D), P)$ is contained in $S(\gamma_1'^{n+1}(D), P)$. But if any three points of a TPL lie on a straight line, the TPL is itself a chain. Note that $\tilde{S} \cap S = S(D, P)$. Now choose D' and γ_1' as before (in particular $D' \not\subset \tilde{S}$). By the same argument as for γ_1, γ_1' acts on a chain $\tilde{S}'$ containing $S(D', P)$. Then $\gamma_1'(\tilde{S}) \subset \tilde{S}$ is impossible unless $\gamma_1'(D) = D$; but in this case $\gamma_1' \mid D$ has to fix Q, Q' and $\pi_D(\gamma_1(x))$, i.e. $\gamma_1' \mid D = \mathrm{id}_D$.

If $\gamma_1'(\tilde{S}) \not\subset \tilde{S}$, then $\pi_{\tilde{S}}(\gamma_1'(\tilde{S}) \backslash \tilde{S})$ is in the Γ-orbit of $\pi_D(\gamma_1(x))$, thus certainly not a fixed point of γ_1. This contradicts lemma 1.18.

3.6 LEMMA. *With notation as before, $C' := \bigcup_{\gamma \in \Gamma} \gamma(S(x, P))$ is a dense subtree of C.*

Proof. C' is connected since P is a common point of all translates of $S(x, P)$, and it is a union of segments of C, thus C' is a subtree of C.

Suppose there is a $\mathbb{P}^1$-component D of C not contained in C'. By stability of the Γ-marking Γx we find $\gamma_1, \gamma_2 \in \Gamma$ such that $D = \mu(x, \gamma_1(x), \gamma_2(x))$. Of course we may assume that $\pi_D(\gamma_2(x)) \neq \pi_D(P)$. But then $D \subset S(P, \gamma_2(x)) \subset C'$, a contradiction.

It follows that C' contains all $\mathbb{P}^1$-components of C, thus is dense in C.

Proof. (Prop. 3.4) $x = \varphi(1)$ is clearly markable, so we only have to check condition (ii) of definition 3.1.

Recall that a surjective morphism of TPL's is a contraction of a closed subforest F such that for any connected component F_0 of F, $C \backslash F_0$ has at most two connected components ([H], lemma 3.3). Such a contraction is Γ-equivariant if and only if F is Γ-invariant. So let F be a closed Γ-invariant subforest of C.

Case 1. $x \in F$.

Assume that F is contractible, and denote by C' the contracted tree. Let $y \in C$ such that the image y' of y in C' has trivial stabilizer. Then by lemma 3.5 and 3.6, y is in the closure of $\Gamma \cdot L$, L being the component of C containing x. But then $y \in F$, and again by lemma 3.5, the stabilizer of the image of F in C' is nontrivial.

Case 2. $x \notin F$.

Then $F \cap \Gamma x = \varnothing$ since F is Γ-invariant. Let D be a $\mathbb{P}^1$-component of F, and let $\gamma_1, \gamma_2 \in \Gamma$ such that $D = \mu(x, \gamma_1(x), \gamma_2(x))$. Then clearly x, $\gamma_1(x)$ and $\gamma_2(x)$ are on three different connected components of $C \backslash F$, thus F is not contractible.

The next step in the proof of theorem 3.3 is

3.7 PROPOSITION. *The map $\alpha_k : B_\Gamma^\Gamma(k) \to A_\Gamma(k)$, $(C, \varphi, \rho) \mapsto (C, \rho)$, is injective.*

Proof. Let $(C, \rho) \in A_\Gamma(k)$.

First suppose that no end point of C has trivial stabilizer. Then it follows from lemma 3.5 and 3.6 that there is only one Γ-orbit of $\mathbb{P}^1$-components of C on which points with trivial stabilizer exist. Let Y be a component in this orbit and $x \in Y(k)$ such that $\Gamma_x = \{1\}$. Denote by $\varphi: \Gamma \to C(k)$ the map given by $\varphi(\gamma) = \gamma(x)$, and assume that φ is a stable Γ-marking on C.

Let φ' be another stable Γ-equivariant Γ-marking of C. Then there is a component $Y' \in \Gamma \cdot Y$ and a point $x' \in Y'(k)$ such that φ' is given by $\varphi'(1) = x'$. Let $\gamma \in \Gamma$ with $Y' = \gamma Y$; then applying the automorphism γ we may assume $Y' = Y$.

From theorem 2.24 we see that Γ_Y has at most two fixed points on Y (and both are, of course, different from x and x'). Let $\alpha: Y \to Y$ be an automorphism with the same set F of fixed points as Γ_Y, and mapping x to x' (in coordinates, we may normalize such that $F = \{0, \infty\}$ or $F = \{\infty\}$, resp.; then $\alpha(z) = (x'/x)z$ or $\alpha(z) = z + x - x'$, resp.). Extend α in the most trivial way to a Γ-equivariant automorphism of C. Then α carries the marking φ into φ', showing that (C, φ, ρ) and (C, φ', ρ) are the same point in $B_\Gamma^\Gamma(k)$.

Now consider the case of markable end points. Observe that by [H], Prop. 8 two Γ-equivariant markings φ, φ' of (C, ρ) give the same point in $B_\Gamma^\Gamma(k)$ if and only if all cross ratios $\lambda_{\gamma_1,\ldots,\gamma_4}$ $(\gamma_i \in \Gamma)$ agree for φ and φ'.

So let $x := \varphi(1)$, $x' := \varphi'(1)$, and let $\gamma_0, \ldots, \gamma_3$ be four distinct elements in Γ; by the Γ-invariance of the cross ratios we may assume $\gamma_0 = 1$. Let L be the median component of x, $\gamma_1(x)$ and $\gamma_2(x)$ in C, and let C' be the subtree of C spanned by L, $\gamma_1^{-1}(L)$, $\gamma_2^{-1}(L)$ and $\gamma_3^{-1}(L)$. Note that C' cannot obtain any end point of C. Since by lemma 3.6 the subtree of C spanned by the orbit of x' is dense in C, we find $\gamma \in \Gamma$ such that $\pi_{C'}(x) = \pi_{C'}(\gamma(x'))$. In particular we have $\pi_{\gamma_i^{-1}(L)}(x) = \pi_{\gamma_i^{-1}(L)}(\gamma(x'))$ for $i = 1, 2, 3$, and this implies $\pi_L(\gamma_i(x)) = \pi_L(\gamma_i \gamma(x'))$. It follows that

$$\lambda_{1,\gamma_1,\gamma_2,\gamma_3}(x) = \lambda_{1,\gamma_1,\gamma_2,\gamma_3}(\gamma(x')) = \lambda_{\gamma,\gamma\gamma_1,\gamma\gamma_2,\gamma\gamma_3}(x') = \lambda_{1,\gamma_1,\gamma_2,\gamma_3}(x')$$

To complete the proof of theorem 3.3 it remains to show

3.8 PROPOSITION. *α_k is surjective.*

For the proof let $(C, \rho) \in A_\Gamma(k)$; we have to show that we can choose a suitable markable $x \in C(k)$ such that $\varphi: \Gamma \to C(k)$, $\gamma \mapsto \gamma(x)$ makes C into a (Γ, Γ)-TPL. This means (see definition 4.1 in [H]):

(a) any $\mathbb{P}^1$-component of C is median component of three marked points
(b) the median of three marked points is never a special point

Property (a) is verified for any markable x, as the following lemma shows:

3.9 LEMMA. *Let $\rho: \Gamma \to \mathrm{Aut}(C)$ be a stable Γ-action on C, and let $x \in C(k)$ with*

$\Gamma_x = \{1\}$. *Then any* $\mathbb{P}^1$*-component of* C *is median component of points in the orbit of* x.

Proof. Let D be a $\mathbb{P}^1$-component of C. Observe that any branch of C emerging at D has nonempty intersection with Γx: otherwise the Γ-orbit of that branch could be contracted.

First suppose $D \cap \Gamma x \neq \emptyset$. Then $\Gamma_D^0 = \{1\}$; if $\Gamma_D \neq \{1\}$ then D contains at least two different images of x, and as Γ is assumed to be different from $\mathbb{Z}/2\mathbb{Z}$, there is a third image of x somewhere in C, projecting onto a third point on D (cf. theorem 2.24). If instead Γ_D is trivial, we are done if $v(D)$ is of valency $\geqslant 2$ in the intersection graph of C. If Γ_D is trivial and $v(D)$ is of valency one, the intersection point y of D with $\overline{C \backslash D}$ cannot be fixed by any $\gamma \in \Gamma$: because $\gamma(y) = y$ would either imply $\gamma(D) = D$, hence $\gamma \in \Gamma_D$, which is absurd; or else $\gamma(D) = D'$, γ of order 2, $C = D \cup D'$, $\Gamma = \{1, \gamma\}$, which is also excluded. Thus we have shown $\Gamma_y = \{1\}$; but then we can contract D to y, contradicting the stability of ρ.

Suppose now $\Gamma x \cap D = \emptyset$. Then $\Gamma \cdot D$ is contractible unless the valency of $v(D)$ is at least 3. But this implies already, as we have seen, that D is median component of three points in the orbit of x.

Property (b) is a little more subtle. An end point cannot be a median, and a special point of valency two can only be the median of three points if it is itself one of the three points. Thus we have to show

3.10 LEMMA. *Let* $\rho\colon \Gamma \to \mathrm{Aut}(C)$ *be a stable* Γ*-action on* C. *Then there exists* $x \in C(k)$ *with* $\Gamma_x = \{1\}$ *which is either nonspecial or an end point of* C.

Proof. By definition $C(k)$ contains x_0 with trivial stabilizer. Assume that x_0 is special, but not an end point of C. Then $C \backslash \{x_0\}$ consists of two connected components C_1, C_2, one of which, say C_1, contains the common fixed point P (resp. the component L on which Γ acts through V_4).

We claim that C_2 contains a $\mathbb{P}^1$-component with trivial stabilizer (on which we then can find x as required). Thus let D be a $\mathbb{P}^1$-component of C_2, and assume $\Gamma_D^0 \neq \{1\}$. Then D cannot contain an image of x_0, i.e. $D \cap \Gamma x_0 = \emptyset$. Using lemma 3.9 we find $\gamma_1, \gamma_2 \in \Gamma$ such that $\pi_D(x_0)$, $\pi_D(\gamma_1(x_0))$ and $\pi_D(\gamma_2(x_0))$ are all three distinct. Note that $x_0 \in S(D, P)$, and P is common fixed point of Γ; therefore γ_i $(i = 1, 2)$ cannot fix D, more precisely $\pi_D(\gamma_i(D)) = \pi_D(\gamma_i(x_0))$, $i = 1, 2$. Noting finally that $\pi_D(\gamma_i^{-1}(D)) = \pi_D(x_0)$, we get a contradiction to theorem 2.24. Note that we have proved a little more than claimed, namely that no $\mathbb{P}^1$-component D of C_2 has nontrivial Γ_D^0.

3.11 REMARK. All arguments in the above proof carry over immediately to the situation of a family C/S of TPL's over a provariety S, except the last lemma: although we have shown that we can choose a suitable markable point x in each fibre it would in general be impossible to modify a given section

$S \to C$ if it happens to pass through a nonmarkable point with trivial stabilizer. For this reason we required in definition 3.2 the section $x: S \to C$ to be markable. Note that inserting $S = \mathrm{Spec}(k)$ in definition 3.2 does not give back definition 3.1, but lemma 3.10 shows that nevertheless both definitions are equivalent.

4. The structure of $B_{\mathbb{Z}}^{\mathbb{Z}}$

In this section we show that the moduli space for stable actions of $\mathbb{Z}$ on TPL's, which according to thm. 3.3 is the provariety $B_{\mathbb{Z}}^{\mathbb{Z}}$, is itself a TPL. It has a "root component" on which all $\mathbb{Z}$-actions on $\mathbb{P}^1$ lie, but this component, as well as any other component of $B_{\mathbb{Z}}^{\mathbb{Z}}$, intersects infinitely many other components.

Recall from section 3.3 (or from [H], sect. 4) that $B_{\mathbb{Z}}^{\mathbb{Z}}$ is the sub-provariety of $(\mathbb{P}^1)^V$, $V = \{(a_1, \ldots, a_4) : a_i \in \mathbb{Z},\ a_i \neq a_j \text{ for } i \neq j\}$, which in the inhomogeneous coordinates $\lambda_{a,b,c,d}$, $(a, b, c, d) \in V$ is given by the equations

(i) $\lambda_{a,b,c,d} = \lambda_{a+n,b+n,c+n,d+n}$

(ii) $\lambda_{a,b,c,d} = \lambda_{a,b,d,c}^{-1}$

(iii) $\lambda_{a,b,c,d} = 1 - \lambda_{b,c,d,a}$

(iv) $\lambda_{a,b,c,e} = \lambda_{a,b,c,d} \cdot \lambda_{a,b,d,e}$

for all $(a, b, c, d) \in V$, $n \in \mathbb{Z}$

Every $\lambda_{a,b,c,d}$ is a morphism $B_{\mathbb{Z}}^{\mathbb{Z}} \to \mathbb{P}^1$ and the equations have to be understood as abbreviations for the corresponding homogeneous equations in a_v and b_v, where $\lambda = (a_v : b_v)$. In particular relation (iv) may be misleading in its inhomogeneous form as $\lambda_{a,b,c,e}$ is only determined by $\lambda_{a,b,c,d}$ and $\lambda_{a,b,d,e}$ if the product is not of the form $0 \cdot \infty$.

One can derive the following formulas ($a, b, c, x, y, n \in \mathbb{N}$, $a < b < c$, $x < y$, $2 < n$):

(1) $\lambda_{0,a,b,c} = \lambda_{0,1,b,c} \cdot \lambda_{0,1,b-1,c-1} \cdot \ldots \cdot \lambda_{0,1,b-a+1,c-a+1}$

(2) $\lambda_{0,1,x,y} = 1 - \dfrac{\lambda_{1,2,y,0}}{1 - \lambda_{1,2,x,0}} \cdot (1 - \lambda_{0,1,x-1,y-1})$

(3) $\lambda_{1,2,n,0} = \dfrac{\lambda_{1,2,3,0}}{1 - \lambda_{1,2,n-1,0}}$

Proof

(1) $\lambda_{0,a,b,c} \overset{(iii)}{=} 1 - \lambda_{a,b,c,0} \overset{(iii)}{=} 1 - (1 - \lambda_{b,c,0,a}) = \lambda_{b,c,0,a} \overset{(iv)}{=} \lambda_{b,c,0,1} \cdot \lambda_{b,c,1,a}$
$\overset{(iii)}{=} \lambda_{0,1,b,c} \cdot \lambda_{1,a,b,c} \overset{(i)}{=} \lambda_{0,1,b,c} \cdot \lambda_{0,a-1,b-1,c-1}$

(2) $\lambda_{0,1,x,y} \overset{(iii)}{=} 1 - \lambda_{y,0,1,x} \overset{(iv)}{=} 1 - \lambda_{y,0,1,2} \cdot \lambda_{y,0,2,x} \overset{(iii)}{=} 1 - \lambda_{1,2,y,0} \cdot \lambda_{2,x,y,0}$
$\overset{(iv)}{=} 1 - \lambda_{1,2,y,0} \cdot \lambda_{2,x,y,1} \cdot \lambda_{2,x,1,0} \overset{(iii)}{=} 1 - \lambda_{1,2,y,0} \cdot \lambda_{2,x,1,0}(1 - \lambda_{1,2,x,y})$

$$\overset{(ii)}{=} 1 - \frac{\lambda_{1,2,y,0}}{\lambda_{2,x,0,1}} \cdot (1 - \lambda_{1,2,x,y}) \overset{(iii)}{=} 1 - \frac{\lambda_{1,2,y,0}}{1 - \lambda_{1,2,x,0}} (1 - \lambda_{1,2,x,y})$$

$$\overset{(i)}{=} 1 - \frac{\lambda_{1,2,y,0}}{1 - \lambda_{1,2,x,0}} (1 - \lambda_{0,1,x-1,y-1})$$

$$(3)\ \lambda_{1,2,n,0} \overset{(iv)}{=} \lambda_{1,2,n,3} \cdot \lambda_{1,2,3,0} \overset{(ii)}{=} \frac{\lambda_{1,2,3,0}}{\lambda_{1,2,3,n}} \overset{(iii)}{=} \frac{\lambda_{1,2,3,0}}{1 - \lambda_{2,3,n,1}} \overset{(i)}{=} \frac{\lambda_{1,2,3,0}}{1 - \lambda_{1,2,n-1,0}}$$

One gets the formulas by induction.

Let $\lambda_n := \lambda_{1,2,n,0}$ and $\lambda := \lambda_{1,2,3,0}$.

4.1 LEMMA. *If $\lambda_n \neq 0, 1, \infty\ \forall n \geqslant 3$ then λ_v, $v \in V$ is determined by λ.*

Proof. If $\lambda_{1,2,n,0} \neq 0, 1, \infty$ then equation (2) makes sense and induction shows that $\lambda_{0,1,x,y} \neq \infty, 1$, since $\lambda_{0,1,2,y} = 1 - \lambda_y$. So equation (1) makes sense and $\lambda_{0,a,b,c} \neq \infty$. Now equations (i)–(iii) give us every coordinate.

4.2 COROLLARY. *There is an irreducible component B_0 of $B_{\mathbb{Z}}^{\mathbb{Z}}$ on which $\lambda_{1,2,3,0}$ is an isomorphism to $\mathbb{P}^1$. The corresponding action of $\mathbb{Z}$ is the action $\rho_\mu(n)(z) = \mu^n z$ on $\mathbb{P}^1$ with $\lambda_{1,2,3,0} =: [\mu/(1+\mu)^2]$ (for general μ).*

Proof. Let $\varphi: \mathbb{Z} \to \mathbb{P}^1$ be defined by $\varphi(n) = \mu^n$, then $(\mathbb{P}^1, \rho_\mu, \varphi)$ is a stable $\mathbb{Z}$-equivariantly marked tree of projective lines, so there is a corresponding point $p \in B_{\mathbb{Z}}^{\mathbb{Z}}$. p is determined by

$$\lambda_{a,b,c,d}(p) = \lambda_{\varphi(a),\dots,\varphi(d)} = \frac{\varphi(a) - \varphi(d)}{\varphi(b) - \varphi(d)} \cdot \frac{\varphi(b) - \varphi(c)}{\varphi(a) - \varphi(c)} =: \mathrm{CR}(\varphi(a),\ \varphi(b), \varphi(c), \varphi(d)).$$

Now by construction

$$\lambda_{1,2,3,0}(p) = \frac{\mu - 1}{\mu^2 - 1} \cdot \frac{\mu^2 - \mu^3}{\mu - \mu^3} = \frac{\mu}{(1+\mu)^2}.$$

The equation (i)–(iv) are exactly the equations that hold for cross-ratios of markings. As $\lambda_{1,2,3,0}$ determines all other coordinates by using these equations (for general μ), all coordinates equal the cross-ratios.

4.3 REMARK. μ and μ^{-1} give the same action by conjugating the action ρ with $z \to 1/z$. μ and μ^{-1} give the same λ as

$$\frac{\mu^{-1}}{(1+\mu^{-1})^2} = \frac{\mu^2}{\mu^2} \cdot \frac{\mu^{-1}}{(1+\mu^{-1})^2} = \frac{\mu}{(\mu+1)^2}.$$

$\lambda = \mu = 0$ implies, that $\lambda_n = 0\ \forall n$, therefore $\lambda_{1,x,y,0} = 0\ \forall x, y$ and $\lambda_{0,a,b,c} = 1\ \forall a$, b, c, so every coordinate can be computed. The corresponding action is the following:

Take a tree of projective lines such that the intersection graph is equal to:

$$* \cdots —\bullet—\bullet—\bullet—\bullet— \cdots *$$

and let $\mathbb{Z}$ act on that tree by translating the components, and fixing the ends.

From now on let $\lambda \neq 0$.

For $\mu = 1$ we get $\lambda = \frac{1}{4}$. The corresponding action ρ is given by $\rho(1)(z) = z + 1$. If $\mu^n = 1$ then $\lambda_{1,2,n,0} = \mathrm{CR}(\mu, \mu^2, \mu^n, 1) = \mathrm{CR}(\mu, \mu^2, 1, 1) = 1$ ($n \geqslant 3$). If $\lambda_n = 0$ then $\lambda_{n-1} = 1 - \lambda/\lambda_n = 1 - \lambda/0 = 1 - \infty = \infty$, as $\lambda \neq 0$. If $\lambda_n = \infty$ then $\lambda_{n-1} = 1 - \lambda/\lambda_n = 1 - \lambda/\infty = 1 - 0 = 1$, if $\lambda \neq \infty$. From this we see that in the case $\lambda_n \in \{0, 1, \infty\}$ for one n and $\lambda \neq \infty$, there is a minimal n_0 such that $\lambda_{n_0} = 1$ and $\lambda_n \neq 0,\ 1,\ \infty\ \forall n < n_0$. Then $\lambda_{n_0+1} = \infty$, $\lambda_{n_0+2} = 0$, $\lambda_{n_0+3} = \lambda$, $\lambda_{n_0+4} = \lambda_4$ and so on by equation (3). As $B_{\mathbb{Z}}^{\mathbb{Z}}$ is compact we have to check now what action we get, if μ tends to a n_0-th primitive root of unity ξ.

4.4 DEFINITION. *Let ξ_n be a n-th primitive root of unity, $n \geqslant 3$. Let C_{ξ_n} be a tree of projective lines with components $C_0, \ldots, C_n$ where the point $(\xi_n)^i$ on C_0 is identified with ∞ on C_i. Define an action ρ_0 of Z on C in the following way:*

$\rho_0(1)$ acts on C_0 by multiplication with ξ_n. $\rho_0(1)$ maps C_i isomorphically onto C_{i+1} for $i = 1, \ldots, n-1$ and C_n isomorphically onto C_1 such that $\rho_0(n)$ acts on C_1 by translation. Then $\rho_0(n)$ acts on every C_i by translation.

4.5 LEMMA *The point $\lambda = [\xi_n/(1 + \xi_n)^2]$ on B_0 corresponds to the action ρ_0 on C_{ξ_n} defined in 4.4.*

Proof. One can check every coordinate explicitly, for example $\lambda_{n,2n,3n,0}(C_{\xi_n}) = \frac{1}{4}$ (see Remark 4.3). Now $\lambda_{n,2n,3n,0}(\mathbb{P}^1, \rho_\mu) = \mathrm{CR}(\mu^n, \mu^{2n}, \mu^{3n}, 1) = [\mu^n/(1 + \mu^n)^2]$ so for $\mu \to \xi_n$ the coordinate gets $\frac{1}{4}$.

4.6 LEMMA. *For every $\xi_n (n > 2)$ there is an irreducible component B_{ξ_n} of $B_{\mathbb{Z}}^{\mathbb{Z}}$. $B_{\xi_n} \cap B_0 = \{p\}$, $\lambda_{1,2,3,0}(p) = [\xi_n/(1 + \xi_n)^2]$. B_{ξ_n} is isomorphic to $\mathbb{P}^1$, $\lambda_{0,1,n,2n}$ is a coordinate on B_{ξ_n} and $\lambda_{0,1,n,2n}(p) = 2$. Let ρ_μ be the action on C_{ξ_n} acting on the components like ρ_0 but let $\rho_\mu(n): C_i \to C_i$ be the multiplication with μ, then $(C_{\xi_n}, \rho_\mu) \in B_{\xi_n}$ for general μ and $\lambda_{0,1,n,2n}(C_{\xi_n}, \rho_\mu) = \mu + 1$.*

4.7 REMARK. $\lambda_{0,n,2n,3n}$ is not a coordinate on B_{ξ_n}, as

$$\lambda_{0,n,2n,3n}(C_{\xi_n}, \rho_\mu) = \lambda_{0,n,2n,3n}(C_{\xi_n}, \rho_{\mu^{-1}})$$

(see Remark 4.3). But in this case the actions are not equivalent, as the fixed points of the action ρ_μ on C_i are not equivalent for one fixed point lies on C_0. So one cannot conjugate with $z \to 1/z$.

Proof. (Lemma 4.6) We have to prove that for $\lambda = [\xi_n/(1 + \xi_n)^2]$ the coordinate $\lambda_{0,1,n,2n}$ determines in general all other coordinates. Let $p \in B_{\mathbb{Z}}^{\mathbb{Z}}$ be a point with $\lambda(p) = [\xi_n/(1+\xi_n)^2]$. Let (C, ρ, φ) be the stable marked tree of projective lines corresponding to p. Let L be the component corresponding to the median of $\varphi(0)$ $\varphi(1)$, $\varphi(2)$. Let $\pi_L: C \to L$ be the projection of C onto L, $P_i = \pi_L(\varphi(i)) \in L$. $\lambda_{1,2,k,0} \neq 0, 1, \infty$ for $3 \leqslant k < n$ so $P_0, \ldots, P_k$ are distinct. $\mathbb{Z}$ operates on L, as $\rho(1)(L) = \rho(1)(\mu(\varphi(0), \varphi(1), \varphi(2))) = \mu(\varphi(1), \varphi(2), \varphi(3)) = L$, because $\pi_L(\varphi(1)) = P_1$, $\pi_L(\varphi(2)) = P_1$, $\pi_L(\varphi(3)) = P_3$ are distinct (the median component of three points in a tree of projective lines is characterized by the fact that the projections of the three points are different). Now $\lambda_{1,2,n,0} = 1$ so $P_0 = P_n$, as $\lambda_{1,2,n,0}(C, \rho, \varphi) = \mathrm{CR}(P_1, P_2, P_n, P_0) = 1$ iff $P_0 = P_n$. Therefore $P_i = P_j \Leftrightarrow i \equiv j \bmod n$. $\rho(n)(P_i) = P_i \, \forall i$ so $\rho(n)|_L = \mathrm{id}_L$ therefore $\rho(1) \in \mathrm{PGL}_2(k(p))$ is an element of order n. After conjugation $\rho(1)$ is of the form $z \to \xi_n z$. From this we get $\lambda_{a,b,c,d}(p) = \mathrm{CR}(\xi_n^a, \xi_n^b, \xi_n^c, \xi_n^d)$ if $\#\{\bar{a}, \bar{b}, \bar{c}, \bar{d}\} \subset \mathbb{Z}/n\mathbb{Z}$ is at least 3. Using equation (iv) one gets $\lambda_{a,b,c,d}(p) = 1$ for $a \equiv b \bmod n$, $c, d \not\equiv a \bmod n$. Using this one gets $\lambda_{a,b,c,d} = \lambda_{a,b,c,d'}$ if $a \equiv b \equiv c \bmod n$, $d, d' \not\equiv a \bmod n$. Now

$$\lambda_{n,2n,3n,0} = \lambda_{n,2n,3n,n+1} \cdot \lambda_{n,2n,n+1,0} = \frac{1 - \dfrac{1}{\lambda_{0,1,n,2n}}}{\lambda_{0,n+1,n,2n}} = \frac{\lambda_{0,1,n,2n} - 1}{\lambda_{0,1,n,2n}^2}$$

so $\lambda_{n,2n,3n,0}$ is always determined by $\lambda_{0,1,n,2n}$. So if $\lambda_{n,2n,3n,0}$ is not a root of 1 every coordinate $\lambda_{a,b,c,d}$ is determined if $a \equiv b \equiv c \equiv d \bmod n$. Now only the coordinates $\lambda_{0,1,in,jn}$ are left to examine. One can compute the following formula:

$$\lambda_{0,1,in,jn} = \frac{1 - \left(1 - \dfrac{1}{\lambda_{0,1,n,2n}}\right) \cdot \lambda_{0,n,in,2n}}{1 - \left(1 - \dfrac{1}{\lambda_{0,1,n,2n}}\right) \cdot \lambda_{0,n,jn,2n}}$$

So for general μ the coordinate $\lambda_{0,1,in,jn}$ is determined by $\lambda_{0,1,n,2n}$ unless

$$\lambda_{0,n,in,2n} = \left(1 - \frac{1}{\lambda_{0,1,n,2n}}\right)^{-1} = \lambda_{0,n,jn,2n} \neq 0, 1, \infty$$

But then $\lambda_{0,n,in,jn} = 1$. Using an equation like (2) where $1, x, y$ is replaced by n, nx, ny it follows that $\lambda_{0,n,2n,(j-i+2)n} = 1$. Then μ must be a root of 1. The remaining statements of Lemma 4.6 are proved by computing the coordinates explicitly. □

4.8 REMARK. Now the only case that is left is $\lambda = \infty$, this corresponds to $\mu \to -1$. In this case the construction in Def. 4.4 does not work as on the component C_0 only two points are special so (C_{-1}, ρ_μ) is not a stable tree.

What happens is that C_0 is contracted to a point, so C_1 and C_2 intersect in one point, $\rho_\mu(1)$ interchanges C_1 and C_2 and $\rho_\mu(2)$ acts on C_i by multiplication with μ. This gives us the correct tree (C_{-1}, ρ_μ).

One has to do similar computations as before to get

4.9 LEMMA. *There is an irreducible component B_{-1} of $B_{\mathbb{Z}}^{\mathbb{Z}}$, $B_{-1} \cap B_0 = \{p\}$, $\lambda_{1,2,3,0}(p) = \infty$, $\lambda_{0,1,2,4}$ is a coordinate on B_{-1}, $\lambda_{0,1,2,4}(p) = 2$. The tree (C_{-1,ρ_μ}) of Remark 4.8 is the tree corresponding to a general point of B_{-1}. $\lambda_{0,1,2,4}(C_{-1}, \rho_\mu) = 1 + \mu$.*

4.10 REMARK. If the coordinate on B_{ξ_n} gets special one has to repeat the previous arguments: so one gets that $B_{\mathbb{Z}}^{\mathbb{Z}}$ has a root component B_0, and for every root ξ_n of 1 there is another component B_{ξ_n} that intersects B_0 in one point. Every new component intersects another component in every root of 1 and so on. As $B_{\mathbb{Z}}^{\mathbb{Z}}$ is compact there have to be points at the end of every sequence of components. These points correspond to trees of projective lines where the fix group of every $\mathbb{P}^1$-component does not act effectively on that component. If for example the fix group of every component acts by multiplication with the same ξ_p, the end points of that tree correspond to the p-adic numbers $\mathbb{Z}_p$. A marking of that tree must always lie on those end points. The fix group of an end point is just $\{1\}$.

4.11 REMARK. For a point $p \in B_{\mathbb{Z}}^{\mathbb{Z}}$ which does not lie on the base component B_0 there is a component L_0 of the corresponding TPL on which the generator of $\mathbb{Z}$ acts through multiplication by a root of unity. So any point of L_0 has infinite stabilizer. Hence discontinuous actions of $\mathbb{Z}$ only occur on B_0. Here we have the coordinate $\lambda = \lambda_{1,2,3,0}$ (see cor. 4.2), and $\mathbb{Z}$ acts on $L_0 = \mathbb{P}^1$ through $z \mapsto \mu z$ with $\lambda = [\mu/(1 + \mu)^2]$. The action is discontinuous if and only if $|\mu| \neq 1$. Since $\lambda(\mu) = \lambda(1/\mu)$ (see remark 4.3) and $\lambda(\bar{\mu}) = \overline{\lambda(\mu)}$ we see that $|\mu|^2 = \mu\bar{\mu} = 1$ is equivalent to $\lambda \in \mathbb{R} \cup \{\infty\}$ and $\frac{1}{4} \leqslant \lambda \leqslant \infty$. So we have shown:

4.12 PROPOSITION. *The domain $D(\mathbb{Z}) := \mathbb{C} \setminus [\frac{1}{4}, \infty)$ is the moduli space of discontinuous actions of $\mathbb{Z}$ on a TPL over $\mathbb{C}$.*

Note that $\lambda = 0$ corresponds to $\mu = 0$ (or $\mu = \infty$) and that the corresponding action (described in remark 4.3) is discontinuous. Via the mapping $\mu \mapsto \lambda(\mu)$, $D(\mathbb{Z})$ is conformally equivalent to the extended Schottky space $\bar{S}_1 = \{\mu \in \mathbb{C} : |\mu| < 1\}$ considered in [GH], §5.

5. Finite abelian groups

The most elementary application of the theory in this paper is to a finite abelian group Γ. Of course, the finiteness of Γ implies that all occuring moduli spaces are zero dimensional projective varieties, thus the machinery of provarieties does not enter the game in this case. Nevertheless we can apply theorem 3.3 in a not completely trivial way to obtain the number of different actons of Γ on a TPL, and to see how they fit together into a scheme over $\mathbb{Z}$.

In a first step we qualitatively describe all possible stable actions of a finite abelian group on a TPL and thus reduce the question (up to combinatorics) to the classification of cyclic actions.

5.1 PROPOSITION. (a) *Any finite abelian group* Γ *admits a stable action on a TPL.*

(b) *for any surjective group homomorphism* $\alpha\colon \Gamma \to V_4$ *there is a stable action of* Γ *on a TPL C and a* $\mathbb{P}^1$*-component L of C such that* $\bar{\Gamma}_L \cong V_4$ *and* $\Gamma_L^0 = \ker\alpha$.

Proof. (a) Write Γ in some way as a direct sum of cyclic groups:

$$\Gamma \cong C_{n_1} \times \cdots \times C_{n_k}$$

(where C_n denotes a cyclic group of order n). Choose generators σ_i of C_{n_i} and begin with a $\mathbb{P}^1$-component L on which σ_1 acts with order n_1 (i.e. through multiplication by an n_1-th root of unity), whereas for $i \geqslant 2$, σ_i acts trivially on L.

Next let $L_0, \ldots, L_{n_1-1}$ be $\mathbb{P}^1$-components intersecting L in $P_0, \ldots, P_{n_1-1}$ such that P_0 is not a fixed point of σ_1, and $P_i = \sigma_1^i(P_0)$ for $i = 1, \ldots, n_1 - 1$. Let σ_2 act on L_0 with order n_2 such that P_0 is a fixed point of σ_2. Extend the action of σ_1 and σ_2 to all L_i by requiring $\sigma_1^i(L_0) = L_i$ and $\sigma_2\sigma_1^i(x) = \sigma_1^i\sigma_2(x)$ for $i = 1, \ldots, n_1 - 1$ and all $x \in L_0$. Let σ_i act trivially on $L_0, \ldots, L_{n_1-1}$ for $i = 3, \ldots, k$. (If $n_1 = 2$, L has to be contracted).

Now continue this construction inductively: in the next step, take $\mathbb{P}^1$-components L_{ij}, $i = 0, \ldots, n_1 - 1$; $j = 0, \ldots, n_2 - 1$ such that L_{ij} intersects L_i in P_{ij} and $P_{ij} = \sigma_1^i\sigma_2^j(P_{00}) \neq P_{00}$ for all $(i, j) \neq (0, 0)$. σ_3 acts with order n_3 on L_{00}, and the action of σ_1, σ_2 and σ_3 is extended to all L_{ij} as in the previous step. Again, σ_j acts trivially for $j \geqslant 4$.

We end up with a TPL C having $1 + n_1 + n_1n_2 + \cdots + n_1 \cdots n_{k-1}$ components. Any point on an end component of C is markable (except for the fixed points of σ_k), and the action is clearly stable since Γ acts transitively on the set of end components (each of which contains $n_k \geqslant 2$ marked points), and all other components by construction have valency $\geqslant 3$ in the intersection tree $T(C)$.

(b) Choose σ_1 and σ_2 in Γ such that $\alpha(\sigma_1)$ and $\alpha(\sigma_2)$ generate V_4. Let σ_1 and σ_2 act on a $\mathbb{P}^1$-component L such that σ_1^2 and σ_2^2 act trivially. This extends in an obvious way to an action of Γ on L ($\gamma \in \Gamma$ acts like the chosen representative of $\alpha^{-1}(\alpha(\gamma))$.) In particular $\ker\alpha$ acts trivially on L. Now write $\ker\alpha$ as a direct sum of cyclic groups and construct a TPL C_0 for $\ker\alpha$ as in (a); choose a point P on L which is not fixed by σ_1, σ_2 or $\sigma_3 := \sigma_1\sigma_2$, and an ordinary point Q on the base component of C_0, and glue C_0 and L by identifying Q and P. Finally take copies C_1, C_2 and C_3 of C_0 intersecting L in $\sigma_1(P)$, $\sigma_2(P)$, and $\sigma_3(P)$, resp., such that $\sigma_i(C_0) = C_i$.

5.2 Recall that the only finite abelian subgroups of PGL_2 (over an arbitrary field) are the cyclic groups, and V_4. Now if an arbitrary finite abelian group Γ acts on a TPL C we know from thm. 2.23 that C has a base component L on which the common fixed point of all elements of Γ lies, or on which Γ acts through V_4. The above remark implies that $\bar{\Gamma}_L$ is cyclic or V_4. Moreover $\bar{\Gamma}_L$ acts transitively on the set $\{L_1, \ldots, L_n\}$ of components of C intersecting L; again $\bar{\Gamma}_{L_i}$ is cyclic, and the commutativity of Γ requires that $\Gamma^0_{L_i}$ is the same group for all i.

The same reasoning applies to the components of C that intersect the L_i, etc. Hence the construction in the proof of prop. 5.1 almost describes the most general case, and the study of all possible Γ-actions on TPL's reduces to cyclic groups, plus the combinatorics of how to decompose a given finite abelian group into cyclic groups (which we shall not treat in this paper).

5.3 Let Γ be a cyclic group of order n. Then B_Γ is the variety B_n of stable n-pointed TPL's, which is embedded into $(\mathbb{P}^1_{\mathbb{Z}})^N$ by the cross ratios $\lambda_{\gamma_1,\ldots,\gamma_4}$, $\gamma_i \in \Gamma$, $\gamma_i \neq \gamma_j$ for $i \neq j$ (N being the number of such quadrupels). Γ acts on B_Γ by left multiplication on the indices of the coordinates, and B_Γ^Γ is the fixed scheme of this action, see [H], remark following Prop. 8. To simplify notation we identify Γ with the integers $0, \ldots, n-1$, the action being addition modulo n, and we write B_n^n instead of B_Γ^Γ. Then B_n^n is the closed subvariety of B_n given by the equations

$$\lambda_{\nu_1,\ldots,\nu_4} = \lambda_{\nu_1+k,\ldots,\nu_4+k} \qquad \text{for all } k \in \mathbb{Z}/n\mathbb{Z}, \text{ and all } \nu_i$$

The inclusion $\{0, \ldots, n-1\} \hookrightarrow \mathbb{Z}$ induces a surjective morphism $B_{\mathbb{Z}} \to B_n$ under which $B_{\mathbb{Z}}^{\mathbb{Z}}$ projects onto B_n^n. Thus we can use the calculations of the previous section, in particular lemma 4.1, to see that the value of

$$t := \lambda_{0123}$$

completely determines a point in B_n^n unless $\lambda_{012k} = 0$, 1 or ∞ for some $k \in \{0, \ldots, n-1\}$. Remark 4.3 tells us that it is sufficient to deal with $\lambda_{012k} = 0$, and that (since $\lambda_{012n} = 0$) this can only occur if k divides n. Thus if n is prime the points in B_n^n are precisely the solutions to

$$\lambda_{0123} = \lambda_{1234} = \cdots = \lambda_{n-4,n-3,n-2,n-1} = \lambda_{n-3,n-2,n-1,0} \tag{1}$$

Now let

$$\lambda_k := \lambda_{k,k+1,k+2,0},\ k = 1, \ldots, n-3$$

Then from the cross ratio relations and (1) we get for $k = 2, \ldots, n-3$:

$$\lambda_k = \lambda_{k,k+1,k+2,k-1} \cdot \lambda_{k,k+1,k-1,0} = \lambda_1 \cdot \frac{1}{1-\lambda_{k-1}}$$

which gives the recursion formula

$$\lambda_1 = 1 - t, \tag{2}$$

$$\lambda_k = \frac{1-t}{1-\lambda_{k-1}},\ k = 2, \ldots, n-3$$

Note that in view of (2) the equations (1) reduce to

$$\lambda_{n-3} = t \tag{3}$$

Now use the recursion formula (2) to write λ_k as a rational function of t with integer coefficients

$$\lambda_k = \frac{p_k(t)}{q_k(t)}, \qquad p_k, q_k \in \mathbb{Z}[t] \tag{4}$$

Equation (3) now yields the polynomial

$$P_n(t) = p_{n-3}(t) - tq_{n-3}(t) \tag{5}$$

5.4 LEMMA. $\deg P_n(t) = [(n-1)/2]$

Proof. Writing (2) in terms of p_k and q_k we find

$$\frac{p_{k+1}}{q_{k+1}} = \frac{1-t}{1-p_k/q_k} = \frac{(1-t)q_k}{q_k - p_k} \tag{2'}$$

now by induction one immediately verifies that for $k = 1, \ldots, n-3$, $(1-t)$ divides p_k but not q_k, and that moreover q_k and p_k are relatively prime. Hence we may from (2′) read off the recursion formulas for the numerator and the denominator of λ_k:

$$p_{k+1} = (1-t)q_k$$

$$q_{k+1} = q_k - p_k \tag{2''}$$

Inserting (2″) into (5) we obtain

$$P_n(t) = p_{n-3} - tq_{n-3} = p_{n-3} - q_{n-3} + p_{n-2} = p_{n-2} - q_{n-2}$$

thus

$$P_n(t) = -q_{n-1}(t) \tag{5'}$$

Since $\lambda_1 = 1 - t$ we have $p_1 = 1 - t$ and $q_1 = 1$. Thus from (2″) we conclude by induction that for all k, the leading coefficient of q_k is positive and that of p_k is negative. It follows that $\deg p_{k+1} = \deg q_k + 1$ and $\deg q_{k+1} = \max(\deg q_k, \deg p_k)$, thus again by induction

$$\deg p_k = \left[\frac{k+1}{2}\right] \quad \text{and} \quad \deg q_k = \left[\frac{k}{2}\right], \tag{6}$$

in particular $\deg P_n = \deg q_{n-1} = [(n-1)/2]$.

The $[(n-1)/2]$ zeroes of $P_n(t)$ are the finite values of t for the points in B_n^n. (One should show that they are all different). $t = \infty$ is a solution of (3) if and only if $\deg p_{n-3} > \deg q_{n-3}$, and this is in view of (6) equivalent to n being even (in fact $t = \infty$ corresponds to the divisor 2 of n). Thus we have proved.

5.5 COROLLARY. *If $n \geqslant 3$ is a prime number, then B_n^n is isomorphic with the affine zero-dimensional scheme* $\mathrm{Spec}(\mathbb{Z}[t]/P_n(t))$, *and this scheme has* $(n-1)/2$ *geometric points* (*at least in characteristic* 0).

Now suppose d is a divisor of n, more precisely $n = d \cdot m$ with $m \geqslant 2$ and $d \geqslant 2$. To determine the points in B_n^n where $\lambda_{012d} = 0$ (i.e. which are not determined by the value of t), we have to use $t_d := \lambda_{0,1,2,2d}$ because this is a coordinate on the corresponding irreducible component of $B_{\mathbb{Z}}^{\mathbb{Z}}$, see lemma 4.6 and remark 4.7.

LEMMA 5.6. *With notation as above, t_d satisfies a polynomial of degree $m - 1$.*

(Hence there are $m - 1$ points of B_n^n corresponding to $\lambda_{012d} = 0$ if d is a prime, otherwise the procedure has to be repeated for the divisors of d, with the same kind of result.)

Proof. For $k \geqslant 1$, let

$$x_k := \lambda_{kd,kd+1,(k+1)d,0}$$

Under the hypothesis $\lambda_{012d} = 0$ (i.e. on the hypersurface of B_n defined by this equation) we shall prove the recursion formula

$$x_k = \frac{1}{1 - t_d\left(1 - \frac{x_{k-1}}{x_{k-1} - 1}\left(1 - \frac{1}{t_d}\right)\right)} \tag{7}$$

which after simplification of the denominator simply reads

$$x_k = \frac{1 - x_{k-1}}{1 - t_d} \tag{7'}$$

Now writing $x_k = f_k/g_k$ with polynomials f_k and g_k we find

$$\frac{f_{k+1}}{g_{k+1}} = \frac{g_k - f_k}{g_k(1 - t_d)}$$

Noting further

$$x_1 = \frac{1}{1 - t_d} \tag{8}$$

we immediately see by induction

$$g_k = (1 - t_d)^k \quad \text{and} \quad \deg f_k = k - 1$$

We also have

$$t_d = \lambda_{0,1,d,2d} = \lambda_{d,d+1,2d,3d} = \cdots = \lambda_{(m-2)d,(m-2)d+1,(m-1)d,0} = x_{m-2}$$

(where all indices have to be taken modulo n). Hence

$$t_d g_{m-2} - f_{m-2} = 0$$

and this is a polynomial of degree $m - 1$ in t_d.

Proof of (8).
We first show

$$\lambda_{d-1,0,d,1,} = 0 \tag{9}$$

This holds because $\lambda_{0,1,d,2} = \infty$ by assumption, and $\lambda_{0,1,2,d-1} \neq 0$ by lemma 4.1

(Otherwise $\lambda_{0,1,2,1}=0$, a contradiction). Thus $\lambda_{0,1,d,d-1}=\lambda_{0,1,d,2}\cdot\lambda_{0,1,2,d-1}=\infty$, hence $\lambda_{d-1,0,d,1}=(1-\lambda_{0,1,d,d-1})^{-1}=0$.

The next step is to show

$$\lambda_{0,d,d+1,1}=1 \tag{10}$$

This is because $\lambda_{d,1,2,0}=1-(1-\lambda_{0,1,2,d})^{-1}=0$ and hence

$$\begin{aligned}\lambda_{0,d,d+1,1}&=(1-\lambda_{d,1,d+1,0})^{-1}=(1-\lambda_{d,1,d+1,2}\cdot\lambda_{d,1,2,0})^{-1}\\&=(1-\lambda_{d-1,0,d,1}\cdot 0)^{-1}=(1-0\cdot 0)^{-1}=1\end{aligned}$$

We use this result to calculate

$$\begin{aligned}x_1&=\lambda_{d,d+1,2d,0}=1-\lambda_{0,d,d+1,2d}=1-\lambda_{0,d,1,2d}\cdot\lambda_{0.d.d+1,1}\\&=1-\lambda_{0,d,1,2d}=1-\left(1-\frac{1}{1-t_d}\right)=\frac{1}{1-t_d}\end{aligned}$$

Proof of (7).

$$\begin{aligned}x_k&=\lambda_{kd,kd+1,(k+1)d,0}=(1-\lambda_{(k+1)d,kd,kd+1,0})^{-1}\\&=(1-\lambda_{(k+1)d,kd,kd+1,(k-1)d}\cdot\lambda_{(k+1)d,kd,(k-1)d,0})^{-1}\\&=(1-\lambda_{2d,d,d+1,0}\cdot(1-\lambda_{kd,(k-1)d,0,(k+1)d}))^{-1}\\&=\left(1-\left(1-\frac{1}{x_1}\right)\cdot(1-\lambda_{kd,(k-1)d,0,(k-1)d+1}\cdot\lambda_{kd,(k-1)d,(k-1)d+1,(k+1)d})\right)^{-1}\\&=\left(1-t_d\cdot\left(1-\frac{x_{k-1}}{x_{k-1}-1}\cdot\lambda_{d,0,1,2d}\right)\right)^{-1}\\&=\left(1-t_d\cdot\left(1-\frac{x_{k-1}}{x_{k-1}-1}\cdot\left(1-\frac{1}{t_d}\right)\right)\right)^{-1}\end{aligned}$$

as claimed.

From this lemma (and its analog for divisors of d etc.) it is clear how to write down explicitly B_n^n as an (affine) scheme and to determine the number of geometric points on it. Since the formulas look somewhat awkward in general we leave this to the interested reader.

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