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ANTONÍN TUZAR

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Remark to a problem on 0-1 matrices

ANTONÍN TUZAR

Institute of Information Theory and Automation, Czechoslovak Academy of Sciences, Pod vodárenskou věží 4, 182 08 Praha 8, Czechoslovakia

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Recently in [8] V. de Valk made a detailed analysis of the maximal and minimal values of the sum of elements of the matrix M^2, where M is a $n \times n$ matrix with all elements from the set $\{0, 1\}$. For the sum of the elements of some matrix we use the symbol $\|.\|$. Our purpose is to prove the inequality

$$n\|M\| - \|M^2\| \leqslant \frac{n^3 - n}{3} \tag{1}$$

by means of elementary tools. The inequality (1) can be deduced from Theorem 2 in [8], but the proof of this more precise theorem is rather complicated. As the inequality (1) is interesting by itself, it makes sense to have a short direct proof for it separately.

We can use the following problem as an example to the use of the inequality (1). Let G be some complete directed graph with n vertices and let every edge (including the loops in vertices) be painted in one of two colours, A and B. We are asking for the exact upper bound of the total number of double-coloured pairs of subsequent edges (paths of length two) in G.

It is sufficient to estimate one half of this quantity, namely the number of the pairs coloured in the sequence (A, B). We introduce the incidence matrix $M = (M_{ij})$ of the edges painted in A ($M_{ij} = 1$ iff the oriented edge from the ith to the jth vertex is painted in A, $M_{ij} = 0$ otherwise) and we denote the row, column and total sums of elements by

$$H_i := \sum_{j=1}^{n} M_{ij}, \quad V_j := \sum_{i=1}^{n} M_{ij}, \quad K := \|M\| := \sum_{i=1}^{n} \sum_{j=1}^{n} M_{ij}. \tag{2}$$

Let the number of A-coloured edges, which terminate in the ith vertex ($i = 1, \ldots, n$), be a_i and let b_i be the number of B-coloured edges, which start in this vertex. The number under consideration is then equal to $\Sigma_{i=1}^{n} a_i b_i$. Obviously $a_i = V_i$ and, as G is a complete graph, $b_i = n - H_i$.

Therefore

$$\sum_{i=1}^{n} a_i b_i = nK - \sum_{i=1}^{n} H_i V_i. \tag{3}$$

Further

$$\begin{aligned}\sum_i H_i V_i &= \sum_i \Big(\sum_j M_{ij}\Big)\Big(\sum_k M_{ki}\Big)\\ &= \sum_j \sum_k \sum_i M_{ki} M_{ij}\end{aligned}$$

and the last expression is equal to the sum of all elements of the matrix M^2. Therefore $\Sigma_i a_i b_i$ is equal to the expression on the left side of inequality (1).

THEOREM. *Let M be an arbitrary $n \times n$ matrix ($n \in N$) with elements M_{ij} from the set $\{0, 1\}$ and let H_i, V_i ($i = 1, \ldots, n$) and K be defined by* (2). *Then*

$$\sum_{i=1}^{n} (H_i - V_i)^2 \leqslant nK - \sum_{i=1}^{n} H_i V_i \leqslant \frac{n^3 - n}{3}. \tag{4}$$

Proof. The first inequality follows immediately from a result by F. Matúš ([5], [6]), namely from the Khintchine-type inequality

$$\sum_{i=1}^{n} H_i^2 + \sum_{i=1}^{n} V_i^2 \leqslant nK + \sum_{i=1}^{n} H_i V_i.$$

In order to prove the second inequality, which is equivalent to (1), we proceed by induction. For $n = 1$ the result is trivial. If $n > 1$, we suppose the validity of (1) for $n - 1$. From the matrix M we remove the nth row and nth column and we denote the remainder by $\bar{M}$. By analogy with (2) we introduce for $(n-1) \times (n-1)$ matrix $\bar{M}$ the corresponding values $\bar{H}_i$, $\bar{V}_i$ and $\bar{K}$. One obtains

$$\begin{aligned}nK - \sum_{i=1}^{n} H_i V_i &= \sum_{i=1}^{n} H_i (n - V_i)\\ &= \sum_{i=1}^{n-1} (\bar{H}_i + M_{in})(n - 1 - \bar{V}_i + 1 - M_{ni}) + H_n (n - V_n)\\ &= \sum_{i=1}^{n-1} \bar{H}_i (n - 1 - \bar{V}_i) + \sum_{i=1}^{n-1} \bar{H}_i (1 - M_{ni})\\ &\quad + \sum_{i=1}^{n-1} M_{in} (n - \bar{V}_i - M_{ni}) + H_n (n - V_n).\end{aligned}$$

The first sum in the last expression is equal to $(n-1)\bar{K}-\Sigma_{i=1}^{n-1}\bar{H}_i\bar{V}_i$ and it can be estimated using the induction hypothesis. The remaining terms will be rewritten as

$$R_n=\sum_{i=1}^{n-1}\bar{H}_i(1-M_{ni})+\sum_{i=1}^{n-1}M_{in}(n-\bar{V}_i-M_{ni})$$
$$+\left(\sum_{i=1}^{n-1}M_{ni}+M_{nn}\right)\left(n-\sum_{j=1}^{n-1}M_{jn}-M_{nn}\right).$$

We introduce the subsets I and J of subscripts $i\in\{1,\dots,n-1\}$ such that

$$i\in I\Leftrightarrow M_{in}=1,\qquad i\in J\Leftrightarrow M_{ni}=1.$$

Let $k:=\operatorname{card}I$ and $l:=\operatorname{card}J$, then apparently $0\leqslant k\leqslant n-1$ and $0\leqslant l\leqslant n-1$. Using the introduced symbols we can rearrange the above expression as follows

$$R_n=\sum_{i\notin J}\bar{H}_i+\sum_{i\in I}(n-\bar{V}_i)-\sum_{i\in I\cap J}M_{in}M_{ni}+(l+M_{nn})(n-k-M_{nn}).$$

Now we estimate the difference $\Sigma_{i\notin J}\bar{H}_i-\Sigma_{i\in I}\bar{V}_i$, which is not greater than $\Sigma_{i\notin J}\bar{H}_i'$, where $\bar{H}_i'$ are the sums of rows of the matrix, which we obtain from $\bar{M}$ turning out the k columns with subscripts in I. We have

$$\sum_{i\notin J}\bar{H}_i'\leqslant(n-1-l)(n-1-k).$$

Further we make use of the obvious identity

$$\sum_{i\in I\cap J}M_{in}M_{ni}=\operatorname{card}(I\cap J).$$

Therefore we have

$$R_N\leqslant(n-1-l)(n-1-k)+kn-\operatorname{card}(I\cap J)+l(n-k)$$
$$+M_{nn}(n-k-l-M_{nn})$$
$$=(n-1)^2+k+l-\operatorname{card}(I\cap J)+M_{nn}(n-k-l-M_{nn}).$$

Using the well known fact, that

$$\operatorname{card}I+\operatorname{card}J-\operatorname{card}(I\cap J)=\operatorname{card}(I\cup J)\leqslant n-1,$$

we obtain the inequality

$$R_n \leqslant n(n-1),$$

which is obvious if $M_{nn}=0$ and for $M_{nn}=1$ it follows from

$$R_n \leqslant (n-1)^2 - \operatorname{card}(I \cap J) + n - 1 \leqslant n(n-1).$$

Therefore finally

$$nK - \sum_{i=1}^{n} H_i V_i \leqslant \frac{(n-1)^3-(n-1)}{3} + n(n-1) = \frac{n^3-n}{3}. \qquad \square$$

REMARK. The inequality (1) generally can not be improved, as for arbitrary n there exist matrices, for which the equality holds. It is easy to verify, that the matrix with the elements $M_{ij}=1$ for $i \leqslant j$ and $M_{ij}=0$ for $i>j$ is such an example. The matrices for which in (4) instead of the first inequality equality holds can be specified using the results in [6].

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