

# COMPOSITIO MATHEMATICA

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*Compositio Mathematica*, tome 89, n° 3 (1993), p. 243-250

[http://www.numdam.org/item?id=CM\\_1993\\_\\_89\\_3\\_243\\_0](http://www.numdam.org/item?id=CM_1993__89_3_243_0)

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## On the topology of holomorphic foliations on Hopf manifolds

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Received 6 January 1992; accepted in final form 19 October 1992

### 1. Introduction

The topological behavior under perturbation of complex linear flows on  $\mathbb{C}^n$  near generic singular points was investigated by J. Guckenheimer (cf. [Gu]). Such flows induce holomorphic foliations on certain Hopf manifolds of dimension  $n$  which we shall call diagonal (see below). In this note we investigate the topological behavior of these foliations under perturbation.

The standard complex coordinates in  $\mathbb{C}^n$  are denoted by  $z = (z_1, \dots, z_n)$ . We identify  $\mathbb{C}^n$  canonically with the tangent space at each of its points. If  $A \in GL(n, \mathbb{C})$ ,  $X_A$  will denote the linear vector field defined on  $\mathbb{C}^n$  by  $X_A(z) = Az$ , and  $\Phi_A$  will denote the complex flow  $\Phi_A: \mathbb{C}^n \times \mathbb{C} \rightarrow \mathbb{C}^n$  obtained by integrating  $X_A$ ,

$$\Phi_A(z, t) = e^{tA}z.$$

**DEFINITION.** A matrix  $A \in GL(n, \mathbb{C})$  lies in the *strong Poincaré domain* if and only if all eigenvalues of  $A$  are different, do not contain the origin in their convex hull and no two eigenvalues lie on the same line through the origin.

A matrix  $\Lambda$  in the strong Poincaré domain may without loss of generality be assumed to be diagonal with diagonal entries  $(\lambda_1, \dots, \lambda_n)$ . By abuse of notation we shall identify  $\Lambda$  with the vector  $(\lambda_1, \dots, \lambda_n)$ . If  $\Lambda \in \mathbb{C}^n$  lies in the strong Poincaré domain, then there is a neighbourhood  $U$  of  $\Lambda$  in  $\mathbb{C}^n$  such that all  $\Lambda' \in U$  lie in the strong Poincaré domain.

There is a complex analog of Hartman's theorem proved by Guckenheimer (cf. [Gu]): *If  $\Phi_\Lambda$  is the flow of the vector field  $X_\Lambda(z)$  on  $\mathbb{C}^n$  with  $\Lambda$  a diagonal matrix in the strong Poincaré domain, then there is a neighbourhood  $U$  of  $\Lambda$  in  $\mathbb{C}^n$  such that for all  $\Lambda' \in U$  a homeomorphism of  $\mathbb{C}^n$  exists mapping  $\Phi_\Lambda$  orbits to the  $\Phi_{\Lambda'}$  orbits.*

The subgroup  $\langle f \rangle$  of automorphisms of  $\mathbb{C}^n$  generated by a contraction  $f: (z_1, \dots, z_n) \rightarrow (\mu_1 z_1, \dots, \mu_n z_n)$  with  $0 < |\mu_1| \leq \dots \leq |\mu_n| < 1$  operates freely and

properly discontinuously on  $\mathbb{C}^n - \{0\}$ . The quotient  $X = \mathbb{C}^n - \{0\}/\langle f \rangle$  is a compact, complex manifold of dimension  $n$  called a (diagonal) Hopf manifold (cf. [Hae], [Ko] §10, [Ma] and [We]).

The structure of the foliations  $\overline{\mathcal{F}}$  induced by the orbits of a flow  $\Phi_\Lambda$  on  $\mathbb{C}^n - \{0\}$ ,  $\Lambda$  in the strong Poincaré domain, has been described by Arnold (cf. [Ar]): the coordinate axes are leaves and the rest of the leaves are  $\mathbb{C}$ -planes which wind around the axes. Because these foliations are invariant under the action of the contraction  $f$  we obtain a foliation  $\mathcal{F}$  on the Hopf manifold  $X$ .

The purpose of this note is to prove the following theorem.

**THEOREM.** *Let  $X$  be a diagonal Hopf manifold,  $\Lambda$  a diagonal matrix  $\Lambda$  in the strong Poincaré domain and  $\mathcal{F}$  the foliation on  $X$  induced by the linear vector field  $X_\Lambda(z) = \Lambda z$  on the universal covering  $\mathbb{C}^n - \{0\}$  of  $X$ . Then there is a neighbourhood  $U \subset \mathbb{C}^n$  of  $\Lambda$ , such that any foliation  $\mathcal{F}'$  on  $X$  induced by a vector field  $X_{\Lambda'}(z) = \Lambda' z$ ,  $\Lambda' \in U$ , is topologically inequivalent to  $\mathcal{F}$ , i.e., there is no homeomorphism  $h: X \rightarrow X$  mapping the leaves of  $\mathcal{F}$  to the leaves of  $\mathcal{F}'$ .*

*Notation:*

$$\mathbb{R}_0^+ := \{x \in \mathbb{R} \mid x \geq 0\},$$

$$W^n := \mathbb{C}^n - \{0\},$$

$pr: W^n \rightarrow X$ , the canonical projection,

if  $\mathcal{F}$  is a foliation on  $X$ ,  $\overline{\mathcal{F}}$  denotes the lifted foliation  $pr^*(\mathcal{F})$  on  $W^n$ ,

$$E_i := pr(\{z \in \mathbb{C}^n \mid z_j = 0, j \neq i\}).$$

*Notation for the two dimensional case:*

$$S_r^3 := \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 = r^2\},$$

$$P := \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1| \leq 1 \text{ and } |z_2| \leq 1\},$$

$\partial P$ : the boundary of  $P$ ,

$$B := \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1| = 1 \text{ and } |z_2| = 1\} \subset \partial P, B \text{ is homeomorphic to } S^1 \times S^1,$$

$$A_j := \{(z_1, z_2) \in \mathbb{C}^2 \mid z_j = 0\}, j = 1, 2, \text{ the coordinate axes,}$$

$$C_j := S_{\sqrt{2}}^3 \cap A_j, j = 1, 2, \text{ the intersection circles,}$$

$$Z_1 := \{(s, e^{it}) \in \mathbb{C}^2 \mid s, t \in \mathbb{R}\},$$

$$Z_2 := \{(e^{it}, s) \in \mathbb{C}^2 \mid s, t \in \mathbb{R}\},$$

$B \cap Z_j$  has two components we fix one and denote it by  $G_j, j = 1, 2$ .

## 2. Proof of the theorem

We shall prove the theorem in the two dimensional case first (Proposition 5) and then reduce the general case to this situation.

**LEMMA 1.** *Let  $X_\Lambda$  be a linear complex vector field on  $\mathbb{C}^2$  with  $\Lambda = (\lambda_1, \lambda_2)$  in the strong Poincaré domain and let  $\overline{\mathcal{F}}$  be the foliation induced by  $X_\Lambda$  on  $W^2$ .*

Then each leaf of  $\overline{\mathcal{F}}$  which is not a coordinate axis has a unique point on  $B$ .

*Proof.* Take any leaf  $L$  which is not a coordinate axis and a point  $p := (z_1(p), z_2(p)) \in L \cap \partial P$ . Without loss of generality we may assume  $|z_1(p)| = 1$ ,  $|z_2(p)| \leq 1$ . The flow through the point  $p$  can be described by

$$z_1(T) = e^{\lambda_1 T} z_1(p), \quad z_2(T) = e^{\lambda_2 T} z_2(p), \quad \text{with } T \in \mathbb{C}.$$

Of course  $|z_1(T)| = 1$  if and only if  $T \in (i/\lambda_1)\mathbb{R}$ . Since  $\lambda_1 \notin \lambda_2\mathbb{R}$ , there is a unique value  $t_0 \in \mathbb{R}$  such that  $|e^{(\lambda_2/\lambda_1)it_0}| = |z_2(p)|^{-1}$ .

Let  $X_\Lambda$  be a linear complex vector field on  $\mathbb{C}^2$  with  $\Lambda = (\lambda_1, \lambda_2)$  in the strong Poincaré domain and let  $\overline{\mathcal{F}}$  be the foliation induced by  $X_\Lambda$  on  $W^2$ . The contraction  $f: (z_1, z_2) \mapsto (\mu_1 z_1, \mu_2 z_2)$  maps  $B$  on  $B_\mu := f(B) \subset \partial(f(P))$  and again each leaf of  $\overline{\mathcal{F}}$  with the exception of the axes intersects  $B_\mu$  in a unique point. There is a bijection *flow*:  $B \rightarrow B_\mu$  defined in the following way: If  $p \in B$  then here is a unique leaf  $L$  of  $\overline{\mathcal{F}}$  such that  $p \in L \cap B$ . The application *flow* maps  $p$  to the unique point  $\tilde{p} \in L \cap B_\mu$ . The gluing of the leaves of  $\overline{\mathcal{F}}$  when we map  $W^2$  onto the Hopf surface  $X$  by the canonical projection  $pr$  is described by the map:

$M: B \xrightarrow{\text{flow}} B_\mu \xrightarrow{f^{-1}} B$ . A short computation shows that  $M$  corresponds to a rotation of the torus  $B$ : Note that the vector fields  $(\lambda_1, \lambda_2)$  and  $c(\lambda_1, \lambda_2)$  with  $c \in \mathbb{C}^*$  induce the same foliation on  $W^2$ . Hence we can restrict ourselves to the case  $(1, \lambda)$ .

Let

$$\lambda = x + iy, \quad T = t_1 + it_2; \quad \mu_1 = \rho_1 e^{i\psi_1}, \quad \mu_2 = \rho_2 e^{i\psi_2}.$$

We take a point  $p := (z_1(p), z_2(p)) \in B$ , i.e.,  $|z_1(p)| = |z_2(p)| = 1$ .

We compute  $M: B \xrightarrow{\text{flow}} B_\mu \xrightarrow{f^{-1}} B$ : choose a value  $T$  such that  $|z_1(T)| = |\mu_1|$ ,  $|z_2(T)| = |\mu_2|$ .

Then  $|z_1(T)| = |e^T| = e^{t_1} = |\mu_1| = \rho_1$  implies that  $t_1 = \log \rho_1$ , and  $|z_2(T)| = |e^{\lambda T}| = e^{xt_1 - yt_2} = |\mu_2| = \rho_2$  implies that  $xt_1 - yt_2 = \log \rho_2$ . Hence

$$t_2 = \frac{xt_1 - \log \rho_2}{y} = \frac{x \log \rho_1 - \log \rho_2}{y}.$$

It follows that

$$(\mu_1^{-1} e^T z_1(p), \mu_2^{-1} e^{\lambda T} z_2(p)) \in B, \quad \text{i.e., } |\mu_1^{-1} e^T| = 1 = |\mu_2^{-1} e^{\lambda T}|.$$

Therefore the map  $M$  corresponds to a rotation about

$$(\alpha, \beta) := (\arg(\mu_1^{-1} e^T), \arg(\mu_2^{-1} e^{\lambda T})).$$

We represent the values of the arg function in  $(-\pi, \pi]$ . If  $a \in \mathbb{R}$ , then  $a \bmod 2\pi$  is represented in the same interval. We obtain

$$\begin{aligned} (\alpha, \beta) &= (t_2 - \psi_1, xt_2 + yt_1 - \psi_2) \bmod 2\pi \\ &= \left( \frac{x \log \rho_1 - \log \rho_2}{y} - \psi_1, \frac{(x^2 + y^2) \log \rho_1 - x \log \rho_2}{y} - \psi_2 \right) \bmod 2\pi. \end{aligned} \tag{1}$$

REMARK 2. It is well known that if  $\phi_1, \phi_2$  are topological maps of  $S^1 \times S^1$  and the induced maps  $\phi_1^*, \phi_2^*$  on the “mapping class group”  $H_1(S^1 \times S^1, \mathbb{Z})$  are equal, then  $\phi_1, \phi_2$  are homotopic (cf. [Ro] p. 26).

LEMMA 3. Let  $\psi: S^3_{\sqrt{2}} \rightarrow S^3_{\sqrt{2}}$  be a topological map which maps the set  $\{C_1, C_2\}$  of intersection circles and the torus  $B$  on themselves, respectively. Then  $\psi^*$  operates as one of the following matrices on  $H_1(G_1 \times G_2, \mathbb{Z})$ :

$$\begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}, \begin{pmatrix} \pm 1 & 0 \\ 0 & \mp 1 \end{pmatrix}, \begin{pmatrix} 0 & \pm 1 \\ \pm 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \pm 1 \\ \mp 1 & 0 \end{pmatrix}.$$

*Proof.* The sphere  $S^3_{\sqrt{2}}$  is the union of two solid tori

$$D_1 := \{(z_1, z_2) \in S^3_{\sqrt{2}} \mid |z_1| \leq 1, |z_2| \geq 1\}, \quad D_2 := \{(z_1, z_2) \in S^3_{\sqrt{2}} \mid |z_1| \geq 1, |z_2| \leq 1\}$$

with  $D_1 \cap D_2 = B$ . The intersection circles  $C_j$  are the souls of  $D_j, j = 1, 2$ . The map  $f: (z_1, z_2) \mapsto (z_2, z_1)$  on  $\mathbb{C}^2$  fixes  $S^3_{\sqrt{2}}$  and  $B$ , but exchanges the intersection circles. We deduce that  $f^* = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ . We will show that if  $\psi(C_j) = C_i$  then

$$\text{either } \psi^* = \begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix} \text{ or } \psi^* = \begin{pmatrix} \pm 1 & 0 \\ 0 & \mp 1 \end{pmatrix}.$$

From this fact the claim will follow immediately because if  $\psi(C_1) = C_2$  we replace  $\psi$  by  $f \circ \psi$  and apply the above result. Hence assume that  $\psi(C_j) = C_j, j = 1, 2$ , and that  $\psi^*([G_1]) = n_1[G_1] + n_2[G_2], n_1, n_2 \in \mathbb{Z}$ . We look at the two natural embeddings  $e_j: B \rightarrow S^3_{\sqrt{2}} - C_j, j = 1, 2$ , and the diagrams

$$\begin{array}{ccc} B & \xrightarrow{e_j} & S^3_{\sqrt{2}} - C_j \\ \psi \downarrow & & \downarrow \psi \\ B & \xrightarrow{e_j} & S^3_{\sqrt{2}} - C_j \end{array}$$

and

$$\begin{array}{ccc} H_1(B, \mathbb{Z}) & \xrightarrow{e_j^*} & H_1(S^3\sqrt{2} - C_j, \mathbb{Z}) \\ \psi^* \downarrow & & \downarrow \psi^* \\ H_1(B, \mathbb{Z}) & \xrightarrow{e_j^*} & H_1(S^3\sqrt{2} - C_j, \mathbb{Z}). \end{array}$$

The circles  $C_1$  and  $G_1$  induce the same generator  $\alpha_1 \in H_1(S^3\sqrt{2} - C_2, \mathbb{Z})$  and we must have  $\psi^*(\alpha_1) = \pm\alpha_1$ . Hence  $\psi^* \circ e_2^*([G_1]) = \pm\alpha_1 = e_2^* \circ \psi^*([G_1]) = e_2^*(n_1[G_1] + n_2[G_2]) = n_1\alpha_1$ .

If we take the embedding  $e_1$  and the intersection circle  $C_2$  instead of  $e_2$ ,  $C_1$  we conclude that  $n_2 = 0$ . The result  $\psi^*(\alpha_2) = \pm\alpha_2$  is obtained by a repetition of the previous argument. □

**PROPOSITION 4.** *Let  $X$  be the diagonal Hopf surface induced by the contraction  $f: (z_1, z_2) \mapsto (\mu_1 z_1, \mu_2 z_2)$  with  $\mu_1 = \rho_1 e^{i\psi_1}$ ,  $\mu_2 = \rho_2 e^{i\psi_2}$ , and let  $\mathcal{F}$  be the foliation on  $X$  induced by the vector field  $X_\Lambda(z) = \Lambda z$  with  $\Lambda = (1, \lambda)$ ,  $\lambda = x + iy \in \mathbb{C} - \mathbb{R}$  on  $W^2$ . Then the set*

$$R_{\mathcal{F}} := \{|\alpha|, |\beta|\} = \left\{ \left| \frac{x \log \rho_1 - \log \rho_2}{y} - \psi_1 \bmod 2\pi \right|, \left| \frac{(x^2 + y^2) \log \rho_1 - x \log \rho_2}{y} - \psi_2 \bmod 2\pi \right| \right\}$$

is a topological invariant of  $\mathcal{F}$  on  $X$ .

*Proof.* Let  $\mathcal{F}, \mathcal{F}'$  be two foliations on  $X$ , induced by  $X_\Lambda, X_{\Lambda'}$  with  $\Lambda, \Lambda'$  in the strong Poincaré domain, and let  $h: X \rightarrow X$  be a homeomorphism such that  $h^*(\mathcal{F}) = \mathcal{F}'$ . Remember that  $\tilde{\mathcal{F}} = \text{pr}^*(\mathcal{F})$  and  $\tilde{\mathcal{F}}' = \text{pr}^*(\mathcal{F}')$ . The map  $h$  induces a topological map  $\tilde{h}: W^2 \rightarrow W^2$  on the universal covering of  $X$  such that  $\tilde{h}^*(\tilde{\mathcal{F}}) = \tilde{\mathcal{F}}'$ . These foliations induce rotation maps  $M, M'$  on  $B$ . Put  $B' := \tilde{h}(B)$ . Now we deform  $\tilde{h}$  continuously along the flow until  $B'$  lies on  $B$  and again call by an abuse of notation the resulting map  $\tilde{h}$ . The following diagram is commutative:

$$\begin{array}{ccc} B & \xrightarrow{\tilde{h}} & B \\ M \downarrow & & \downarrow M' \\ B & \xrightarrow{\tilde{h}} & B \end{array}$$

which means that  $M$  and  $M'$  are topologically conjugate.

According to a result of Herman (cf. [He] XIII. Prop. 1.4, remark 1.5), the rotation number  $(\alpha, \beta)$  of a rotation on  $S^1 \times S^1$  is unchanged by conjugation with a topological map which is homotopic to the identity on  $S^1 \times S^1$ . This implies that the change of the rotation number  $(\alpha, \beta)$  by conjugation with a topological map of  $B$  depends only on its homotopy class. Lemma 3 applies to our map  $\tilde{h}$ . Hence, if the rotation number of  $M$  equals  $(\alpha, \beta)$  then the rotation number of  $M'$  must be one of the following

$$(\pm\alpha, \pm\beta), (\mp\alpha, \pm\beta), (\pm\beta, \pm\alpha), (\mp\beta, \pm\alpha). \quad \square$$

**PROPOSITION 5.** *Let  $X$  be a diagonal Hopf surface and  $\mathcal{F}$  a foliation on  $X$  induced by the linear complex vector field  $X_\Lambda(x) = \Lambda z$ , with  $\Lambda = (\lambda_1, \lambda_2)$  in the strong Poincaré domain. Then there is a neighbourhood  $U \subset \mathbb{C}^2$  of  $(\lambda_1, \lambda_2)$ , such that for any foliation  $\mathcal{F}'$  induced by a vector field  $X_{\Lambda'}$ , with  $(\lambda'_1, \lambda'_2) \in U$ , there is no homeomorphism  $h: X \rightarrow X$  with  $h^*(\mathcal{F}') = \mathcal{F}$ .*

*Proof.* Without loss of generality we may assume that  $\Lambda = (1, \lambda)$ . Hence our claim follows from Proposition 4 and the following fact: By (1) we have a differentiable map

$$\begin{aligned} \text{rot}: \quad \mathbb{C} - \mathbb{R} &\rightarrow \mathbb{R}/2\pi\mathbb{Z} \times \mathbb{R}/2\pi\mathbb{Z} \\ \lambda = x + iy &\mapsto (\alpha, \beta) \end{aligned}$$

which is locally a diffeomorphism:

$$\det \begin{pmatrix} \partial(\alpha, \beta) \\ \partial(x, y) \end{pmatrix} = \frac{(y^2 + x^2) \log^2 \rho_1 - 2x \log \rho_1 \log \rho_2 + \log^2 \rho_2}{y^3} = 0$$

has for a given real  $y \neq 0$  only non real  $x$  as solutions. □

**REMARK 6.** The neighbourhood  $U$  around  $\Lambda$  in Proposition 5 can be chosen such that for all  $\mathcal{F}'$  induced by  $\Lambda' \in U$  there are only finitely many  $\mathcal{F}''$  induced by a  $\Lambda'' \in U$  which are topologically equivalent to  $\mathcal{F}'$ .

*Proof of the theorem:* (a) The complex planes  $P_{ij} := \{z \in \mathbb{C}^n \mid z_l = 0, l \neq i, j\}$  are mapped by  $pr$  on Hopf surfaces  $X_{ij}$  in  $X$ . The foliation  $\mathcal{F}_{ij} := \mathcal{F} \mid X_{ij}$  is induced by  $X(z) = X_\Lambda \mid P_{ij} \cong (\lambda_i z_i, \lambda_j z_j)$  on  $\mathbb{C}^2 - \{0\}$ .

(b) If we have two foliations  $\mathcal{F}$  and  $\mathcal{F}'$  on  $X$  and a homeomorphism  $h: X \rightarrow X$  with  $h^*(\mathcal{F}') = \mathcal{F}$ , then we claim that for  $i, j$  there are  $k, l$  such that  $h(X_{ij}) = X_{kl}$  and  $h^*(\mathcal{F}'_{kl}) = \mathcal{F}_{ij}$ .

The leaves  $E_i$  are compact for all  $i$ . Therefore  $h$  maps the set of the  $E_i$ 's on itself. The set of leaves of  $\mathcal{F}'_{i,j}$  consists of  $E_i, E_j$  and the  $\mathbb{C}$ -planes which wind

around  $E_i, E_j$  and have exactly these two compact leaves in its closure. If  $h$  maps  $E_i, E_j$  on  $E_k, E_l$ , respectively, then a leaf which has  $E_i$  and  $E_j$  in its closure is mapped on a leaf which has  $E_k$  and  $E_l$  in its closure. Hence the claim follows.

(c) We denote the set of embedded Hopf surfaces  $X_{ij} \subset X$  by  $HS$ . It follows from (b) that each homeomorphism  $h: X \rightarrow X$  with  $h^*(\mathcal{F}') = \mathcal{F}$  induces a permutation  $Perm(h)$  of  $HS$ .

(d) Assume there is no neighbourhood of  $\Lambda$  in  $\mathbb{C}^n$  with the properties claimed in the theorem. Then there exists a sequence  $\{\Lambda^r\}$  which converges to  $\Lambda$  in  $\mathbb{C}^n$ , for which each  $\Lambda^r \neq \Lambda$ , and a sequence of homeomorphisms  $h^r: X \rightarrow X$  with  $(h^r)^*(\mathcal{F}^r) = \mathcal{F}$ , where  $\mathcal{F}^r$  denotes the foliation induced by  $X_{\Lambda^r}$ . The set  $HS$  is finite and  $\{h^r\}$  is infinite. Choosing an appropriate subsequence of  $\{h^r\}$  we may assume therefore without loss of generality that  $Perm(h^r)$  is independent of  $r$ . This implies that  $\{h^r\}$  induces for each pair  $i, j$  a sequence of homeomorphisms  $\{h^r_{ijkl}\}$  with  $h^r_{ijkl}: X_{ij} \rightarrow X_{kl}$  and  $(h^r_{ijkl})^*(\mathcal{F}^r_{kl}) = \mathcal{F}_{ij}$ . Proposition 4 implies that  $R_{\mathcal{F}_{ij}} = R_{\mathcal{F}_{kl}}$  for all  $i, j$ . On the other side, the sequence  $\{(\lambda'_k, \lambda'_l)\}$  converges to  $\{(\lambda_k, \lambda_l)\}$ . Therefore by Proposition 5 and Remark 6 we may assume without loss of generality that  $(\lambda'_k, \lambda'_l)$  is independent of  $r$  for all  $k, l$ . Hence  $\Lambda^r$  is constant. Since  $\{\Lambda^r\}$  converges to  $\Lambda$  we obtain  $\Lambda^r = \Lambda$  for all  $r$ , a contradiction.  $\square$

### 3. An observation on the structure of $B$

Given a transversally holomorphic foliation  $\mathcal{F}$  on a manifold  $X$  and a submanifold  $S$ . If the leaves of  $\mathcal{F}$  intersect  $S$  transversally in all points of  $S$ , then  $\mathcal{F}$  induces on  $S$  a complex structure (cf. [GHS]).

In our discussion of the two dimensional case we have introduced a differentiable torus  $B$  in  $W^2$  which is intersected transversally in all points by the leaves of the foliation  $\bar{\mathcal{F}}$  induced by the vector field  $X_\Lambda(z)$ ,  $\Lambda$  in the strong Poincaré domain. The image  $pr(B)$  is again a torus now in the Hopf surface  $X^2$  and transversally intersected in all points by the leaves of the corresponding foliation  $\mathcal{F}$ . We inquire into the relation between the foliation  $\mathcal{F}$  and the complex structure of  $pr(B)$ .

**PROPOSITION 6.** *Let  $\mathcal{F}$  be the foliation on  $X^2$  induced by the vector field  $X_\Lambda(z) = \Lambda z$  where  $\Lambda = (1, \lambda)$ ,  $\lambda = x + iy \in \mathbb{C} - \mathbb{R}$  on  $W^2$ . Then the complex structure induced on  $pr(B)$  is conformally equivalent to  $\mathbb{C}^*/\langle e^{2\pi i \lambda} \rangle$ .*

*Proof.* Apart from the coordinate axes every leaf has some intersection points with the punctured plane  $Pl = \{(z_1, z_2) \in \mathbb{C}^2 \mid z_1 = 1, z_2 \in \mathbb{C}^*\}$ . Take a leaf  $L$  and a point  $p = (1, z_2(0)) \in L \cap Pl$ . The flow through this point is given by  $z_1(T) = e^T z_1(0) = e^T$ ,  $z_2(T) = e^{\lambda T} z_2(0)$ . We obtain the other intersection points of  $L$  with  $Pl$ ,  $(1, z_2(T))$ , by the equation  $1 = e^T$ , and hence

$$(1, e^{\lambda 2\pi i k} z_2(0)) = (1, (e^{2\pi i \lambda})^k z_2(0)).$$

We map these isolated points in  $Pl$  onto the unique intersection point of  $L$  with  $B$  and obtain eventually a holomorphic covering map:

$$Pl \cong \mathbb{C}^*$$

$$\text{exp} \downarrow \quad \searrow$$

$$\mathbb{C}^*/\langle e^{2\pi i \lambda} \rangle \cong B.$$

□

### Acknowledgement

The author thanks Prof. A. Haefliger for helpful discussions.

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