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# Approximate dilations

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## 1. Introduction

The concept of *dilation* was introduced and investigated by several important mathematicians [2]. Given probability measures $P$, $Q$ on the $\sigma$-field of Borel subsets of a topological space $S$, we say that *$Q$ is a dilation of $P$ relatively to a set $K$ of functions $S \to R$, and write $P \prec_K Q$, iff $\int f\,\mathrm{d}P \leqslant \int f\,\mathrm{d}Q$ for all $f \in K$.* The set of functions $K$ is usually a cone. It is possible that, although $Q$ does not dilate $P$ relatively to $K$, it nearly does so in some sense, giving rise to an *approximate dilation* of $P$. A natural approach is to employ a 'distance' of type

$$\delta(P, Q) := \inf \left\{ \varepsilon \geqslant 0 \,\middle|\, \int f\,\mathrm{d}P \leqslant \int f\,\mathrm{d}Q + \varepsilon L(f), f \in K \right\}$$

where $L(f) \geqslant 0$ measures the 'size' of $f$.

We allow any cone of bounded functions which is *admissible*, i.e., *a convex cone of continuous functions containing the constants and being invariant under the operation* $\vee$. The latter means that $\max\{f, g\} \in K$ whenever $f, g \in K$. Initially $L(f)$ will be taken as the oscillation of $f$. Afterwards, other approximate dilations will also be discussed. Theorem 10, summarized in Fig. 1, is our main result.

## 2. Notations

In this paper $A^c$ denotes the complement of the set $A$; $\mathscr{B} = \mathscr{B}(S)$ the $\sigma$-field of Borel subsets of a topological space $S$; $C(S)$ the set of all continuous functions $S \to R$; $C_b(S)$ the set of all functions in $C(S)$ which are bounded; 'distribution function' is abbreviated as df; $K'$ is the set of all $f \in K$ ($K$ is a cone of functions) with $\inf f = 0$ and $\sup f = 1$; $\mathscr{M}(S)$ the set of all probability measures on the $\sigma$-field of Borel subsets of $S$; osc $f$ stands for 'oscillation of the function $f$', i.e., $\operatorname{osc} f := \sup f - \inf f$; $\delta_s$ represents the Dirac measure at the point $s$; the symbols $\vee$, $\wedge$ have the usual meaning, i.e., they denote the maximum and the minimum operation, respectively; lsc abbreviates 'lower semicontinuous'; and, finally, iff stands for 'if and only if'.

We begin with a lemma, essential for the fundamental Theorem 7. It was suggested by Lemma 4 in [2], to which it reduces when $\varepsilon = 0$.

3. LEMMA. *Let $S$ be a compact Hausdorff space and $K \subset C(S)$ an admissible cone. Let $P, Q \in \mathcal{M}(S)$ be such that $\int f \,\mathrm{d}P \leqslant \int f \,\mathrm{d}Q + \varepsilon \operatorname{osc} f$ for all $f \in K$. Let us fix bounded functions $\alpha, \beta, \phi_i : S \to R$, where $\alpha$ and $\beta$ are Borel measurable and $\phi_i \geqslant 0$, $i = 1, \ldots, n$. Further let us fix $f_i \in K$, $i = 1, \ldots, n$. Then*

$$\inf_{s,t \in S} \left[ \alpha(s) + \beta(t) + \sum_{i=1}^{n} (f_i(s) - f_i(t))\phi_i(s) \right] \geqslant 0 \tag{1}$$

*implies*

$$\int \alpha \,\mathrm{d}P + \int \beta \,\mathrm{d}Q + \varepsilon \operatorname{osc} \beta \geqslant 0. \tag{2}$$

*Proof.* The proof is patterned after that of Lemma 3 in [2]. As in that lemma, the crucial step consists of defining an auxiliary function $\hat{\beta}: R^n \to \bar{R} := [-\infty, +\infty]$ having convenient properties. The Euclidean space $R^n$ will be equipped with the usual coordinatewise partial ordering. Throughout the rest of the proof we will use the notation $f := (f_1, \ldots, f_n)$. Also $\boldsymbol{\beta}: S \to R$ will be the lsc regularization of $\beta$. It is given by $\boldsymbol{\beta}(t) := \underline{\lim}_{s \to t} \beta(s)$. Of course (1) holds true with $\boldsymbol{\beta}$ in place of $\beta$.

Let $x \in R^n$ and consider the sequences

$$(p_1, p_2, \ldots) \in [0, 1]^\infty \quad \text{with } p_1 + p_2 + \cdots = 1, \tag{3}$$

$$(t_1, t_2, \ldots) \in S^\infty \quad \text{with } x \leqslant \sum_j p_j f(t_j). \tag{4}$$

Set

$$T_x := \left\{ \sum_j p_j \boldsymbol{\beta}(t_j) \mid (3) \text{ and } (4) \text{ hold} \right\}$$

and define

$$\hat{\beta}(x) := \inf T_x \quad \text{if } T_x \neq \varnothing, \qquad \text{and} \qquad \hat{\beta}(x) := +\infty \quad \text{if } T_x = \varnothing.$$

It is easy to see that $\hat{\beta}(x)$ is finite on and only on the set $U := \{x \in R^n \mid x \leqslant y$ for some $y \in \operatorname{conv} f(S)\}$. Here the notation $\operatorname{conv} f(S)$ indicates the convex hull of $f(S)$. The properties of $\hat{\beta}$ that we are interested in are: (i) $-\alpha \leqslant \hat{\beta} \circ f \leqslant \beta$, (ii) $\hat{\beta}$ is increasing, (iii) $\hat{\beta}$ is convex, and (iv) $\hat{\beta}$ is lsc. The last one is the more important; it is the Lemma 4 in [2], where we need the lower semicontinuity of $\boldsymbol{\beta}$.

Let us prove the property (i). Taking $(p_n) := (1, 0, \ldots)$ and $(t_n) := (t, t, \ldots) \in S^\infty$, we see that $\hat{\beta}(t) \in T_{f(t)}$, hence $\hat{\beta}(f(t)) \leqslant \boldsymbol{\beta}(t)$, that is,

$$\hat{\beta} \circ f \leqslant \boldsymbol{\beta} \leqslant \beta \quad \text{on } S. \tag{5}$$

For the first inequality in (i), fix $s \in S$, set $x := f(s)$ and take sequences $(p_j)$, $(t_j)$ verifying (3) and (4), respectively. In particular

$$f(s) \leqslant \sum_j p_j f(t_j). \tag{6}$$

Let us apply (1) with $t := t_j$; afterwards, we multiply by $p_j$ and sum over $j$ obtaining

$$\alpha(s) + \sum_j p_j \boldsymbol{\beta}(t_j) + \sum_{i=1}^{n} \left[ f_i(s) - \sum_j p_j f_i(t_j) \right] \phi_i(s) \geqslant 0,$$

which gives, using (6), $\alpha(s) + \Sigma_j p_j \boldsymbol{\beta}(t_j) \geqslant 0$. This together with the definition of $\hat{\beta}$ yield $\alpha(s) + \hat{\beta} \circ f(s) \geqslant 0$ so that, by (5),

$$-\alpha \leqslant \hat{\beta} \circ f \leqslant \beta \quad \text{on } S. \tag{7}$$

That $\hat{\beta}$ is increasing is immediate.

The convexity is easy: let $p$, $q \in [0, 1]$ with $p + q = 1$, $x$, $y \in R^n$ and

$$\sum_j p_j \boldsymbol{\beta}(t_j) \in T_x, \qquad \sum_j q_j \boldsymbol{\beta}(t_j) \in T_y.$$

Therefore it is readily seen that

$$\left[ \sum_j p p_j \boldsymbol{\beta}(t_j) + \sum_j q q_j \boldsymbol{\beta}(t_j) \right] \in T_{px+qy},$$

hence

$$\hat{\beta}(px + qy) \leqslant p \sum_j p_j \boldsymbol{\beta}(t_j) + q \sum_j q_j \boldsymbol{\beta}(t_j),$$

which produces

$$\hat{\beta}(px + qy) \leqslant p \inf T_x + q \inf T_y = p\hat{\beta}(x) + q\hat{\beta}(y),$$

so $\hat{\beta}$ is convex indeed.

It is known that a convex lsc function like $\hat{\beta}$ restricted to $U$, which is a convex set with non-empty interior, is the limit of an increasing sequence $(h_{(\nu)})$ of functions $h_{(\nu)} := h_1 \vee \cdots \vee h_\nu$, where, for $i = 1, \ldots, \nu$, $h_i$ is the restriction to $U$ of an affine function on $R^n$ given by $h_i(x) := \langle A_i, x \rangle + a_i$, $A_i \in R^n$, $a_i \in R$. Here $\langle \cdot, \cdot \rangle$ is the usual inner product. Since $\hat{\beta}$ is increasing, we can suppose that all the $h_i$'s are increasing, equivalently, that $A_i \geqslant 0$. As $K$ contains the constants, the linear combinations $h_i \circ f \in K$, thus also $h_{(\nu)} \circ f \in K$ for all $\nu \in N$, because $K$ is invariant under the operation $\vee$, so that

$$\int h_{(\nu)} \circ f \, \mathrm{d}P \leqslant \int h_{(\nu)} \circ f \, \mathrm{d}Q + \varepsilon \operatorname{osc}(h_{(\nu)} \circ f) \quad \text{for all } \nu \in N.$$

Therefore by the Monotone Convergence Theorem

$$\int \hat{\beta} \circ f \, \mathrm{d}P \leqslant \int \hat{\beta} \circ f \, \mathrm{d}Q + \varepsilon \overline{\lim} \operatorname{osc}(h_{(\nu)} \circ f).$$

It is obvious that $\sup h_{(\nu)} \circ f \leqslant \sup \hat{\beta} \circ f$. Further $\lim_\nu (\inf h_{(\nu)} \circ f) = \inf \hat{\beta} \circ f$ by Dini's lemma. Thus $\overline{\lim} \operatorname{osc}(h_{(\nu)} \circ f) \leqslant \operatorname{osc}(\hat{\beta} \circ f) \leqslant \operatorname{osc} \beta$. Putting all together, one arrives at the inequality

$$\int \hat{\beta} \circ f \, \mathrm{d}P \leqslant \int \hat{\beta} \circ f \, \mathrm{d}Q + \varepsilon \operatorname{osc} \beta. \tag{8}$$

Finally, using (7) and (8), we conclude that

$$\int \alpha \, \mathrm{d}P + \int \beta \, \mathrm{d}Q \geqslant \int \alpha \, \mathrm{d}P + \int \hat{\beta} \circ f \, \mathrm{d}Q$$
$$\geqslant \int (\alpha + \hat{\beta} \circ f) \, \mathrm{d}P - \varepsilon \operatorname{osc} \beta \geqslant -\varepsilon \operatorname{osc} \beta. \qquad \square$$

Let $P$, $Q \in \mathscr{M}(S)$. We will describe the property $\int f \, \mathrm{d}P \leqslant \int f \, \mathrm{d}Q + \varepsilon \operatorname{osc} f$, for all $f$ in a subset $L$ of $C_b(S)$ also by saying that $Q$ is an *approximate dilation* or an *ε-dilation* of $P$ relatively to $L$.

The following theorem supplies an equivalent definition of 'ε-dilation' relatively to an admissible cone $K \subset C(S)$ for the case that $S$ is a compact metric space. It says that a necessary and sufficient condition for $Q$ to be an ε-dilation of $P$ relatively to $K$ is that one can find a probability measure $\lambda \in \mathscr{M}(S^2)$ that satisfies

$$\int (f(s) - f(t)) \phi(s) \lambda(\mathrm{d}s, \mathrm{d}t) \leqslant 0 \quad \text{for all } f \in K, \ \phi \in C^+(S) \tag{9}$$

and whose first marginal is $P$ and second marginal is 'ε-close' to $Q$.

4. THEOREM. *Let $S$ be a compact metric space, $K \subset C(S)$ an admissible cone, $\varepsilon \geqslant 0$, and $P, Q \in \mathcal{M}(S)$. Then $\int f\,\mathrm{d}P \leqslant \int f\,\mathrm{d}Q + \varepsilon \operatorname{osc} f$ for all $f \in K$ iff there exists $\lambda \in \mathcal{M}(S^2)$ satisfying* (9) *and, in addition,*

$$\int \alpha(s)\lambda(\mathrm{d}s, \mathrm{d}t) \leqslant \int \alpha\,\mathrm{d}P \quad \textit{for all } \alpha \in C(S), \tag{10}$$

$$\int \beta(t)\lambda(\mathrm{d}s, \mathrm{d}t) \leqslant \int \beta\,\mathrm{d}Q + \varepsilon \operatorname{osc} \beta \quad \textit{for all } \beta \in C(S). \tag{11}$$

*Proof.* 'If': Fix $f \in K$. Applying (10) with $\alpha = -f$, (9) with $\phi = 1$, (11) with $\beta = f$, one finds that $\int f\,\mathrm{d}P \leqslant \int f(s)\lambda(\mathrm{d}s, \mathrm{d}t) \leqslant \int f(t)\lambda(\mathrm{d}s, \mathrm{d}t) \leqslant \int f\,\mathrm{d}Q + \varepsilon \operatorname{osc} f$.

'Only if': By Theorem 7 in [4], the existence of a measure $\lambda \in M(S^2)$ satisfying (9), (10) and (11) is equivalent to the implication (1) ⇒ (2). Thus the 'only if' part follows from Lemma 3. □

In the following lemma the equivalence (b) ⇔ (c) is known. See for example [3].

5. LEMMA. *Assume $S$ is a metric space, $\varepsilon \geqslant 0$ and $P$, $Q \in \mathcal{M}(S)$. Then the following are equivalent:*

(a) $\int \alpha\,\mathrm{d}P \leqslant \int \alpha\,\mathrm{d}Q + \varepsilon \operatorname{osc} \alpha$ *for all* $\alpha \in C_b(S)$;
(b) $|P(B) - Q(B)| \leqslant \varepsilon$ *for all* $B \in \mathscr{B}(S)$;
(c) $\|P - Q\| \leqslant 2\varepsilon$.

*Proof.* We will show that (a) ⇒ (b) ⇒ (c) ⇒ (a).

(a) ⇒ (b): Since the indicator function $1_A$ of an open set $A \subset S$ is lsc, it is the pointwise limit of an increasing sequence of non-negative functions in $C_b(S)$. So (a) implies through the Monotone Convergence Theorem that $P(A) \leqslant Q(A) + \varepsilon$ for all open sets $A \subset S$. Now (b) follows by regularity of $P$.

(b) ⇒ (c): Let $\mu := (P+Q)/2$ and consider $f := \mathrm{d}P/\mathrm{d}\mu$, $g := \mathrm{d}Q$, the Radon-Nikodym derivatives. We have, using (b), $\|P - Q\| = \int |f - g|\,\mathrm{d}\mu \leqslant 2\varepsilon$.

(c) ⇒ (a): Let $\mu$, $f$ and $g$ be as in the proof of (b) ⇒ (c), $\alpha \in C_b(S)$ and $c := -(\sup \alpha + \inf \alpha)/2$. Therefore $2\|\alpha + c\| = \operatorname{osc} \alpha$ and

$$\int \alpha\,\mathrm{d}P - \int \alpha\,\mathrm{d}Q = \int (\alpha + c)(f - g)\,\mathrm{d}\mu \leqslant \|\alpha + c\| \int |f - g|\,\mathrm{d}\mu$$
$$= \|\alpha + c\| \cdot \|P - Q\| \leqslant \varepsilon \operatorname{osc} \alpha. \qquad \square$$

6. DEFINITIONS. In view of Theorem 4 and Lemma 5 it becomes natural to study the five quantities $\varepsilon_i(P, Q)$, $i = 1, \ldots, 5$, defined as follows.

Let $S$ be a topological space, $K \subset C_b(S)$ an admissible cone and $P, Q \in \mathcal{M}(S)$.

Here the dilations will be relative to $K$. Let us define

$$E_1 := \left\{ \varepsilon \geqslant 0 \mid \int f \, dP \leqslant \int f \, dQ + \varepsilon \operatorname{osc} f \text{ for all } f \in K \right\},$$

$$E_2 := \left\{ \varepsilon \geqslant 0 \mid \text{there exists } Q' \in \mathscr{M}(S) \text{ with } P \prec Q' \text{ and } \|Q' - Q\| \leqslant 2\varepsilon \right\},$$

$$E_3 := \left\{ \varepsilon \geqslant 0 \mid \text{there exists } P' \in \mathscr{M}(S) \text{ with } P' \prec Q \text{ and } \|P' - P\| \leqslant 2\varepsilon \right\},$$

$$E_4 := \left\{ \varepsilon \geqslant 0 \mid \text{there exists } P', Q' \in \mathscr{M}(S) \text{ with } P' \prec Q', \right.$$
$$\left. \|P' - P\| \leqslant 2\varepsilon \text{ and } \|Q' - Q\| \leqslant 2\varepsilon \right\},$$

$$E_5 := \left\{ \varepsilon \geqslant 0 \mid \text{there exists } P', Q' \in \mathscr{M}(S) \text{ with } P' \prec Q' \right.$$
$$\left. \text{and } \|P' - P\| + \|Q' - Q\| \leqslant 2\varepsilon \right\}.$$

Now we define

$$\varepsilon_i(P, Q) := \inf E_i, \quad i = 1, \dots, 5. \tag{12}$$

It is trivial to see that $E_2 \subset E_1$, $E_3 \subset E_1$ and $E_2 \subset E_5 \subset E_4$. Now, if $S$ is a compact metric space, taking $Q'$ as the second marginal of the measure $\lambda$, it follows from Theorem 4 that $E_1 \subset E_2$. To summarize, whenever $S$ is compact metric space $E_3 \subset E_1 = E_2 \subset E_5 \subset E_4$, thus we have proved the important

7. THEOREM. *If $S$ is a compact metric space,* $\varepsilon_4 \leqslant \varepsilon_5 \leqslant \varepsilon_2 = \varepsilon_1 \leqslant \varepsilon_3$.

8. REMARKS. (i) Later on it will be seen that $\varepsilon_5 = \varepsilon_1$ and that the first and last inequalities in Theorem 7 are frequently strict.

(ii) If $P \prec Q$, then $\varepsilon_i(P, Q) = 0$, $i = 1, \dots, 5$.

(iii) We always have $0 \leqslant \varepsilon_i(P, Q) \leqslant 1$, $i = 1, \dots, 5$.

(iv) Obviously

$$\varepsilon_1(P, Q) = \sup_{\substack{\operatorname{osc} f \leqslant 1 \\ f \in K}} \left[ \int f \, dP - \int f \, dQ \right]. \tag{13}$$

(v) Theorem 4 is false for non-compact spaces. For such spaces the condition

$\int f\,dP \leqslant \int f\,dQ + \varepsilon \operatorname{osc} f$ for all $f \in K$ is obviously necessary but no longer sufficient for (9), (10) and (11). To see that the named condition fails to be sufficient, consider $S := [0,1)$, take $P := \delta_{1/2}$ and $Q := \delta_0$ and let $K$ consist of all increasing convex functions on $S$. One can show that $\varepsilon_1(P,Q) = 1/2$ and that there is no $Q' \in \mathscr{M}(S)$ dilating $P$ with $\|Q' - Q\| \leqslant 2\varepsilon$. This contradicts the inclusion $E_1 \subset E_2$, thus Theorem 4. □

From (13) it follows immediately that $\varepsilon_1$ satisfies the triangle inequality. But $\varepsilon_1$ is not symmetric. The mapping $(P,Q) \mapsto \delta_1(P,Q) := \varepsilon_1(P,Q) + \varepsilon_1(Q,P)$ is a pseudo-metric on $\mathscr{M}(S)$, in fact a metric when $K$ is a determining class for $\mathscr{M}(S)$ (for instance, $S$ a convex compact metrizable subset of a topological vector space and $K \subset C(S)$ the cone of convex functions). It is not difficult to prove that a sequence $(P_n)$ in $\mathscr{M}(S)$ converges with respect to $\delta_1$, i.e., $\delta_1(P_n, P) \to 0$ for some $P \in \mathscr{M}(S)$ iff the sequence of linear functionals $f \mapsto \int f\,dP_n$ converges uniformly on $K \cap \{f \in C(S) \mid \|f\| = 1\}$. As a consequence, if $K$ is a determining class for $\mathscr{M}(S)$, the $\delta_1$-topology on $\mathscr{M}(S)$ is finer than the weak topology.

Neither $\varepsilon_3$ nor $\varepsilon_4$ satisfy the triangle inequality as Examples 9 and 13 will show. On the other hand it is easy to see that $\varepsilon_4(P,R) \leqslant 2[\varepsilon_4(P,Q) + \varepsilon_4(Q,R)]$.

9. EXAMPLE. A case where $\varepsilon_3(P,R) > \varepsilon_3(P,Q) + \varepsilon_3(Q,R)$. Let $S := [0,1]$, $K \subset C(S)$ be the cone of all convex functions, $P := \delta_{1/2}$, $Q := (1/2)(\delta_0 + \delta_1)$ and $R := \delta_0$. For each $f \in K$, $f(1/2) \leqslant (1/2)f(0) + (1/2)f(1)$, so that $P \prec Q$, hence $\delta_3(P,Q) = 0$. Also $\delta_3(Q,R) \leqslant \|Q - R\|/2 = 1/2$. Since $P' \prec R$ requires $P' = \delta_0$, it follows that $\delta_3(P,R) = \|\delta_0 - \delta_{1/2}\|/2 = 1$. □

Probably there is no easy formula for computing the value $\varepsilon_i$, $i = 1, \ldots, 5$, but the next theorem and corollary are an important step in this direction.

10. THEOREM. *Let $S$ be a compact space, $K \subset C(S)$ an admissible cone, $P$, $Q \in \mathscr{M}(S)$ and $u$, $v \geqslant 0$ constants. Then that there exist $P'$, $Q' \in \mathscr{M}(S)$ such that*

$$\|P' - P\| \leqslant 2u,\ \|Q' - Q\| \leqslant 2v,\ P' \prec_K Q' \tag{14}$$

*if and only if, for all $f \in K$ with* $\inf f = 0$ *and all $c \in R$ with* $0 < c \leqslant \sup f$, we have

$$\int f \wedge c\,dP \leqslant \int f\,dQ + uc + v \sup f. \tag{15}$$

*Proof.* By the very definition of $\varepsilon_2$, (14) is equivalent to the existence of $P' \in \mathscr{M}(S)$ such that

$$\|P' - P\| \leqslant 2u,\ \varepsilon_2(P', Q) \leqslant v. \tag{16}$$

By Lemma 5 and the equality $\varepsilon_2 = \varepsilon_1$, condition (16) on $P'$ is equivalent to

$$\int \alpha \, dP' \leqslant \int \alpha \, dP + u \operatorname{osc} \alpha, \quad \text{for all } \alpha \in C(S)$$

$$\int f \, dP' \leqslant \int f \, dQ + v \operatorname{osc} f, \quad \text{for all } f \in K. \tag{17}$$

Since $C(S)$ and $K$ are cones, Theorem 5 in [4] tells us that a $P' \in \mathscr{M}(S)$ satisfying (17) exists iff, for all $f_j \in K$, $\alpha_i \in C(S)$, and $m, n \in N$, we have that

$$\inf\left(\sum_{i=1}^{m} \alpha_i + \sum_{j=1}^{n} f_j\right) \geqslant 0 \tag{18}$$

implies

$$\sum_{i=1}^{m} \left(\int \alpha_i \, dP + u \operatorname{osc} \alpha_i\right) + \sum_{j=1}^{n} \left(\int f_j \, dQ + v \operatorname{osc} f_j\right) \geqslant 0. \tag{19}$$

Letting $\alpha := \Sigma \alpha_i$ and $f := \Sigma f_j$, then $\alpha \in C(S)$ and $f \in K$, since the cones $C(S)$ and $K$ are convex. As $\operatorname{osc} \alpha \leqslant \Sigma \operatorname{osc} \alpha_i$ and $\operatorname{osc} f \leqslant \Sigma \operatorname{osc} f_j$, it suffices to establish the implication

$$\alpha \in C(S),\ f \in K,\ \inf(\alpha + f) \geqslant 0$$
$$\Rightarrow \int \alpha \, dP + \int f \, dQ + u \operatorname{osc} \alpha + v \operatorname{osc} f \geqslant 0. \tag{20}$$

Introducing $h := \alpha + f$, this is equivalent to the requirement that

$$\int f \, dP - \int f \, dQ \leqslant \int h \, dP + u \operatorname{osc}(f - h) + v \operatorname{osc} f,$$
$$\text{if } f \in K,\ h \in K,\ h \in C^+(S). \tag{21}$$

Given $f \in K$, we want to choose $h \in C^+(S)$ so as to minimize the right-hand side of (21). Put $a := \inf(f - h)$ and $c := \sup(f - h)$ so that $\operatorname{osc}(f - h) = c - a$ and $a \leqslant f - h \leqslant c$, or $f - c \leqslant h \leqslant f - a$. As $h \geqslant 0$, setting $h_0 := (f - c)^+ := (f - c) \vee 0$, we have $f - c \leqslant h_0 \leqslant h \leqslant f - a$. Further $f - c \leqslant h_0 \leqslant f - a$, or $a \leqslant f - h_0 \leqslant c$, which shows that $\operatorname{osc}(f - h_0) \leqslant c - a = \operatorname{osc}(f - h)$. Since $0 \leqslant h_0 = (f - c)^+ \leqslant h$ and $\operatorname{osc}(f - h_0) \leqslant \operatorname{osc}(f - h)$, it is clear from (21) that it suffices to consider only functions of the form $h := (f - c)^+$, where $c$ is a constant. Observing that $f - (f - c)^+ = f \wedge c$, (21) is equivalent to

$$\int f \wedge c\,\mathrm{d}P - \int f\,\mathrm{d}Q \leqslant u \operatorname{osc}(f \wedge c) + v \operatorname{osc} f, \quad \text{for all } f \in K,\ c \in R. \tag{22}$$

Let us show that in (22) we only need

$$\inf f < c \leqslant \sup f. \tag{23}$$

For, the choice $c > \sup f$ is the same as the choice $c = \sup f$, because in both cases $f \wedge c = f$. If $c \leqslant \inf f$, then $\int f \wedge c\,\mathrm{d}P = c$ and $\int f\,\mathrm{d}Q \geqslant \inf f \geqslant c$ so that (23) is always true.

Since $K$ contains the constants we can always take $\inf f = 0$, in which case $\operatorname{osc} f = \sup f$. Thus the proof will be complete if we show that $\operatorname{osc}(f \wedge c) = c$. Indeed, by (23) $\inf(f \wedge c) = \inf f = 0$ and $\sup(f \wedge c) = 0$. □

Besides using only functions $f \in K$ with $\inf f = 0$ in (15) one may also assume without loss of generality that $\sup f = 1$. Hence (15), thus also (14), is equivalent to

$$tu + v \geqslant \phi(t), \quad \text{for all } 0 \leqslant t \leqslant 1. \tag{24}$$

Here $\phi(t) := \sup\{\int f \wedge t\,\mathrm{d}P - \int f\,\mathrm{d}Q \mid f \in K, \inf f = 0, \sup f = 1\}$.

The set of relations (24) represents a family $(H_t)_{t\in[0,1]}$ of closed half planes. The intersection

$$A := A(P, Q, K) := \left(\bigcap_{t\in[0,1]} H_t\right) \cap \{(u, v) \in R^2 \mid u \geqslant 0, v \geqslant 0\}$$

is a closed convex subset of $R^2$. The pairs $(u, v) \in A$ are precisely the pairs for which there exist $P', Q' \in \mathscr{M}(S)$ satisfying (14).

Considering the definitions of $\varepsilon_i(P, Q)$ it is clear that

$$\varepsilon_2(P, Q) = \inf\{v \mid (0, v) \in A\},$$

$$\varepsilon_3(P, Q) = \inf\{u \mid (u, 0) \in A\},$$

$$\varepsilon_4(P, Q) = \inf\{u \mid (u, u) \in A\},$$

$$\varepsilon_5(P, Q) = \inf\{u + v \mid (u, v) \in A\}.$$

The geometric meaning of $\varepsilon_1 = \varepsilon_2$, $\varepsilon_3$, $\varepsilon_4$ and $\varepsilon_5$ is clear. So putting all together we have the situation described in Fig. 1.

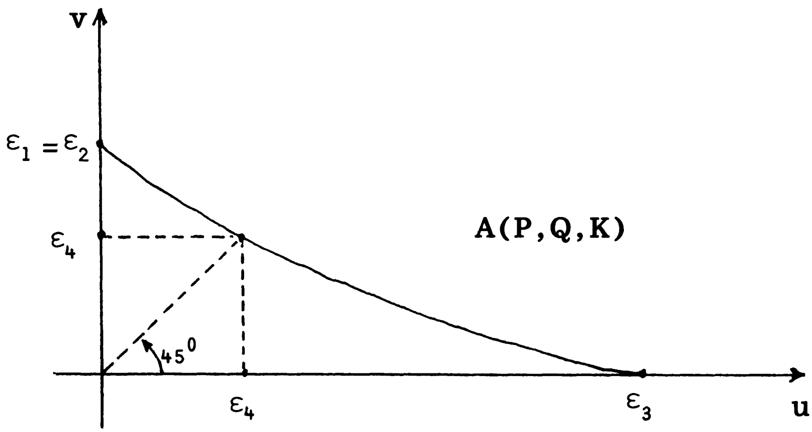


Fig. 1

The only thing that is not clear is how $\varepsilon_5$ fits into the picture. In fact one has:

11. COROLLARY. $\varepsilon_5 = \varepsilon_2$.

*Proof.* The function $t \mapsto \phi(t)$ in (24) is increasing. Hence $\varepsilon_2(P, Q) = \phi(1)$. Therefore, taking $t = 1$ in (24), all points $(u, v) \in A$ satisfy $u + v \geqslant \varepsilon_2(P, Q)$. The equality sign is attained at $(0, \varepsilon_2(P, Q))$. This proves that $\varepsilon_5 = \varepsilon_2$. □

Let $S$ be a compact metric space with a partial order, and $K$ the cone of all continuous increasing functions that assume their minimum at every point of $U := \operatorname{supp} Q$, the support of $Q$. Note that such a cone $K$ is not only invariant under the operation $\vee$ but also under $\wedge$. Letting $P \in \mathscr{M}(S)$ be arbitrary, we have as $\phi(t)$ in (24)

$$\phi(t) = \int t \wedge 1_{U^c}(s) P(\mathrm{d}s) = tP(U^c),$$

which leads to $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = 2\varepsilon_4 = \varepsilon_5 = P(U^c)$ (see Fig. 2).

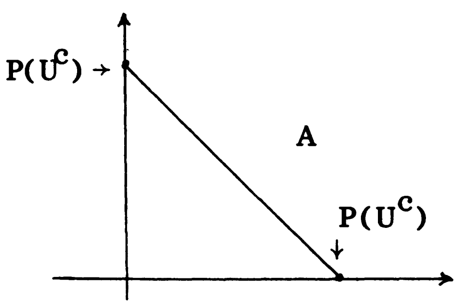


Fig. 2

The above expression for $\phi(t)$ was possible because $K'$ (see Notations) is filtering from the right (see [1], p. 145), i.e., given $f, g \in K'$, there exists $h \in K'$ with $f, g \leqslant h$. In general, if $S$ is a compact space with a partial ordering, $K \subset C(S)$ an admissible cone such that $K'$ is filtering from the right, and $Q \in \mathscr{M}(S)$ is such that each $f \in K'$ assumes its minimum at every point of supp $Q$, then (24) takes the form

$$tu + v \geqslant \phi(t) = t - \int_0^t F(s)\,\mathrm{d}s, \quad t \in [0, 1], \tag{25}$$

where $F$ is the $P$-distribution functions of $s \mapsto \sup_{f \in K'} f(s) := f^*(s)$.

It is true in general that the right slope $r$ of the lower boundary of a region $A(P, Q)$ at $(0, \varepsilon_1)$ is given by the formula $r = -\inf\{t \in [0, 1] \mid \phi$ is constant on $[t, 1]\}$, where $\phi$ is as in (24). Now, if $\phi(t)$ is the right-hand side in (25), then, as it is easy to see, the formula for $r$ specializes to

$$r = -\inf\{t \in [0, 1] \mid F(t) = 1\}. \tag{26}$$

Similarly, it is true in general that the left slope $l$ of the lower boundary of $A(P, Q)$ at $(\varepsilon_3, 0)$ is obtained by the formula $l = -\sup\{t_1 \in [0, 1] \mid \phi(t)/t$ is constant on $(0, t_1]\}$, which, in the situation of (25), becomes

$$l = -\sup\{t_1 \in [0, 1] \mid F \text{ is constant on } [0, t_1]\}. \tag{27}$$

12. EXAMPLE. Let $S$ be the interval $[0, 1]$, $K \subset C(S)$ the cone of convex increasing functions, $P \in \mathscr{M}(S)$ the measure with density $[1/(b-a)]1_{[a,b]}(s)\,\mathrm{d}s$ where $0 \leqslant a < b \leqslant 1$, and $Q := \delta_0$. The corresponding df $F$ is given by $F(s) := (s-a)/(b-a)$ if $s \in [a, b]$. Hence here $r = -b$ and $l = -a$, which show that the right slope of the lower boundary of $A$ at $(0, \varepsilon_1)$ can be any number in $[-1, 0)$ and its left slope at $(\varepsilon_3, 0)$ any number in $(-1, 0]$. We observe also that here $\varepsilon_1(P, \delta_0) = 1 - \int_0^1 F(s)\,\mathrm{d}s = (a+b)/2$, so that $\varepsilon_1$ *can be close to* 0 *or* 1. □

## 13. Measures $P_t'$, $Q_t'$ realizing the boundary of $A(P, Q)$

As it was already observed, $A(P, Q)$ is a closed subset of $R^2$. This means that, for each point $(u, v)$ on the boundary of $A(P, Q)$, one can attain both equality signs in (14) by a suitable choice of $P'$ and $Q'$. Let us now give an example where $P', Q'$ can be explicitly described.

Let $S$ be a compact metric space and $K \subset C(S)$ an admissible cone. Suppose $K'$ possesses a largest element $f^*$. Choose $P \in \mathscr{M}(S)$ and let $F$ be the $P$-distribution function of $f^*$. Suppose there is a unique point $y$ in $S$ with $f^*(y) = 0$

and a unique point $y'$ in $S$ with $f^*(y') = 1$. (Example: let $S$ be a compact space with a partial ordering, a least element $y$ and a greatest element $y'$, and let $K \subset C(S)$ be the cone of all convex increasing functions.) Choose $Q = \delta_y$. A little calculation readily shows that here $\phi$ in (24) is given by $\phi(t) = t - \int_0^t F(s)\,ds$ so that (24) reads $tu + v \geqslant t - \int_0^t F(s)\,ds$ for all $t \in [0, 1]$. Hence we obtain that the part of the lower boundary of $A(P, \delta_y, K)$ not contained in the coordinate axes is a smooth curve (envelope) with parametric equations $u(t) = 1 - F(t)$, $v(t) = tF(t) - \int_0^t F(s)\,ds$, $t \in [0, 1]$. Here we are assuming that $P$ has no atom.

Define $P'_t$, $Q'_t \in \mathscr{M}(S)$ by

$$P'_t(E) := P[E \cap (f^* \leqslant t)] + u(t)\delta_y(E),$$

$$Q'_t(E) := v(t)\delta_{y'}(E) + (1 - v(t))\delta_y(E).$$

Certainly $\|P'_t - P\| = 2u(t)$ and $\|Q'_t - Q\| = 2v(t)$. Moreover, given $f \in K'$,

$$\int f\,dP'_t \leqslant \int f^*\,dP'_t = \int_{[0,t]} s\,dF(s) + uf^*(y)$$

$$= \int_{[0,t]} s\,dF(s) = tF(t) - \int_0^t F(s)\,ds = v(t),$$

and

$$\int f\,dQ'_t = v(t)f(y') + [1 - v(t)]f(y) = v(t).$$

Thus $\int f\,dP'_t \leqslant \int f\,dQ'_t$. This proves that $P'_t \prec Q'_t$.

## 14. The triangle inequality fails for $\varepsilon_4$

Let $S := [0, 1]$, $K \subset C(S)$ be the cone of decreasing convex functions and $Q := (1/2)(\delta_0 + \delta_1)$. We want to show that,

$$\varepsilon_4(\delta_{1/2}, \delta_1) > \varepsilon_4(\delta_{1/2}, Q) + \varepsilon_4(Q, \delta_1). \tag{28}$$

Let us first compute $\varepsilon_4(\delta_{1/2}, \delta_1)$. Here (25) applies. The function $s \mapsto -s + 1$ is the largest element in $K$ and its $\delta_{1/2}$-distribution function is $F = 1_{[1/2,\infty)}$. Using (25) we obtain the following family of half planes

$$tu + v \geqslant \begin{cases} t, & \text{if } t \leqslant 1/2 \\ 1/2, & \text{if } t \geqslant 1/2. \end{cases}$$

Thus $u + 2v = 1$ is the equation of the lower boundary of $A(\delta_{1/2}, \delta_1)$. Letting

$v = u$ in that equation, we conclude that $\varepsilon_4(\delta_{1/2}, \delta_1) = 1/3$.

Next consider $\varepsilon_4(\delta_{1/2}, Q)$. Here it is easier going back to (24). We have

$$\phi(t) = \sup_{f \in K'} \left[ \int (f \wedge t) \, \mathrm{d}\delta_{1/2} - \int f \, \mathrm{d}Q \right] = \begin{cases} t - 1/2, & t \leqslant 1/2 \\ 0, \ t \geqslant 1/2. \end{cases}$$

The equation of the important part of the lower boundary of $A(\delta_{1/2}, Q)$ is $u + 2v = 0$, from which, letting $v = u$, we obtain $\varepsilon_4(\delta_{1/2}, Q) = 0$.

As to $\varepsilon_4(Q, \delta_1)$, here again (25) applies. The $Q$-distribution function $F$ of $s \mapsto -s + 1$ has values $F(s) = 0$ if $s < 0$, $F(s) = 1/2$ if $0 \leqslant s < 1$ and $F(s) = 1$ if $s \geqslant 1$. By (25)

$$tu + v \geqslant t - \int_0^t F(s) \, \mathrm{d}s = t - \tfrac{1}{2}t = \tfrac{1}{2}t, \ t \in [0, 1].$$

So the part of the lower boundary of $A(Q, \delta_1)$ we are interested in is given by $u + v = 1/2$, $u \in [0, 1/2]$, showing that $\varepsilon_4(Q, \delta_1) = 1/4$. Therefore $\varepsilon_4(\delta_{1/2}, Q) + \varepsilon_4(Q, \delta_1) = 1/4$. Thus (28) is proved. □

When we dealt with cones both invariant under max and min operation, the corresponding picture, Fig. 2, was very peculiar. In particular $\varepsilon_2 = \varepsilon_3 = 2\varepsilon_4$ in that situation. Let us show that this is always so whenever the cone has the mentioned property through the following proposition.

15. PROPOSITION. *Let $S$ be a compact metric space, $K \subset C(S)$ an admissible cone which is invariant under the operation $\wedge$ and let $P$, $Q \in \mathscr{M}(S)$. Then the portion of the boundary of $A(P, Q, K)$ not contained in the u-axis is a line segment with slope $-1$. In particular $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = 2\varepsilon_4 = \varepsilon_5$ at $(P, Q)$.*

*Proof.* The lower boundary of $A(P, Q, K)$ has slope $\leqslant 1$ (in absolute value). But so has the corresponding set $A(P, Q, -K)$, where $-K := \{f \mid -f \in K\}$. Since $A(P, Q, -K)$ is simply the reflexion $\{(v, u) \mid (u, v) \in A(P, Q, K)\}$ of $A(P, Q, K)$, the lower boundary of the latter is a straight line of slope $-1$. □

Before ending this article it is worthwhile to make the following

16. REMARK. Let $S$ be a compact metric space, $K \subset C(S)$ an admissible cone and $P$, $Q \in \mathscr{M}(S)$. Using the definition of $\varepsilon_1(P, Q)$, Theorems 7 and 10 and Corollary 11, we have

$$\varepsilon_i(P, Q) = \sup_f \left[ \int f \, \mathrm{d}P - \int f \, \mathrm{d}Q \right], \quad i = 1, 2, 5;$$

$$\varepsilon_3(P, Q) = \sup_{f,t} \left[ \frac{1}{t} \int f \wedge t \, \mathrm{d}P - \frac{1}{t} \int f \, \mathrm{d}Q \right];$$

$$\varepsilon_4(P, Q) = \sup_{f,t} \left[ \frac{1}{1+t} \int f \wedge t \, \mathrm{d}P - \frac{1}{1+t} \int f \, \mathrm{d}Q \right];$$

where $f$ runs over $K'$ and $t$ over $(0, 1)$. It follows that, endowing $\mathscr{M}(S)$ with the weak topology, the function $(P, Q) \mapsto \varepsilon_1(P, Q)$, $i = 1, \ldots, 5$, is lsc and convex. It is easy to produce examples showing that those functions are not (weakly) continuous. □

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