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Canonical heights on varieties with morphisms

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Let A be an abelian variety defined over a number field K and let D be a symmetric divisor on A. Néron and Tate have proven the existence of a canonical height $\hat{h}_{A,D}$ on $A(\bar{K})$ characterized by the properties that $\hat{h}_{A,D}$ is a Weil height for the divisor D and satisfies $\hat{h}_{A,D}([m]P) = m^2\hat{h}_{A,D}(P)$ for all $P \in A(\bar{K})$. Similarly, Silverman [19] proved that on certain K3 surfaces S with a non-trivial automorphism $\phi: S \to S$ there are two canonical height functions $\hat{h}_S^{\pm}$ characterized by the properties that they are Weil heights for certain divisors $E^{\pm}$ and satisfy $\hat{h}_S^{\pm}(\phi P) = (7 + 4\sqrt{3})^{\pm 1}\hat{h}_S^{\pm}(P)$ for all $P \in S(\bar{K})$. In this paper we will generalize these examples to construct a canonical height on an arbitrary variety V possessing a morphism $\phi: V \to V$ and a divisor class η which is an eigenclass for ϕ with eigenvalue strictly greater than 1. We will also prove a number of results about these canonical heights which should be useful for arithmetic applications and numerical computations. We now describe the contents of this paper in more detail.

Let V be a variety defined over a number field K, let $\phi: V \to V$ be a morphism, and suppose that there is a divisor class $\eta \in \mathrm{Div}(V) \otimes \mathbb{R}$ such that $\phi^*\eta = \alpha\eta$ for some $\alpha > 1$. Our first main result (Theorem 1.1) says that there is a canonical height function

$$\hat{h}_{V,\eta,\phi}: V(\bar{K}) \to \mathbb{R}$$

characterized by the two properties that $\hat{h}_{V,\eta,\phi}$ is a Weil height function for the divisor class η and

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$$\hat{h}_{V,\eta,\phi}(\phi P) = \alpha \hat{h}_{V,\eta,\phi}(P) \qquad \text{for all } P \in V(\bar{K}).$$

As an application of the canonical height, we show that if η is ample, then $V(K)$ contains only finitely many points which are ϕ-periodic, generalizing results of Narkiewicz [13] and Lewis [10].

Néron (see [8]) has shown how to decompose the canonical height $\hat{h}_{A,D}$ on an abelian variety into a sum of local heights, $\hat{h}_{A,D}(P) = \sum n_v \hat{\lambda}_{A,D}(P, v)$, where the sum is over the distinct places v of $K(P)$. We likewise show that the canonical height $\hat{h}_{V,\eta,\phi}$ constructed in Theorem 1.1 can be decomposed as a sum

$$\hat{h}_{V,\eta,\phi}(P) = \sum_{v \in M_{K(P)}} n_v \hat{\lambda}_{V,E,\phi}(P, v) \qquad \text{for all } P \in V(\bar{K}).$$

Here E is any divisor in the divisor class η, and

$$\hat{\lambda}_{V,E,\phi}\colon (V \setminus |E|) \times M \to \mathbb{R}$$

is a Weil local height function for the divisor E with the additional property that if f is any function satisfying $\phi^* E = \alpha E + \operatorname{div}(f)$, then there is a constant a so that

$$\hat{\lambda}_{V,\eta,\phi}(\phi P, v) = \alpha \hat{\lambda}_{V,\eta,\phi}(P, v) + v(af(P))$$
$$\text{for all } (P, v) \in (V \setminus (|E| \cup |\phi^* E|)) \times M.$$

The existence of the canonical local height $\hat{\lambda}_{V,\eta,\phi}$ is proven in Theorem 2.1, and the fact that the canonical height $\hat{h}_{V,\eta,\phi}$ is the sum of the local heights is given in Theorem 2.3.

Any two Weil heights for a given divisor differ by a bounded amount, so in particular the difference of the canonical height $\hat{h}_{V,\eta,\phi}$ and any given Weil height $h_{V,\eta}$ is bounded by a constant depending on V, η, ϕ and $h_{V,\eta}$. For many applications it is important to have an explicit bound for this constant. Such a bound was given by Dem'janenko [4] and Zimmer [25] for Weierstrass families of elliptic curves, by Manin and Zarhin [12] for Mumford families of abelian varieties, and by Silverman and Tate [15] for arbitrary families of abelian varieties. We follow the approach in [15] and consider a family $\mathcal{V} \to T$ of varieties with a map $\phi\colon \mathcal{V} \to \mathcal{V}$ over T and a divisor class η satisfying $\phi^* \eta = \alpha \eta$. Then on almost all fibers $\mathcal{V}_t$ there is a canonical height $\hat{h}_{\mathcal{V}_t,\eta_t,\phi_t}$, and we can ask to bound the difference between this height and a given Weil height $h_{\mathcal{V},\eta}$ in terms of the parameter t. In Theorem 3.1 we show that there are constants c_1, c_2 so that

$$|\hat{h}_{\mathscr{V}_t,\eta_t,\phi_t}(x) - h_{\mathscr{V},\eta}(x)| \le c_1 h_T(t) + c_2 \qquad \text{for all } t \in T^0 \text{ and all } x \in \mathscr{V}_t.$$

We also give (Theorem 3.2) a similar estimate for the difference between the canonical local height $\hat{\lambda}_{\mathscr{V}_t,E_t,\phi_t}$ and a given Weil local height $\lambda_{\mathscr{V},E}$. This generalizes Lang's result [8] for abelian varieties.

Suppose now that the base T of the family $\mathscr{V} \to T$ is a curve, let h_T be a Weil height on T corresponding to a divisor of degree 1, and let $P: T \to \mathscr{V}$ be a section. The generic fiber V of $\mathscr{V}$ is a variety over the global function field $\bar{K}(T)$, the section P corresponds to a rational point $P_V \in V(\bar{K}(T))$, and Theorem 1.1 gives a canonical height $\hat{h}_{\mathscr{V}_t},\eta_{V,\phi_V}$ which can be evaluated at the point P_V. There are then three heights $\hat{h}_{V,\eta_V,\phi_V}$, $\hat{h}_{\mathscr{V}_t,\eta_t,\phi_V}$, and h_T which may be compared. Generalizing a result of Silverman [15], we show in Theorem 4.1 that

$$\lim_{\substack{t \in T(\bar{K}) \\ h_T(t) \to \infty}} \frac{\hat{h}_{\mathscr{V}_t,\eta_t,\phi_t}(P_t)}{h_T(t)} = \hat{h}_{V,\eta_V,\phi_V}(P_V).$$

Silverman's original result has been generalized and strengthened in various ways by Call [2], Green [6], Lang [8, 9], Silverman [20] and Tate [22]. We have not yet been able to prove any of these stronger results in our general situation.

In the fifth section we take up the question of how one might efficiently compute the canonical local heights $\hat{\lambda}_{V,E,\phi}$, and thereby eventually the canonical global height $\hat{h}_{V,E,\phi}$. In the case that V is an elliptic curve, Tate (unpublished) gave a rapidly convergent series for $\hat{\lambda}_{V,(O),[2]}(P, v)$ provided that the completion K_v of K at v is not algebraically closed, and Silverman [18] described a modification of Tate's series which works for all v. We give series for our canonical local heights $\hat{\lambda}_{V,E,\phi}$ generalizing the series of Tate (Proposition 5.1) and Silverman (Theorem 5.3) and briefly discuss how such series could be implemented in practice.

The final section is devoted to a description of canonical local heights for non-archimedean places in terms of intersection theory. In the case of abelian varieties it is known that the local heights can be computed using intersection theory on the Néron model. We show in general that if V has a model $\mathscr{V}$ over a complete local ring O_v such that every rational point extends to a section and such that the morphism $\phi: V \to V$ extends to a finite morphism $\Phi: \mathscr{V} \to \mathscr{V}$, then the canonical local height is given by a certain intersection index on $\mathscr{V}$. We leave for future study the question of whether such models exist.

To summarize, in this paper we develop a theory of global and local canonical heights on varieties possessing morphisms with non-unit divisorial eigenclasses. We describe how these heights vary in algebraic families and give algorithms which may be used for computational purposes.

The theory of canonical heights on abelian varieties has had profound applications throughout the field of arithmetic geometry. Likewise, several arithmetic applications and many open questions for K3 canonical heights are described in [19]. It is our hope that the general theory of canonical heights described in this paper will likewise prove useful in studying the arithmetic properties of varieties.

1. Global canonical heights

In this section we fix the following data:

- K a global field with a complete set of proper absolute values satisfying a product formula. We will call such a field a *global height field*, since it is for such fields that one can define a height function on $\mathbb{P}^n(\bar{K})$. (For example, K could be a number field or a one variable function field.)
- V/K a smooth, projective variety.
- ϕ a morphism $\phi: V \to V$ defined over K.
- η a divisor class $\eta \in \operatorname{Pic}(V) \otimes \mathbb{R}$.
- $h_{V,\eta}$ a Weil height function $h_{V,\eta}: V(\bar{K}) \to \mathbb{R}$ corresponding to η.

(See [8], Chapters 2, 3, 4, for details about global height fields and height functions on varieties.) We further assume that η is an eigenclass for ϕ with eigenvalue greater than 1. In other words, we assume that

$$\phi^*\eta = \alpha\eta \qquad \text{for some } \alpha \in \mathbb{R} \text{ with } \alpha > 1. \tag{1}$$

It follows from functoriality of height functions [8] that

$$h_{V,n} \circ \phi = \alpha h_{V,\eta} + O_V(1). \tag{2}$$

The $O_V(1)$ depends on the variety V, the map ϕ, and the choice of Weil height function $h_{V,\eta}$, but it is bounded independently of $P \in V(\bar{K})$. Our first result says that there exists a Weil function associated to η for which the $O_V(1)$ entirely disappears.

THEOREM 1.1. *With notation as above, there exists a unique function*

$$\hat{h}_{V,\eta,\phi}: V(\bar{K}) \to \mathbb{R}$$

satisfying the following two conditions:

(i) $\hat{h}_{V,\eta,\phi} = h_{V,\eta} + O_V(1)$.
(ii) $\hat{h}_{V,\eta,\phi} \circ \phi = \alpha \hat{h}_{V,\eta,\phi}$.

We call the function $\hat{h}_{V,\eta,\phi}$ described in Theorem 1.1 the *canonical height on V associated to the divisor class η and the morphism ϕ*. If one or more of the elements of the triple (V, η, ϕ) is clear, we will sometimes omit it from the notation.

Example 1. Let A be an abelian variety, let $[n]: A \to A$ be the multiplication-by-n map for some $n \geq 2$, and let $\eta \in \mathrm{Pic}(A)$ be a symmetric divisor. (That is, $[-1]^*\eta = \eta$.) As is well-known, one then has the relation $[n]^*\eta = n^2\eta$, so one obtains a canonical height $\hat{h}_\eta = \hat{h}_{A,\eta,[n]}$ satisfying $\hat{h}_\eta(nP) = n^2\hat{h}_\eta(P)$. This is the classical canonical height constructed by Néron and Tate. (See, for example, [8, Chapter 5].) Our proof of Theorem 1.1 is an easy extension of Tate's argument to a slightly more general setting.

Example 2. Let $S \subset \mathbb{P}^2 \times \mathbb{P}^2$ be a smooth K3 surface given by the intersection of a $(2,2)$-form and a $(1,1)$-form. The two projections $S \to \mathbb{P}^2$ are double covers, so each gives an involution on S, say $\sigma_1, \sigma_2: S \to S$. Let $\xi_1, \xi_2 \in \mathrm{Pic}(S)$ be hyperplane sections of type $(1,0)$ and $(0,1)$ respectively, let $\beta = 2 + \sqrt{3}$, and define divisor classes on S by the formulas

$$\eta^+ = \beta\xi_1 - \xi_2 \in \mathrm{Pic}(S) \otimes \mathbb{R}, \qquad \eta^- = -\xi_1 + \beta\xi_2 \in \mathrm{Pic}(S) \otimes \mathbb{R}.$$

Further let $\phi = \sigma_2 \circ \sigma_1$. Then one can check that

$$\phi^*\eta^+ = \beta^2\eta^+ \qquad \text{and} \qquad (\phi^{-1})^*\eta^- = \beta^2\eta^-,$$

so we obtain two canonical heights h_{S,ϕ,η^+} and h_{S,ϕ^{-1},η^-}. It is worth noting that η^+ and η^- each lie on the boundary of the effective cone in $\mathrm{Pic}(S) \otimes \mathbb{R}$, but that their sum $\eta^+ + \eta^-$ is ample. This observation is useful for studying the arithmetic of S. For more details concerning this example, including a proof of the facts we have stated and explicit formulas that can be used to compute the canonical height, see [19] and [3].

Example 3. Let $\phi: \mathbb{P}^n \to \mathbb{P}^n$ be a morphism of degree $d \geq 2$. Then any divisor class $\eta \in \mathrm{Pic}(\mathbb{P}^n) \cong \mathbb{Z}$ satisfies $\phi^*\eta = d\eta$, so we can apply Theorem 1.1. (This had earlier been observed by Tate, but never published.) Notice that just as in the case of abelian varieties, one can take η to be ample, which means that Corollary 1.1.1 below is applicable.

Using the canonical height, we can easily prove a strong rationality result for pre-periodic points. We recall that a point P is called *pre-periodic for* ϕ if its orbit

$$\{P, \phi(P), \phi^2(P), \phi^3(P), \ldots\}$$

is finite. (Equivalently, P is pre-periodic if some iterate $\phi^i(P)$ is periodic.) The following corollary gives a strong rationality bound for pre-periodic points. It generalizes results of Narkiewicz [13] and Lewis [10], who prove that there are only finitely many K-rational pre-periodic points in the cases $V = \mathbb{A}^n$ and $V = \mathbb{P}^n$ respectively. Lewis' proof, in particular, uses basic properties of Weil height functions, but our use of the canonical height reduces the proof to just a few lines.

COROLLARY 1.1.1. *Let $\phi\colon V/K \to V/K$ be a morphism defined over a number field K, and suppose that there is an ample divisor class η satisfying* (1). *Let $P \in V(\bar{K})$.*
(a) *P is pre-periodic for ϕ if and only if $\hat{h}_{V,\eta,\phi}(P) = 0$.*
(b) *$V(K)$ contains only finitely many pre-periodic points for ϕ. More generally,*

$$\{P \in V(\bar{K})\colon\ P \text{ is pre-periodic for } \phi\}$$

is a set of bounded height, so in particular it contains only finitely many points defined over all extensions of K of a bounded degree.

Proof. (a) If P is pre-periodic for ϕ, then $h_{V,\eta}(\phi^n(P))$ takes on only finitely many distinct values, and so

$$\hat{h}_{V,\eta,\phi}(P) = \frac{1}{\alpha^n}\hat{h}_{V,\eta,\phi}(\phi^n(P)) = \frac{1}{\alpha^n}(h_{V,\eta}(\phi^n(P)) + O(1)) \xrightarrow[n\to\infty]{} 0.$$

Conversely, if $\hat{h}_{V,\eta,\phi}(P) = 0$, then

$$h_{V,\eta}(\phi^n(P)) = \hat{h}_{V,\eta,\phi}(\phi^n(P)) + O(1) = \alpha^n\hat{h}_{V,\eta,\phi}(P) + O(1) = O(1).$$

Hence the set $\{P, \phi(P), \phi^2(P), \ldots\}$ is a set of bounded height, so it is finite. (Note this is where we use the fact that η is ample.) Therefore P is pre-periodic.

(b) If P is pre-periodic for ϕ, then $\hat{h}_{V,\eta,\phi}(P) = 0$ from (a), so $h_{V,\eta}(P) = \hat{h}_{V,\eta,\phi}(P) + O(1)$ is bounded. This shows that the pre-periodic points form a set of bounded height. The rest of (b) is then immediate, since a set of bounded height contains only finitely many points defined over fields of bounded degree. □

The proof of Theorem 1.1 uses the following result, whose clever telescoping-sum argument is due to Tate.

PROPOSITION 1.2. *With notation as above, let $c(V) = c(V, \phi, h_{V,\eta})$ be a bound for the $O_V(1)$ in (2). In other words,*

$$|h_{V,\eta}(\phi P) - \alpha h_{V,\eta}(P)| \leq c(V) \qquad \text{for all } P \in V(\bar{K}). \tag{3}$$

Then for any point $P \in V(\bar{K})$ the limit

$$\hat{h}_{V,\eta,\phi}(P) \stackrel{\text{def}}{=} \lim_{n\to\infty} \frac{1}{\alpha^n} h_{V,\eta}(\phi^n(P)) \tag{4}$$

exists and satisfies

$$|\hat{h}_{V,\eta,\phi}(P) - h_{V,\eta}(P)| \leq \frac{c(V)}{\alpha - 1}. \tag{5}$$

Proof. Let $P \in V(\bar{K})$, and let $n \geq m \geq 0$ be integers. Then

$$\begin{aligned}
&|\alpha^{-n}h_\eta(\phi^n P) - \alpha^{-m}h_\eta(\phi^m P)| \\
&\quad = \left|\sum_{i=m+1}^{n} \alpha^{-i}h_\eta(\phi^i P) - \alpha^{-i+1}h_\eta(\phi^{i-1}P)\right| \\
&\quad \leq \sum_{i=m+1}^{n} \alpha^{-i}|h_\eta(\phi Q_i) - \alpha h_\eta(Q_i)|, \qquad \text{where } Q_i = \phi^{i-1}P, \\
&\quad \leq \sum_{i=m+1}^{n} \alpha^{-i}c(V), \quad \text{from (3)}, \\
&\quad \leq \frac{\alpha^{-m}c(V)}{\alpha - 1}.
\end{aligned} \tag{6}$$

This inequality shows that the sequence $\alpha^{-n}h_\eta(\phi^n P)$ is Cauchy, so the limit (4) defining $\hat{h}_{V,\eta,\phi}(P)$ exists. Now put $m = 0$ in (6) and let $n \to \infty$ to obtain the estimate

$$|\hat{h}_{V,\eta,\phi}(P) - h_{V,\eta}(P)| \leq \frac{c(V)}{\alpha - 1},$$

which completes the proof of Proposition 1.2. □

Proof. (of Theorem 1.1). We define $\hat{h}_{V,\eta,\phi}$ by the formula (4) in Proposition

1.2. Then (5) tells us that $\hat{h}_{V,\eta,\phi}$ satisfies property (i) of Theorem 1.1, while property (ii) is immediate from the definition (4):

$$\begin{aligned}\hat{h}_{V,\eta,\phi}(\phi P) &= \lim_{n\to\infty} \frac{1}{\alpha^n} h_{V,\eta}(\phi^n(\phi P)) \\ &= \alpha \lim_{n\to\infty} \frac{1}{\alpha^{n+1}} h_{V,\eta}(\phi^{n+1}(P)) = \alpha \hat{h}_{V,\eta,\phi}(P).\end{aligned}$$

It remains to check that $\hat{h}_{V,\eta,\phi}$ is unique. Suppose that $\hat{h}'_{V,\eta,\phi}$ is another function satisfying (i) and (ii), and let $g = \hat{h}_{V,\eta,\phi} - \hat{h}'_{V,\eta,\phi}$. Then (i) implies that g is bounded, while (ii) says that $g(\phi P) = \alpha g(P)$. Hence

$$O_V(1) \geqslant |g(\phi^n P)| = \alpha^n |g(P)| \qquad \text{for all } n = 1, 2, \ldots .$$

Since $\alpha > 1$ by assumption, it follows that $g(P) = 0$, and since $P \in V(\bar{K})$ was arbitrary, there is only one function satisfying (i) and (ii). This completes the proof of Theorem 1.1. □

2. Canonical local heights

In this section we are going to develop a theory of canonical local heights, analogous to the theory of Néron local heights on abelian varieties. Summing the results of this section over all absolute values, we then recover Theorem 1.1, albeit with a far more complicated proof. We will use the following notation, much of it carried over from Section 1:

- K a global height field with set of absolute values M_K. (See Section 1.)
- $M = M_{\bar{K}}$ the set of absolute values on $\bar{K}$ extending those on K.
- V/K a smooth projective variety.
- ϕ a morphism $\phi: V \to V$ defined over K.
- E a divisor $E \in \operatorname{Div}(V) \otimes \mathbb{R}$.
- $\lambda_{V,E}$ a (Weil) local height function $\lambda_{V,E}: (V \setminus |E|) \times M \to \mathbb{R}$ associated to the divisor E. (Here to ease notation we write V in place of $V(\bar{K})$.)

For basic facts about local height functions, M_K-bounded functions and M_K-constants, see [8], Chapter 10. We will freely use terminology from [8] without further reference.

We further assume that the divisor class of E is an eigenclass for ϕ:

$$\phi^* E \sim \alpha E \qquad \text{for some } \alpha \in \mathbb{R} \text{ with } \alpha > 1. \tag{7}$$

Here $\sim$ denotes linear equivalence of divisors. Thus the divisor class $[E] \in \operatorname{Pic}(V) \otimes \mathbb{R}$ of E satisfies the condition (1) imposed on η in Section 1.

THEOREM 2.1. (a) *With notation as above, there exists a function*

$$\hat{\lambda}_{V,E,\phi}\colon (V \setminus |E|) \times M \to \mathbb{R}$$

with the following properties:

(i) $\hat{\lambda}_{V,E,\phi}$ *is a Weil local height function corresponding to the divisor* E.

(ii) *Let* $f \in \bar{K}(V)^* \otimes \mathbb{R}$ *be a function such that*

$$\phi^*E = \alpha E + \operatorname{div}(f). \tag{8}$$

Then there is a constant $a \in \bar{K}^* \otimes \mathbb{R}$, *depending on* f *and* $\hat{\lambda}_{V,E,\phi}$, *such that*

$$\hat{\lambda}_{V,E,\phi}(\phi P, v) = \alpha\hat{\lambda}_{V,E,\phi}(P, v) + v(af(P)) \tag{9}$$

as functions on $(V \setminus (|E| \cup |\phi^*E|)) \times M$.

If $\hat{\lambda}'_{V,E,\phi}$ *is another function satisfying* (*i*) *and* (*ii*), *then there is a constant* $b \in \bar{K}^* \otimes \mathbb{R}$ *such that*

$$\hat{\lambda}_{V,E,\phi}(P, v) = \hat{\lambda}'_{V,E,\phi}(P, v) + v(b) \qquad \text{for all } (P, v) \in (V \setminus |E|) \times M.$$

(b) *Equivalently, given any function* $f \in \bar{K}(V)^* \otimes \mathbb{R}$ *satisfying* (8) *there exists a unique function*

$$\hat{\lambda}_{V,E,\phi,f}\colon (V \setminus |E|) \times M \to \mathbb{R}$$

which is a Weil local height for the divisor E and which satisfies

$$\hat{\lambda}_{V,E,\phi,f}(\phi P, v) = \alpha\hat{\lambda}_{V,E,\phi,f}(P, v) + v(f(P)). \tag{10}$$

Remark. A "function" $f \in \bar{K}(V)^* \otimes \mathbb{R}$ is really a formal product $f = \prod f_i^{\epsilon_i}$ with each $f_i \in \bar{K}(V)$ a rational function on V and each $\epsilon_i \in \mathbb{R}$. The "value" of f at a point $P \in V(\bar{K})$ is the formal product $\prod f_i^{\epsilon_i}(P)$. (Note for example that the field K may have positive characteristic, so raising to a real power may not make sense.) However, it makes sense to define the divisor of f to be

$$\operatorname{div}(f) = \sum \epsilon_i \operatorname{div}(f_i) \in \operatorname{Div}(V) \otimes \mathbb{R};$$

and similarly if $v \in M$ is an absolute value on $\bar{K}$, then we can define $v(f(P))$ to be the real number

$$v(f(P)) = \sum \epsilon_i v(f_i(P)) \in \mathbb{R}.$$

This explains the symbols used in part (ii) of Theorem 2.1.

We begin by proving a variant of a lemma of Tate described in [8], Chapter 11, Lemma 1.2.

LEMMA 2.2. *Let $\mathfrak{X}$ be a topological space, let $\phi: \mathfrak{X} \to \mathfrak{X}$ be a continuous function, let $\alpha \in \mathbb{R}$ be a real number satisfying $\alpha > 1$, and let*

$$\gamma: \mathfrak{X} \to \mathbb{R}$$

be a bounded continuous function. Then there exists a unique bounded continuous function $\hat{\gamma}: \mathfrak{X} \to \mathbb{R}$ satisfying

$$\gamma(x) = \hat{\gamma}(\phi x) - \alpha\hat{\gamma}(x) \qquad \text{for all } x \in \mathfrak{X}.$$

Further, $\hat{\gamma}$ satisfies

$$\sup_{x \in \mathfrak{X}} |\hat{\gamma}(x)| \leq \frac{4\alpha}{\alpha - 1} \sup_{x \in \mathfrak{X}} |\gamma(x)|. \tag{11}$$

Proof. Let $BC(\mathfrak{X}, \mathbb{R})$ be the Banach space of bounded continuous functions on $\mathfrak{X}$, and let $\|\cdot\|$ be the sup norm on $BC(\mathfrak{X}, \mathbb{R})$. Consider the operator

$$S: BC(\mathfrak{X}, \mathbb{R}) \to BC(\mathfrak{X}, \mathbb{R}), \qquad S\delta(x) = \frac{1}{\alpha}[\delta(\phi x) - \gamma(x)].$$

Then for any $\delta_1, \delta_2 \in BC(\mathfrak{X}, \mathbb{R})$ and any $x \in \mathfrak{X}$ we have

$$\begin{aligned} &(S\delta_1 - S\delta_2)(x) = \frac{1}{\alpha}[\delta_1(\phi x) - \delta_2(\phi x)], \quad \text{so} \\ &\|S\delta_1 - S\delta_2\| \leq \frac{1}{\alpha}\|\delta_1 - \delta_2\|. \end{aligned} \tag{12}$$

Since $\alpha > 1$, we see that S is a shrinking map, so by standard fixed point theorems on Banach spaces we know that S has a unique fixed point $\hat{\gamma} \in BC(\mathfrak{X}, \mathbb{R})$. Now the definition of S and the fact that $S\hat{\gamma} = \hat{\gamma}$ gives the desired relationship

$$\frac{1}{\alpha}[\hat{\gamma}(\phi x) - \gamma(x)] = \hat{\gamma}(x).$$

To get a precise estimate, we first note that for any $x \in \mathfrak{X}$,

$$\begin{aligned} |(S\gamma - \gamma)(x)| &= \left|\frac{\gamma(\phi x) - \gamma(x)}{\alpha} - \gamma(x)\right| \\ &\leq \left|\frac{\gamma(\phi x)}{\alpha}\right| + \left(\frac{1}{\alpha} + 1\right)|\gamma(x)| \leq \frac{1}{\alpha}\|\gamma\| + \left(\frac{1}{\alpha} + 1\right)\|\gamma\| \leq 3\|\gamma\|, \end{aligned}$$

so taking the supremum over x gives

$$\|S\gamma - \gamma\| \leq 3\|\gamma\|. \tag{13}$$

Now we compute

$$\begin{aligned} \|S^n\gamma - S^m\gamma\| &= \left\|\sum_{i=m+1}^{n} S^i\gamma - S^{i-1}\gamma\right\| \\ &\leq \sum_{i=m+1}^{n} \|S^i\gamma - S^{i-1}\gamma\| \\ &\leq \sum_{i=m+1}^{n} \frac{1}{\alpha^{i-1}}\|S\gamma - \gamma\|, \quad \text{from (12)}, \\ &\leq \sum_{i=m+1}^{\infty} \frac{1}{\alpha^{i-1}} \cdot 3\|\gamma\|, \quad \text{from (13)}, \\ &= 3\|\gamma\|\frac{\alpha^{-m+1}}{\alpha - 1}. \end{aligned} \tag{14}$$

This shows that the sequence $(S^n\gamma)(x)$ is Cauchy for any $x \in \mathfrak{X}$, so we can define a function

$$\hat{\delta}: \mathfrak{X} \to \mathbb{R}, \qquad \hat{\delta}(x) = \lim_{n\to\infty}(S^n\gamma)(x). \tag{15}$$

Clearly the function $\hat{\delta}$ satisfies $S\hat{\delta} = \hat{\delta}$, and it is not hard to verify that $\hat{\delta}$ is continuous, so $\hat{\delta}$ is just the function $\hat{\gamma}$ from above. Then putting $m = 0$ in (14) and letting n go to infinity gives

$$|\hat{\gamma}(x) - \gamma(x)| \leq \frac{3\alpha\|\gamma\|}{\alpha - 1} \qquad \text{for all } x \in \mathfrak{X},$$

which is clearly stronger than (11). $\square$

Proof. (of Theorem 2.1). Let $\lambda_{V,E}$ be a fixed local height function associated to the divisor E. The divisor relation (8) and standard properties of local height functions ([8], Chapter 10) imply that there is an M_K-bounded and M_K-continuous function $\gamma: V \times M \to \mathbb{R}$ such that

$$\lambda_{V,E}(\phi P, v) = \alpha\lambda_{V,E}(P, v) + v(f(P)) + \gamma(P, v) \tag{16}$$

for all $v \in M$ and all $P \in V$ outside of some Zariski closed subset. Note that γ itself actually extends to all of V by [8], Chapter 10, Proposition 2.3 and Corollary 2.4.

To ease notation, we will write $\gamma_v(P)$ instead of $\gamma(P, v)$. For each $v \in M_K$ we know that $\gamma_v: V(\bar{K}) \to \mathbb{R}$ is v-continuous (since γ is M_K-continuous) and bounded (since γ is M_K-bounded). Applying Lemma 2.2 to the function γ_v, the morphism $\phi: V \to V$, and the real number $\alpha > 1$ appearing in (8), we produce a new v-continuous and bounded function

$$\hat{\gamma}_v: V(\bar{K}) \to \mathbb{R}.$$

The function $\hat{\gamma}_v$ satisfies

$$\gamma_v(P) = \hat{\gamma}_v(\phi P) - \alpha\hat{\gamma}_v(P) \tag{17}$$

$$\sup_{P \in V(\bar{K})} |\hat{\gamma}_v(P)| \leq \frac{4\alpha}{\alpha - 1} \sup_{P \in V(\bar{K})} |\gamma_v(P)|. \tag{18}$$

Now we observe that since γ is M_K-bounded, the functions γ_v are identically 0 for all but finitely many $v \in M_K$. It follows from (18) that the same is true for the $\hat{\gamma}_v$'s. In other words, the map

$$\hat{\gamma}: V \times M \to \mathbb{R}, \qquad \hat{\gamma}(P, v) = \hat{\gamma}_v(P),$$

is M_K-continuous and M_K-bounded.

We now define $\hat{\lambda}_{V,E,\phi,f}$ by the formula

$$\hat{\lambda}_{V,E,\phi,f}(P, v) = \lambda_{V,E}(P, v) - \hat{\gamma}(P, v). \tag{19}$$

The fact that $\hat{\gamma}$ is M_K-continuous and M_K-bounded means that it is a local height function associated to the zero divisor, so $\hat{\lambda}_{V,E,\phi,f}$ is a local height function associated to E. Further, combining the relations (16) and (17) with the definition (19) gives

$$\begin{aligned}\hat{\lambda}_{V,E,\phi,f}(\phi P, v) &= \lambda_{V,E}(\phi P, v) - \hat{\gamma}(\phi P, v) \\ &= \{\alpha\lambda_{V,E}(P, v) + v(f(P)) + \gamma(P, v)\} \\ &\quad - \{\gamma(P, v) + \alpha\hat{\gamma}(P, v)\} \\ &= \alpha\{\lambda_{V,E}(P, v) - \hat{\gamma}(P, v)\} + v(f(P)) \\ &= \alpha\hat{\lambda}_{V,E,\phi,f}(P, v) + v(f(P)).\end{aligned} \tag{20}$$

This proves the existence half of (b).

To show that $\hat{\lambda}_{V,E,\phi,f}$ is unique, we suppose that $\hat{\lambda}'_{V,E,\phi,f}$ is another such function, and let $\Lambda = \hat{\lambda}_{V,E,\phi,f} - \hat{\lambda}'_{V,E,\phi,f}$ be the difference. Then Λ is a local height corresponding to the divisor $E - E = 0$, so it extends to an M_K-bounded and M_K-continuous function on all of $V \times M$. Next from (10) we see that Λ satisfies

$$\Lambda(\phi P, v) = \alpha\Lambda(P, v), \quad \text{and so by iteration} \quad \Lambda(P, v) = \alpha^{-m}\Lambda(\phi^m P, v).$$

But $\Lambda(\cdot, v)$ is bounded on $V(\bar{K})$, we can let $m \to \infty$ to obtain $\Lambda(P, v) = 0$. This gives the uniqueness half of (b).

Next we claim that any such $\hat{\lambda}_{V,E,\phi,f}$ from (b) will have the properties (i) and (ii) in (a). It is clear that $\hat{\lambda}_{V,E,\phi,f}$ has property (i), since it is a Weil local height for E. Similarly, if $f' \in \bar{K}(V)^* \otimes \mathbb{R}$ is another function satisfying (8), then $\operatorname{div}(f/f') = 0$, so $f = af'$ for some constant $a \in \bar{K}^* \otimes \mathbb{R}$. Hence (20) becomes the desired result

$$\hat{\lambda}_{V,E,\phi,f}(\phi P, v) = \alpha\hat{\lambda}_{V,E,\phi,f}(P, v) + v(af(P)).$$

This proves the existence part of (a).

The uniqueness part of (a) can then be proven exactly as we proved (b). Or, alternatively, we can observe that if $\hat{\lambda}_{V,E,\phi}$ satisfies (i) and (ii) and if we pick a function f satisfying (8) and the corresponding a in (9), then the function

$$\Lambda(P, v) = \hat{\lambda}_{V,E,\phi}(P, v) + \frac{1}{\alpha - 1} v(a)$$

satisfies (9), so is uniquely determined from (b). Hence $\hat{\lambda}_{V,E,\phi}$ is uniquely determined up to addition of a function of the form $(P, v) \mapsto v(b)$ for a constant b. □

We conclude this section by showing that the global height from Section 1 is the sum of the local heights constructed in this section.

THEOREM 2.3. *Fix notation as in Theorem* 1.1 *and* 2.1, *and let* $\hat{\lambda}_{V,E,\phi}$ *be a canonical local height associated to* E *and* ϕ. *Then for all finite extensions* L/K *and all points* $P \in V(L) \setminus |E|$,

$$\hat{h}_{V,[E],\phi}(P) = \frac{1}{[L:K]} \sum_{v \in M_L} [L_v : K_v] \hat{\lambda}_{V,E,\phi}(P, v).$$

(*The absolute values in* M_L *are to be normalized as described in* [8].)

Proof. The canonical local height $\hat{\lambda}_{V,E,\phi}$ is in particular a Weil local height associated to the divisor E, so [8], Chapter 10, Section 4, tells us that the function

$$V \setminus |E| \to \mathbb{R},$$

$$P \mapsto \frac{1}{[L:K]} \sum_{v \in M_L} [L_v : K_v] \hat{\lambda}_{V,E,\phi}(P, v) \quad \text{for } P \in V(L),$$

extends to a global Weil height

$$F_\eta : V \to \mathbb{R}$$

which is well-defined on all of V and depends only on the linear equivalence class η of E. In particular, F_η differs from any given $h_{V,\eta}$ by a bounded function.

Next let $f \in \bar{K}(V)^* \otimes \mathbb{R}$ be a function satisfying

$$\phi^* E = \alpha E + \operatorname{div}(f).$$

Then Theorem 2.1. tells us that there is a constant $a \in \bar{K}^* \otimes \mathbb{R}$ such that

$$\hat{\lambda}_{V,E,\phi}(\phi P, v) = \alpha \hat{\lambda}_{V,E,\phi}(P, v) + v(af(P)) \tag{21}$$

for all $(P, v) \in V \times M$ with $P, \phi P \notin |E|$. Taking a finite extension L/K with $P \in V(L)$ and $a \in L$, we multiply (21) by $[L_v : K_v]$, sum over $v \in M_L$, and divide by $[L:K]$. Note that the product formula gives

$$\sum_{v \in M_L} [L_v : K_v] v(af(P)) = 0,$$

so we obtain $F_\eta(\phi P) = \alpha F_\eta(P)$. In other words, we have shown that F_η satisfies

$$F_\eta = h_{V,\eta} + O(1) \quad \text{and} \quad F_\eta \circ \phi = \alpha F_\eta \tag{22}$$

for all points P with $P, \phi P \notin |E|$. But F_η depends only on the linear equivalence class η of E, so by varying E in this class we find that (22) is valid on all of V. It follows from (22) and the uniqueness assertion in Theorem 1.1 that $F_\eta = \hat{h}_{V,\eta,\phi}$, which completes the proof of Theorem 2.3. □

3. Variation of the canonical height in families of varieties

Theorem 1.1(i) says that the canonical height and the Weil height on a variety differ by a bounded amount, where the bound depends (among other things) on the variety. In this section we will consider an algebraic family of varieties and will show how the bound varies as one moves along the family. The following notation will be used for this section and the next section.

K a global height field (cf. section 1).

T/K a smooth projective variety.

h_T a fixed Weil height function on T associated to an ample divisor, chosen to satisfy $h_T \geqslant 0$.

$\mathcal{V}/K$ a smooth projective variety.

π a morphism $\pi: \mathcal{V} \to T$ defined over K whose generic fiber is smooth and irreducible.

ϕ a rational map $\phi: \mathcal{V}/T \to \mathcal{V}/T$, defined over K, such that ϕ is a morphism on the generic fiber of $\mathcal{V}/T$. Note our assumption that ϕ is defined on $\mathcal{V}/T$ means that $\pi \circ \phi = \pi$.

η a divisor class $\eta \in \mathrm{Pic}(\mathcal{V}) \otimes \mathbb{R}$.

T^0 the subset of T having "good" fibers,

$$T^0 = \{t \in T: \mathcal{V}_t \text{ is smooth and } \phi_t: \mathcal{V}_t \to \mathcal{V}_t \text{ is a morphism}\}.$$

[Here and in what follows we use a subscript t to denote restriction to the fiber $\mathcal{V}_t \overset{\text{def}}{=} \pi^{-1}(t)$.]

We further make the assumption that there is a real number $\alpha > 1$ such that the divisor class $\phi^*\eta - \alpha\eta$ is fibral. (23)

By this we mean that the divisor class $\phi^*\eta - \alpha\eta$ is represented by a divisor Δ with the property that $\pi(|\Delta|) \neq T$. Equivalently, there exists a divisor $D \in \mathrm{Div}(T)$ such that $\pi^*D > \Delta > -\pi^*D$. (Note we write $A > B$ to mean that the divisor $A - B$ is positive, while we will say that a divisor is effective if its divisor class contains a positive divisor.)

By definition, the fiber $\mathscr{V}_t$ is irreducible for each $t \in T^0$. If the support of a fibral divisor includes an irreducible fiber, we can always find a linearly equivalent divisor which does not include that fiber. This shows that

$$\phi_t^*\eta_t = \alpha\eta_t \in \mathrm{Pic}(\mathscr{V}_t) \otimes \mathbb{R} \qquad \text{for all } t \in T^0. \tag{24}$$

It follows from Theorem 1.1 and (24) that for each $t \in T^0$ there is a canonical height

$$\hat{h}_{\mathscr{V}_t,\eta_t,\phi_t}\colon \mathscr{V}_t(\bar{K}) \to \mathbb{R}.$$

Next we fix a Weil height

$$h_{\mathscr{V},\eta}\colon \mathscr{V}(\bar{K}) \to \mathbb{R}$$

associated to η. For any $t \in T^0$ we let $i_t\colon \mathscr{V}_t \hookrightarrow \mathscr{V}$ be the natural inclusion, and then by definition $i_t^*\eta = \eta_t$. It follows from Theorem 1.1(i) and functoriality of heights that

$$\hat{h}_{\mathscr{V}_t,\eta_t,\phi_t} = h_{\mathscr{V}_t,\eta_t} + O(1) = h_{\mathscr{V},\eta} \circ i_t + O(1),$$

where the $O(1)$ depends (at least) on t. Our main result in this section makes explicit this dependence on t.

THEOREM 3.1. *With notation as above, there exist constants c_1, c_2 depending on the family $\mathscr{V} \to T$, the map ϕ, the divisor class η, and the choice of Weil height functions $h_{\mathscr{V},\eta}$ and h_T, so that*

$$|\hat{h}_{\mathscr{V}_t,\eta_t,\phi_t}(x) - h_{\mathscr{V},\eta}(x)| \leq c_1 h_T(t) + c_2 \qquad \textit{for all } t \in T^0 \textit{ and all } x \in \mathscr{V}_t.$$

Remark. The first results of this sort were proven by Dem'janenko [4] and Zimmer [25] for the canonical height on the family of elliptic curves $E\colon y^2 = x^3 + ax + b$. In the notation of Theorem 3.1 we would set $\mathscr{V} = E$, $T = \mathbb{P}^1$, $t = [a, b, 1]$, and $T^0 = \{[\alpha, b, 1]\colon 4a^3 + 27b^2 \neq 0\}$. This was generalized to families of abelian varieties by Manin and Zarhin [12] for a certain universal family and by Silverman and Tate [15] in general. Our proof of Theorem 3.1 uses the methods of Silverman–Tate.

Lang [8] reworked the proof in [15] to estimate the variation of canonical local heights in families of abelian varieties. At the end of this section we will likewise prove a local version of Theorem 3.1. We could, of course, then deduce Theorem 3.1 by simply adding up the local contributions. However, the proof of the local version requires considerably more machinery than the global case, so we felt it was worthwhile to prove Theorem 3.1. directly.

Proof. (of Theorem 3.1). During the proof of Theorem 3.1 we will encounter one technical problem for which we know the following three solutions:

(1) We can apply Hironaka's resolution of singularities to our varieties, as was done in the original Silverman–Tate proof. The advantage of this approach is that it is quick and easy, the disadvantage is that it requires using a very deep result and that it is not currently applicable in positive characteristic.

(2) We can apply normalization to our varieties. This is a comparatively elementary procedure and works in any characteristics ([7], II Exercise 3.8). Further, the theory of Weil divisors and Weil heights carries over to normal varieties, but unfortunately the height theory is not well documented in the literature.

(3) We can replace T by T^0, replace $\mathcal{V}$ by $\pi^{-1}(T^0)$, and use the theory of heights on quasi-projective families developed in [17]. This has the advantage of working in complete generality, but the disadvantage of requiring the rather complicated machinery from [17].

We will opt to solve our technical problem by using solution (1), since we feel that this makes the proof most accessible to the reader. Those who object to using resolution of singularities are invited to skip to the proof of Theorem 3.2 below which requires only the results in [17]. They may then obtain Theorem 3.1 by adding Theorem 3.2 over all absolute values of K and applying Theorem 2.3.

With this preamble, we are now ready to begin the proof of Theorem 3.1. Let

$$\mathcal{V}^0 = \pi^{-1}(T^0).$$

From the definition of T^0 we see that $\mathcal{V}^0$ is smooth, so applying resolution of singularities to $\mathcal{V}$ allows us to assume that $\mathcal{V}$ is smooth without changing $\mathcal{V}^0$. Next we observe that although the map $\phi: \mathcal{V} \to \mathcal{V}$ is merely rational, it is a morphism on $\mathcal{V}^0$. This means that we can blow-up $\mathcal{V}$ to produce:

(i) a smooth projective variety $\tilde{\mathcal{V}}$,

(ii) a birational morphism $\psi: \tilde{\mathcal{V}} \to \mathcal{V}$ which is an isomorphism on $\mathcal{V}^0$,

(iii) a morphism $\xi\colon \tilde{\mathcal{V}} \to \mathcal{V}$ which extends the rational map $\phi \circ \psi\colon \tilde{\mathcal{V}} \to \mathcal{V}$.
The existence of a $\tilde{\mathcal{V}}$ with these properties follows from [7] II.7.17.3 and II.7.16, except that $\tilde{\mathcal{V}}$ might be singular. We then use resolution of singularities to make $\tilde{\mathcal{V}}$ smooth. In summary, we have a commutative diagram.

$$\begin{tikzcd}
\tilde{\mathcal{V}}^0 \arrow[r, phantom, "\subset"] \arrow[d, "\wr"] & \tilde{\mathcal{V}} \arrow[d, "\psi"] \arrow[drr, "\xi"] & & \\
\mathcal{V}^0 \arrow[r, phantom, "\subset"] & \mathcal{V} \arrow[rr, dashed, "\phi"] \arrow[dr, "\pi"] & & \mathcal{V} \arrow[dl, "\pi"] \\
& & T &
\end{tikzcd}$$

where the dashed arrow is a rational map and the solid arrows are morphisms. This distinction is of crucial importance, because the functoriality of Weil height functions only works for morphisms, not for rational maps.

Next we choose a divisor $E \in \mathrm{Div}(\mathcal{V}) \otimes \mathbb{R}$ in the divisor class of η, and we let $H \in \mathrm{Div}(T)$ be the ample divisor used to define h_T. Our assumption (23) says that $\phi^*\eta - \alpha\eta$ is fibral, so there is a divisor $D \in \mathrm{Div}(T) \otimes \mathbb{R}$ with

$$\pi^*D > \phi^*E - \alpha E > -\pi^*D. \tag{25}$$

We also choose an integer $n > 0$ so that the divisors

$$nH + D \text{ and } nH - D \text{ are both ample on } T. \tag{26}$$

The height function with respect to a positive divisor is bounded below off of the support of the divisor, and for an ample divisor is everywhere bounded below. So (25) and (26) imply that

$$|h_{\mathcal{V},\phi^*E-\alpha E}| \leqslant nh_{\mathcal{V},\pi^*H} + O(1) = nh_{T,H} \circ \pi + O(1) \qquad \text{for all points in } \mathcal{V}^0 \setminus |D|. \tag{27}$$

Now let $x \in \mathcal{V}^0$ be any point, and let $\tilde{x} \in \tilde{\mathcal{V}}$ satisfy $\psi(\tilde{x}) = x$. In the following computation, we write $O(1)$ for a quantity that is boundable in terms of the family $\mathcal{V} \to T$, the map ϕ, the divisor class η, and the choice of Weil height functions $h_{\mathcal{V},\eta} = h_{\mathcal{V},E}$ and $h_T = h_{T,H}$. The crucial point is that each $O(1)$ bound is independent of $x \in \mathcal{V}^0$.

$$
\begin{aligned}
&|h_{\mathcal{V},E}(\phi x) - \alpha h_{\mathcal{V},E}(x)| \\
&\quad = |h_{\mathcal{V},E}(\phi\circ\psi\tilde{x}) - \alpha h_{\mathcal{V},E}(x)| && \text{since } x = \psi\tilde{x}, \\
&\quad = |h_{\mathcal{V},E}(\xi\tilde{x}) - \alpha h_{\mathcal{V},E}(x)| && \text{since } \xi = \phi\circ\psi \text{ on } \mathcal{V}^0, \\
&\quad = |h_{\mathcal{V},\xi^*E}(\tilde{x}) - \alpha h_{\mathcal{V},E}(x)| + O(1) && \text{since } \xi \text{ is a morphism,} \\
&\quad = |h_{\mathcal{V},\psi^*\phi^*E}(\tilde{x}) - \alpha h_{\mathcal{V},E}(x)| + O(1) \\
&\quad = |h_{\mathcal{V},\phi^*E}(\psi\tilde{x}) - \alpha h_{\mathcal{V},E}(x)| + O(1) && \text{since } \psi \text{ is a morphism,} \\
&\quad = |h_{\mathcal{V},\phi^*E}(x) - \alpha h_{\mathcal{V},E}(x)| + O(1) && \text{since } \psi\tilde{x} = x, \\
&\quad = |h_{\mathcal{V},\phi^*E-\alpha E}(x)| + O(1) \\
&\quad \leqslant n h_{T,H}(\pi x) + O(1) && \text{from (27).}
\end{aligned}
$$

This inequality is valid off of the support of the divisor D. It is now a standard matter to choose different divisors E in the divisor class of η so as to move D around and obtain this inequality at all points $\mathcal{V}^0$. (See, e.g., [15], pages 203–204.) This proves

$$
|h_{\mathcal{V},E}(\phi x) - \alpha h_{\mathcal{V},E}(x)| \leqslant n h_{T,H}(\pi x) + O(1) \qquad \text{for all } x \in \mathcal{V}^0. \tag{28}
$$

In order to complete the proof of Theorem 3.1 we need merely apply Proposition 1.2. More precisely, we match the estimate (28) with (3) in Proposition 1.2. This allows us to apply inequality (5) from Proposition 1.2, which yields the desired result

$$
|\hat{h}_{\mathcal{V}_t,\eta_t,\phi_t}(x) - h_{\mathcal{V},\eta}(x)| \leqslant \frac{n h_{T,H}(\pi x) + O(1)}{\alpha - 1}.
$$

□

We will now prove the following local version of Theorem 3.1. We will make extensive use of the notation and results from [17]. We especially refer the reader to [17], Theorem 7.3 and Corollary 7.4, whose proofs we have transcribed to our more general setting.

THEOREM 3.2. *With notation as above, fix a divisor* $E \in \mathrm{Div}(\mathcal{V}) \otimes \mathbb{R}$ *in the class of* η *and a Weil local height function* $\lambda_{\mathcal{V},E}$. *Let*

$$
U = \left\{ t \in T \colon \begin{array}{l} \mathcal{V}_t \text{ is smooth, } \phi_t\colon \mathcal{V}_t \to \mathcal{V}_t \text{ is a morphism,} \\ E_t \text{ is a divisor on } \mathcal{V}_t, \text{ and } \phi_t^* E_t \sim \alpha E_t \end{array} \right\}.
$$

(The condition that E_t *be a divisor on* $\mathcal{V}_t$ *means that* $|E|$ *contains no components of* $\mathcal{V}_t$, *or equivalently that* $E|_U \to U$ *is a flat family of divisors, see* [7],

III.9.8.5.). *Let* $\partial U = T \setminus U$ *be the complement of* U, *and let* $\lambda_{\partial U}$ *be a local height function associated to* ∂U *as described in* [17].

It is possible to choose canonical local heights $\hat{\lambda}_{\mathcal{V}_t, E_t, \phi_t}$ *as described in Theorem 2.1, one for each* $t \in U$, *in such a way that*

$$|\hat{\lambda}_{\mathcal{V}_t, E_t, \phi_t}(x, v) - \lambda_{\mathcal{V}, E}(x, v)| \leq c\lambda_{\partial U}(t, v) \qquad \text{for all } (x, v) \in (\mathcal{V} \setminus |E|) \times M \text{ with } \pi(x) = t \in U.$$

Here the constant c *depends on the family* $\mathcal{V} \to T$, *the map* ϕ, *the divisor* E, *and the choice of local height functions* $\lambda_{\mathcal{V}, E}$ *and* $\lambda_{\partial U}$.

Proof. We replace $\mathcal{V}$ by the quasi-projective variety $\pi^{-1}(U)$, and replace E by its restriction to this new $\mathcal{V}$. This does not affect the statement of the theorem because [17] Section 5 says that our old $\lambda_{\mathcal{V}, E}$ and our new $\lambda_{\mathcal{V}, E}$ differ by $O(\lambda_{\partial U})$. It now follows from the definition of U that $\phi: \mathcal{V} \to \mathcal{V}$ is a morphism, and that on every fiber we have $\phi_t^* E_t \sim \alpha E_t$. Hence there is a function $f \in \bar{K}(\mathcal{V})^* \otimes \mathbb{R}$ and a fibral divisor $F \in \operatorname{Div}(\mathcal{V}) \otimes \mathbb{R}$ such that

$$\phi^* E = \alpha E + \operatorname{div}(f) + F.$$

But the definition of U says that every fiber of $\mathcal{V}$ is irreducible, so we can write $F = \pi^* D$ for a divisor $D \in \operatorname{Div}(U) \otimes \mathbb{R}$, which gives

$$\phi^* E = \alpha E + \operatorname{div}(f) + \pi^* D. \tag{29}$$

Now standard properties of local heights, specifically [17] Theorem 5.4, transforms the divisorial relation (29) into the height relation

$$\lambda_{\mathcal{V}, E}(\phi x, v) = \alpha \lambda_{\mathcal{V}, E}(x, v) + v(f(x)) + \lambda_{U, D}(\pi x, v) + O(\lambda_{\partial U}(\pi x, v)). \tag{30}$$

We can now repeat the proof of Theorem 2.1, using (30) in place of (16). This yields

$$\hat{\lambda}_{\mathcal{V}_t, E_t, \phi_t} = \lambda_{\mathcal{V}, E} + O(\lambda_{U, D} \circ \pi) + O(\lambda_{\partial U} \circ \pi),$$

which is almost what we want. To conclude, we note that $\phi_t^* E_t \sim \alpha E_t$ on every fiber, so we can repeat the above argument with functions $f_1, \ldots, f_n$ and divisors $D_1, \ldots, D_n$ having the property that $\cap |D_i| = \emptyset$. Then

$$\min\{\lambda_{U, D_i}\} = \lambda_{U, \cap D_i} = \lambda_{U, \emptyset}$$

is M_K-bounded, so

$$\hat{\lambda}_{\mathscr{V}_t,E_t,\phi_t} = \lambda_{\mathscr{V},E} + \min_i O(\lambda_{U,D_i} \circ \pi) + O(\lambda_{\partial U} \circ \pi) = \lambda_{\mathscr{V},E} + O(\lambda_{\partial U} \circ \pi). \qquad \square$$

4. Variation of the canonical height along sections

we studied how the canonical height and the Weil height differ in a family of varieties. In this section we give a more precise result for a one-parameter algebraic family of points. We thus retain the notation from the previous section with the following additional notation and assumptions:

T/K we assume that the base variety T has dimension 1, so T is a smooth projective curve.

h_T we assume that the Weil height function on T corresponds to a divisor of degree 1.

P a section P: $T \to \mathscr{V}$ to the fibration π: $\mathscr{V} \to T$. Equivalently, we can think of the generic fiber V of $\mathscr{V}$ as a variety over the function field $\bar{K}(T)$, and then the section P corresponds to a point $P_V \in V(\bar{K}(T))$.

The function field $\bar{K}(T)$ is itself a global height field in the usual way, namely for each point $t \in T$ there is an absolute value ord_t on $\bar{K}(T)$ defined by

$$\mathrm{ord}_t(f) = \text{order of vanishing of } f \text{ at } t.$$

Further, the rational map ϕ: $\mathscr{V} \to \mathscr{V}$ induces a morphism ϕ_V: $V \to V$ on the generic fiber, and we have $\phi_V^* \eta_V = \alpha \eta_V$, where η_V is the restriction of η to the generic fiber V. This is all the data that we need to use Theorem 1.1 to construct the canonical height

$$\hat{\lambda}_{V,\eta_V,\phi_V}\colon\ V(\overline{K(T)}) \to \mathbb{R}.$$

We are now ready to state our main result.

THEOREM 4.1. *With notation as above,*

$$\lim_{\substack{t \in T^0(\bar{K}) \\ h_T(t) \to \infty}} \frac{\hat{h}_{\mathscr{V}_t,\eta_t,\phi_t}(P_t)}{h_T(t)} = \hat{h}_{V,\eta_V,\phi_V}(P_V).$$

Remark. In the case that $\mathscr{V} \to T$ is a family of elliptic curves, Tate [22] proved the stronger result

$$\hat{h}_{\mathscr{V}_t}(P_t) = h_{T,D_P}(t) + O_P(1) \qquad \text{for all } t \in T^0(\bar{K}),$$

where $D_P \in \mathrm{Div}(T) \otimes \mathbb{Q}$ is a certain divisor with the property $\deg D_P = \hat{h}_V(P_V)$. This was strengthened by Silverman [20] and was generalized to families of abelian varieties and to local heights under various hypotheses by Call [2], Green [6], and Lang [8, 9]. It is natural to ask whether Theorem 4.1 is true in the stronger form

$$\hat{h}_{\mathcal{V}_t,\eta_t,\phi_t}(P_t) = h_{T,D_P}(t) + O_P(1), \tag{31}$$

where the divisor D_P depends on $\mathcal{V}$, T, η, ϕ, P and satisfies $\deg D_P = \hat{h}_{V,\eta_V,\phi_V}(P_V)$. Using current methods, the existence of such a formula would seem to depend on the construction of a "good" model for $\mathcal{V} \to T$ such that the rational map ϕ behaves nicely on $\mathcal{V}$. It is not at all clear whether such good models will exist. It would be interesting to test (31) numerically, say for the K3 surfaces described in [19] and Example 2. This is one reason we are interested in developing an efficient algorithm to compute canonical heights, similar to the algorithms for elliptic curves described in [18]. We will describe such an algorithm theoretically in Section 5. For a more detailed discussion and implementation for K3 surfaces, see [3].

Proof. (of Theorem 4.1). We begin by gathering together several results. First, from Theorem 3.1 we have

$$|\hat{h}_{\mathcal{V}_t,\eta_t,\phi_t}(x) - h_{\mathcal{V},\eta}(x)| \leq c_1 h_T(t) + c_2 \qquad \text{for all } t \in T^0 \text{ and all } x \in \mathcal{V}_t. \tag{32}$$

In particular, (32) is true for $x = P_t$. Note that the constants c_1 and c_2 are independent of both t and x. Second, we apply functoriality of Weil heights to the morphism $P\colon T \to \mathcal{V}$. Note that P will be a morphism because we have assumed that T is a smooth curve, so any rational map from T to a variety is automatically a morphism. This gives

$$|h_{\mathcal{V},\eta}(P_t) - h_{T,P^*\eta}(t)| \leq c_3(P) \qquad \text{for all } t \in T. \tag{33}$$

As our notation indicates, the constant $c_3(P)$ will depend on the section P, but it is independent of t. Third, we use [8], Chapter 3, Proposition 3.2 to describe the Weil height h_{V,η_V} on the generic fiber in terms of intersection theory,

$$|h_{V,\eta_V}(Q_V) - \deg Q^*\eta| \leq c_4 \qquad \text{for all sections } Q\colon T \to \mathcal{V}. \tag{34}$$

Fourth, we apply Theorem 1.1(i) to the height $\hat{h}_{V,\eta_V,\phi_V}$ on the generic fiber, which gives

$$|\hat{h}_{V,\eta_V,\phi_V}(Q_V) - h_{V,\eta_V}(Q_V)| \leqslant c_5 \qquad \text{for all } Q_V \in V(\overline{K(T)}). \tag{35}$$

Using the four estimates (32), (33), (34), (35) and the triangle inequality, we compute

$$\begin{aligned}
&|\hat{h}_{V_t,\eta_t,\phi_t}(P_t) - \hat{h}_{V,\eta_V,\phi_V}(P_V)h_T(t)| \\
&\quad \leqslant |\hat{h}_{\mathcal{V}_t,\eta_t,\phi_t}(P_t) - h_{\mathcal{V},\eta}(P_t)| + |h_{\mathcal{V},\eta}(P_t) - h_{T,P^*\eta}(t)| \\
&\qquad + |h_{T,P^*\eta}(t) - (\deg P^*\eta)h_T(t)| \\
&\qquad + |(\deg P^*\eta)h_T(t) - h_{V,\eta_V}(P_V)h_T(t)| \\
&\qquad + |h_{V,\eta_V}(P_V)h_T(t) - \hat{h}_{V,\eta_V,\phi_V}(P_V)h_T(t)| \\
&\qquad \leqslant (c_1 h_T(t) + c_2) + c_3(P) + |h_{T,P^*\eta}(\mathrm{t}) - (\deg P^*\eta)h_T(t)| \\
&\qquad\qquad + c_4 h_T(t) + c_5 h_T(t).
\end{aligned}$$

We now divide this inequality by $h_T(t)$ and let $h_T(t) \to \infty$. This gives

$$\begin{aligned}
&\limsup_{h_T(t)\to\infty} \left| \frac{\hat{h}_{V,\eta_t,\phi_t}(P_t)}{h_T(t)} - \hat{h}_{V,\eta_V,\phi_V}(P_V) \right| \\
&\qquad \leqslant c_1 + c_4 + c_5 + \limsup_{h_T(t)\to\infty} \left| \frac{h_{T,P^*\eta}(t)}{h_T(t)} - (\deg P^*\eta) \right|.
\end{aligned}$$

Notice that the term $c_3(P)$ which depended on P has disappeared. Further, it follows from [8], Chapter 4, Corollary 3.5 that the heights $h_{T,P^*\eta}$ and $(\deg P^*\eta)h_T(t)$ are quasi-equivalent, and so

$$\lim_{h_T(t)\to\infty} \frac{h_{T,P^*\eta}(t)}{h_T(t)} = (\deg P^*\eta).$$

This gives the fundamental estimate

$$\limsup_{h_T(t)\to\infty} \left| \frac{\hat{h}_{\mathcal{V}_t,\eta_t,\phi_t}(P_t)}{h_T(t)} - \hat{h}_{V,\eta_V,\phi_V}(P_V) \right| \leqslant c_1 + c_4 + c_5. \tag{36}$$

Note that the remaining constants c_1, c_4, c_5 in (36) are independent of both the section P and the point t. So we can apply (36) with P replaced by $\phi^n P$. Theorem 1.1 says that the canonical heights satisfy

$$\hat{h}_{\mathcal{V}_t,\eta_t,\phi_t}(\phi^n x) = \alpha^n \hat{h}_{\mathcal{V}_t,\eta_t,\phi_t}(x) \quad \text{and} \quad \hat{h}_{V,\eta_V,\phi_V}(\phi^n P_V) = \alpha^n \hat{h}_{V,\eta_V,\phi_V}(P_V),$$

so we finally obtain

$$\alpha^n \limsup_{h_T(t)\to\infty} \left| \frac{\hat{h}_{\mathcal{V}_t,\eta_t,\phi_t}(P_t)}{h_T(t)} - \hat{h}_{V,\eta_V,\phi_V}(P) \right| \leqslant c_1 + c_4 + c_5. \tag{37}$$

The right-hand side of (37) is independent of n, while $\alpha > 1$, so letting $n \to \infty$ gives the desired result

$$\limsup_{h_T(t)\to\infty} \left| \frac{\hat{h}_{\mathcal{V}_t,\eta_t,\phi_t}(P_t)}{h_T(t)} - \hat{h}_{V,\eta_V,\phi_V}(P_V) \right| = 0.$$

□

5. A convergent series for the canonical local height

In this section we will derive a rapidly convergent series which can be used to compute the canonical local heights $\hat{\lambda}_{V,\phi,E}$ described in Section 2. The basic methods we use are generalizations of the ideas developed by Tate [21] and Silverman [18] for computing canonical heights on elliptic curves.

Remark. One can always find a finite collection of functions $f_{ij} \in \bar{K}(V) \otimes \mathbb{R}$ so that

$$\max_j \min_i v(f_{ij}(P))$$

is a Weil local height function for E. (See [8], Chapter 10, Proposition 3.2 and Theorem 3.5.) Since any two Weil local heights for the same divisor are M_K-bounded, we see that for all but finitely many $v \in M_K$ there is an exact formula

$$\hat{\lambda}_{V,E,\phi}(P, v) = \max_j \min_i v(f_{ij}(P)). \tag{38}$$

The series we give below is thus useful for computing $\hat{\lambda}_{V,E,\phi}$ for the finitely many exceptional v, which includes especially all archimedean v. Then by adding (38) over the "good" v and the series given below for the "bad" v, one is able to efficiently compute the canonical height $\hat{\lambda}_{V,\phi,\eta}$. An alternative method for computing the "bad" non-archimedean v in terms of intersection theory is described in Section 6.

Tate's original series for $\hat{\lambda}$ on the elliptic curve only converges under a fairly stringent hypothesis, but it has the advantage of being easy to describe and implement. We will begin by giving the analogous series in our more general situation. This will help explain the main ideas before we get enmeshed in the fairly complicated modifications needed to ensure convergence in all cases.

We continue with the notation set in Section 2. In particular, $\phi: V \to V$ is a morphism and $E \in \mathrm{Div}(V) \otimes \mathbb{R}$ is a divisor satisfying $\phi^*E \sim \alpha E$. We will assume that E is defined over K, and we fix a rational function $f \in K(V)^* \otimes \mathbb{R}$ satisfying

$$\phi^*E = \alpha E + \mathrm{div}(f). \tag{39}$$

We also fix an absolute value v on K, and throughout this section we will omit both v and V from our notation for local height functions. For example, we will write $\lambda_D(P)$ for the local height function $\lambda_{V,D}(P, v)$ on $V(K_v)$.

Next we choose a function $t \in K(V)^* \otimes \mathbb{R}$ and write its divisor as

$$\mathrm{div}(t) = E - D \qquad \text{with } D \in \mathrm{Div}(V) \otimes \mathbb{R}. \tag{40}$$

We use t to define two other functions $z, w \in K(V)^* \otimes \mathbb{R}$ by the formulas

$$w = f \cdot t^\alpha, \qquad z = \frac{f \cdot t^\alpha}{t \circ \phi}, \tag{41}$$

or equivalently,

$$\frac{w}{z} = t \circ \phi, \qquad f = \frac{w}{t^\alpha} = \frac{t \circ \phi}{t^\alpha} z. \tag{42}$$

Using (39), (40), and (41) we can compute the divisors of w and z:

$$\mathrm{div}(w) = \phi^*E - \alpha D \qquad \mathrm{div}(z) = \phi^*D - \alpha D. \tag{43}$$

Note in particular that the support of div(z) is contained in $|D| \cup |\phi^*D|$. This means that $|v(z(P))|$ will be bounded as long as P and ϕP are not v-adically close to $|D|$. In order to implement the series for $\hat{\lambda}_{E,\phi}$ it is necessary to find an expression for z which reflects this fact. In other words, z is defined in (41) as a quotient, and one must explicitly write out $t \circ \phi$ in such a way that it partially cancels out the $f \cdot t^\alpha$. One should not to try to compute $v(z)$ by computing each of the three terms in $v(f) + \alpha v(t) - v(t \circ \phi)$, since if either P or ϕP is v-adically close to E then two of the terms may be large while their sum/difference could be very small. We will illustrate this remark below when we discuss the case of elliptic curves.

Having fixed the function f satisfying (39), we can use Theorem 2.1(b) to produce a canonical height $\hat{\lambda}_{E,\phi}$ satisfying

$$\hat{\lambda}_{E,\phi} \circ \phi = \alpha \hat{\lambda}_{E,\phi} + v \circ f. \tag{44}$$

(This is the function denoted by $\hat{\lambda}_{V,E,\phi,f}$ in Theorem 2.1, but having fixed V and f, we will drop them from our notation.) We use (42) to rewrite (44) as

$$\{\hat{\lambda}_{E,\phi} \circ \phi - v \circ t \circ \phi\} = \alpha\{\hat{\lambda}_{E,\phi} - v \circ t\} + v \circ z. \tag{45}$$

Following Tate, this suggests we define a function μ by the formula

$$\mu(P) = \alpha\{\hat{\lambda}_{E,\phi}(P) - v(t(P))\}, \tag{46}$$

and then (45) becomes

$$\mu \circ \phi = \alpha\mu + \alpha v \circ z. \tag{47}$$

It is now easy to verify that

$$\mu(P) = \sum_{n=0}^{N-1} - \alpha^{-n} v(z(\phi^n P)) + \alpha^{-N}\mu(\phi^N P), \tag{48}$$

since if we substitute (47) into the right-hand side of (48) we get a telescoping sum. It is tempting to let $N \to \infty$ in (48), but some additional hypothesis is necessary to ensure convergence.

PROPOSITION 5.1. (aprés Tate) *With notation as above, suppose that the divisor D has the property that*

$$|D| \cap V(K_v) = \emptyset.$$

(a) *The series*

$$\mu(P) = \sum_{n=0}^{\infty} - \alpha^{-n} v(z(\phi^n P))$$

is absolutely convergent for all $P \in V(K_v) \backslash |E|$. More precisely, the n^{th} term of this series has magnitude less than $O(\alpha^{-n})$, where the big-O constant is independent of both P and n.

(b)

$$\hat{\lambda}_{E,\phi}(P) = v(t(P)) + \frac{1}{\alpha}\mu(P).$$

Proof. The fact that $|D| \cap V(K_v)=\emptyset$ and that $V(K_v)$ is compact means that there is a strictly positive v-adic "distance" between $|D|$ and $V(K_v)$. In particular, if we fix a local height λ_D, then

$$\lambda_D(Q) = O(1) \qquad \text{for all } Q \in V(K_v). \tag{49}$$

Here and in what follows the $O(1)$'s are bounded independently of the choice of points in $V(K_v)$. Next we use (43) to write

$$v(z(Q)) = \lambda_{\phi^*D}(Q) - \alpha\lambda_D(Q) + O(1) = \lambda_D(\phi Q) - \alpha\lambda_D(Q) + O(1).$$

Applying (49) to both Q and ϕQ, we deduce that

$$v(z(Q)) = O(1) \quad \text{for all } Q \in V(K_v). \tag{50}$$

Similarly, (40) tells us that

$$v(t(Q)) = \hat{\lambda}_{E,\phi}(Q) - \lambda_D(Q) + O(1).$$

(Note that $\hat{\lambda}_{E,\phi}$ is a Weil local height for E by Theorem 2.1.) It follows from (46) and (49) that

$$\mu(Q) = \alpha\lambda_D(Q) + O(1) = O(1) \quad \text{for all } Q \in V(K_v). \tag{51}$$

We now consider the sum (48) giving $\mu(P)$ for a point $P \in V(K_v)$. It follows from (50) and (51) that

$$|v(z(\phi^n P))| \quad \text{and} \quad |\mu(\phi^N P)| \qquad \text{are bounded independently of } P,\ n, \text{ and } N.$$

Hence if we let $N \to \infty$ in (48), the remainder term $\alpha^{-N}\mu(\phi^N P)$ will go to 0 and we get an absolutely convergent series

$$\mu(P) = \sum_{n=0}^{\infty} -\alpha^{-n} v(z(\phi^n P))$$

whose individual terms satisfy

$$|\alpha^{-n} v(z(\phi^n P))| \leq O(\alpha^{-n}).$$

This proves (a), and (b) then follows immediately from the definition (46) of the function μ. $\square$

Example 4. We illustrate Proposition 5.1 by considering an elliptic curve

$$V: y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6 \tag{52}$$

together with it duplication map $\phi(P) = [2]P$. We let $E = (O)$ be our divisor and note that

$$\phi^* E = 4E + (2y + \alpha_1 x + a_3),$$

so in the notation of Proposition 5.1 we have $\alpha = 4$ and $f = 2y + a_1 x + a_3$. We choose our function t to be $t = x^{-1/2}$, so

$$\operatorname{div}(t) = -\tfrac{1}{2}\operatorname{div}(x) = (O) - \tfrac{1}{2}((R) + (R')) = E - D.$$

Here R and R' are the two points of V with $x = 0$ and y satisfying $y^2 + a_3 y - a_6 = 0$. So the hypothesis of Proposition 5.1 will be satisfied if $R, R' \notin V(K_v)$, or equivalently if $a_3^2 + 4a_6$ is not a square in K_v. Clearly, if $K_v = \mathbb{C}$, then we are stuck. But, for example, if $K_v = \mathbb{R}$, then Tate observed that by making a change of variables $x = x' - r$ with r sufficiently large, one can ensure that the new Weierstrass equation satisfies $a_3'^2 + 4a_6' < 0$, and so Proposition 5.1 can be used.

However, in order to use Proposition 5.1 in practice it is necesasry to write the function z in such a way that $v(z(Q))$ remains bounded even when the point Q is v-adically close to $|E|$. Now $z = f \cdot t^\alpha / t \circ \phi$. In our situation it is easier notationally to work with t^2 and z^2, so we write

$$z^2 = \frac{f^2 \cdot t^{2\alpha}}{t^2 \circ \phi} = \frac{(2y + a_1 x + a_3)^2 \cdot (x \circ [2])}{x^4}. \tag{53}$$

Now we use the duplication formula [16] III.2.3, which says that

$$x \circ [2] = \frac{x^4 - b_4x^2 - 2b_6x - b_8}{4x^3 + b_2x^2 + 2b_4x + b_6} = \frac{x^4 - b_4x^2 - 2b_6x - b_8}{(2y + a_1x + a_3)^2}.$$

Here b_2, b_4, b_6, b_8 are the usual quantities associated to the Weierstrass equation, see [16] III §1. Substituting this into (53) gives

$$z^2 = \frac{x^4 - b_4x^2 - 2b_6x - b_8}{x^4} = 1 - b_4t^4 - 2b_6t^6 - b_8t^8. \tag{54}$$

Similarly,

$$w^2 = f^2 \cdot t^{2\alpha} = \frac{(2y + a_1x + a_3)^2}{x^4} = \frac{4x^3 + b_2x^2 + 2b_4x + b_6}{x^4}$$
$$= 4t^2 + b_2t^4 + 2b_4t^6 + b_6t^8. \tag{55}$$

These two formulas (54) and (55) let one easily compute $z^2(Q)$ and $w^2(Q)$ from the value of $t^2(Q)$, and then (42) says that $t^2(\phi Q) = w^2(Q)/z^2(Q)$, so one immediately gets the value of t^2 for the next term in the series for $\mu(P)$. The key here is that t is uniformly bounded away from ∞ and z is uniformly bounded away from 0 and ∞ on all of $V(\mathbb{R})$. The algorithm we have just described for computing the canonical local height on the $\mathbb{R}$ points of an elliptic curve is exactly Tate's method (unpublished [21]) as described in [18].

Proposition 5.1 provides a rapidly convergent series for $\hat{\lambda}_{E,\phi}$ provided we can find a function t with $\operatorname{div}(t) = E - D$ satisfying $|D| \cap V(K_v) = \emptyset$. Unfortunately, this is often impossible, either because of the particular form of the divisor E or because the field K_v is algebraically closed. We are now going to describe a modification of the series in Proposition 5.1 which will converge even without the hypothesis that $|D| \cap V(K_v) = \emptyset$. This generalizes the case of elliptic curves that was treated in [18].

In order to describe this modification we need a way of measuring the v-adic distance from a point $P \in V(K_v)$ to the support of a divisor D. We begin by writing D as

$$D = \sum_{k=1}^{m} a_kA_k \quad \text{with } a_k \in \mathbb{R} \text{ and } A_k \in \operatorname{Div}(V) \text{ irreducible divisors.}$$

Then the support of D is $|D| = \cup|A_k|$, so it is reasonable to define the local height corresponding to the support of D by the formula

$$\lambda_{|D|}(P) = \sum_{k=1}^{m} \lambda_{A_k}(P). \tag{56}$$

Notice in particular that $\lambda_{|D|}(P)$ will be large if and only if P is v-adically close to at least one of the A_k's.

This serves to define $\lambda_{|D|}$ theoretically, but in order to implement our algorithm for the canonical local height we will need to actually compute the value of $\lambda_{|D|}$ at particular points. From (56) it suffices to compute λ_A for a given irreducible divisor $A \in \mathrm{Div}(V)$, and to do this we need merely follow the prescription given in [8], Chapter 10, Section 3. If A is very ample, or more generally if the linear system associated to A has no base points, this is particularly easy. Namely choose divisors $A_1, \ldots, A_n$ linearly equivalent to A satisfying $\cap |A_i| = \emptyset$, and write $A - A_i = \mathrm{div}(f_i)$. Then one can take

$$\lambda_A(P) = \max_{1 \le i \le n} v(f_i(P)).$$

And of course if A is merely ample, then mA is very ample for some $m > 0$ and one can take $\lambda_A = m^{-1}\lambda_{mA}$. In general, one finds a collection of positive divisors $B_1, \ldots, B_n$ and $C_1, \ldots, C_m$ satisfying $\cap|B_i| = \cap|C_j| = \emptyset$ and $A + C_j \sim B_i$. Then one chooses rational functions $f_{ij} \in K(V)$ with $\mathrm{div}(f_{ij}) = A - B_i + C_j$ and uses them to define the local height

$$\lambda_A(P) = \max_{1 \le i \le n} \min_{1 \le j \le m} v(f_{ij}(P)).$$

The series in Proposition 5.1 leads to difficulties if some multiple $\phi^n P$ is v-adically close to $|D|$. The solution to this problem is to take several functions $t_1, \ldots, t_r$ with divisors $\mathrm{div}(t_i) = E - D_i$ satisfying $\cap|D_i| = \emptyset$. Then each $\phi^n P$ is v-adically far away from at least one D_i, so we need to modify the series to allow us to use different z_i's for different $\phi^n P$'s. With this brief motivation, we now give the details of how this is implemented.

We begin by choosing rational functions $t_1, \ldots, t_r \in K(V)^* \otimes \mathbb{R}$ satisfying

$$\mathrm{div}(t_i) = E - D_i \quad \text{and} \quad \bigcap_{i=1}^{r} |D_i| = \emptyset. \tag{57}$$

(N.B. We are not requiring that the D_i's be effective.) For each i we define as before the functions

$$w_i = f \cdot t_i^\alpha, \qquad z_i = \frac{f \cdot t_i^\alpha}{t_i \circ \phi}, \tag{58}$$

or equivalently,

$$\frac{w_i}{z_i} = t_i \circ \phi, \qquad f = \frac{w_i}{t_i^\alpha} = \frac{t_i \circ \phi}{t_i^\alpha} z_i. \tag{59}$$

Using (39), (57) and (58) we compute the divisors of w_i and z_i to be

$$\operatorname{div}(w_i) = \phi^* E - \alpha D_i \qquad \operatorname{div}(z_i) = \phi^* D_i - \alpha D_i. \tag{60}$$

For each i we define a function μ_i by the formula

$$\mu_i(P) = \alpha\{\hat{\lambda}_{E,\phi}(P) - v(t_i(P))\}. \tag{61}$$

Then the functional equation (44) for $\hat{\lambda}_{E,\phi}$ combined with the expression (59) for f gives the functional equation for μ_i,

$$\mu_i \circ \phi = \alpha \mu_i + \alpha v \circ z_i. \tag{62}$$

The next step is to relate the μ_i's to one another, because we want to replace the earlier sum (48) for μ with a sum in which the terms switch between the various μ_i's. The first observation is that $\hat{\lambda}_{E,\phi}$ does not depend on i, so the definition (61) of μ_i gives

$$\mu_i - \mu_j = -\alpha v \circ \frac{t_i}{t_j}. \tag{63}$$

Next we want a functional equation like (62) but involving μ_i and μ_j.

$$\begin{aligned} \mu_i &= \frac{1}{\alpha} \mu_i \circ \phi - v \circ z_i && \text{from (62)}, \\ &= \frac{1}{\alpha} \mu_j \circ \phi - v \circ \frac{t_i}{t_j} \circ \phi - v \circ z_i && \text{from (63)}, \\ &= \frac{1}{\alpha} \mu_j \circ \phi - v \circ \frac{z_j w_i}{w_j} && \text{from (59)}. \end{aligned}$$

To ease notation we write this as

$$\mu_i = \frac{1}{\alpha}\mu_j \circ \phi - v \circ s_{ij} \qquad \text{with} \quad s_{ij} = \frac{z_j w_i}{w_j}. \tag{64}$$

Using (60) we find that s_{ij} has divisor

$$\operatorname{div}(s_{ij}) = \phi^* D_j - \alpha D_i. \tag{65}$$

For each divisor D_i we fix a distance function $\lambda_{|D_i|}$ as described above, and then for any constant κ we define subsets of $V(\bar{K}_v)$,

$$U_i(\kappa) = \{Q \in V(\bar{K}_v) : \lambda_{|D_i|}(Q) \leqslant \kappa\}.$$

The following lemma provides the estimates we will need to prove that our series for $\hat{\lambda}_{E,\phi}$ converges.

LEMMA 5.2. (a) *There exists a constant* κ_1 *such that*

$$V(\bar{K}_v) = \bigcup_{i=1}^{r} U_i(\kappa_1).$$

(b) *There exist constants* κ_2, κ_3 *such that*

$$|\mu_i(Q)| \leqslant \kappa_2 \lambda_{|D_i|}(Q) + \kappa_3 \qquad \textit{for all } Q \in V(\bar{K}_v).$$

In particular,

$$|\mu_i(Q)| \leqslant \kappa_2 \kappa + \kappa_3 \qquad \textit{for all } Q \in U_i(\kappa).$$

(c) *There exist constants* κ_4, κ_5 *such that if* $Q \in V(\bar{K}_v)$ *is any point satisfying*

$$Q \in U_i(\kappa) \quad \textit{and} \quad \phi Q \in U_j(\kappa),$$

then

$$s_{ij} \textit{ is defined at } Q \quad \textit{and} \quad |v(s_{ij}(Q))| \leqslant \kappa_4 \kappa + \kappa_5.$$

Proof. (a) We know that $\cap |D_i| = \emptyset$ from (57), so [8], Chapter 10, Corollary 3.3 says that there is a constant κ_1 such that

$$\min_{1 \leq i \leq r} \lambda_{|D_i|}(Q)| \leq \kappa_1 \qquad \text{for all } Q \in V(\bar{K}_v).$$

Hence every point of $V(\bar{K}_v)$ is in at least one $U_i(\kappa_1)$.

(b) Theorem 2.1 says that $\hat{\lambda}_{E,\phi}$ is a Weil local height for the divisor E, while (57) tells us that $v \circ t_i$ is a Weil local height for the divisor $E - D_i$. It follows from (61) that μ_i is a Weil local height for the divisor αD_i, so

$$|\mu_i| = |\alpha\lambda_{D_i}| + O(1) \ll \lambda_{|D_i|} + O(1).$$

This proves the first half of (b), and the second half follows from this and the definition of the $U_i(\kappa)$'s.

(c) The assumption that $Q \in U_i(\kappa)$ and $\phi Q \in U_j(\kappa)$ implies in particular that $Q \notin |D_i|$ and $\phi Q \notin |D_j|$, so (65) shows that $Q \notin |\mathrm{div}(s_{ij})|$. In particular, s_{ij} is defined at Q (and not equal to 0 or ∞). Now we compute

$$\begin{aligned} |v(s_{ij}(Q))| &\leq \alpha^{-1}|\mu_j(\phi Q)| + |\mu_i(Q)| && \text{from (64)} \\ &\leq \alpha^{-1}(\kappa_2\kappa + \kappa_3) + (\kappa_2\kappa + \kappa_3) && \text{from (b), since } Q \in U_i(\kappa) \text{ and } \phi Q \in U_j(\kappa), \\ &= \kappa_4\kappa + \kappa_5. && \square \end{aligned}$$

Now let $i_0, i_1, i_2, \ldots$ be any sequence of indices with each i_n between 1 and r. We claim that for any $N \geq 0$ there is an identity

$$\mu_{i_0}(P) = \sum_{n=0}^{N-1} -\alpha^{-n}v(s_{i_n i_{n+1}}(\phi^n P)) + \alpha^{-N}\mu_{i_N}(\phi^N P) \tag{66}$$

analogous to our earlier identity (48). To verify (66) we use (64) to write

$$v(s_{i_n i_{n+1}}(\phi^n P)) = \alpha^{-1}\mu_{i_{n+1}}(\phi^{n+1}P) - \mu_{i_n}(\phi^n P). \tag{67}$$

Substituting (67) into the right-hand side of (66) gives a telescoping sum whose only remaining term is $\mu_{i_0}(P)$.

In order to use (66) to compute $\mu_{i_0}(P)$, we want to choose the indices i_0, $i_1, \ldots$ so that the remainder term $\alpha^{-N}\mu_{i_N}(\phi^N P)$ goes rapidly to 0. Lemma 5.2(b) tells us what to do, namely choose i_N so that $\phi^N P$ is in $U_{i_N}(\kappa)$. By Lemma 5.2(a), this will be possible if we take $\kappa = \kappa_1$ sufficiently large, but in practice we do not have an explicit value for κ_1. However, this is irrelevant, since what we really want to do is choose i_N so that $\phi^N P$ is in $U_{i_N}(\kappa)$ for the smallest possible κ, which means finding the index i which minimizes $\lambda_{|D_i|}(\phi^N P)$. This is what we do in the following theorem which describes an explicit procedure for choosing the i_n's to give a rapidly convergent series for $\hat{\lambda}_{E,\phi}$.

THEOREM 5.3. *Let $P \in V(\bar{K}_v) \setminus |E|$ be given. With notation as above, define a sequence of indices $i_0, i_1, \ldots$ (depending on P) by the prescription*

$$\lambda_{|D_{i_n}|}(\phi^n P) = \min_{1 \leqslant i \leqslant r} \lambda_{|D_i|}(\phi^n P). \tag{68}$$

(If there is any ambiguity, take the smallest possible i_n.)

(a) *For every $n \geqslant 0$, the function $s_{i_n i_{n+1}}$ is defined at $\phi^n P$. Further, the sequence of real numbers*

$$c_n = -v(s_{i_n i_{n+1}}(\phi^n P)), \qquad n = 0, 1, 2, \ldots,$$

is bounded independently of n and P.

(b)

$$\hat{\lambda}_{E,\phi}(P) = v(t_{i_0}(P)) + \sum_{n=0}^{N-1} \alpha^{-n-1} c_n + O(\alpha^{-N}),$$

where the big-O constant is independent of both P and N.

Remark. Just as with Proposition 5.1, in order to successfully implement Theorem 5.3 it is necessary to write out the functions s_{ij} explicitly so that they can actually be evaluated at every point at which they are defined. There are also various ways of speeding up the computation without affecting the basic convergence of the series. For example, it might make sense to set i_{n+1} equal to the first i for which $\lambda_{|D_i|}(\phi^{n+1}P)$ is not too large, say less than 10. This may obviate the need to compute most of the $\lambda_{|D_i|}(\phi^{n+1}P)$'s. Further, one should probably start by checking if $i = i_n$ will do, since if one can take $i_{n+1} = i_n$, then $s_{i_n i_{n+1}} = s_{i_n i_n} = z_{i_n}$ is easier to compute.

Proof. (of Theorem 5.3). Let κ_1 be the constant described in Lemma 5.2(a). Then Lemma 5.2(a) and the definition of $U_i(\kappa)$ tell us that

$$\min_{1 \leqslant i \leqslant r} \lambda_{|D_i|}(Q) \leqslant \kappa_1 \qquad \text{for all } Q \in V(\bar{K}_v).$$

It follows from the definition (68) of the i_n's that for all $n \geqslant 0$,

$$\lambda_{|D_{i_n}|}(\phi^n P) \leqslant \kappa_1, \quad \text{or equivalently,} \quad \phi^n P \in U_{i_n}(\kappa_1).$$

Now Lemma 5.2(b) implies that

$$|\mu_{i_n}(\phi^n P)| \leqslant \kappa_2\kappa_1 + \kappa_3 \qquad \text{for all } n \geqslant 0. \tag{69}$$

Similarly, Lemma 5.2(c) tells us that

$$|c_n| = |v(s_{i_n i_{n+1}}(\phi^n P))| \leqslant \kappa_4\kappa_1 + \kappa_5$$

and that $s_{i_n i_{n+1}}$ is defined at $\phi^n P$. This proves (a).

Next comparing the definition of the c_n's with the formula (66) for μ_i, we see that

$$\mu_{i_0}(P) = \sum_{n=0}^{N-1} \alpha^{-n} c_n + \alpha^{-N}\mu_{i_N}(\phi^N P). \tag{70}$$

Now (69), (70), and the definition (61) of μ_i give

$$\begin{aligned}\hat{\lambda}_{E,\phi}(P) &= v(t_{i_0}(P)) + \frac{1}{\alpha}\mu_{i_0}(P)\\ &= v(t_{i_0}(P)) + \sum_{n=0}^{N-1} \alpha^{-n-1} c_n + \alpha^{-N-1}\mu_{i_N}(\phi^N P)\\ &= v(t_{i_0}(P)) + \sum_{n=0}^{N-1} \alpha^{-n-1} c_n + O(\alpha^{-N}).\end{aligned}$$

This completes the proof of (b). □

Example 4 (*continued*). We again let V be an elliptic curve given by a Weierstrass equation (52), E be the divisor (O), and $\phi = [2]$ be the duplication map. To illustrate Theorem 5.3 we take the two functions

$$t_1 = x^{-1/2} \quad \text{and} \quad t_2 = (x-1)^{-1/2}.$$

These functions have divisors

$$\begin{aligned}\operatorname{div}(t_1) &= (O) - \tfrac{1}{2}((R_1) + (R_1')) = E - D_1 \quad \text{and}\\ \operatorname{div}(t_2) &= (O) - \tfrac{1}{2}((R_2) + (R_2')) = E - D_2.\end{aligned}$$

Here R_1 and R_1' are the points with $x = 0$, and R_2 and R_2' are the points with $x = 1$, so clearly $|D_1| \cap |D_2| = \emptyset$. The formulas (54) and (55) from above give

$$z_1^2 = 1 - b_4 t_1^4 - 2b_6 t_1^6 - b_8 t_1^8 \quad \text{and} \quad w_1^2 = 4t_1^2 + b_2 t_1^4 + 2b_4 t_1^6 + b_6 t_1^8.$$

Similarly one finds

$$z_2^2 = 1 - b_4' t_2^4 - 2b_6' t_2^6 - b_8' t_2^8 \quad \text{and} \quad w_2^2 = 4t_2^2 + b_2' t_2^4 + 2b_4' t_2^6 + b_6' t_2^8,$$

where b_2', b_4', b_6', b_8' are the quantities associated to the Weierstrass equation (52) after making the substitution $x = x' + 1$. Using these and the relation $t_2^2 = t_1^2/(1 - t_1^2)$, it is not hard to derive formulas for the s_{ij}'s. For example,

$$\begin{aligned} s_{12}^2 &= \left(\frac{z_2 w_1}{w_2}\right)^2 = \left(\frac{t_1}{t_2} \circ \phi\right)^2 z_1^2 = (1 - t_1^2 \circ \phi) z_1^2 \\ &= \left(1 - \frac{w_1^2}{z_1^2}\right) z_1^2 = z_1^2 - w_1^2, \end{aligned}$$

and similarly $s_{21}^2 = z_2^2 + w_2^2$. It is now an easy matter to implement the algorithm described in Theorem 5.3 to compute the local height on an elliptic curve, even in the case $K_v = \mathbb{C}$ where Proposition 5.1 is not applicable. Details of this example may be found in [18].

Example 5. We return to the K3 surface S described in Example 2. In this case the automorphism $\phi\colon S \to S$ is the composition of two involutions $\phi = \sigma_2 \circ \sigma_1$. Further, the divisor classes η^+ and η^- are interchanged by σ_1, σ_2, for example $\sigma_2^* \eta^+ = \beta \eta^-$. Applying first σ_1 and then σ_2, it is not hard to make all of the constructions in this section completely explicit. For a description of the resulting series suitable for computer implementation, together with additional formulas and several numeric examples, see [3].

6. Canonical local heights as intersection multiplicities

Fix an absolute value v on K and let O_v denote the ring of v-integers in K. If A is an abelian variety over K, Néron (see [23] and [8], Chapter 11, Section 5) showed that any canonical local height $\hat{\lambda}_{A,D}(\cdot, v)$ can be interpreted as an intersection multiplicity on the special fiber of the Néron model of A over $\mathrm{Spec}(O_v)$. In Theorem 6.1 we extend Néron's result to show that if the pair (V, ϕ) has a certain weak Néron model over $\mathrm{Spec}(O_v)$, then the canonical local height $\hat{\lambda}_{V,E,\phi}(\cdot, v)$ can be computed as an intersection multiplicity.

We continue with the notation used in previous sections, but we add the assumption that the morphism $\phi\colon V \to V$ is finite. As in Section 5, we assume that E is defined over K, where $E \in \mathrm{Div}(V) \otimes \mathbb{R}$ is a divisor satisfying $\phi^* E \sim \alpha E$ with $\alpha > 1$.

Let $S = \operatorname{Spec}(O_v)$. We will say that a smooth scheme $\mathcal{V}/S$ is a *weak Néron model* for $(V/K, \phi)$ over S if it satisfies the following axioms:
(1) The generic fiber of $\mathcal{V}/S$, denoted $\mathcal{V}_K$, is V.
(2) $\mathcal{V}(S) \cong V(K)$. In other words, every point $P \in V(K)$ extends to a section $\mathbf{P}\colon S \to \mathcal{V}$.
(3) There exists a finite morphism $\Phi\colon \mathcal{V}/S \to \mathcal{V}/S$ whose restriction to the generic fiber is ϕ.
We remark that axioms 2 and 3 impose opposing requirements on $\mathcal{V}/S$. Axiom 2 demands that $\mathcal{V}$ has enough points over S so that every K-rational point on V extends to a section. On the other hand, axiom 3 asks that $\mathcal{V}/S$ not be so large that the rational map on $\mathcal{V}$ induced by ϕ cannot be extended to a finite morphism. Note that the Néron model of an abelian variety A/K is a weak Néron model for $(A/K, [n])$ for all $n \geq 2$. For an alternative notion of weak Néron model, see Bosch–Lütkebohmert–Raynaud [1].

Henceforth we will assume that (V, ϕ) has a weak Néron model $\mathcal{V}/S$. Let $\mathcal{V}_s$ denote the special fiber of $\mathcal{V}$ and write

$$\mathcal{V}_s = \sum_{j=1}^{n} \mathcal{V}_s^j,$$

where $\mathcal{V}_s^1, \ldots, \mathcal{V}_s^n \in \operatorname{Div}(\mathcal{V})$ are the irreducible components of $\mathcal{V}_s$. Let W be a prime divisor on V which is rational over K. Observe that the closure of W in $\mathcal{V}$, denoted $\bar{W}$, is a prime divisor on $\mathcal{V}$. Extending this process by linearity, we obtain a natural injection

$$\operatorname{Div}(V)_K \to \operatorname{Div}(\mathcal{V}), \quad D \mapsto \bar{D},$$

called the thickening map. Similarly, given a point $P \in V(K)$, we write $\bar{P} = \mathbf{P}(S)$ to denote the image of the section $\mathbf{P}$ in $\mathcal{V}$. Note that the divisor group on S is a cyclic group generated by the special point (s). Hence, for any $D \in \operatorname{Div}(V)_K$ and any $P \in V(K)$ which does not lie on the support of D, we may define the intersection multiplicity $i(D, P)$ (also denoted $\bar{P} \cdot \bar{D}$) by

$$\mathbf{P}^*\bar{D} = i(D, P)(s).$$

With these notations in hand, we can now state the main result of this section.

THEOREM 6.1. *Suppose $\mathcal{V}/S$ is a weak Néron model for $(V/K, \phi)$ over O_v. Let $\hat{\lambda}_{V,E,\phi}$ be a canonical local height as constructed in Theorem 2.1. Then there exist real numbers $\gamma_1, \ldots, \gamma_n$ so that for all $P \in V(K) \setminus |E|$,*

$$\hat{\lambda}_{V,E,\Phi}(P, v) = \bar{P} \cdot \left(\bar{E} + \sum_{j=1}^{n} \gamma_j \mathcal{V}_s^j\right). \tag{71}$$

A key point in the proof of Theorem 6.1 is to describe the action of Φ on the set of irreducible components $\{\mathcal{V}_s^1, \ldots, \mathcal{V}_s^n\}$ of $\mathcal{V}_s$. Since Φ is a finite morphism, it maps each irreducible component of $\mathcal{V}_s$ onto another irreducible component (possibly the same component) of $\mathcal{V}_s$. Let $N = \{1, \ldots, n\}$. Then

$$A = A_\Phi\colon N \to N \quad \text{defined by} \quad \Phi\colon \mathcal{V}_s^j \to \mathcal{V}_s^{A(j)} \quad \text{for } j \in N.$$

Note that A need not be a permutation of N, since Φ may map several components onto the same component. However, we can identify A with a matrix of the following type.

DEFINITION. A square matrix M is a *permutation-type matrix* if every column of M has exactly one 1 and all other entries are 0.

Identify each $j \in N$ with the n-by-1 column vector $\mathbf{e}_j$ which has a 1 as its jth entry and zeros elsewhere. Then the map $A\colon N \to N$ corresponds to a unique n-by-n permutation-type matrix, which by abuse of notation we will also denote by A. Thus, $A(j) = A\mathbf{e}_j$.

LEMMA 6.2. (a) *Fix $m \geq 1$. Then the set of all m-by-m permutation-type matrices forms a finite monoid under matrix multiplication.*

(b) *Every eigenvalue of a permutation-type matrix is either zero or a root of unity.*

Proof. It is clear that there are only finitely many (in fact m^m) m-by-m permutation-type matrices. Suppose $B = (b_{ij})$ and $C = (c_{ij})$ are m-by-m permutation-type matrices. Fix k, $1 \leq k \leq m$. By the definition of permutation-type matrix, there is a (unique) j_0, $1 \leq j_0 \leq m$, such that $c_{j_0 k} = 1$ and $c_{jk} = 0$ for all $j \neq j_0$. Then

$$(BC)_{ik} = \sum_{j=1}^{m} b_{ij} c_{jk} = b_{ij_o}.$$

Hence, the j_0th column of B becomes the kth column of BC. Therefore, the set of m-by-m permutation-type matrices is closed under multiplication, and thus it is a monoid.

To prove (b), let M be any permutation-type matrix. Since M lies in a finite monoid, there exist integers $q, r > 0$ such that $M^{r+q} = M^r$. Suppose $\mathbf{v} \neq \mathbf{0}$ is an eigenvector of M with eigenvalue μ. Then $\mu^{r+q}\mathbf{v} = \mu^r\mathbf{v}$, so $\mu^r(\mu^q - 1)\mathbf{v} = \mathbf{0}$. Since $\mathbf{v} \neq \mathbf{0}$, we have $\mu = 0$ or $\mu^q = 1$ with $q > 0$. $\square$

Proof. (of Theorem 6.1). Since E is assumed to be rational over K, we may fix a rational function $f \in K(V)^* \otimes \mathbb{R}$ so that

$$\phi^*E = \alpha E + \operatorname{div}_V(f). \tag{72}$$

Since $K(V) \cong K(\mathcal{V})$, we may also regard f as an element of $K(\mathcal{V})^* \otimes \mathbb{R}$. Then the divisors of f on V and $\mathcal{V}$ differ by a divisor supported on the special fiber, say

$$\overline{\operatorname{div}_V(f)} = \operatorname{div}_{\mathcal{V}}(f) + Z_f, \quad \text{where} \quad Z_f = \sum_{j=1}^{n} m(j,f)\mathcal{V}_s^j, \tag{73}$$

for some constants $m(j,f) \in \mathbb{R}$.

By Theorem 2.1(b), there is a unique canonical local height $\hat{\lambda}_E = \hat{\lambda}_{V,E,\phi,f}$ which satisfies

$$\hat{\lambda}_E(\phi P, v) = \alpha\hat{\lambda}_E(P, v) + v(f(P)). \tag{74}$$

Recall from Theorem 2.1(a) that the difference between any canonical local height $\hat{\lambda}_{V,E,\phi}$ and the particular canonical local height $\hat{\lambda}_E$ is a constant of the form $v(b)$, where $b \in \bar{K}^* \otimes \mathbb{R}$. Hence, it suffices to prove that there are real numbers $\gamma_1, \ldots, \gamma_n$ so that the function $\hat{\lambda}_E$ satisfies (71).

Consider the map $V(K) \setminus |E| \to \mathbb{R}$ defined by $P \mapsto i(E, P) = \bar{P} \cdot \bar{E}$. Given any $P \in V(K) \setminus |E|$, there is a pair (U, g) representing $\bar{E}$ such that $U \subseteq \mathcal{V}$ is an open neighborhood of P and $g(P) \neq 0, \infty$. Then, by definition, $i(E, P) = v(g(P))$. In particular, $i(E, P)$ is independent of the chosen pair (U, g). Thus, the map $P \mapsto i(E, P)$ is a Weil local height for E on $V(K)$.

Note that $\Phi^*\bar{E}$ and $\overline{\phi^*E}$ differ by a divisor supported on the special fiber, since Φ and ϕ are the same on the generic fiber. Combining this fact with (72), we have

$$\Phi^*\bar{E} = \alpha\bar{E} + \overline{\operatorname{div}_V(f)} + \sum_{j=1}^{n} n_j\mathcal{V}_s^j, \tag{75}$$

for some constants $n_j \in \mathbb{R}$. Further, $\Phi_*\bar{P} = \Phi_*\mathbf{P}(S) = \Phi \circ \mathbf{P}(S) = \phi\mathbf{P}(S) = \overline{\phi P}$, where $\phi\mathbf{P}$ is the section corresponding to ϕP. Hence,

$$\bar{P} \cdot \Phi^*\bar{E} = \Phi_*\bar{P} \cdot \bar{E} = \overline{\phi P} \cdot \bar{E} = i(E, \phi P). \tag{76}$$

Intersecting both sides of (73) with $\bar{P}$ yields:

$$\begin{aligned}\bar{P}\cdot\overline{\operatorname{div}_V(f)} &= \bar{P}\cdot\operatorname{div}_{\mathcal{V}}(f) + \bar{P}\cdot Z_f\\ &= v(f(P)) + \bar{P}\cdot\sum_{j=1}^{n} m(j,f)\mathcal{V}_s^j.\end{aligned} \tag{77}$$

Now, intersecting both sides of (75) with $\bar{P}$ and applying (76) and (77), we conclude

$$i(E,\phi P) = \alpha i(E,P) + v(f(P)) + \bar{P}\cdot\sum_{j=1}^{n} c_j\mathcal{V}_s^j, \tag{78}$$

where $c_j = m(j,f) + n_j$ are constants which depend on E, ϕ and f, but are independent of P. In particular, we see that (78) holds for all $P \in V(K)$ for which the intersection multiplicities $i(E,\phi P)$ and $i(E,P)$ are defined; i.e., for all $P \notin |E| \cup |\phi^* E|$.

Next, we will show that one can choose real numbers $x_1,\ldots,x_n$ so that the function

$$\Lambda_E(P) = i(E,P) + \bar{P}\cdot\sum_{j=1}^{n} x_j\mathcal{V}_s^j \tag{79}$$

satisfies

$$\Lambda_E(\phi P) = \alpha\Lambda_E(P) + v(f(P)) \tag{80}$$

for all $P \in V(K)\setminus(|E| \cup |\phi^* E|)$. Using (79) and (78), we compute

$$\begin{aligned}&\Lambda_E(\phi P) - \alpha\Lambda_E(P) - v(f(P))\\ &\quad = \overline{\phi P}\cdot\sum_{j=1}^{n} x_j\mathcal{V}_s^j - \alpha\bar{P}\cdot\sum_{j=1}^{n} x_j\mathcal{V}_s^j + \bar{P}\cdot\sum_{j=1}^{n} c_j\mathcal{V}_s^j.\end{aligned}$$

Recall that Φ determines a permutation-type matrix $A = A_\Phi$ defined by

$$\Phi\colon \mathcal{V}_s^j \to \mathcal{V}_s^{A(j)}.$$

Since $\bar{P}$ and $\overline{\phi P} = \Phi(\bar{P})$ intersect the components of $\mathcal{V}_s$ transversally, it follows from the definition of A that if $\mathbf{P}(s) \in \mathcal{V}_s^k$, then

$$\bar{P} \cdot \sum_{j=1}^{n} x_j \mathcal{V}_s^j = x_k \quad \text{and} \quad \overline{\phi P} \cdot \sum_{j=1}^{n} x_j \mathcal{V}_s^j = x_{A(k)}. \tag{81}$$

Therefore, it suffices to find constants $x_1, \ldots, x_n$ such that

$$x_{A(k)} - \alpha x_k + c_k = 0, \quad \text{for } k = 1, \ldots, n.$$

Writing $x_1, \ldots, x_n$ and $c_1, \ldots, c_n$ as column vectors, we can combine these n equations into a matrix equation

$$(\alpha I - A)\mathbf{x} = \mathbf{c}.$$

Lemma 6.2(b) tells us that the eigenvalues of the permutation-type matrix A are zero or roots of unity, while $\alpha > 1$ by hypothesis. Therefore $\det(\alpha I - A) \neq 0$, so $\alpha I - A$ is invertible and we may take $\mathbf{x} = (\alpha I - A)^{-1}\mathbf{c}$. This finishes the proof that we can choose $x_1, \ldots, x_n$ so that the function Λ_E defined by (79) satisfies the relation (80).

To complete the proof of Theorem 6.1 we will show that $\hat{\lambda}_E(P, v) = \Lambda_E(P)$ for all $P \in V(K) \setminus |E|$. Since $\hat{\lambda}_E(\cdot, v)$ and $i(E, \cdot)$ are both Weil local heights for E, their difference $\hat{\lambda}_E(\cdot, v) - i(E, \cdot)$ has a unique v-continuous extension to a bounded v-continuous function defined on all of $V(K)$ (see [8], Chapter 10, Propositions 1.5 and 2.3). Hence, by (79), we see that the map $L_E(P) = \hat{\lambda}_E(P, v) - \Lambda_E(P)$ extends to a bounded function on $V(K)$. Furthermore, since $\hat{\lambda}_E$ and Λ_E were chosen to satisfy (74) and (80), it follows that

$$L_E(\phi P) = \alpha L_E(P) \quad \text{for all } P \in V(K).$$

Therefore, for any $P \in V(K)$,

$$|L_E(P)| = |\alpha^{-N} L_E(\phi^N P)| \ll \alpha^{-N},$$

where the $\ll$ constant is independent of N. We conclude $L_E \equiv 0$, so $\hat{\lambda}_E(P, v) = \Lambda_E(P)$ for all $P \in V(K)$, $P \notin |E|$. □

We can use Theorem 6.1 to show that the difference between a canonical local height and a Weil local height varies more-or-less periodically with ϕ. This had been noted earlier for abelian varieties (e.g., [8], Chapter 11, Theorem 5.2) and has been verified experimentally in some cases for K3 surfaces [3].

COROLLARY 6.3. *Suppose that $(V/K, \phi)$ has a weak Néron model $\mathcal{V}/S$ over O_v. Choose a canonical local height $\hat{\lambda}_{V,E,\phi}$ and define a correction factor*

$$\delta(P) = \hat{\lambda}_{V,E,\phi}(P, v) - \bar{P} \cdot \bar{E}.$$

Then there are integers $t \geq 0$ and $r \geq 1$ such that

$$\delta(\phi^{t+r}P) = \delta(\phi^t P) \qquad \text{for all } P \in V(K).$$

Further, if ϕ is an automorphism, then one can take $t = 0$, so

$$\delta(\phi^r P) = \delta(P) \qquad \text{for all } P \in V(K).$$

Proof. From Theorem 6.1 and the definition of δ we have

$$\delta(P) = \bar{P} \cdot \sum_{j=1}^{n} \gamma_j \mathcal{V}_s^j$$

for certain real numbers $\gamma_1, \ldots, \gamma_n$. The section $\bar{P}$ hits exactly one of the components on the special fiber, say $\mathbf{P}(s) \in \mathcal{V}_s^k$. Then $\delta(P) = \gamma_k$. Further, as in the proof of Theorem 6.1 there is a map $A: \{1, \ldots, n\} \to \{1, \ldots, n\}$ corresponding to a permutation-type matrix such that $\delta(\phi P) = \gamma_{A(k)}$. Applying this repeatedly, we find

$$\delta(\phi^m P) = \gamma_{A^m(k)} \qquad \text{for all } m \geq 0. \tag{82}$$

From Lemma 6.2(a) we know that A belongs to a finite monoid, so there are integers $t \geq 0$ and $r \geq 1$ such that $A^{t+r} = A^t$. Using this in (82) gives the first half of Corollary 6.3. Finally, if ϕ is an automorphism, then we can replace P by $\phi^{-t}P$ to get the second half. □

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References

1. Bosch, S., Lütkebohmert, W., and Raynaud, M.: *Néron Models*. Springer-Verlag, Berlin, (1990).
2. Call, G.: Variation of local heights on an algebraic family of abelian varieties. Théorie des Nombres, Berlin, (1989).
3. Call, G. and Silverman, J.: Computing canonical heights on K3 surfaces, in preparation.
4. Dem'janenko, V.A.: An estimate of the remainder term in Tate's formula (Russian), *Mat. Zametki* **3** (1968) 271–278.
5. Dem'janenko, V.A.: Rational points of a class of algebraic curves, *AMS Translations* (2) 66 (1968) 246–272.
6. Green, W.: Heights in families of abelian varieties, *Duke Math. J.* **58** (1989) 617–632.
7. Hartshorne, R.: *Algebraic Geometry*, Springer-Verlag, New York, (1977).
8. Lang, S.: *Fundamentals of Diophantine Geometry*, New York, (1983).
9. Lang, S.: Number Theory III: Diophantine Geometry. *Encycl. Math. Sci. v. 60*, Springer-Verlag, Berlin, (1991).
10. Lewis, D.J.: Invariant sets of morphisms on projective and affine number spaces, *J. Algebra* **20** (1972) 419–434.
11. Manin, Ju.: The p-torsion of elliptic curves is uniformly bounded. *Izv. Akad. Nauk. SSSR* **33** (1969) 433–438.
12. Manin, Ju. and Zarhin, Ju.: Height on families of abelian varieties, *Math. USSR Sbor.* **18** (1972) 169–179.
13. Narkiewicz, W.: On polynomial transformations in several variables, *Acta Arith.* **11** (1965) 163–168.
14. Néron, A.: Quasi-fonctions et hauteurs sur les variétés abéliennes, *Annals of Math.* **82** (1965) 249–331.
15. Silverman, J.H.: Heights and the specialization map for families of abelian varieties, *J. Reine Angew. Math.* **342** (1983) 197–211.
16. Silverman, J.H.: *The Arithmetic of Elliptic Curves*, Springer, New York, (1986).
17. Silverman, J.H.: Arithmetic distance functions and height functions in Diophantine geometry, *Math. Ann.* **279** (1987) 193–216.
18. Silverman, J.H.: Computing heights on elliptic curves, *Math. Comp.* **51** (1988) 339–358.
19. Silverman, J.H.: Rational points on K3 surfaces: A new canonical height, *Invent. Math.* **105** (1991) 347–373.
20. Silverman, J.H.: Variation of the canonical height on elliptic surfaces I: Three examples, *J. Reine Angew. Math.* **426** (1992) 151–178.
21. Tate, J.: Letter to J.-P. Serre, Oct. 1, (1979).
22. Tate, J.: Variation of the canonical height of a point depending on a parameter, *Amer. J. Math.* **105** (1983) 287–294.
23. Wehler, J.: K3-surfaces with Picard number 2, *Arch. Math.* **50** (1988) 73–82.
24. Wehler, J.: Hypersurfaces of the Flag Variety, *Math. Zeit.* **198** (1988) 21–38.
25. Zimmer, H.: On the difference of the Weil height and the Néron–Tate height, *Math. Z.* **174** (1976) 35–51.

Added in proof

After this paper was accepted for publication, the authors became aware of two related papers. D. G. Northcott, *Annals Math* **5** (1950) 167–177, gives a proof of the finiteness result (Corollary 1.1.1b) which we had attributed to Narkiewicz and Lewis. L. Denis, *Math. Ann.* **294** (1992) 213–223, constructs canonical heights on Drin'feld modules similar to our construction in Theorem 1.1.