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A comparison theorm for $\mathfrak{n}$-homology

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Introduction

The purpose of this paper is to compare homological properties of an analytic representation of a semisimple Lie group and of its Harish-Chandra module.

Throughout the paper G_0 denotes a connected semisimple Lie group with finite center. Let π be an admissible representation of G_0 on a complete, locally convex Hausdorff topological vector space M_π. Vectors in M_π transforming finitely under the action of a maximal compact subgroup K_0 of G_0 are analytic and they form a subspace M invariant under the action of the Lie algebra $\mathfrak{g}_0$ of G_0 – this subspace is, by definition, the Harish-Chandra module of π. In the reverse direction, we refer to π as a *globalization* of M. The advantages of working with the underlying Harish-Chandra module, rather than with the global representation itself, are multifold. Although not a G_0-invariant subspace of M_π, M retains the essential features of the representation. For example, it fully determines the (distributional) character of π. On the other hand, M is a much smaller object than M_π: it is stripped of the often cumbersome functional analytic superstructure of the latter, and enjoys finiteness properties which make it amenable to algebraic, and, in particular, homological methods.

In the study of M a special role is played by the homology groups $H_p(\mathfrak{n}, M)$, with respect to maximal nilpotent subalgebras $\mathfrak{n}$ of the complexification $\mathfrak{g}$ of $\mathfrak{g}_0$. They carry information not only about the module itself, but also about the global representation. Two examples, somewhat interconnected, may serve as evidence: (1) For a generic $\mathfrak{n}$, $H_0(\mathfrak{n}, M)$ determines the growth of matrix coefficients of M_π ([4], [6], [12], [17]). (2) The value of the character of M_π at a regular point $g \in G_0$ is equal to the "Euler characteristic" of the $\mathfrak{n}$-homology of M, where the choice of $\mathfrak{n}$ depends on g ([11], see also [22]).

These examples suggest that the groups $H_p(\mathfrak{n}, M)$ are global invariants, and a natural question which arises is whether for a suitably chosen M_π the map

$$H_p(\mathfrak{n}, M) \to H_p(\mathfrak{n}, M_\pi) \tag{1}$$

induced by the embedding $M \to M_\pi$ is an isomorphism.

To make sense of this statement we have to assume that the globalization M_π of M consists of smooth vectors. Even then we cannot expect (1) to be true for every $\mathfrak{n}$, as M and M_π have different invariance properties. On the one hand M_π is a G_0-module. On the other, M is a module for the complexification K of K_0.

The main result of this paper asserts that the answer to the above question is positive assuming that M_π is the minimal globalization of M, in the sense defined below, *and* $\mathfrak{n}$ is chosen appropriately. In fact, it is positive for a set of $\mathfrak{n}$'s which is sufficiently rich to allow one (using the invariance properties) to compute $H_p(\mathfrak{n}, M)$ for all $\mathfrak{n}$'s if one knows $H_p(\mathfrak{n}, M_\pi)$ for all $\mathfrak{n}$'s and vice-versa.

In order to state this result precisely we first need to dispose of some preliminaries.

Every Harish-Chandra module M can be globalized: that is, it arises as the space of K_0-finite vectors of a continuous representation π of G_0 on a topological vector space M_π. One can choose M_π to be Banach, or even Hilbert; the choice is by no means unique. However, there exist several canonical globalization functors ([5], [19], [23]). The one we consider in this paper is the functor $M \mapsto \tilde{M}$ of *minimal globalization* introduced by Schmid in [19]. The G_0-module $\tilde{M}$ carries the topology of a dual of a nuclear Fréchet space and it embeds continuously into every globalization of M to G_0-module. If M is the Harish-Chandra module of a Banach representation (π, M_π) of G_0, then $\tilde{M}$ coincides with the space of analytic vectors in M_π. Because of the above we also refer to $\tilde{M}$ as *the analytic globalization* or *analytic completion* of M.

Let X be the flag variety of $\mathfrak{g}$. As usual we denote by $\mathfrak{b}_x$ the Borel subalgebra corresponding to $x \in X$, and by $\mathfrak{n}_x$ its nilpotent radical. Both G_0 and K act on X with finitely many orbits and there is one-to-one correspondence between G_0-orbits S and K-orbits Q, uniquely characterized by the property that $Q \cap S$ is compact and nonempty; $Q \cap S$ is in fact a K_0-orbit [16]. If S and Q correspond to each other, we call a point $x \in Q \cap S$ *special.*

We now state the main result of the paper.

COMPARISON THEOREM. *Let $x \in X$ be special. Then the map $H_p(\mathfrak{n}_x, M) \to H_p(\mathfrak{n}_x, \tilde{M})$, induced by the embedding of a Harish-Chandra module M* into its minimal globalization *$\tilde{M}$, is an isomorphism.*

The main idea is to use a local geometric characterization of $\mathfrak{n}_x$-homology for modules with regular infinitesimal character θ. Let $\mathfrak{h}$ denote the abstract Cartan subalgebra of $\mathfrak{g}$. Via the Harish-Chandra isomorphism we can think of θ as a Weyl group orbit of a regular linear form on $\mathfrak{h}$. Then $\mathfrak{h}$ acts on the $\mathfrak{n}_x$-homology groups semisimply, with possible weights of the form $\lambda + \rho$, $\lambda \in \theta$ (as usual, ρ denotes one-half of the sum of positive roots). Therefore to prove the comparison theorem in this situation it is enough to show that

$$H_p(\mathfrak{n}_x, M)_{\lambda+\rho} \to H_p(\mathfrak{n}_x, \tilde{M})_{\lambda+\rho} \tag{2}$$

is an isomorphism, whenever x is special and $\lambda \in \theta$.

Let $\mathscr{D}_\lambda$ be the twisted sheaf of differential operators on X attached to λ, and $\mathscr{M}$ a complex of $\mathscr{D}_\lambda$-modules, which is Γ-*acyclic*, i.e. hypercohomology groups $H^p(X, \mathscr{M})$ vanish except in degree zero. The $\mathfrak{g}$-module $H^0(X, \mathscr{M})$ has infinitesimal character θ. Denote by T_x the functor of geometric fiber at x, and by $L_p T_x$ its p-th left derived functor. Then

$$H_p(\mathfrak{n}_x, H^0(X, \mathscr{M}))_{\lambda+\rho} \text{ is naturally isomorphic to } L_p T_x(\mathscr{M}).$$

Suppose now that there exists a morphism $\mathscr{M} \to \tilde{\mathscr{M}}$ of Γ-acyclic complexes of $\mathscr{D}_\lambda$-modules which on the level of zero-th hypercohomology induces the embedding $M \to \tilde{M}$. Then (2) is equivalent to showing

$$L_p T_x(\mathscr{M}) \cong L_p T_x(\tilde{\mathscr{M}}). \tag{3}$$

The fact that such a morphism always exists is provided by the localization theories ([1], [13]). For our purposes we need to work with the analytic variant of localization, developed in [13], as it better reflects the structure of minimal globalizations. We prove (3) directly when M is a standard Harish-Chandra module, and λ has antidominant real part, using the very explicit description of $\mathscr{M}$ and $\tilde{\mathscr{M}}$ available in this situation. The comparison theorem follows from this special case by a series of formal reductions.

We adhere to the following notational conventions: Complex groups are denoted by capital Roman letters, and their Lie algebras by the corresponding lower case german letters. For any Lie algebra $\mathfrak{m}$, $\mathscr{U}(\mathfrak{m})$ and $\mathscr{Z}(\mathfrak{m})$ denote, respectively, the universal enveloping algebra of $\mathfrak{g}$, and its center.

By f_* and f^{-1} we denote the functors of direct image and inverse image in the category of sheaves, induced by a continuous map f. If $(X, \mathcal{O}_X) \xrightarrow{f} (Y, \mathcal{O}_Y)$ happens to be a morphism of ringed spaces then f^* denotes the inverse image in the category of $\mathcal{O}$-modules.

1. Localization and standard modules

In this section we collect some basic facts on localization of $\mathfrak{g}$-modules ([1], [2], [10], [13], [18]).

We regard the flag variety X of $\mathfrak{g}$ both as an algebraic variety and a complex manifold. Any connected Lie group G with Lie algebra $\mathfrak{g}$ acts transitively on X. Given x, we can therefore identify X with G/B_x, where B_x is the Borel subgroup of G corresponding to x.

Let $\mathfrak{B}$ be the tautological vector bundle of X: the fiber over x is $\mathfrak{b}_x$. Similarly, let $\mathfrak{N}$ be the subbundle of $\mathfrak{B}$ with typical fiber $\mathfrak{n}_x$. The quotient bundle $\mathfrak{B}/\mathfrak{N}$ is trivial: B_x acts on its fiber $\mathfrak{b}_x/\mathfrak{n}_x$ over x as the identity transformation. The space $\mathfrak{h}$ of constant sections of $\mathfrak{B}/\mathfrak{N}$ is, by definition, the *abstract Cartan subalgebra* of $\mathfrak{g}$. If $\mathfrak{c}$ is a Cartan subalgebra of $\mathfrak{g}$ contained in a particular $\mathfrak{b}_x$, then $\mathfrak{h}$ can be related to $\mathfrak{c}$ as follows. There are natural isomorphisms: $\mathfrak{h} \to \mathfrak{b}_x/\mathfrak{n}_x$, obtained by evaluating a section at x, and $\mathfrak{c} \to \mathfrak{b}_x/\mathfrak{n}_x$, induced by the inclusion $\mathfrak{c} \to \mathfrak{b}_x$. The composition of the first with the inverse of the second results in an isomorphism $\mathfrak{h} \to \mathfrak{c}$, which we refer to as the *specialization* of $\mathfrak{h}$ to $\mathfrak{c}$.

Let $\Delta(\mathfrak{c})$ be the set of roots of $\mathfrak{c}$ in $\mathfrak{g}$, and $\Delta^+(\mathfrak{c})$ the subset of positive roots cut out by $\mathfrak{b}_x$. Let $\Delta \subset \mathfrak{h}^*$ denote the image of $\Delta(\mathfrak{c})$ under the transpose of $\mathfrak{h} \to \mathfrak{c}$. Similarly, let $\Delta^+ \subset \Delta$ be the image of $\Delta^+(\mathfrak{c})$. We call $\Delta \subset \mathfrak{h}^*$ the *abstract root system* (of $\mathfrak{g}$) and $\Delta^+ \subset \Delta$ the *abstract positive root system.* This construction is independent of the choice of $\mathfrak{c}$ and $\mathfrak{b}_x$. In this context the (unnormalized) Harish-Chandra homomorphism $\gamma: \mathscr{Z}(\mathfrak{g}) \to \mathscr{U}(\mathfrak{h})$ is defined as follows. Each $z \in \mathscr{Z}(\mathfrak{g})$ agrees, modulo the right ideal in $\mathscr{U}(\mathfrak{g})$ generated by $\mathfrak{n}_x$, with a unique element of $\mathscr{U}(\mathfrak{b}_x)$. Consequently, z determines a collection $\{\gamma_x(z) \in \mathscr{U}(\mathfrak{b}_x/\mathfrak{n}_x)\}_{x \in X}$, which is G-equivariant, and hence defines an element $\gamma(z)$ in $\mathscr{U}(\mathfrak{h})$. Denote by W the Weyl group of Δ, and by ρ one-half of the sum of roots in Δ^+. Composing γ with the automorphism of $\mathscr{U}(\mathfrak{h})$ equal to $h \to h + \rho(h)$ on $\mathfrak{h}^*$ results in the Harish-Chandra isomorphism of $\mathscr{Z}(\mathfrak{g})$ onto W-invariants in $\mathscr{U}(\mathfrak{h})$. Consequently, characters of $\mathscr{Z}(\mathfrak{g})$ can be identified with Weyl group orbits in $\mathfrak{h}^*$. Set $\mathscr{U}_\theta = \mathscr{U}(\mathfrak{g})/\mathscr{U}(\mathfrak{g})\,\mathrm{Ker}(\theta)$.

As we have mentioned before, X can be regarded both as a complex analytic manifold X^{an} and as an algebraic variety X^{alg}. We denote by $\mathcal{O}_X$ the sheaf of holomorphic functions on X and by $\mathcal{O}_X^{\mathrm{alg}}$ the sheaf of regular functions. Let $\varepsilon: X^{an} \to X^{\mathrm{alg}}$ be the identity map. Denote by $(-)^{an}$ the functor $\mathcal{O}_X \otimes_{\varepsilon^{-1}\mathcal{O}_X^{\mathrm{alg}}} \varepsilon^{-1}(-)$ from the category of $\mathcal{O}_X^{\mathrm{alg}}$-modules to the category of $\mathcal{O}_X$-modules ([21], [8]). Let $\mathscr{F}$ be a quasicoherent $\mathcal{O}_X^{\mathrm{alg}}$-module. According to ([8], Lemme 6.5)

$$\text{the canonical morphism } H^p(X, \mathscr{F}) \to H^p(X, \mathscr{F}^{an}) \text{ is an isomorphism.} \tag{1.1}$$

Beilinson and Bernstein associate to each $\lambda \in \mathfrak{h}^*$ a sheaf $\mathscr{D}_\lambda^{\mathrm{alg}}$ on X containing $\mathcal{O}_X^{\mathrm{alg}}$, which is locally isomorphic to the sheaf of differential operators on X with regular coefficients. In particular $\mathscr{D}_\lambda^{\mathrm{alg}}$ is quasicoherent when regarded as an $\mathcal{O}_X^{\mathrm{alg}}$-module with respect to either left, or right, multiplication. Moreover, $\Gamma(X, \mathscr{D}_\lambda^{\mathrm{alg}}) \cong \mathscr{U}_\theta$ ([1]). Similarly, define the sheaf of algebras $\mathscr{D}_\lambda = (\mathscr{D}_\lambda^{\mathrm{alg}})^{an}$. By (1.1) its space of global sections is also isomorphic to $\mathscr{U}_\theta$. Denote by $\mathscr{M}(\mathscr{U}_\theta)$ the category of $\mathscr{U}_\theta$-modules. Let $\mathscr{M}(\mathscr{D}_\lambda^{\mathrm{alg}})$ be the category of $\mathscr{D}_\lambda^{\mathrm{alg}}$-modules, i.e. sheaves of *left* modules on X over $\mathscr{D}_\lambda^{\mathrm{alg}}$, and $D(\mathscr{D}_\lambda^{\mathrm{alg}})$ its derived category. We identify $\mathscr{M}(\mathscr{D}_\lambda^{\mathrm{alg}})$ with a full subcategory of $D(\mathscr{D}_\lambda^{\mathrm{alg}})$ consisting of complexes with cohomology concentrated in degree zero. If $\mathscr{V}$ is in $D(\mathscr{D}_\lambda^{\mathrm{alg}})$ then the hypercohomology groups of $\mathscr{V}$ are objects in $\mathscr{M}(\mathscr{U}_\theta)$. The same is true in the context of the category $\mathscr{M}(\mathscr{D}_\lambda)$ of $\mathscr{D}_\lambda$-modules, and its derived category $D(\mathscr{D}_\lambda)$. This is a very general method of geometric construction of representations, which lies at the heart of localization theory. Of particular importance to us are two "standard objects" in $D(\mathscr{D}_\lambda)$, which, after passing to the cohomology, lead to a realization of standard Harish-Chandra modules and their analytic completions. At this point we do not give a precise definition of these objects but rather state some of their relevant properties. More details are provided in section 2.

We assume

$$\lambda \text{ is regular, and } \operatorname{Re}\lambda(\alpha^\vee) \leqslant 0, \text{ for } \alpha\in\Delta^+ \tag{1.2}$$

Let Q be a K-orbit in X, $x\in Q$ a special point, and S the G_0-orbit of x. According to [14], $\mathfrak{b}_x$ contains a Cartan subalgebra $\mathfrak{c}$ stable both under the Cartan involution determined by $\mathfrak{t}$, and under the conjugation with respect to $\mathfrak{g}_0$. Via the specialization map we identify the triple $(\mathfrak{c}, \Delta(\mathfrak{c}), \Delta^+(\mathfrak{c}))$ with the abstract triple $(\mathfrak{h}, \Delta, \Delta^+)$. Let C_0 be the Cartan subgroup of G_0 with Lie algebra $\mathfrak{c}\cap\mathfrak{g}_0$, and (χ, L) an irreducible finite dimensional representation of C_0 with differential $\lambda+\rho$. Via a *$\mathscr{D}$-module direct image construction* Beilinson and Bernstein attach to χ a K-equivariant module $\mathscr{I}^{\text{alg}}_{\chi,Q}$ in $\mathscr{M}(\mathscr{D}^{\text{alg}}_\lambda)$, supported on the closure of Q ([1], ([10], Appendix)). Moreover, $\mathscr{I}^{\text{alg}}_{\chi,Q}$ is Γ-acyclic, i.e. its nonzero hypercohomology groups vanish. The Harish-Chandra module

$$I_{\chi,Q} = \Gamma(X, \mathscr{I}^{\text{alg}}_{\chi,Q}) \tag{1.3}$$

is usually referred to as *standard*. Let

$$\mathscr{I}_{\chi,Q} = (\mathscr{I}^{\text{alg}}_{\chi,Q})^{an}. \tag{1.4a}$$

It follows from (1.1) that $\mathscr{I}_{\chi,Q}$ is also Γ-acyclic and $\Gamma(X, \mathscr{I}_{\chi,Q}) = I_{\chi,Q}$.

By definition, the standard Harish-Chandra module $I_{\chi,Q}$ is geometrically defined: it arises as the space of global sections of $\mathscr{I}_{\chi,Q}$. It turns out that the same is true for the minimal globalization $\tilde{I}_{\chi,Q}$ of $I_{\chi,Q}$. Namely, χ determines in a constructive manner a Γ-acyclic object

$$\mathscr{I}_{\chi,S} \tag{1.4b}$$

in $D(\mathscr{D}_\lambda)$ supported on S such that $H^0(X, \mathscr{I}_{\chi,S})$ is isomorphic to the minimal globalization of $\tilde{I}_{\chi,Q}$ ([13], Proposition 10.8).[1]

1.1 PROPOSITION. *There exists a morphism $\Phi: \mathscr{I}_{\chi,Q} \to \mathscr{I}_{\chi,S}$ in $D(\mathscr{D}_\lambda)$, which, on the level of zero-th hypercohomology, induces the natural embedding $\varphi: I_{\chi,Q} \to \tilde{I}_{\chi,Q}$.*

Proof. The statement of this propostion can be best explained in the context

[1] An analogous geometric construction of the *maximal* globalization of the dual standard module is carried out in [20].

of localization. Let $\mathcal{M}_{qc}(\mathcal{D}_\lambda^{\mathrm{alg}})$ be the category of quasicoherent $\mathcal{D}_\lambda^{\mathrm{alg}}$-modules. A celebrated theorem of Beilinson and Bernstein ([1]) asserts that the *localization functor*

$$\Delta_\lambda^{\mathrm{alg}}: \mathcal{M}(\mathcal{U}_\theta) \to \mathcal{M}_{qc}(\mathcal{D}_\lambda^{\mathrm{alg}})$$

defined by $\Delta_\lambda^{\mathrm{alg}}(M) = \mathcal{D}_\lambda^{\mathrm{alg}} \otimes_{\mathcal{U}_\theta} M$, and the functor of global sections

$$\Gamma: \mathcal{M}_{qc}(\mathcal{D}_\lambda^{\mathrm{alg}}) \to \mathcal{M}(\mathcal{U}_\theta)$$

are exact and inverses of each other. It follows that $\mathcal{D}_\lambda^{\mathrm{alg}} \otimes_{\mathcal{U}_\theta} I_{\chi,Q} \cong \mathcal{I}_{\chi,Q}^{\mathrm{alg}}$ and that $\mathcal{I}_{\chi,Q}^{\mathrm{alg}}$ is Γ-acyclic.

Applying (1.1) we get:

$$\mathcal{D}_\lambda \otimes_{\mathcal{U}_\theta} I_{\chi,Q} \cong \mathcal{I}_{\chi,Q} \text{ is } \Gamma\text{-acyclic and } \Gamma(X, \mathcal{I}_{\chi,Q}) \cong I_{\chi,Q}.$$

We now recall some facts about analytic localization from [13]. We use the abbreviation "DNF-space" (resp. module, sheaf, etc.) to denote a continuous dual of a nuclear Fréchet space (resp. module, sheaf, etc.). Denote by $\mathcal{M}(\mathcal{U}_\theta)_{\mathrm{DNF}}$ the category whose objects are DNF $\mathcal{U}_\theta$-modules, and morphisms are the continuous homomorphisms. Minimal globalizations of Harish-Chandra modules are in this category, as are Harish-Chandra modules themselves: every finitely generated $\mathcal{U}_\theta$-module can be given the topology of an inductive limit of finite dimensional subspaces, which is DNF. Also, $\mathcal{D}_\lambda$ is a DNF-sheaf, i.e. its sections over any compact set form a DNF-space. Denote by $\mathcal{M}(\mathcal{D}_\lambda)_{\mathrm{DNF}}$ the category of DNF $\mathcal{D}_\lambda$-modules.

Let $\hat{\otimes}$ denote the operator of completed tensor product. The formula

$$\Delta_\lambda(M) = \mathcal{D}_\lambda \hat{\otimes}_{\mathcal{U}_\theta} M$$

does not define a functor from $\mathcal{M}(\mathcal{U}_\theta)_{\mathrm{DNF}}$ to $\mathcal{M}(\mathcal{D}_\lambda)_{\mathrm{DNF}}$: the (only) obstruction is the possibility that sections of $\mathcal{D}_\lambda \hat{\otimes}_{\mathcal{U}_\theta} M$ over a compact set may not form a Hausdorff space. However, it induces a functor on the derived level. Let $D(\mathcal{U}_\theta)_{\mathrm{DNF}}$ and $D(\mathcal{D}_\lambda)_{\mathrm{DNF}}$ be the *derived categories of* $\mathcal{M}(\mathcal{U}_\theta)_{\mathrm{DNF}}$ and $\mathcal{M}(\mathcal{D}_\lambda)_{\mathrm{DNF}}$ as defined in ([13], §5). Analogously to the Abelian case, they contain $\mathcal{M}(\mathcal{U}_\theta)_{\mathrm{DNF}}$ and $\mathcal{M}(\mathcal{D}_\lambda)_{\mathrm{DNF}}$ as full subcategories consisting of complexes with cohomology concentrated in degree zero. Define $L\Delta_\lambda: D(\mathcal{U}_\theta)_{\mathrm{DNF}} \mapsto D(\mathcal{D}_\lambda)_{\mathrm{DNF}}$ as follows: let $F(M)$ be the Hochschild resolution of an object M in $\mathcal{M}(\mathcal{U}_\theta)_{\mathrm{DNF}}$. Then

$$L\Delta_\lambda(M) = \Delta_\lambda(F(M)).$$

We refer to $L\Delta_\lambda$ as the functor of *analytic localization.*

It is shown in ([13], Theorem 5.4) that $L\Delta_\lambda$ is an equivalence of categories, with the derived functor $R\Gamma$ of Γ serving as an inverse. The complex $\mathscr{I}_{\chi,S}$ of D_λ-modules can be regarded as an object in $D(\mathscr{D}_\lambda)_{\text{DNF}}$. According to ([13], Proposition 10.8) $R\Gamma(X, \mathscr{I}_{\chi,S}) \cong \tilde{I}_{\chi,Q}$ and hence by ([13], Theorem 5.4)

$$L\Delta_\lambda(\tilde{I}_{\chi,Q}) \cong \mathscr{I}_{\chi,S}. \tag{1.5a}$$

In what follows we regard $L\Delta_\lambda$ as a functor into $D(\mathscr{D}_\lambda)$, by composing it with the forgetful functor $D(\mathscr{D}_\lambda)_{\text{DNF}} \mapsto D(\mathscr{D}_\lambda)$. By definition, (1.5a) still holds in this context and, although the equivalence of categories statement fails, it is still true that the composition $R\Gamma \circ L\Delta_\lambda$ is equivalent to the identity on $\mathscr{M}(\mathscr{U}_\theta)_{\text{DNF}}$.

For a finitely generated $\mathscr{U}_\theta$-module M, $\Delta_\lambda(M) = \mathscr{D}_\lambda \otimes_{\mathscr{U}_\theta} M = (\Delta_\lambda^{\text{alg}}(M))^{an}$. The exactness of $\Delta_\lambda^{\text{alg}}$ ([1]) and of the functor $(-)^{an}$ ([21], §6, Corollaire 1) imply that

$$L\Delta_\lambda(I_{\chi,Q}) = \mathscr{I}_{\chi,Q}. \tag{1.5b}$$

We now set $\Phi = L\Delta_\lambda(\varphi)$, where $\varphi: I_{\chi,Q} \to \tilde{I}_{\chi,Q}$ is the natural embedding. □

The importance of standard modules is explained by the following well known proposition. For a Harish-Chandra module M let

$$\ell(M) = \dim(\operatorname{supp}(\Delta_\lambda^{\text{alg}}(M)))$$

1.2 PROPOSITION ([1]). *$I_{\chi,Q}$ contains a unique irreducible submodule $J_{\chi,Q}$. This module has the property $\ell(J_{\chi,Q}) = \ell(I_{\chi,Q})$, and $\ell(I_{\chi,Q}/J_{\chi,Q}) < \ell(I_{\chi,Q})$. Every irreducible Harish-Chandra module with infinitesimal character θ arises in this manner.* □

2. A local computation of $\mathfrak{n}$-homology

In this section we prove the comparison theorem for the standard Harish-Chandra module $I_{\chi,Q}$, in a particular weight λ.

Recall first the definition of $\mathfrak{n}_x$-homology groups. Let M be a $\mathscr{U}(\mathfrak{n}_x)$-module or, more generally, a complex of such modules. Then

$$H_p(\mathfrak{n}_x, M) = \operatorname{Tor}_p^{\mathscr{U}(\mathfrak{n}_x)}(\mathbb{C}, M) = h_p(\mathbb{C} \overset{L}{\otimes}_{\mathscr{U}(\mathfrak{n}_x)} M). \tag{2.1}$$

Here $\overset{L}{\otimes}$ denotes the left derived functor of $\otimes$. In most of our applications each term in M is at least a $\mathscr{U}(\mathfrak{b}_x)$-module. In this case the action of $\mathfrak{b}_x$ on M descends to an action of $\mathfrak{b}_x/\mathfrak{n}_x$ on $H_p(\mathfrak{n}_x, M)$. Now, the abstract Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is naturally isomorphic to $\mathfrak{b}_x/\mathfrak{n}_x$. Therefore we can, and we shall, view $H_p(\mathfrak{n}_x, M)$

as a $U(\mathfrak{h})$-module. If M is, moreover, a complex of $\mathscr{U}(\mathfrak{g})$-modules, the action of $\mathscr{Z}(\mathfrak{g})$ descends to the homology groups. According to [7], this action factors through $\mathscr{U}(\mathfrak{h})$, via the Harish-Chandra homomorphism γ (see §1). Consequently, if M is $\mathscr{Z}(\mathfrak{g})$-finite then $H_p(\mathfrak{n}_x, M)$ is $\mathfrak{h}$-finite. In particular, if M is a complex of $\mathscr{U}_\theta$-modules, its $\mathfrak{n}_x$-homology groups decompose under $\mathfrak{h}$ into finite direct sums

$$H_p(\mathfrak{n}_x, M) = \bigoplus_{\lambda\in\theta} H_p(\mathfrak{n}_x, M)_{\lambda+\rho}$$

of generalized weight spaces.

Until further notice we assume that $\lambda \in \mathfrak{h}^*$ is regular. Each complex $\mathscr{M}$ in $D(\mathscr{D}_\lambda)$ can be regarded as a complex of sheaves of $\mathscr{U}_\theta$-modules, thus we can consider the $\mathfrak{n}_x$-homology sheaves $H_p(\mathfrak{n}_x, \mathscr{M})$. We note that for each $y \in X$, $H_p(\mathfrak{n}_x, \mathscr{M})_y \cong H_p(\mathfrak{n}_x, \mathscr{M}_y)$.

Let $i_x: \{x\} \to X$ be the inclusion map. Denote by T_x the functor of geometric fiber at x: $T_x(\mathscr{M}) = i_x^* \mathscr{M} = \mathbb{C} \otimes_{\mathscr{O}_{X,x}} \mathscr{M}_x$, and by LT_x its left derived functor. Occasionally we regard T_x as a functor on stalks at x only. It follows from the definition of $\mathscr{D}_\lambda$ that whenever $\mathscr{M}$ is a $\mathscr{D}_\lambda$-module, $T_x(\mathscr{M})$ has a structure of a $\mathfrak{b}_x/\mathfrak{n}_x$-module, and the natural projection $\mathbb{C} \otimes \mathscr{M}_x \to \mathbb{C} \otimes_{\mathscr{O}_{X,x}} \mathscr{M}_x$ induces a morphism

$$H_0(\mathfrak{n}_x, \mathscr{M}_x)_{\lambda+\rho} \to T_x(\mathscr{M})$$

We first recall the following result from [13] (cf. ([13], Lemma 4.5)).

2.1 LEMMA. *$H_p(\mathfrak{n}_x, \mathscr{D}_\lambda) = 0$ for $p > 0$, and $H_0(\mathfrak{n}_x, \mathscr{D}_{\lambda,y})_{\lambda+\rho} = 0$ if $y \neq x$. Moreover, $H_0(\mathfrak{n}_x, D_{\lambda,x})_{\lambda+\rho} \cong T_x(\mathscr{D}_\lambda)$.* □

The complex of sheaves $\mathscr{M}_{(x)}$ supported on $\{x\}$, with stalks $\mathscr{M}_x$ at x has a natural structure of a complex of $\mathscr{D}_\lambda$-modules.

2.2 COROLLARY. *The natural morphism $\mathscr{M} \to \mathscr{M}_{(x)}$ induces an isomorphism*

$$H_p(\mathfrak{n}_x, \mathscr{M})_{\lambda+\rho} \cong H_p(\mathfrak{n}_x, \mathscr{M}_{(x)})_{\lambda+\rho}.$$

Moreover

$$H_p(\mathfrak{n}_x, \mathscr{M}_x)_{\lambda+\rho} \cong L_p T_x(\mathscr{M}).$$

Proof. All the functors involved have finite cohomological dimension. Thus without loss of generality we may assume that $\mathscr{M}$ is bounded from above. Let $y \in X$. Let $\mathscr{P}_y$ be a resolution of $\mathscr{M}_y$ consisting of free $\mathscr{D}_{\lambda,y}$-modules. By the above lemma, this resolution is acyclic for $\mathfrak{n}_y$-homology. It is also acyclic for the functor of the geometric fiber at y, as $\mathscr{D}_{\lambda,y}$ is a free $\mathcal{O}_{X,y}$-module. Thus, appealing to the lemma again, we conclude:

$$H_p(\mathfrak{n}_x, \mathscr{M}_y)_{\lambda+\rho} = h_p(\mathbb{C} \otimes_{U(\mathfrak{n}_x)} \mathscr{P}_y)_{\lambda+\rho} = h_p(H_0(\mathfrak{n}_x, \mathscr{P}_y))_{\lambda+\rho}$$

which is equal to zero, if $y \neq x$, and is isomorphic to $L_p T_x(\mathscr{P}_y)$, if $y = x$. This completes the proof. □

Let $\mathscr{M}$ be a Γ-acyclic $\mathscr{D}_\lambda$-module.

2.3 LEMMA. (a) *There exists a spectral sequence converging to* $H_k(\mathfrak{n}_x, \Gamma(X, \mathscr{M}))$, *such that* $E_2^{-p,q} = H^q(X, H_p(\mathfrak{n}_x, \mathscr{M}))$;

(b) *The natural map* $\Gamma(X, \mathscr{M}) \to \mathscr{M}_x$ *induces an isomorphism* $H_k(\mathfrak{n}_x, \Gamma(X, \mathscr{M}))_{\lambda+\rho} \cong H_k(\mathfrak{n}_x, \mathscr{M}_x)_{\lambda+\rho}$.

Proof. Let $\mathscr{F}(\mathscr{M})$ be the canonical flabby resolution of $\mathscr{M}$. Consider the double complex

$$\mathscr{A}^{-p,q} = \Gamma(X, \textstyle\bigwedge^p \mathfrak{n}_x \otimes_{\mathbb{C}} \mathscr{F}^q(\mathscr{M}))$$

Each row $(\mathscr{A}^{\cdot,q}, d_I)$ is the Koszul complex, with respect to $\mathfrak{n}_x$, of $\mathscr{F}^q(\mathscr{M})$. On the other hand, the vertical differential d_{II} is induced by that in $\mathscr{F}(\mathscr{M})$. For the second filtration we have

$$''E_2^{-p,q} = H^q(X, H_p(\mathfrak{n}_x, \mathscr{M}))$$

and for the first

$$'E_2^{-p,q} = H_p(\mathfrak{n}_x, H^q(X, \mathscr{M}))$$

which, by Γ-acyclicity, is equal to zero, unless $q = 0$. This proves (a).

The morphism $\mathscr{M} \to \mathscr{M}_{(x)}$ induces $\mathfrak{h}$-invariant maps

$$H^q(X, H_p(\mathfrak{n}_x, \mathscr{M})) \to H^q(X, H_p(\mathfrak{n}_x, \mathscr{M}_{(x)}))$$

which according to Corollary 2.2 become isomorphisms when restricted to the weight $\lambda + \rho$. Therefore by (a) $H_k(\mathfrak{n}_x, \Gamma(X, \mathscr{M}))_{\lambda+\rho} \cong H_k(\mathfrak{n}_x, \mathscr{M}_x)_{\lambda+\rho}$ is an isomorphism. □

Suppose now that $\mathscr{M}$ is a Γ-acyclic object in $D(\mathscr{D}_\lambda)$.

2.4 PROPOSITION. *There exists a natural* $\mathfrak{h}$*-module isomorphism*

$$H_p(\mathfrak{n}_x, H^0(X, \mathscr{M}))_{\lambda+\rho} \cong L_p T_x(\mathscr{M}).$$

In the case when $\mathcal{M}$ is a complex of quasicoherent $\mathcal{D}_\lambda^{\mathrm{alg}}$-modules, this proposition is alluded to in [2], and shown in [9]. It also follows from arguments in [13] for complexes of DNF $\mathcal{D}_\lambda$-modules.

Proof. As all the functors involved have finite cohomological dimension, we can assume that $\mathcal{M}$ is bounded from below, and consists of Γ-acyclic $\mathcal{D}_\lambda$-modules. Let $\mathcal{A}$ denote the double complex

$$\mathcal{A}^{-p,q} = \bigwedge^p \mathfrak{n}_x \otimes_{\mathbb{C}} \Gamma(X, \mathcal{M}^q).$$

Also, let $\mathcal{B}$ be the double complex

$$\mathcal{B}^{-p,q} = \bigwedge^p \mathfrak{n}_x \otimes_{\mathbb{C}} \mathcal{M}_x^q.$$

The natural morphism $\Gamma(X, \mathcal{M}) \to \mathcal{M}_x$ of $\mathcal{U}_\theta$-modules induces a morphism $\mathcal{A} \to \mathcal{B}$ of double complexes of $\mathcal{U}(\mathfrak{h})$-modules.

Since $\mathcal{M}$ is Γ-acyclic the spectral sequence $'E(\mathcal{A})^{-p,q}$ associated to the first filtration of $\mathcal{A}$ degenerates at the E_2 stage: we get

$$'E_2^{-p,q}(\mathcal{A}) = H_p(\mathfrak{n}_x, H^q(X, \mathcal{M})) = 0 \text{ unless } q = 0.$$

and hence

$$h_p(\mathrm{Tot}(\mathcal{A})) \cong H_p(\mathfrak{n}_x, H^0(X, \mathcal{M})).$$

On the other hand,

$$h_p(\mathrm{Tot}(\mathcal{B})) \cong H_p(\mathfrak{n}_x, \mathcal{M}_x).$$

We have

$$''E(\mathcal{A})_1^{-p,q} = H_p(\mathfrak{n}_x, \Gamma(X, \mathcal{M}^q)), \text{ and } ''E(\mathcal{B})_1^{-p,q} = H_p(\mathfrak{n}_x, \mathcal{M}_x^q).$$

Now, $''E(\mathcal{A})_1^{-p,q} \to ''E(\mathcal{B})_1^{-p,q}$ induces an isomorphism

$$(''E(\mathcal{A})_1^{-p,q})_{\lambda+\rho} \to (''E(\mathcal{B})_1^{-p,q})_{\lambda+\rho}.$$

This follows from Lemma 2.3(b). We conclude that

$$h_p(\mathrm{Tot}(\mathcal{A}))_{\lambda+\rho} \cong h_p(\mathrm{Tot}(\mathcal{B}))_{\lambda+\rho}$$

or equivalently,

$$H_p(\mathfrak{n}_x, H^0(X, \mathcal{M}))_{\lambda+\rho} \to H_p(\mathfrak{n}_x, \mathcal{M}_x)_{\lambda+\rho}$$

is an isomorphism. The proposition now follows from second part of Corollary 2.2. □

The above proposition explains the local nature of $\mathfrak{n}_x$-homology and is an important tool in the proof of the comparison theorem. We apply it now to standard modules. Assume now that y is special and λ satisfies (1.2).

Recall the definition of the standard Harish-Chandra module $I_{\chi,Q}$ from Section 1.

2.5 PROPOSITION. *$\varphi: I_{\chi,Q} \to \tilde{I}_{\chi,Q}$ induces an isomorphism*

$$H_p(\mathfrak{n}_y, I_{\chi,Q})_{\lambda+\rho} \to H_p(\mathfrak{n}_y, \tilde{I}_{\chi,Q})_{\lambda+\rho}.$$

The proof of this proposition occupies the remainder of this section.

According to Proposition 1.1, both $I_{\chi,Q}$ and $\tilde{I}_{\chi,Q}$ arise as zero-th hypercohomology of Γ-acyclic objects $\mathscr{I}_{\chi,Q}$ and $\mathscr{I}_{\chi,S}$ in $D(\mathscr{D}_\lambda)$. Moreover, there is a morphism $\Phi: \mathscr{I}_{\chi,Q} \to \mathscr{I}_{\chi,S}$ in $D(\mathscr{D}_\lambda)$ which induces the natural embedding φ of $I_{\chi,Q}$ into $\tilde{I}_{\chi,Q}$. Therefore, in view of Proposition 2.4, it is enough to show that Φ induces an isomorphism

$$L_p T_y(\mathscr{I}_{\chi,Q}) \cong L_p T_y(\mathscr{I}_{\chi,S}) \tag{2.2}$$

for all special points $y \in X$. All the ingredients of the proof of (2.2) are well known. However, we could not find them presented in a convenient form and decided to sketch the argument here.

We need to describe in some detail the construction of $\mathscr{I}_{\chi,Q}$ and $\mathscr{I}_{\chi,S}$. Recall from Section 1 that Q and S are, respectively, the K-, and G_0-orbit of a special point x, C_0 is a Cartan subalgebra of G_0 with complexified Lie algebra contained in $\mathfrak{b}_x$, and χ is an irreducible representation of C_0 with differential $\lambda + \rho \in \mathfrak{h}^*$. Then χ uniquely extends to a representation of the stabilizer of x in G_0, trivial on its unipotent radical. Therefore the datum (χ, S) determines a G_0-equivariant locally free $\mathscr{O}_{X|S}$-module $\mathscr{L}$ with geometric fiber L over x. Let j be the inclusion $S \to X$ and $j_!$ the corresponding "extension by zero" functor. By definition

$$\mathscr{I}_{\chi,S} = j_! \mathscr{L}[c]$$

where $c = \operatorname{codim}_{\mathbb{C}} Q$, and $A \to A[c]$ denotes the usual shift operator on complexes ([13], §8, §10).

Denote by i the embedding of Q into X, and let $\mathscr{O}_Q$ and $\mathscr{O}_Q^{\mathrm{alg}}$ be the structure sheaves of Q, regarded, respectively, as an analytic and an algebraic variety. The restriction of χ to $K_0 \cap C_0$ extends uniquely to an algebraic representation of the stabilizer of x in K, trivial on its unipotent radical, and hence defines a locally free $\mathcal{O}_Q^{\mathrm{alg}}$-module $\mathcal{L}_Q^{\mathrm{alg}}$, equipped with an algebraic action of K, with geometric fiber L over x. In the analytic category we get a locally free $\mathcal{O}_Q$-module $\mathcal{L}_Q = (\mathcal{L}_Q^{\mathrm{alg}})^{an}$. Then

$$\mathcal{I}_{\chi,Q}^{\mathrm{alg}} = i_+ \mathcal{L}_Q^{\mathrm{alg}}$$

and hence

$$\mathcal{I}_{\chi,Q} = (i_+ \mathcal{L}_Q^{\mathrm{alg}})^{an}.$$

Here i_+ denotes the functor of *direct image in the category of $\mathcal{D}$-modules.* This is a K-equivariant holonomic $\mathcal{D}_\lambda^{\mathrm{alg}}$-module supported on the closure of Q ([1], [2] ([10], Appendix)).

Assume first that $y \notin Q \cap S$. Since y is special, $y \notin Q$ and $y \notin S$. Clearly

$$L_p T_y(\mathcal{I}_{\chi,S}) = 0$$

for all p, as the stalk $\mathcal{I}_{\chi,S,y}$ vanishes. The fact that also

$$L_p T_y(\mathcal{I}_{\chi,Q}) = 0$$

for all p is less trivial: it follows from a special case of base change for the direct image functor ([3], VI, §8.4).

Without loss of generality we may therefore assume that $y = x$. Thus the proof of the proposition reduces to showing that Φ induces an isomorphism

$$L_p T_x(\mathcal{I}_{\chi,Q,x}) \cong L_p T_x(\mathcal{I}_{\chi,S,x}). \tag{2.3}$$

Clearly

$$\mathcal{I}_{\chi,S,x} = \mathcal{L}_x[c].$$

The explicit description of $\mathcal{I}_{\chi,Q,x}$, which we recall below, is more involved. We alter the notation slightly, and let i denote the embedding of the germ Q_x of Q at x into the germ X_x of X at x.

Regard $\mathcal{D}_{\lambda,x}$ as a $\mathcal{O}_{X,x}$-module with respect to left multiplication. Set

$$(\mathcal{D}_\lambda)_{Q\to X,x} = \mathcal{O}_{Q,x} \otimes_{\mathcal{O}_{X,x}} \mathcal{D}_{\lambda,x}.$$

Let $\mathcal{D}_{\lambda,x}^i$ be the ring of (left) differential operators of the $\mathcal{O}_{Q,x}$-module $(\mathcal{D}_\lambda)_{Q\to X,x}$, which commute with the right action of $\mathcal{D}_{\lambda,x}$. The functor $i^* = \mathcal{O}_{Q,x} \otimes_{\mathcal{O}_{X,x}} (-)$ maps $\mathcal{D}_{\lambda,x}$-modules to $\mathcal{D}_{\lambda,x}^i$-modules. Similarly, regard $\mathcal{D}_{\lambda,x}$ as a $\mathcal{O}_{X,x}$-module with respect to *right* multiplication, set

$$(\mathscr{D}_\lambda)_{X\leftarrow Q,x} = \mathscr{D}_{\lambda,x} \otimes_{\mathscr{O}_{X,x}} \mathscr{O}_{Q,x}$$

and define $\mathscr{D}^{(i)}_{\lambda,x}$ to be the ring of *right* differential operators of the $\mathscr{O}_{Q,x}$-module $(\mathscr{D}_\lambda)_{X\leftarrow Q,x}$, which commute with the left action of $\mathscr{D}_{\lambda,x}$. Let $\Omega_{X|Q,x}$ be the stalk at x of the sheaf of top degree relative differential forms on Q. One checks that $\mathscr{N}_x$ is a $\mathscr{D}^i_{\lambda,x}$-module if and only if $\mathscr{N}^\vee_{Q,x} = \mathscr{N}_x \otimes_{\mathscr{O}_{Q,x}} \Omega^{-1}_{X|Q,x}$ is a $\mathscr{D}^{(i)}_{\lambda,x}$-module. In the above notation

$$\mathscr{I}_{\chi,Q,x} = (\mathscr{D}_\lambda)_{X\leftarrow Q,x} \otimes_{\mathscr{D}^{(i)}_{\lambda,x}} \mathscr{L}^\vee_{Q,x}. \tag{2.4}$$

Denote by $D(\mathscr{D}_{\lambda,x})$, $D(\mathscr{D}^i_{\lambda,x})$, $D(\mathscr{D}^{(i)}_{\lambda,x})$ the derived categories of modules over $\mathscr{D}_{\lambda,x}$, $\mathscr{D}^i_{\lambda,x}$, and $\mathscr{D}^{(i)}_{\lambda,x}$, respectively, and by $Li^*: D(\mathscr{D}_{\lambda,x}) \to D(\mathscr{D}^i_{\lambda,x})$ the left derived functor of i^*.

2.6 LEMMA. *Let $\mathscr{M}_x$ be in $D(\mathscr{D}_{\lambda,x})$. Then*

$$\mathrm{Hom}_{D(\mathscr{D}_{\lambda,x})}(\mathscr{I}_{\chi,Q,x}, \mathscr{M}_x) \cong \mathrm{Hom}_{D(\mathscr{D}^i_{\lambda,x})}(Li^*\mathscr{I}_{\chi,Q,x}, Li^*\mathscr{M}_x).$$

Proof. Let $\mathscr{J}$ be the ideal of germs of functions in $\mathscr{O}_{X,x}$ vanishing on Q_x. Also let $\mathscr{J}$ denote the (right exact) functor of $\mathscr{J}$-invariants: for any $\mathscr{D}_{\lambda,x}$-module $\mathscr{M}_x$

$$\mathscr{J}(\mathscr{M}_x) = \mathrm{Hom}_{\mathscr{O}_{X,x}}(\mathscr{O}_{Q,x}, \mathscr{M}_x).$$

Note that, via the isomorphism with $\mathrm{Hom}_{\mathscr{D}_{\lambda,x}}((\mathscr{D}_\lambda)_{X\leftarrow Q,x}, \mathscr{M}_x)$, this is naturally a $\mathscr{D}^{(i)}_{\lambda,x}$-module.

By change of rings we get isomorphisms

$$\begin{aligned}\mathrm{Hom}_{\mathscr{D}_{\lambda,x}}(\mathscr{I}_{\chi,Q,x}, \mathscr{M}_x) &= \mathrm{Hom}_{\mathscr{D}_{\lambda,x}}((\mathscr{D}_\lambda)_{X\leftarrow Q,x} \otimes_{\mathscr{D}^{(i)}_{\lambda,x}} \mathscr{L}^\vee_{Q,x}, \mathscr{M}_x)\\ &\cong \mathrm{Hom}_{\mathscr{D}^{(i)}_{\lambda,x}}(\mathscr{L}^\vee_{Q,x}, \mathrm{Hom}_{\mathscr{D}_{\lambda,x}}((\mathscr{D}_\lambda)_{X\leftarrow Q,x}, \mathscr{M}_x))\\ &\cong \mathrm{Hom}_{\mathscr{D}^{(i)}_{\lambda,x}}(\mathscr{L}^\vee_{Q,x}, \mathrm{Hom}_{\mathscr{O}_{X,x}}(\mathscr{O}_{Q,x}, \mathscr{M}_x))\\ &\cong \mathrm{Hom}_{\mathscr{D}^{(i)}_{\lambda,x}}(\mathscr{L}^\vee_{Q,x}, \mathscr{J}(\mathscr{M}_x)).\end{aligned}$$

Denote by $R\mathscr{J}$ the right derived function of $\mathscr{J}$. We have

$$Li^*\mathscr{M}_x \cong R\mathscr{J}(\mathscr{M}_x) \otimes_{\mathscr{O}_{Q,x}} \Omega_{X|Q,x}[c]$$

and

$$R\mathscr{J}(\mathscr{I}_{\chi,Q,x}) \cong \mathscr{L}^\vee_{Q,x}$$

([3], VI, §7). Also, $(\mathscr{D}_\lambda)_{X\leftarrow Q,x}$ is a free $\mathscr{D}^{(i)}_{\lambda,x}$-module and hence $\mathscr{J}$ maps injective objects into injective objects. A standard argument implies now that the above string of isomorphisms descends to the level of derived categories. Therefore

$$\begin{aligned}\mathrm{Hom}_{D(\mathscr{D}_{\lambda,x})}(\mathscr{I}_{\chi,Q,x}, \mathscr{M}_x) &\cong \mathrm{Hom}_{D(\mathscr{D}^{(i)}_{\lambda,x})}(\mathscr{L}^{\vee}_{Q,x}, R\mathscr{J}(\mathscr{M}_x))\\ &\cong \mathrm{Hom}_{D(\mathscr{D}^{(i)}_{\lambda,x})}(R\mathscr{J}(\mathscr{I}_{\chi,Q,x}), R\mathscr{J}(\mathscr{M}_x))\\ &\cong \mathrm{Hom}_{D(\mathscr{D}^{i}_{\lambda,x})}(Li^*\mathscr{I}_{\chi,Q,x}, Li^*\mathscr{M}_x)\end{aligned}$$

which proves the lemma. □

The following lemma asserts that every morphism between $\mathscr{I}_{\chi,Q}$ and $\mathscr{I}_{\chi,S}$ is uniquely determined by the induced morphism on geometric fibers.

2.7 LEMMA.

$$\begin{aligned}\mathrm{Hom}_{D(\mathscr{D}_{\lambda,x})}(\mathscr{I}_{\chi,Q,x}, \mathscr{I}_{\chi,S,x}) &\cong \mathrm{Hom}_{\mathfrak{b}_x\cap\mathfrak{k}}(LT_x(\mathscr{I}_{\chi,Q}), LT_x(\mathscr{I}_{\chi,S}))\\ &\cong \mathrm{Hom}_{\mathfrak{b}_x\cap\mathfrak{k}}(L, L).\end{aligned}$$

Proof. We have

$$\begin{aligned}\mathrm{Hom}_{D(\mathscr{D}_{\lambda,x})}(\mathscr{I}_{\chi,Q,x}, \mathscr{I}_{\chi,S,x}) &\cong \mathrm{Hom}_{D(\mathscr{D}^{i}_{\lambda,x})}(Li^*\mathscr{I}_{\chi,Q,x}, Li^*\mathscr{I}_{\chi,S,x})\\ &\cong \mathrm{Hom}_{D(\mathscr{D}^{i}_{\lambda,x})}(\mathscr{L}_{Q,x}[c], \mathscr{L}_{Q,x}[c])\\ &\cong \mathrm{Hom}_{D(\mathscr{D}^{i}_{\lambda,x})}(\mathscr{L}_{Q,x}, \mathscr{L}_{Q,x})\\ &\cong \mathrm{Hom}_{\mathscr{D}^{i}_{\lambda,x}}(\mathscr{L}_{Q,x}, \mathscr{L}_{Q,x}).\end{aligned}$$

The natural map

$$\mathrm{Hom}_{\mathscr{D}^{i}_{\lambda,x}}(\mathscr{L}_{Q,x}, \mathscr{L}_{Q,x}) \to \mathrm{Hom}_{\mathfrak{b}_x\cap\mathfrak{k}}(T_x(\mathscr{L}_{Q,x}), T_x(\mathscr{L}_{Q,x}))$$

induced by the functor T_x is an isomorphism ($\mathscr{L}_{Q,x}$ is isomorphic to a direct sum of copies of $\mathscr{O}_{Q,x}$ and the proof reduces to the fact that $\mathrm{Hom}_{\mathscr{D}_{Q,x}}(\mathscr{O}_{Q,x}, \mathscr{O}_{Q,x}) \cong \mathbb{C}$). Also, $T_x(\mathscr{L}_{Q,x})$ is naturally isomorphic as a $(\mathfrak{b}_x \cap \mathfrak{k})$-module to L. The lemma follows, since $LT_x(\mathscr{I}_{\chi,Q}) \cong T_x(Li^*\mathscr{I}_{\chi,Q,x})$ and $LT_x(\mathscr{I}_{\chi,S}) \cong T_x(Li^*\mathscr{I}_{\chi,S,x})$. □

We are now ready to prove (2.3). Since Φ is nonzero, for some $y \in X$ the induced morphism Φ_y on the stalks must be nonzero. As this y must lie in the intersection of supports of $\mathscr{I}_{\chi,Q}$ and $\mathscr{I}_{\chi,S}$, it belongs to $\bar{Q} \cap S$ (we denote by $\overline{(\cdot)}$ the closure of $(\cdot)$). We claim

$$\bar{Q} \cap S = Q \cap S$$

In fact, assume that a K-orbit Q_b in the boundary of Q has a nonzero intersection with S. Denote by S_b the G_0-orbit corresponding to Q_b via the Matsuki correspondence. According to ([16], Theorem (i)) $S \cap Q_b \neq 0$ is equivalent to $S_b \subset \bar{S}$. On the other hand, the Matsuki correspondence reverses the closure relations: $S_b \subset \bar{S}$ means that Q is a subset of $\bar{Q}_b$ ([15]). This contradicts the fact that Q_b is a boundary component of Q.

Thus $y \in Q \cap S$ and hence is special. By K_0-invariance, we may therefore assume as well that Φ induces a nonzero morphism in $\mathrm{Hom}_{D(\mathscr{D}_{\lambda,x})}(\mathscr{I}_{\chi,Q,x}, \mathscr{I}_{\chi,S,x})$. As φ is K_0-equivariant, the corresponding morphism in $\mathrm{Hom}_{\mathfrak{b}_x \cap \mathfrak{k}}(L, L)$ is also invariant under the action of the stabilizer K_x of x in K, and hence is an isomorphism, since L is irreducible as a K_x-module. Consequently, $LT_x(\varphi)$: $LT_x(\mathscr{I}_{\chi,Q}) \cong LT_x(\mathscr{I}_{\chi,S})$. $LT_x(\mathscr{I}_{\chi,Q}) \cong LT_x(\mathscr{I}_{\chi,S})$. This implies (2.3) and completes the roof of Proposition 2.5. □

3. A proof of the comparison theorem

As in the introduction, we denote by M a Harish-Chandra module, and by $\tilde{M}$ its minimal globalization. Both M and $\tilde{M}$ are finite $\mathscr{Z}(\mathfrak{g})$-modules and hence their $\mathfrak{n}_x$-homology groups decompose into finite direct sums

$$H_p(\mathfrak{n}_x, M) = \bigoplus_{\lambda \in \mathfrak{h}^*} H_p(\mathfrak{n}_x, M)_{\lambda+\rho};$$

$$H_p(\mathfrak{n}_x, \tilde{M}) = \bigoplus_{\lambda \in \mathfrak{h}^*} H_p(\mathfrak{n}_x, \tilde{M})_{\lambda+\rho}$$

of generalized weight-spaces. The proof of the comparison theorem amounts therefore to showing

$$\begin{aligned} &H_p(\mathfrak{n}_x, M)_{\lambda+\rho} \to H_p(\mathfrak{n}_x, \tilde{M})_{\lambda+\rho} \text{ is an isomorphism} \\ &\quad \text{for all } \lambda \in \mathfrak{h}^*, \text{ and all special points } x \in X. \end{aligned} \tag{3.1}$$

In what follows we assume that x is special.

3.1 LEMMA. *Let* $0 \to M' \to M \to M'' \to 0$ *be a short exact sequence of Harish-Chandra modules, and suppose that the statement* (3.1) *holds for any two of these modules. Then it holds also for the third.*

Proof. Consider the commutative diagram

$$\begin{array}{ccccccccc} \cdots \to & H_p(\mathfrak{n}_x, M')_{\lambda+\rho} & \to & H_p(\mathfrak{n}_x, M)_{\lambda+\rho} & \to & H_p(\mathfrak{n}_x, M'')_{\lambda+\rho} & \to & H_{p+1}(\mathfrak{n}_x, M')_{\lambda+\rho} & \to \cdots \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ \cdots \to & H_p(\mathfrak{n}_x, \tilde{M}')_{\lambda+\rho} & \to & H_p(\mathfrak{n}_x, \tilde{M})_{\lambda+\rho} & \to & H_p(\mathfrak{n}_x, \tilde{M}'')_{\lambda+\rho} & \to & H_{p+1}(\mathfrak{n}_x, \tilde{M}')_{\lambda+\rho} & \to \cdots \end{array}$$

Since the sequences $0 \to M' \to M \to M'' \to 0$ and $0 \to \tilde{M}' \to \tilde{M} \to \tilde{M}'' \to 0$ are exact (the minimal globalization functor is exact [19]) the top and the bottom rows in this diagram are exact. By assumption, two out of three consecutive columns are exact. The proof of the lemma now follows from the five-lemma. □

3.2 PROPOSITION. *Suppose* λ *satisfies condition* (1.2). *Then*

$$H_p(\mathfrak{n}_x, M)_{\lambda+\rho} \to H_p(\mathfrak{n}_x, \tilde{M})_{\lambda+\rho}$$

is an isomorphism

Proof. By the above lemma, we may assume that M is irreducible. Moreover, we may assume that M has infinitesimal character $\theta = W\lambda$: otherwise both sides in (3.1) are zero ([7]). Thus M is of the form $J_{\chi,Q}$, where the differential of χ is equal to $\lambda + \rho$. By Proposition 1.2 we have an exact sequence

$$0 \to M \to I_{\chi,Q} \to M' \to 0$$

where $\ell(M') < \ell(M)$. We argue by induction on ℓ. We assume that the proposition holds for all irreducible subquotients of M'. By Lemma 3.1 it holds for M'. It holds also for $I_{\chi,Q}$ as this is precisely the statement of Proposition 2.5. Thus, appealing to Lemma 3.1 again, we conclude that it is true for M. □

Proof of Comparison Theorem. Let M be an arbitrary Harish-Chandra module. Denote by S the set of all $\lambda \in \mathfrak{h}^*$ such that for some p either $H_p(\mathfrak{n}_x, M)_{\lambda+\rho}$ or $H_p(\mathfrak{n}_x, \tilde{M})_{\lambda+\rho}$ is nonzero. Let $\leqslant$ be the partial ordering on $\mathfrak{h}^*$ defined by $\mu \leqslant \nu$ if $\nu - \mu$ is a sum of roots in Δ^+. We proceed by induction with respect to $<$. In order to prove (3.1) in general it is enough to show that, given $\lambda \in S$, if (3.1) holds for all $\gamma' \in S$ less than λ, then it also holds for λ. Let F be an irreducible finite dimensional representation of G_0 such that its lowest weight $\mu \in \mathfrak{h}^*$ has the property:

$\lambda + \mu$ satisfies condition (1.2)

Let

$$0 = F_0 \subset \cdots F_i \subset F_{i+1} \subset \cdots \subset F_N = F \tag{3.2}$$

be a $\mathfrak{b}_x$-module filtration of F of maximal length. Note that for $0 < i \leqslant N$, F_i/F_{i-1} is a one dimensional $\mathfrak{b}_x$-module $\mathbb{C}_\nu$ determined by a weight ν of F, and $\nu = \mu$ precisely when $i = N$.

Regard $M \otimes_{\mathbb{C}} F$ as a $\mathcal{U}(\mathfrak{g})$-module with respect to the tensor product action.

Then (3.2) induces a $\mathfrak{b}_x$-module filtration

$$0 = M \otimes F_0 \subset \cdots M \otimes F_i \subset M \otimes F_{i+1} \subset \cdots \subset M \otimes F_N = M \otimes F$$

of $M \otimes F$.

3.3 LEMMA. *The map*

$$H_p(\mathfrak{n}_x, M \otimes_{\mathbb{C}} F_i)_{\lambda+\mu+\rho} \to H_p(\mathfrak{n}_x, \tilde{M} \otimes_{\mathbb{C}} F_i)_{\lambda+\mu+\rho}$$

is an isomorphism for all $i \leqslant N$.

Proof. We note that $\tilde{M} \otimes_{\mathbb{C}} F_N$ is the minimal globalization of $M \otimes_{\mathbb{C}} F_N$. Thus for $i = N$ the lemma follows from Proposition 3.2. Now, assume $i < N$. We proceed by induction on i. The statement is obviously true for $i = 0$. Suppose now that $0 < i < N$, and assume the statement for $i - 1$. We note that the induced map

$$H_p(\mathfrak{n}_x, M \otimes_{\mathbb{C}} F_i/F_{i-1})_{\lambda+\mu+\rho} \to H_p(\mathfrak{n}_x, \tilde{M} \otimes_{\mathbb{C}} F_i/F_{i-1})_{\lambda+\mu+\rho} \tag{3.3}$$

is an isomorphism. In fact, $F_i/F_{i-1} = \mathbb{C}_\nu$, where ν is a weight of F different from μ, and hence equal to $\mu + A$, where A is a nonempty sum of positive roots. Also, F_i/F_{i-1} is a trivial $\mathfrak{n}_x$-module. Consequently, (3.3) is equivalent to

$$H_p(\mathfrak{n}_x, M)_{\lambda-A+\rho} \otimes_{\mathbb{C}} \mathbb{C}_\nu \to H_p(\mathfrak{n}_x, \tilde{M})_{\lambda-A+\rho} \otimes_{\mathbb{C}} \mathbb{C}_\nu$$

which is an isomorphism, since, by assumption, $H_p(\mathfrak{n}_x, M)_{\lambda-A+\rho} \to H_p(\mathfrak{n}_x, \tilde{M})_{\lambda-A+\rho}$ is an isomorphism. Therefore, in the commutative diagram

$$\begin{array}{ccccc} H_p(\mathfrak{n}_x, M \otimes F_{i-1})_{\lambda+\mu+\rho} & \to & H_p(\mathfrak{n}_x, M \otimes F_i)_{\lambda+\mu+\rho} & \to & H_p(\mathfrak{n}_x, M \otimes F_i/F_{i-1})_{\lambda+\mu+\rho} \\ \downarrow & & \downarrow & & \downarrow \\ H_p(\mathfrak{n}_x, \tilde{M} \otimes F_{i-1})_{\lambda+\mu+\rho} & \to & H_p(\mathfrak{n}_x, \tilde{M} \otimes F_i)_{\lambda+\mu+\rho} & \to & H_p(\mathfrak{n}_x, \tilde{M} \otimes F_i/F_{i-1})_{\lambda+\mu+\rho} \end{array}$$

the first and third vertical arrows are isomorphisms for all p. Arguing as in Lemma 3.1 we conclude that also the middle arrows is an isomorphism. □

Hence in the commutative diagram

$$\begin{array}{ccccc} H_p(\mathfrak{n}_x, M \otimes F_{N-1})_{\lambda+\mu+\rho} & \to & H_p(\mathfrak{n}_x, M \otimes F_N)_{\lambda+\mu+\rho} & \to & H_p(\mathfrak{n}_x, M \otimes F_N/F_{N-1})_{\lambda+\mu+\rho} \\ \downarrow & & \downarrow & & \downarrow \\ H_p(\mathfrak{n}_x, \tilde{M} \otimes F_{N-1})_{\lambda+\mu+\rho} & \to & H_p(\mathfrak{n}_x, \tilde{M} \otimes F_N)_{\lambda+\mu+\rho} & \to & H_p(\mathfrak{n}_x, \tilde{M} \otimes F_N/F_{N-1})_{\lambda+\mu+\rho} \end{array}$$

the first and second vertical arrows are isomorphisms and, consequently,

$$H_p(\mathfrak{n}_x, M \otimes F_N/F_{N-1})_{\lambda+\mu+\rho} \to H_p(\mathfrak{n}_x, \tilde{M} \otimes F_N/F_{N-1})_{\lambda+\mu+\rho}$$

is an isomorphism. Since $F_N/F_{N-1} = \mathbb{C}_\mu$, this implies

$$H_p(\mathfrak{n}_x, M)_{\lambda+\rho} \to H_p(\mathfrak{n}_x, \tilde{M})_{\lambda+\rho}$$

is an isomorphism, which completes the induction.[2] □

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