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Dunkl operators, Bessel functions and the discriminant of a finite Coxeter group

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1. Introduction

Let G be a finite Coxeter group acting on a Euclidean space \mathfrak{a} as a real reflection group. In his paper [8] Dunkl defines a commuting set of first order differential-difference operators related to such a G , involving a parameter $k \in \mathbf{C}^m$ (where m is the number of conjugacy classes of reflections in G). In this paper we study these Dunkl operators, by means of local analysis (even though these operators themselves are not local operators since they do not preserve the support of functions). The crucial observation that enables us to apply local methods was made by Heckman in his paper [17]. His result says that G -invariant compositions of Dunkl's commuting operators are partial differential operators when restricted to G -invariant polynomials on \mathfrak{a} . This yields a polynomial algebra of partial differential operators on $G \backslash \mathfrak{a}_{\mathbf{C}}$, and in the Sections 3–6 we study the system of differential equations that arises from the spectral problem for this algebra of differential operators (the Bessel equations) both locally and globally. It turns out that locally on $G \backslash \mathfrak{a}_{\mathbf{C}}^{\text{reg}}$ this system of Bessel equations has a finite dimensional solution space (of dimension $|G|$), which depends holomorphically on the parameter $k \in \mathbf{C}^m$, and the spectral parameter $\lambda \in \mathfrak{a}_{\mathbf{C}}^*$. An important piece of information on the global behaviour of such a system of equations is given by the so-called monodromy representation on the space of local solutions at some base point $x_0 \in G \backslash \mathfrak{a}_{\mathbf{C}}^{\text{reg}}$. For generic values of λ and k we determine this monodromy representation in Section 5. However, the structure of the monodromy representation can change dramatically for special values of λ and k . For instance, generically the representation is both semisimple and cyclic, but for special values of λ and k the representation may fail to possess either one of these properties. (For example, generically the space of global holomorphic solutions is one dimensional but for special values of (λ, k) this space can be bigger). It is of course not surprising that such phenomena arise, but it is remarkable that it is

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possible to obtain fairly detailed information as to what kind of degenerations occur for which specializations of (λ, k) . In Section 5 we also exactly locate the set of “singular” values for the parameter k , i.e. those values for which the above degenerations of the monodromy representation occur.

In Section 6 we introduce and study the Bessel function for G , which is the unique solution of the Bessel equations that is holomorphic and globally defined on $G \backslash \mathfrak{a}_C$, normalized such that its value at the origin equals 1. We also indicate in this section how one may use this Bessel function to solve the simultaneous eigenfunction problem for the Dunkl operators themselves (within the space of holomorphic functions on \mathfrak{a}_C). We thus recover some of Dunkl’s results on this subject ([11]).

In the Sections 7, 8 and 9 we investigate the relations between

- (a) The degenerations of the monodromy representation for certain specializations of (λ, k) .
- (b) The singular locus of the Bessel function (as a function of $k \in \mathbb{C}^m$).
- (c) The conjecture of Yano and Sekiguchi (see [35]) on the explicit form of the Bernstein Sato polynomial for the discriminant of G .
- (d) Macdonald’s conjecture on the Mehta type integral associated with G .

(The interplay between (b), (c) and (d) was already noted in [28], and is a consequence of the behaviour of the so-called Bessel shift operators. For Weyl groups this leads to a proof of (c) and (d)). It is not hard to see that there must be some relation between (a) and (b), but for later use we are interested in the precise description of (b), including multiplicities of poles. The key tool we use is the linear functional “evaluation at the origin” on the local system of local solutions of the Bessel equations (Section 8). This functional exists in the interior of a certain polygon of parameter values, thanks to the results of Section 7. It behaves very nicely: it commutes with monodromy, is non-zero and depends holomorphically on the parameters (λ, k) . Using these properties (in Section 9) we are able to describe (b) explicitly. In turn this solves (c) for all G (and not only for Weyl groups). Our results are not sufficient to solve (d) completely for non-Weyl groups, but we reduce Macdonald’s conjecture to the verification that the conjecture holds for one choice of the parameter k (other than $k=0$). F. Garvan has informed me that he has been able to do this for $k=1$ by making use of certain symmetries and with the help of a computer. His calculations complete the verification of the Macdonald-Mehta conjecture. The considerations in Section 9 will also give some information on the nature of the degenerations (a) for different values of (λ, k) , but a lot of questions one might ask about this topic remain unclear.

Some of the proofs in this paper are a bit tedious (especially in Sections 4 and 7). This is mainly due to the fact that we have no explicit knowledge of the connection matrices of the connection ∇ we introduce (in Section 3) in relation

to the system of Bessel differential equations. From this point of view the results of Matsuo (see [25]) for an analogous system of equations are very interesting. These results seem to indicate that it should be possible to rewrite the system of Bessel equations as a first order system in a simple explicit way.

Finally I would like to give another motivation for the study (in Sections 7 and 8) of the evaluation map of local solutions (evaluation at the origin). Namely there exists another system of differential equations, the hypergeometric equations associated with a root system ([18], [19], [27]), to which the results of Sections 7 and 8 apply (almost without a change). In this case however, one disposes of an explicit basis of power series solutions for the space of local solutions. It turns out that one can obtain an explicit summation formula for these series at a certain point on the boundary of the domain of convergence. In the case of one variable this formula is equivalent to Gauss' summation formula for the hypergeometric series at $z=1$. These results will be published in a forthcoming paper.

2. Preliminaries

This section serves to fix notation and to recall certain results of Dunkl and Heckman ([8], [17]). If G is a finite reflection group we will define a set of differential operators which will be called the set of Bessel differential operators for G .

Let \mathfrak{a} be a real inner product space of dimension n . For $\alpha \in \mathfrak{a} \setminus \{0\}$ we let r_α be the orthogonal reflection in α^\perp .

DEFINITION 2.1. A root system in \mathfrak{a} is a finite subset R of $\mathfrak{a} \setminus \{0\}$ such that

- (1) $r_\alpha(R) = R \ \forall \alpha \in R$.
- (2) If both $\alpha, \lambda\alpha \in R$ then $\lambda = \pm 1$.
- (3) $R \cdot R = \mathfrak{a}$.

A root system is called normalized if $(\alpha, \alpha) = 2 \ \forall \alpha \in R$.

REMARK 2.2. If R is a normalized root system then the group $G(R) = \langle r_\alpha \rangle_{\alpha \in R}$ is a finite reflection group in \mathfrak{a} . Conversely, any finite reflection group in E arises from a (uniquely determined) normalized root system in this way.

DEFINITION 2.3. If R is a root system, then a function $k: R \rightarrow \mathbb{C}$ which is $G(R)$ -invariant is called a multiplicity function. The vector space of all multiplicity functions is denoted by $K = K(R)$, and its dimension is denoted by $m = m(R)$. If $S \subset R$ is a G -invariant subset then $1_S \in K$ is its characteristic function.

Fix R , a normalized root system. Let $\mathfrak{h} = \mathfrak{a} + \sqrt{-1} \mathfrak{a}$ be the complexification of \mathfrak{a} . For $\xi \in \mathfrak{h}$ let ∂_ξ be the directional derivative $\partial_\xi f(x) =$

$\lim_{t \rightarrow 0} t^{-1}(f(x+t\xi) - f(x))$ (with $f \in \mathbf{C}[\mathfrak{h}]$), and let $\xi^* \in \mathbf{C}[\mathfrak{h}]$ be the function $\{\mu \rightarrow (\xi, \mu)\}$. If $\xi \in \mathfrak{a} \setminus \{0\}$ we define $\Delta_\xi \in \text{End}(\mathbf{C}[\mathfrak{h}])$ by

$$\Delta_\xi(f) = (\xi^*)^{-1}(f - f \circ r_\xi) \quad (f \in \mathbf{C}[\mathfrak{h}])$$

DEFINITION 2.4. Let R be a normalized root system, and $k \in K(R)$. Then for $\xi \in \mathfrak{h}$ the operator $T_\xi \in \text{End}(\mathbf{C}[\mathfrak{h}])$ defined by

$$T_\xi(k) = \partial_\xi + \sum_{\alpha \in R_+} k_\alpha(\alpha, \xi) \Delta_\alpha$$

is called the Dunkl operator. We also let $T_\xi \in \text{End}(\mathbf{C}[K \times \mathfrak{h}])$ denote the family $\{T_\xi(k) \mid k \in K\}$.

THEOREM 2.5 (Dunkl, see [8]). $\forall \xi, \mu \in \mathfrak{h}: T_\xi T_\mu = T_\mu T_\xi$.

THEOREM 2.6 (Dunkl, see [8]). Let ξ_1, \dots, ξ_n be an orthonormal basis for \mathfrak{a} . Then

$$\sum_i T_{\xi_i}^2 = \sum_i \partial_{\xi_i}^2 + 2 \sum_{\alpha \in R_+} k_\alpha(\alpha^*)^{-1}(\partial_\alpha - \Delta_\alpha).$$

As a consequence of Theorem 2.5 one can define a homomorphism of algebras $\mathbf{C}[\mathfrak{h}^*] \rightarrow \text{End}(\mathbf{C}[K \times \mathfrak{h}])$ by sending $\xi \in \mathbf{C}[\mathfrak{h}^*]$ to T_ξ . Clearly the result is an injective homomorphism, and we denote the image of $p \in \mathbf{C}[\mathfrak{h}^*]$ by T_p . (The specialization to $k = 0$ gives the constant coefficient differential operator $T_p(0)$ which we usually denote by ∂_p).

A well known result of Chevalley is that $\mathbf{C}[\mathfrak{h}]^G = \mathbf{C}[p_1, \dots, p_n]$, where p_1, \dots, p_n are homogeneous G -invariant polynomials. The homogeneous degrees of the p_i are uniquely determined by R . These are called the primitive degrees of R , and denoted by $d_i = \text{deg}(p_i)$. We fix a choice of fundamental G -invariant polynomials p_1, \dots, p_n such that: $d_1 \leq d_2 \leq \dots \leq d_n$. In general, if X is a complex manifold, then $\mathbf{A}[X]$ will denote the algebra of (algebraic) differential operators on X , but (due to its predominant role) we will write \mathbf{A} for the Weyl algebra on $G \setminus \mathfrak{h}$, thus

$$\mathbf{A} = \mathbf{A}[G \setminus \mathfrak{h}] = \mathbf{C} \left\langle p_1, \dots, p_n, \frac{\partial}{\partial p_1}, \dots, \frac{\partial}{\partial p_n} \right\rangle.$$

The next theorem was an important observation by Heckman.

THEOREM 2.7 (Heckman [17]). Suppose D is an element of the associative algebra of endomorphisms of $\mathbf{C}(\mathfrak{h})$, the rational functions on \mathfrak{h} , generated by (i) the operators $T_\xi(k)(\xi \in \mathfrak{h}, k \in K \text{ fixed})$ and (ii) multiplication with elements of $\mathbf{C}(\mathfrak{h})$. If D

commutes with the G -action on \mathfrak{h} , then its restriction to $\mathbf{C}(\mathfrak{h})^G = \mathbf{C}(p_1, \dots, p_n)$ is a differential operator on $G \backslash \mathfrak{h}$ with rational coefficients.

Since one obviously has

$$g \cdot T_\xi \cdot g^{-1} = T_{g(\xi)} \quad \forall \xi \in \mathfrak{h}, g \in G,$$

we obtain:

COROLLARY 2.8 (Heckman [17]). *There exists an injective homomorphism of algebras*

$$\begin{aligned} \mathbf{C}[K] \otimes \mathbf{C}[\mathfrak{h}^*]^G &\rightarrow \mathbf{C}[K] \otimes \mathbf{A} \\ p &\rightarrow D_p = T_p|_{\mathbf{C}[\mathfrak{h}]^G}. \end{aligned}$$

Put

$$\mathbf{S} = \{D_p \mid p \in \mathbf{C}[K] \otimes \mathbf{C}[\mathfrak{h}^*]^G\}$$

and

$$\mathbf{S}(k) = \{D_p(k) \mid p \in \mathbf{C}[\mathfrak{h}^*]^G\}$$

for its specialization to $k \in K$. Let $\gamma(k): \mathbf{S}(k) \rightarrow \mathbf{C}[\mathfrak{h}^*]^G$ be the isomorphism of algebras given by $D_p(k) \rightarrow p$.

Let $R_+ \subset R$ be a choice of positive roots. If $S \subset R$ is G -invariant, then

$$p_S^* = \prod_{\alpha \in R_+ \cap S} \alpha^* \in \mathbf{C}[\mathfrak{h}] \quad \text{and} \quad p_S = \prod_{\alpha \in R_+ \cap S} \alpha \in \mathbf{C}[\mathfrak{h}^*]$$

are semi-invariants transforming according to the same one dimensional character χ_S of G . Hence, again using Theorem 2.7, we have:

$$\begin{aligned} G_S^+(k) &:= ((p_S^*)^{-1} \circ T_{p_S}(k))|_{\mathbf{C}[\mathfrak{h}]^G} \in \mathbf{A} \\ G_S^-(k) &:= (T_{p_S}(k - 1_S) \circ (p_S^*))|_{\mathbf{C}[\mathfrak{h}]^G} \in \mathbf{A} \end{aligned} \tag{2.1}$$

(The fact that G_S^\pm has polynomial coefficients follows from the following result: $\forall f \in \mathbf{C}[\mathfrak{h}]$ such that $f \circ g = \chi_S(g)f, \forall g \in G: (p_S^*)^{-1}f \in \mathbf{C}[\mathfrak{h}]$. This is well known in the case $S = R$, and the general case can be proven similarly, see e.g. [31], Lemma 2.2.)

THEOREM 2.9 (Heckman [17]). *The operators $G_S^\pm(k)$ are shift operators, i.e.*

they satisfy: $\forall p \in \mathbf{C}[\mathfrak{h}^*]^G, k \in K$:

$$G_S^\pm(k) \circ D_p(k) = D_p(k \pm 1_S) \circ G_S^\pm(k). \tag{2.2}$$

Consider the map $\pi: \mathfrak{h} \rightarrow G \backslash \mathfrak{h}$ (canonical projection). This map is a covering, branched along the set

$$\{p = (p_1, \dots, p_n) \in G \backslash \mathfrak{h} \mid p_R^{*2}(p) = 0\}.$$

The polynomial $I = p_R^{*2} \in \mathbf{C}[p_1, \dots, p_n]$ is called the discriminant of R (or G). Note that the map π induces an isomorphism of the algebras $\mathbf{A}[\mathfrak{h}^{\text{reg}}]^G$ (where $\mathfrak{h}^{\text{reg}} = \{X \in \mathfrak{h} \mid p_R(X) \neq 0\}$) and $\mathbf{A}[G \backslash \mathfrak{h}^{\text{reg}}]$. (However, the image of $\mathbf{A}[\mathfrak{h}]^G$ is, of course, strictly smaller than $\mathbf{A} = \mathbf{A}[G \backslash \mathfrak{h}]$). Via this isomorphism we often consider $\mathbf{A} \subset \mathbf{A}[\mathfrak{h}^{\text{reg}}]^G$.

REMARK 2.10. Take $p = \sum_i \xi_i^2 \in \mathbf{C}[\mathfrak{h}^*]^G$ then we see from Theorem 2.6 that

$$D_p(k) = \sum_i \partial_{\xi_i}^2 + 2 \sum_{\alpha \in R_+} k_\alpha (\alpha^*)^{-1} \partial_\alpha \in \mathbf{A}.$$

This operator was studied in ([27]), and the results Corollary 2.8 and Theorem 2.9 were obtained there for the case where G is a crystallographic finite reflection group. However, the above method of Heckman to use the Dunkl operators as building blocks for an algebra of commuting differential operators is much simpler, and gives a better understanding.

3. The system of Bessel differential equations

Let R be a normalized root system.

DEFINITION 3.1. Given $\lambda \in \mathfrak{h}^*, k \in K$ we will call

$$(D - \gamma(k)(D)(\lambda))f = 0 \quad \forall D \in \mathbf{S}(k) \tag{3.1}$$

the system of Bessel differential equations on $G \backslash \mathfrak{h}$.

Put $\mathbf{D} = \mathbf{C}[\mathfrak{h}^*] \otimes \mathbf{C}[K] \otimes \mathbf{A}$ and let $\mathbf{I} \subset \mathbf{C}[\mathfrak{h}^*] \otimes \mathbf{S}$ denote the kernel of the natural extension of γ to $\mathbf{C}[\mathfrak{h}^*] \otimes \mathbf{S}$. Let \mathbf{J} be the left ideal in \mathbf{D} generated by \mathbf{I} , and write $M = \mathbf{D}/\mathbf{J}$ for the associated cyclic \mathbf{D} -module. Let C be the ring of coefficient functions of operators in \mathbf{D} , so $C = \mathbf{C}[\mathfrak{h}^*] \otimes \mathbf{C}[K] \otimes \mathbf{C}[\mathfrak{h}]^G$. Also introduce the following notations. If N is a $\mathbf{C}[\mathfrak{h}]^G$ -module then $N^{\text{reg}} = I^{-1}N$ (with $I = \prod_{\alpha \in R_+} (\alpha^*)^2 \in \mathbf{C}[\mathfrak{h}]^G$, the discriminant of G), the localization of N at $I \neq 0$. If N is a $\mathbf{C}[\mathfrak{h}^*] \otimes \mathbf{C}[K]$ module and $(\lambda, k) \in \mathfrak{h}^* \times K$, then $N(\lambda, k)$ denotes

the specialization

$$N(\lambda, k) = N \otimes_{\mathbb{C}[\mathfrak{h}^*] \otimes \mathbb{C}[K]} \{(\mathbb{C}[\mathfrak{h}^*] \otimes \mathbb{C}[K]) / \text{Ker } \phi\},$$

where $\phi: \mathbb{C}[\mathfrak{h}^*] \otimes \mathbb{C}[K] \rightarrow \mathbb{C}$ denotes the character $\phi(f) = f(\lambda, k)$.

Recall the notion of harmonic polynomials

$$\mathcal{H}^* := \{q^* \in \mathbb{C}[\mathfrak{h}] \mid \partial_p(q^*) = 0 \ \forall p \in \mathbb{C}[\mathfrak{h}^*]^G \ \text{with } p(0) = 0\}.$$

Let $\mathcal{H} := \{\partial_q \in \mathcal{S}(\mathfrak{h}) \mid q^* \in \mathcal{H}^*\}$ (here $q^* \in \mathbb{C}[\mathfrak{h}]$ and $q \in \mathbb{C}[\mathfrak{h}^*]$ are corresponding elements if we identify $\mathbb{C}[\mathfrak{h}]$ with $\mathbb{C}[\mathfrak{h}^*]$ via the inner product) be the space of harmonic constant coefficient differential operators. Put $V = (\mathcal{H}^* \otimes_{\mathbb{C}} \mathcal{H})^G$. It is well known that, as a G -module, $\mathcal{H} \simeq \mathbb{C}[G]$ and hence $\dim_{\mathbb{C}} V = |G|$.

PROPOSITION 3.2. *Consider $\mathcal{H}^* \otimes_{\mathbb{C}} \mathcal{H}$ as a subspace of $\mathbf{A}[\mathfrak{h}]$ via $q_1^* \otimes \partial_{q_2} \rightarrow q_1^* \partial_{q_2}$. By taking G -invariants we consider V as a linear subspace of \mathbf{A} . Also identify $\mathbb{C} \otimes_{\mathbb{C}} V$ with a \mathbb{C} -submodule of \mathbf{D} via $f \otimes v \rightarrow f \cdot v$. Then we have a direct sum decomposition of C^{reg} -modules:*

$$\mathbf{D}^{\text{reg}} = (C^{\text{reg}} \otimes V) \oplus (C^{\text{reg}} \otimes V)\mathbf{I}.$$

Proof. Observe that $\mathbf{A}[\mathfrak{h}^{\text{reg}}]^G = \mathbf{A}^{\text{reg}}$ and that $(\mathbb{C}[\mathfrak{h}] \otimes \mathcal{H})^G \simeq \mathbb{C}[\mathfrak{h}]^G \otimes V$. Therefore the above statement follows by taking G -invariants from:

$$\begin{aligned} &\mathbb{C}[\mathfrak{h}^*] \otimes \mathbb{C}[K] \otimes \mathbf{A}[\mathfrak{h}^{\text{reg}}] \\ &\simeq \mathbb{C}[\mathfrak{h}^*] \otimes \mathbb{C}[K] \otimes \mathbb{C}[\mathfrak{h}^{\text{reg}}] \otimes \mathcal{H} \oplus (\mathbb{C}[\mathfrak{h}^*] \otimes \mathbb{C}[K] \otimes \mathbb{C}[\mathfrak{h}^{\text{reg}}] \otimes \mathcal{H})\mathbf{I}. \end{aligned} \quad (3.2)$$

This is proved in exactly the same way as Proposition 3.2 of [19] if one replaces the coefficient ring R by $\mathbb{C}[\mathfrak{h}^*] \otimes \mathbb{C}[K] \otimes \mathbb{C}[\mathfrak{h}^{\text{reg}}]$. □

COROLLARY 3.3. *As a C^{reg} -module, $M^{\text{reg}} \simeq C^{\text{reg}} \otimes_{\mathbb{C}} V$. The \mathbf{A} -module structure on $M(\lambda, k)$ induces an integrable connection $\nabla(\lambda, k)$ on $\mathbb{C}[\mathfrak{h}^{\text{reg}}]^G \otimes V$ such that the connection matrices are elements of $C^{\text{reg}} \otimes_{\mathbb{C}} \text{End}_{\mathbb{C}}(V)$.*

COROLLARY 3.4. *$M^{\text{reg}}(\lambda, k)$ is holonomic ($\forall \lambda \in \mathfrak{h}^*, k \in K$).*

Proof. This follows since by Corollary 3.3, $M^{\text{reg}}(\lambda, k)$ is a free $\mathbb{C}[\mathfrak{h}^{\text{reg}}]^G$ -module of rank $|G|$. □

DEFINITION 3.5. Let $x \in G \backslash \mathfrak{h}^{\text{reg}}$ and let \mathcal{O}_x be the space of holomorphic germs at x . Define $\mathcal{L}_x(\lambda, k) = \{f \in \mathcal{O}_x \mid f \text{ satisfies the equations (3.1)}\}$.

COROLLARY 3.6. *$\mathcal{L}_x(\lambda, k) \simeq (\mathcal{O}_x \otimes V^*)^{\nabla^*(\lambda, k)}$. In particular, $\mathcal{L}(\lambda, k)$ is a local system on $G \backslash \mathfrak{h}^{\text{reg}}$ of rank $|G|$.*

Proof. Let $b_1, \dots, b_{|G|}$ be a basis for V , and assume that $b_1 = 1 \otimes 1$ ($= 1$ if we

consider $V \subset \mathbf{A}$). Clearly, b_1 is an \mathbf{A}^{reg} -generator for $M^{\text{reg}}(\lambda, k)$. Hence if $x \in G \setminus \mathfrak{h}^{\text{reg}}$ then the map

$$(\mathcal{O}_x \otimes V^*)^{\nabla^*(\lambda, k)} \rightarrow \mathcal{L}_x(\lambda, k)$$

$$v \rightarrow v(1)$$

is an isomorphism of \mathbf{C} -vector spaces, since

$$(\mathcal{O}_x \otimes V^*)^{\nabla^*(\lambda, k)} = \text{Hom}_{\mathbf{A}}(\mathcal{O}_x \otimes V, \mathcal{O}_x) \simeq \text{Hom}_{\mathbf{A}}(M_x(\lambda, k), \mathcal{O}_x).$$

Note that the inverse map is given by

$$\mathcal{L}_x(\lambda, k) \rightarrow (\mathcal{O}_x \otimes V^*)^{\nabla^*(\lambda, k)}$$

$$f \rightarrow \sum_i (b_i f) \cdot b_i^*$$

where $\{b_i^*\}$ is the dual basis in V^* of $\{b_i\}$. □

Put $\mathcal{U} = \widetilde{G \setminus \mathfrak{h}^{\text{reg}}}$, the universal covering space, and $\Phi: \mathcal{U} \rightarrow G \setminus \mathfrak{h}^{\text{reg}}$ the covering map.

COROLLARY 3.7. *There exist functions $\phi_1, \dots, \phi_{|G|}$, holomorphic on $\mathfrak{h}^* \times K \times \mathcal{U}$, such that $\forall (\lambda, k) \in \mathfrak{h}^* \times K$ the specializations $\phi_1(\lambda, k), \dots, \phi_{|G|}(\lambda, k)$ form a basis of global sections of $\Phi^*(\mathcal{L}(\lambda, k))$.*

Proof. Let $x_0 \in G \setminus \mathfrak{h}^{\text{reg}}$, and $(\lambda_0, k_0) \in \mathfrak{h}^* \times K$. Let $\{b_i\}$: a basis for V as in the proof of Corollary 3.6. By standard theory of first order linear systems of differential equations with a parameter there exist open neighbourhoods $\Omega_2 \ni x_0$ and $\Omega_1 \ni (\lambda_0, k_0)$, and holomorphic functions $v_i: \Omega_1 \times \Omega_2 \rightarrow V^*$ such that $v_i(\lambda, k, x_0) = b_i^*$ and $v_i(\lambda, k, \cdot)$ is $\nabla^*(\lambda, k)$ flat ($\forall i$). Put $\phi_i = v_i(b_1) = v_i(1)$. Of course, when lifted to \mathcal{U} , $\phi_i(\lambda, k, \cdot)$ extends holomorphically to all of \mathcal{U} . By Lemma 2.6 of [27] we obtain that this extension of ϕ_i is holomorphic on $\Omega_1 \times \mathcal{U}$. But since (λ_0, k_0) was chosen arbitrarily this readily implies that $\phi_1, \dots, \phi_{|G|}$ can be extended holomorphically to $\mathfrak{h}^* \times K \times \mathcal{U}$. □

COROLLARY 3.8. *The monodromy matrices of the system \mathcal{L} can be represented (with respect to a suitable basis of solutions) as elements in $GL(|G|, \mathcal{O}(\mathfrak{h}^* \times K))(|G| \times |G|$ matrices with entries that are entire functions on $\mathfrak{h}^* \times K$).*

At this point we make some remarks on two different forms of duality that play a role in the theory of Bessel differential equations.

PROPOSITION 3.9. *Let, for $D \in \mathbf{A}$, tD denote the formal transpose of D (i.e.*

$D \rightarrow {}^tD$ is the unique anti-automorphism of \mathbf{A} generated by

$${}^t(p_i) = p_i \quad \text{and} \quad {}^t\left(\frac{\partial}{\partial p_i}\right) = -\frac{\partial}{\partial p_i}.$$

Let $I(k) = \prod_{\alpha \in \mathbf{R}_+} (\alpha^*)^{2k_\alpha} (k \in K)$ and let $p \in \mathbf{C}[\mathfrak{h}^*]^G$ be homogeneous. Then:

- (1) $D_p(1-k) = I(k - \frac{1}{2}) \circ D_p(k) \circ I(\frac{1}{2} - k)$
- (2) ${}^tD_p(k) = (-1)^{\deg(p)} D_p(1-k).$

Proof. For both (1) and (2) it is sufficient to show that the statement holds in the case $p = p_1^* = \sum_i \zeta_i^2$ (the G -invariant of degree two), since one has (see [17], [26])

$$D_p = (2^m m!)^{-1} (\text{ad } D_{p_1^*})^m (p^*) \quad \forall p \in \mathbf{C}[\mathfrak{h}^*]^G. \tag{3.3}$$

In this special case, both (1) and (2) are easy consequences of the formula

$$I(\frac{1}{2}k) \circ D_{p_1^*}(k) \circ I(-\frac{1}{2}k) = \sum_{i=1}^n \partial_{\zeta_i}^2 + \sum_{\alpha \in \mathbf{R}_+} \frac{1}{2} k_\alpha (1 - k_\alpha) (\alpha^*)^{-2}.$$

The proof of this formula is similar to the proof of ([19], Proposition 2.2). \square

COROLLARY 3.10

- (1) $G_R^-(k) = I(\frac{3}{2} - k) \circ G_R^+(1-k) \circ I(k - \frac{1}{2})$
- (2) ${}^tG_R^+(k) = (-1)^{|R|+1} G_R^+(-k).$

Proof. Let us denote the left-hand side of identity (1) by $\tilde{G}_R^-(k)$. Clearly $\tilde{G}_R^-(k)$ satisfies the same shift relation (2.2) as $G_R^-(k)$. Moreover $\tilde{G}_R^-(k)$ and $G_R^-(k)$ have the same homogeneous degree, and the same highest order part. Using $\mathfrak{sl}(2)$ theory (as in [17], [26]) and formula (3.3), this implies that $G_R^+(k-1) \circ G_R^-(k) = G_R^+(k-1) \circ \tilde{G}_R^-(k)$, and since there are no zero divisors in \mathbf{A} this completes the proof of (1). Identity (2) is proved by a similar argument. \square

COROLLARY 3.11. *Let α be the $\mathbf{C}[\mathfrak{h}^*] \otimes \mathbf{C}[K]$ automorphism corresponding to the map $(\lambda, k) \rightarrow (-\lambda, 1-k)$, and let α also denote the extension of this automorphism to \mathbf{D}^{reg} . Define \tilde{M}^{reg} as the \mathbf{D}^{reg} -module obtained from M^{reg} by composing the multiplication with α (so $D \cdot m \stackrel{\text{def}}{=} \alpha(D)m$ in \tilde{M}^{reg}). Then $\tilde{M}^{\text{reg}} \simeq (M^{\text{reg}})^*$ (the dual connection).*

Proof. We compute $(\text{Ext}_{\mathbf{D}^{\text{reg}}}^n(M^{\text{reg}}, \mathbf{D}^{\text{reg}}))^\circ$ using two different free \mathbf{D}^{reg} -resolutions for M^{reg} (where M° denotes the left \mathbf{D}^{reg} -module obtained from the right \mathbf{D}^{reg} -module M by composing the multiplication of M with the anti-automorphism $D \rightarrow {}^tD$ of \mathbf{D}^{reg}). Consider the operator $\delta_i: \mathbf{D}^{\text{reg}} \rightarrow \mathbf{D}^{\text{reg}}$ given by $D \rightarrow D(D_{p_i} - p_i)$. These operators commute, and from Proposition 3.2 one easily

obtains that the δ_i form a regular sequence (here we use the algebraic independence of the $(D_{p_i} - p_i)$). Hence we may compute $(\text{Ext}_{\mathbf{D}^{\text{reg}}}^n(M^{\text{reg}}, \mathbf{D}^{\text{reg}}))^\circ$ by means of the Koszul resolution for these operators δ_i . If we use Proposition 3.9(2) in addition we obtain that $(\text{Ext}_{\mathbf{D}^{\text{reg}}}^n(M^{\text{reg}}, \mathbf{D}^{\text{reg}}))^\circ \simeq \tilde{M}^{\text{reg}}$. On the other hand, by Corollary 3.3 we have $M^{\text{reg}} \simeq C^{\text{reg}} \otimes V$, endowed with a connection. Let the \mathbf{D}^{reg} -module structure be given by the formula

$$\frac{\partial}{\partial p_i}(f \otimes v) = \frac{\partial f}{\partial p_i} \otimes v + \sum_k r_k^i \otimes \varepsilon_k^i v$$

(with $r_k^i \in C^{\text{reg}}$, and $\varepsilon_k^i \in \text{End}_{\mathbb{C}}(V)$). Define operators

$$\delta'_i: \mathbf{D}^{\text{reg}} \otimes V \rightarrow \mathbf{D}^{\text{reg}} \otimes V$$

by

$$\delta'_i(a \otimes v) = a \frac{\partial}{\partial p_i} \otimes v - \sum_k ar_k^i \otimes \varepsilon_k^i v.$$

The δ'_i commute and form a regular sequence and

$$M^{\text{reg}} \simeq (\mathbf{D}^{\text{reg}} \otimes V) \Big/ \sum_i \delta'_i(\mathbf{D}^{\text{reg}} \otimes V)$$

as a left \mathbf{D}^{reg} -module. The corresponding resolution gives:

$$(\text{Ext}_{\mathbf{D}^{\text{reg}}}^n(M^{\text{reg}}, \mathbf{D}^{\text{reg}}))^\circ \simeq (M^{\text{reg}})^* \quad \square$$

COROLLARY 3.12. *Suppose that $\{b_i\}_{i=1}^{|G|}$ is a basis for $C^{\text{reg}} \otimes V$. Then there exist elements $\alpha_{ij} \in C^{\text{reg}}$ ($i, j \in \{1, \dots, |G|\}$) such that if $\phi \in \mathcal{L}(\lambda, k)$, $\psi \in \mathcal{L}(-\lambda, 1-k)$, then*

$$\{\phi, \psi\} = \sum_{i,j} \alpha_{ij}(\lambda, k)(b_i \phi)(b_j \psi)$$

(where we consider $C^{\text{reg}} \otimes V \subset \mathbf{D}^{\text{reg}}$ as usual) is independent of $x \in G \setminus \mathfrak{h}^{\text{reg}}$, and such that the resulting pairing $\{\cdot, \cdot\}$ of $\mathcal{L}(\lambda, k)$ with $\mathcal{L}(-\lambda, 1-k)$ is perfect.

Proof. This is a reformulation of Corollary 3.11. □

4. Regular singularities

The object of this section is to prove that $M^{\text{reg}}(\lambda, k)$ has regular singularities at the discriminant $\{I=0\}$.

PROPOSITION 4.1. *Let $k \in K$ and let $S \subset R$ be G -invariant. The left $\mathbf{C}[\mathfrak{h}^*] \otimes \mathbf{A}$ -module morphism*

$$i_S^\pm(k): \mathbf{C}[\mathfrak{h}^*] \otimes \mathbf{A} \rightarrow \mathbf{C}[\mathfrak{h}^*] \otimes \mathbf{A}$$

$$D \rightarrow D \cdot G_S^\mp(k \pm 1_S)$$

induces a morphism of left $\mathbf{C}[\mathfrak{h}^] \otimes \mathbf{A}$ -modules*

$$i_S^\pm(k): M(k) \rightarrow M(k \pm 1_S).$$

If $\lambda \in (\mathfrak{h}^)^{\text{reg}}$ then i_S^\pm specializes to an isomorphism of \mathbf{A} -modules $i_S^\pm(\lambda, k): M(\lambda, k) \xrightarrow{\sim} M(\lambda, k \pm 1_S)$.*

Proof. This is a simple reformulation of the properties of the shift operators G_S^\pm . The fact that $i_S^\pm(\lambda, k)$ is an isomorphism is a consequence of the identity (in $M(\lambda, k)$): $i_S^+(\lambda, k - 1_S) \circ i_S^-(\lambda, k)(\bar{D}) = \overline{D \cdot G_S^+(k - 1_S) \cdot G_S^-(k)} = \overline{D \cdot D_{p_S^+}(k)} = p_S^2(\lambda)\bar{D}$. □

COROLLARY 4.2. *If $(\lambda, k) \in (\mathfrak{h}^*)^{\text{reg}} \times \{k \mid k_\alpha \in \mathbf{Z} \forall \alpha\}$ then $M^{\text{reg}}(\lambda, k)$ has regular singularities (R.S.) at $\{I=0\}$.*

Proof. Using Proposition 4.1 it is enough to show that $M^{\text{reg}}(\lambda, 0)$ has R.S. at $\{I=0\}$. This is clearly equivalent with showing that $\pi^*M^{\text{reg}}(\lambda, 0)$ has R.S. at $\{p_R=0\}$ (where $\pi: \mathfrak{h}^{\text{reg}} \rightarrow G \backslash \mathfrak{h}^{\text{reg}}$ is the canonical projection). But

$$\begin{aligned} \pi^*M^{\text{reg}}(\lambda, 0) &= \mathbf{C}[\mathfrak{h}^{\text{reg}}] \otimes_{\mathbf{C}[\mathfrak{h}^{\text{reg}}]^G} M^{\text{reg}}(\lambda, 0) = \mathbf{A}[\mathfrak{h}^{\text{reg}}] \Big/ \sum_{p \in \mathbf{C}[\mathfrak{h}^*]^G} \mathbf{A}[\mathfrak{h}^{\text{reg}}](\partial_p - p(\lambda)) \\ &= \mathbf{C}[\mathfrak{h}^{\text{reg}}] \otimes_{\mathbf{C}[\mathfrak{h}]} \left(\mathbf{A}[\mathfrak{h}] \Big/ \sum_{p \in \mathbf{C}[\mathfrak{h}^*]^G} \mathbf{A}[\mathfrak{h}](\partial_p - p(\lambda)) \right) \end{aligned}$$

and by the argument of the proof of Proposition 3.2, this equals (as $\mathbf{C}[\mathfrak{h}^{\text{reg}}]$ -module) $\mathbf{C}[\mathfrak{h}^{\text{reg}}] \otimes_{\mathbf{C}[\mathfrak{h}]} (\mathbf{C}[\mathfrak{h}] \otimes_{\mathbf{C}} \mathcal{H})$. In other words: $\pi^*M^{\text{reg}}(\lambda)$ is the restriction to $\mathfrak{h}^{\text{reg}}$ of a connection on $\mathbf{C}[\mathfrak{h}] \otimes_{\mathbf{C}} \mathcal{H}$ (so without any singularities in \mathfrak{h}). □

LEMMA 4.3. *Let \mathcal{O} be the ring of germs of holomorphic functions at the origin of \mathbf{C} , and K its quotient field. Let ∇ be a germ of a meromorphic connection on $K^n = V$, and suppose that ∇ depends polynomially on a parameter $\alpha \in \mathbf{C}$. Then the set $\{\alpha \in \mathbf{C} \mid \nabla(\alpha) \text{ has R.S.}\}$ is Zariski closed.*

Proof. Denote by $v: K^* \rightarrow \mathbf{Z}$ the valuation determined by \mathcal{O} . If $f \in K \otimes \mathbf{C}[\alpha]$, then the map $\mathbf{C} \rightarrow \mathbf{Z}, \alpha \rightarrow v(f(\alpha))$ is lower half continuous (with respect to the Zariski topology on \mathbf{C}) since $f = \sum_{n \geq N} a_n z^n, a_n \in \mathbf{C}[\alpha]$, so that

$$\{\alpha \mid v(f(\alpha)) \geq k\} = \{\alpha \mid a_N(\alpha) = \dots = a_{k-1}(\alpha) = 0\}$$

is closed ($\forall k \in \mathbf{Z}$). Choose, for some $\alpha_0 \in \mathbf{C}$, a cyclic vector $\sigma \in V^*$ for $\nabla(\alpha_0)$. Clearly σ will be cyclic for $\nabla(\alpha)$ if α is in the nonempty open set

$$\Omega = \{\alpha \mid \sigma \wedge \nabla(\alpha)\sigma \wedge \dots \wedge \nabla^{n-1}(\alpha)\sigma \neq 0\}.$$

Choose $a_i \in \mathcal{O} \otimes \mathbf{C}[\alpha]$ such that $\sum_{i=0}^n a_i \nabla^i \sigma = 0$, then the Fuchsian condition for R.S. says: if $\alpha \in \Omega$, then $\nabla(\alpha)$ has R.S. if and only if

$$v(a_i(\alpha)) \geq v(a_n(\alpha)) + i - n \quad \forall i \in \{0, 1, \dots, n\}.$$

By shrinking Ω a bit we may assume that $v(a_n(\alpha))$ is constant on Ω , and thus we obtain that $\{\alpha \in \Omega \mid \nabla(\alpha) \text{ has R.S.}\}$ is closed in Ω . Hence $\{\alpha \mid \nabla(\alpha) \text{ has R.S.}\}$ is either open or closed in \mathbf{C} . Now suppose that $\{\alpha \mid \nabla(\alpha) \text{ has R.S.}\}$ is nonempty and open. We will show that in this case $\{\alpha \mid \nabla(\alpha) \text{ has R.S.}\} = \mathbf{C}$. Namely assume $\exists \alpha_0 \in \mathbf{C}$, $\nabla(\alpha_0)$ is not R.S. According to a result of Katz [21] this is equivalent with the following: if $A \in K^{n \times n} \otimes \mathbf{C}[\alpha]$ is the matrix of ∇ with respect to the derivation $\theta = z(d/dz)$, then the set $\{v(A^n(\alpha_0))\}_{n \in \mathbf{Z}_+}$ is not bounded from below. Put $K_n = \{\alpha \mid v(A^n(\alpha)) > v(A^n(\alpha_0))\}$, then K_n is finite. Hence $\bigcup_{n \in \mathbf{Z}_+} K_n$ is at most countable, while $\bigcap_{n \in \mathbf{Z}_+} K_n^c$ consists of points $\alpha \in \mathbf{C}$ for which $\nabla(\alpha)$ is not R.S. This contradicts the assertion that there are only finitely many $\alpha \in \mathbf{C}$ such that $\nabla(\alpha)$ is not R.S. □

REMARK 4.4. Along the same lines one can prove (more generally) that if the connection matrix has coefficients in $K \otimes \mathbf{C}[S]$ (S affine variety, $\mathbf{C}[S]$ the ring of regular functions on S) then $\{s \in S \mid \nabla(s) \text{ has R.S.}\}$ is Zariski closed in S . □

COROLLARY 4.5. For all $(\lambda, k) \in \mathfrak{h}^* \times K$, $M^{\text{reg}}(\lambda, k)$ has R.S. at $\{I=0\}$.

Proof. It is enough to prove that for any map $i: D^\times \rightarrow (G \backslash \mathfrak{h}^{\text{reg}})$ (D^\times the punctured disk) the connection $i^*(M^{\text{reg}}(\lambda, k))$ has R.S. at $\{0\}$. By Corollary 4.2 this is true for (λ, k) in a certain dense set, hence by Lemma 4.3 we are done. □

REMARK 4.6. $M^{\text{reg}}(\lambda, k)$ has R.S. at infinity $\Leftrightarrow \lambda = 0$. Namely if $\lambda = 0$ then the Euler vector field $\theta = \sum_i x_i (\partial/\partial x_i)$ acts on the solution space $\text{Hom}_{\mathcal{A}}(M_{p_0}, \mathcal{O}_{p_0})(\forall p_0 \in G \backslash \mathfrak{h}^{\text{reg}})$ since $I(0, k)$ has a basis (over \mathbf{C}) of eigenvectors for $\text{ad } \theta$. Thus, since the solution space is finite dimensional there exists a polynomial $\chi \in \mathbf{C}[T]$ such that all (local) solutions of (3.1) (for $\lambda=0$) satisfy $\chi(\theta)f = 0$. This implies the moderate growth of solutions of (3.1) (in case $(\lambda=0)$) towards infinity. For the other implication see Remark 5.7.

5. The monodromy representation

Let G be a finite reflection group with generators r_1, \dots, r_n , satisfying the relations $r_i^2 = 1$ ($\forall i$) and $(r_i r_j)^{m_{ij}} = 1$ ($i \neq j$). Recall the notion of the associated Artin group A_G , which is the group generated by elements $\delta_1, \dots, \delta_n$ satisfying the relations $\delta_i \delta_j \delta_i \dots = \delta_j \delta_i \delta_j \dots$ ($i \neq j$, m_{ij} factors on both sides).

Consider $X_0 \in \mathfrak{a}_+^{\text{reg}}$ as base point for the orbit space $G \backslash \mathfrak{h}^{\text{reg}}$ and let $s_i \in \pi_1(G \backslash \mathfrak{h}^{\text{reg}}, X_0)$ be defined by the loop

$$s_i(t) = (1 - t)X_0 + t(r_i X_0) + \sqrt{-1} \varepsilon(t) \alpha_i$$

where $\varepsilon: [0, 1] \rightarrow [0, 1]$ is continuous, $\varepsilon(0) = \varepsilon(1) = 0$ and $\varepsilon(\frac{1}{2}) > 0$. The next theorem is due to Artin ($G = S_n$) and Brieskorn (general case) (see [4]).

THEOREM 5.1. *The fundamental group $\pi_1(G \backslash \mathfrak{h}^{\text{reg}}, X_0)$ is isomorphic with A_G via $\delta_i \rightarrow s_i$.*

If $q \in K$ we define the Hecke algebra related to G as the complex algebra $H_G(q)$ generated by elements T_i satisfying $(T_i - 1)(T_i - q_{\alpha_i}) = 0$ ($\forall i$) and $T_i T_j T_i \dots = T_j T_i T_j \dots$ ($i \neq j$, m_{ij} factors on both sides). Hence there exists an epimorphism of complex algebras:

$$\begin{aligned} \tau_q: \mathbf{C}A_G &\rightarrow H_G(q) \\ \delta_i &\rightarrow T_i. \end{aligned}$$

LEMMA 5.2. *The monodromy representation of the local system*

$$\text{Hom}_\Lambda(M(\lambda, k), \mathcal{O})(\simeq \mathcal{L}(\lambda, k))$$

on $G \backslash \mathfrak{h}^{\text{reg}}$ with respect to the base point X_0 , $\mu(\lambda, k)$ say, factors through τ_q for $q = -\exp(-2\pi\sqrt{-1}k)$.

$$\begin{array}{ccc} \mathbf{C}\pi_1(G \backslash \mathfrak{h}^{\text{reg}}, X_0) & \xrightarrow{\mu(\lambda, k)} & \text{End}(\mathcal{L}(\lambda, k)) \\ \cong \downarrow & & \uparrow \nu(\lambda, k) \\ \mathbf{C}A_G & \xrightarrow{\tau_q} & H_G(q) \end{array}$$

Proof. We have to show that the relations

$$(\mu(s_i) - 1)(\mu(s_i) - q_i) = 0 \quad (i = 1, \dots, n) \tag{5.1}$$

hold. Choose coordinates (x_1, \dots, x_n) in a neighbourhood of the orthogonal projection of X_0 onto $\{\alpha_i^* = 0\}$ such that

$$x_1 = \frac{1}{\sqrt{2}} \alpha_i^*,$$

and let

$$U = \{(x_1, \dots, x_n) \mid |x_i| < r\}$$

be such that

$$U \cap \{I = 0\} = U \cap \{x_1 = 0\}.$$

Let

$$\bar{U} = \pi(U) \subset (G \setminus \mathfrak{h}),$$

thus \bar{U} is a polydisk with coordinates (y_1, \dots, y_n) where $y_1 = x_1^2$, and $y_i = x_i$ ($i > 1$). Moreover,

$$\bar{U} \cap \{I = 0\} = \bar{U} \cap \{y_1 = 0\}.$$

According to Corollary 4.5 we know that $M^{\text{reg}}(\lambda, k)$ has R.S. at $y_1 = 0$ when restricted to $\bar{U}^{\text{reg}} = \bar{U} \cap \{y_1 \neq 0\}$. Hence the connection $\nabla^*(\lambda, k)$ on $\mathcal{O}_{\bar{U}}[y_1^{-1}] \otimes V^*$ has, with respect to a suitable basis in $\mathcal{O}_{\bar{U}}[y_1^{-1}] \otimes V^*$, the following connection matrix: $M = y_1^{-1} M_1 dy_1$, where M_1 is a constant matrix (see for instance ([6], Remarques 5.5(ii)). Thus the columns of the matrix $\exp(\log(y_1)M_1)$ form a basis for the flat sections. By (the proof of) Corollary 3.6 we therefore see that all sections of $\mathcal{L}(\lambda, k)$ are of the form (on \bar{U}):

$$f = \sum_{\varepsilon \in \mathbb{C}} \sum_{n \in \mathbb{Z}_+} (y_1)^\varepsilon (\log y_1)^n f_{\varepsilon, n}$$

(finite sum, and $f_{\varepsilon, n} \in \mathcal{O}_{\bar{U}} \forall \varepsilon, n$). Hence there is a basis of sections of $\mathcal{L}(\lambda, k)$ of the form (on \bar{U}):

$$f = y_1^\varepsilon \sum_{l \in \mathbb{Z}_+} \sum_{n \in \mathbb{Z}_+} y_1^l (\log y_1)^n f_{l, n} \tag{5.2}$$

(finite sum, and $f_{l, n} \in \mathcal{O}_{\bar{U}}$ such that $f_{l, n}|_{y_1=0} \equiv 0 \Leftrightarrow f_{l, n} \equiv 0$). The second order

operator in the system (3.1) (see Remark 2.10) can be written on \bar{U} as:

$$L = \sum_i \frac{\partial^2}{\partial x_i^2} + 2 \sum_{\alpha \in R_+} k_\alpha (\alpha^*)^{-1} \partial_\alpha = 4y_1^{-1}(\theta^2 + (k_i - \frac{1}{2})\theta + R)$$

with

$$\theta = y_1 \frac{\partial}{\partial y_1} \quad \text{and} \quad R = y_1 P \left(y_1, \dots, y_n; \theta; \frac{\partial}{\partial y_2}, \dots, \frac{\partial}{\partial y_n} \right)$$

where P is an operator with holomorphic coefficients on \bar{U} . So from the equation $(L - (\lambda, \lambda))f = 0$ we obtain the following conclusion when we insert the expansion (5.2): if $(k_i - \frac{1}{2}) \notin \mathbf{Z}$, then logarithmic terms never occur in (5.2) (for any section f of $\mathcal{L}(\lambda, k)$), and the values for ε that can occur are $\varepsilon = 0$ or $\varepsilon = \frac{1}{2} - k_i$. (Details are left to the reader.) Hence in the case $(k_\alpha - \frac{1}{2}) \notin \mathbf{Z} (\forall \alpha \in R)$ we obtain the relation (5.1). Finally we use Corollary 3.8 in order to see that this implies the general case. □

DEFINITION 5.3. Let $q = -\exp(-2\pi\sqrt{-1}k)$, and put $K^s = \{k \in K \mid H_G(q) \text{ is a semisimple algebra}\}$.

REMARK 5.4. If we write $\Sigma = K \setminus K^s$, then $\Sigma \subset K$ is an analytic subset of codimension 1.

Proof. It is well known that (see e.g. [2], Ch. IX, §2, Ex. 1) $\Sigma = \{k \in K \mid \det(\text{Trace}(T_g T_{g^{-1}})) = 0\}$ (where the trace refers to the trace map on $\text{End}(H_G(q))$). Also note that $\Sigma \neq \emptyset$. □

COROLLARY 5.5. Let $v(\lambda, k): H_G(q) \rightarrow \text{End}(\mathcal{L}_{x_c}(\lambda, k))$ be the representation so that $\mu(\lambda, k) = v(\lambda, k) \circ \tau_q$ (cf. Lemma 5.2). If $k \in K^s$ then $v(\lambda, k)$ is equivalent with the regular representation of $H_G(q)$.

Proof. Let \mathcal{R} be the ring $\mathcal{O}(\mathfrak{h}^* \times K)$ of entire functions on $\mathfrak{h}^* \times K$, Q be its quotient field, and let \mathfrak{H} be the \mathcal{R} -algebra with \mathcal{R} -basis $T_g (g \in G)$ satisfying the relations: $\forall g \in G, s_i \in G$ simple reflection: $(q = -\exp(-2\pi\sqrt{-1}k)$ as always)

$$T_{s_i} T_g = \begin{cases} T_{s_i g} & \text{if } l(s_i g) > l(g) \\ -q_i T_{s_i g} + (1 + q_i) T_g & \text{else.} \end{cases}$$

Clearly the specialization $\mathfrak{H}(0, 0) = \mathfrak{H} \otimes_{\mathcal{R}} \mathbf{C}$ (and \mathbf{C} the \mathcal{R} -module $f.z := f(0, 0)z$ ($\forall f \in \mathcal{R}, z \in \mathbf{C}$)) is isomorphic to the group algebra $\mathbf{C}G$, which is semisimple. Hence by ([2], Ex. 26, Ch. IV, §2) we know that $\mathfrak{H}_Q = \mathfrak{H} \otimes_{\mathcal{R}} Q$ is separable over

Q. By Corollary 3.8, Theorem 5.1 and Lemma 5.2 we have

$$\begin{array}{ccc}
 \mathcal{R}A_G & \xrightarrow{\mu} & \mathcal{R}^{|G| \times |G|} \\
 \downarrow \tau & & \uparrow \nu \\
 \mathfrak{H} & \xrightarrow{\text{id}} & \mathfrak{H}.
 \end{array}$$

Hence ν induces a representation, also denoted by ν , of the semisimple algebra $\mathfrak{H}_{\bar{Q}}$ on the space $\bar{Q}^{|G|}$ (where \bar{Q} is an algebraic closure of Q). It is easy to see that for each character of $\mathfrak{H}_{\bar{Q}}$, ψ say, one has: $\psi(T_g) \in \mathcal{R}^*$ ($\forall g \in G$), where \mathcal{R}^* is the integral closure of \mathcal{R} in \bar{Q} . Fix an extension $\phi^*: \mathcal{R}^* \rightarrow \mathbf{C}$ of the homomorphism $\phi: \mathcal{R} \rightarrow \mathbf{C}$ given by $f \rightarrow f(0, 0)$. From the proof of the deformation lemma in ([32], Lemma 8.5) we see that (see also the proof of Proposition 7.1 in [5]) the application $\psi \rightarrow \psi_{\phi^*}$ (ψ_{ϕ^*} defined by $\psi_{\phi^*}(T_g) = \phi^*(\psi(T_g))$) gives a bijection between the set of (irreducible) characters of $\mathfrak{H}_{\bar{Q}}$ and those of $\mathbf{C}G$. So we just have to show that ν is the regular representation of $\mathfrak{H}_{\bar{Q}}$ (since the same deformation lemma can then be applied for any specialization $\mathfrak{H} \rightarrow \mathfrak{H}(\lambda, k)$, $\nu \rightarrow \nu(\lambda, k)$ if $k \in K^s$ (since $\mathfrak{H}(\lambda, k) (\simeq H_q(G))$ is semisimple in that case) and for this it is enough to show that $\nu(0, 0)$ is the left regular representation of $\mathbf{C}G$. But this is clear since for $(\lambda, k) = (0, 0)$, the solution space of (3.1) equals the space of harmonic polynomials on \mathfrak{h} , and the monodromy specializes to the ordinary G -action on this space. □

PROPOSITION 5.6. *Let (as usual) $\mathcal{O}(X)$ denote the space of holomorphic functions on X .*

- (1) $\exists f \in \mathcal{O}(\mathfrak{h}^* \times K \times G \setminus \mathfrak{h})$, $f \neq 0$, such that the specialization $f(\lambda, k)$ satisfies (3.1) ($\forall (\lambda, k) \in \mathfrak{h}^* \times K$).
- (2) $\forall (\lambda, k) \in \mathfrak{h}^* \times K$: $\dim_{\mathbf{C}}\{\phi \in \mathcal{O}(\mathfrak{h})^G \mid \phi \text{ satisfies (3.1)}\} \geq 1$, and equality holds if $k \in K^s$ (see Definition 5.3).
- (3) $\forall (\lambda, k) \in \mathfrak{h}^* \times K^s$, if $\phi \in \mathcal{O}(\mathfrak{h})^G$ satisfies (3.1) then $\phi(0) \neq 0$ if $\phi \neq 0$.

Proof. (1) Recall the basis of solutions $\phi_1, \dots, \phi_{|G|}$ of (3.1). The equations to solve in order to obtain a solution with trivial monodromy are:

$$(\mu(s_i) - 1) \left(\sum_{k=1}^{|G|} a_k \phi_k \right) = 0 \quad i = 1, \dots, n.$$

By Corollary 3.8, this is a set of homogeneous, linear equations in $\{a_k\}_{k=1}^{|G|}$ with coefficients in $\mathcal{O}(\mathfrak{h}^* \times K)$, and by Corollary 5.5 it has rank $|G| - 1$ when specialized generically (since the left regular representation contains the trivial

representation once). Hence there exists a nontrivial solution with a_k meromorphic ($\forall k$), and by homogeneity of the equations we may even assume that $a_k \in \mathcal{O}(\mathfrak{h}^* \times K)$ ($\forall k$). Next we show that if $(\frac{1}{2} - k_\alpha) \notin \mathbf{Z}$ then a solution of (3.1) (for $(\lambda, k) \in \mathfrak{h}^* \times K$ fixed) which has trivial monodromy will automatically extend holomorphically to $G \setminus \mathfrak{h}$ (and not just to $G \setminus \mathfrak{h}^{\text{reg}}$). In fact, in the neighbourhood \bar{U} of a point in the smooth part of $\{I=0\}$ it follows from the description of solutions as in the proof of Lemma 5.2 that a solution which has trivial monodromy extends (if $\frac{1}{2} - k_\alpha \notin \mathbf{Z}, \forall \alpha \in R$) holomorphically to \bar{U} . Thus the singular set of the solution is contained in the singular part of $\{I=0\}$, which is an analytic set of codimension ≥ 2 . So by Hartog's theorem the singular set of the solution is empty. Finally we consider $\sum_{k=1}^{|G|} a_k \phi_k$ as function on $\mathfrak{h}^* \times K \times G \setminus \mathfrak{h}$. Using Corollary 3.7 and the remarks above we see that this function is holomorphic outside the set $\mathfrak{h}^* \times \{k \mid \frac{1}{2} - k_\alpha \in \mathbf{Z} \forall \alpha\} \times \{I=0\}$, which is again an analytic set of codimension ≥ 2 . Hence $f = \sum_{k=1}^{|G|} a_k \phi_k (\neq 0)$ has the required properties.

Let us now consider (2). To prove that $\dim_{\mathbf{C}}\{\phi \in \mathcal{O}(G \setminus \mathfrak{h}) \mid \phi \text{ satisfies (3.1)}\} \geq 1, \forall (\lambda_0, k_0) \in \mathfrak{h}^* \times K$ fixed, we use (1). If $f(\lambda_0, k_0, \cdot) \neq 0$ (f as in (1)) then there is nothing to prove, so assume that $f(\lambda_0, k_0, \cdot) \equiv 0$. It is easy to see that we can choose an embedding $i: D \hookrightarrow \mathfrak{h}^* \times K$ (where $D = \{\alpha \in \mathbf{C} \mid |\alpha| < 1\}$) such that $i(0) = (\lambda_0, k_0)$, and $f|_{(i(D) \times G \setminus \mathfrak{h})}$ is not identically zero. Let $l \in \mathbf{N}$ be the largest number such that if g is the function $g(\alpha; x) = f(i(\alpha); x) \cdot \alpha^{-l}$ then $g \in \mathcal{O}(D \times G \setminus \mathfrak{h})$. Clearly, $\phi(x) = g(0; x)$ is a nonzero solution of (3.1) (for the parameters (λ_0, k_0)). Finally note that if $k \in K^s$ then $\dim_{\mathbf{C}}\{\phi \in \mathcal{O}(G \setminus \mathfrak{h}) \mid \phi \text{ satisfies (3.1)}\} \leq 1$ because of Corollary 5.5. Hence we have proven (2) completely.

Finally consider (3). Suppose that for some $(\lambda, k) \in \mathfrak{h}^* \times K^s$ we have a solution $0 \neq \phi \in \mathcal{O}(\mathfrak{h})^G$ of (3.1) with $\phi(0) = 0$. Then the lowest homogeneous part ϕ_0 of ϕ clearly satisfies (3.1) for the parameters $(0, k) \in \mathfrak{h}^* \times K$. But $1 \in \mathbf{C}[\mathfrak{h}]^G$ also satisfies this system of differential equations, contradicting (2). □

REMARK 5.7. In addition to Remark 4.6, we show that if $\lambda \neq 0$ then $M^{\text{reg}}(\lambda, k)$ has irregular singularities at infinity, as a consequence of Proposition 5.6. Namely, by Proposition 5.6(2) there exists a solution $0 \neq \phi \in \mathcal{O}(\mathfrak{h})^G$ of (3.1). Now assume that $M^{\text{reg}}(\lambda, k)$ has R.S. at infinity. Then Corollary 3.6 implies that ϕ is a polynomial on \mathfrak{h} . But if $p \in \mathbf{C}[\mathfrak{h}^*]^G$ of positive homogeneous degree d , then $D_p(k)$ has homogeneous degree $-d$. So the equation

$$(D_p(k) - p(\lambda))\phi = 0$$

implies that $p(\lambda) = 0$ in this situation (by looking at the highest degree part). Since this must hold for all nonconstant homogeneous p (by equation (3.1)), we see that $\lambda = 0$.

6. The Bessel function and the exponential function for G

In this section we make a first study of the holomorphic eigenfunctions on \mathfrak{h} for the Bessel differential operators and for the Dunkl operators and their mutual interdependence. It turns out that it is useful to consider the relationship between these functions and I am indebted to G. J. Heckman for suggesting this idea to me.

DEFINITION 6.1. Let $K_+ = \{k \in K \mid \text{all eigenvalues of the operator } \sum_{i=1}^n \xi_i^* T_{\xi_i}(k) \text{ on the space } \{f \in C[\mathfrak{h}] \mid f(0) = 0\} \text{ have strictly positive real part}\}$. Following ([8], p. 176) we note that K_+ is the interior of a polytope, containing the cone $\{k \in K \mid \text{Re}(k_\alpha) \geq 0 \forall \alpha\}$.

LEMMA 6.2. *If $k \in K_+$ then $\{f \in C[\mathfrak{h}] \mid T_\xi(k)(f) = 0 \forall \xi \in \mathfrak{h}\} = C \cdot 1$ (the constants).*
Proof. Clear by Definition 6.1. □

COROLLARY 6.3. *If $k \in K_+$ and $\lambda \in \mathfrak{h}^*$ arbitrary, then*

$$\dim_{\mathbb{C}}\{f \in \mathcal{O}(\mathfrak{h}) \mid (T_\xi(k) - (\xi, \lambda))f = 0 \forall \xi \in \mathfrak{h}\} \leq 1$$

and if $f \neq 0$ is in this simultaneous eigenspace, then $f(0) \neq 0$.

Proof. Clear by the previous lemma. □

The next lemma is essentially due to Harish-Chandra ([14], [15]). It is the key result for the description of the relationship between eigenfunctions for Bessel differential operators and those for Dunkl operators.

LEMMA 6.4. *There exists a unique rational function*

$$Q \in C(\mathfrak{h}^*) \otimes \mathcal{H} \subset C(\mathfrak{h}^* \times \mathfrak{h}^*)$$

such that $\forall \lambda \in (\mathfrak{h}^)^{\text{reg}}, Q(\lambda, g\lambda) = |G|\delta_{1,g} (\forall g \in G)$. The function Q has the following properties:*

- (1) $(\lambda, \mu) \rightarrow (\prod_{\alpha > 0} \alpha(\lambda))Q(\lambda, \mu)$ is a polynomial in $(\lambda, \mu) \in \mathfrak{h}^* \times \mathfrak{h}^*$.
- (2) Given $\lambda \in \mathfrak{h}^*$, let \mathcal{I}_λ denote the ideal in $C[\mathfrak{h}^*]$ generated by $\{p \in C[\mathfrak{h}^*]^G \mid p(\lambda) = 0\}$. If $\lambda \in (\mathfrak{h}^*)^{\text{reg}}$, and $\xi \in \mathfrak{h} (\subset C[\mathfrak{h}^*])$ then $(\xi - \xi(\lambda))Q(\lambda, \cdot) \in \mathcal{I}_\lambda$.
- (3) $\forall g \in G: Q(g\lambda, g\mu) = Q(\lambda, \mu)$.
- (4) $\sum_{g \in G} Q(\lambda, g\mu) = |G|$.

Proof. We refer the reader to ([33], Ex. 70, Ch. 4) for existence of Q , and property (1). The properties (2) and (3) are clear, so let us prove (4). It is enough to show that $\sum_{g \in G} Q(\lambda, g\mu)$ is independent of μ , since we have $Q(\lambda, g\lambda) = |G|\delta_{1,g}$.

But this follows from the formula

$$\frac{1}{|G|} \sum_{g \in G} f^g = f(0) \quad (\forall f \in \mathcal{H}). \quad \square$$

DEFINITION 6.5. We denote by $T(\lambda, k)$ the operator $T_{Q(\lambda, \cdot)}(k)$ ($\lambda \in (\mathfrak{h}^*)^{\text{reg}}, k \in K$), acting on the space $\mathcal{O}(\mathfrak{h})$ of entire functions on \mathfrak{h} .

COROLLARY 6.6. Let $k \in K$, $\lambda \in (\mathfrak{h}^*)^{\text{reg}}$, and let $\phi \in \mathcal{O}(\mathfrak{h})^G$ satisfy the Bessel differential equation (3.1). Then $\psi = T(\lambda, k)\phi \in \mathcal{O}(\mathfrak{h})$ satisfies $(T_\xi(k) - (\xi, \lambda))\psi = 0 \quad \forall \xi \in \mathfrak{h}$, and $\psi(0) = \phi(0)$.

Proof. Clearly $(T_\xi(k) - (\xi, \lambda))\psi = 0 \quad \forall \xi \in \mathfrak{h}$, by Lemma 6.4(2). In order to see that $\psi(0) = \phi(0)$, note that

$$\psi(0) = \frac{1}{|G|} \sum_{g \in G} \psi^g(0) = \frac{1}{|G|} \left(\sum_{g \in G} T^g(\lambda, k)\phi \right)(0) = \phi(0),$$

by Lemma 6.4(4). □

PROPOSITION 6.7. There exists a unique meromorphic function Exp_G on $\mathfrak{h}^* \times K \times \mathfrak{h}$ characterized by:

- (1) $\text{Exp}_G(\lambda, k; 0) = 1 \quad \forall (\lambda, k) \in \mathfrak{h}^* \times K$.
- (2) $(T_\xi - (\xi, \lambda)) \text{Exp}_G(\lambda, k) = 0 \quad \forall (\lambda, k) \in \mathfrak{h}^* \times K, \xi \in \mathfrak{h}$.

Moreover, this function satisfies the following properties:

- (3) Exp_G is holomorphic on $\mathfrak{h}^* \times K_+ \times \mathfrak{h}$.
- (4) $\text{Exp}_G(g\lambda, k; gx) = \text{Exp}_G(\lambda, k; x)$.

Proof. Uniqueness is clear by Corollary 6.3, so it suffices to construct a function that satisfies (1) and (2). Let f be the function constructed in Proposition 5.6(1). Put $g(\lambda, k; x) = T(\lambda, k)f(\lambda, k; x)$. This is a meromorphic function with singular set contained in $\{\prod_{\alpha > 0} \alpha(\lambda) = 0\} \times K \times \mathfrak{h}$ (see Lemma 6.4(1)), and by Corollary 6.6 this function satisfies (2). By Proposition 5.6(3) and Corollary 6.5 we also see that $g(\lambda, k; 0) \neq 0$, and thus we can define

$$\text{Exp}_G(\lambda, k; x) = \frac{g(\lambda, k; x)}{g(\lambda, k; 0)}.$$

Property (4) is a direct consequence of Lemma 6.4(3), so let us finally prove (3). By construction, the singular set of Exp is of the form $Z \times \mathfrak{h}$ for some analytic set of codimension 1 in $\mathfrak{h}^* \times K$. Let $i: D \hookrightarrow \mathfrak{h}^* \times K$ be any embedding of the disk $D = \{\alpha \in \mathbb{C} \mid |\alpha| < 1\}$ such that $i(D^\times) = i(D - \{0\}) \subset Z^c$. If the function

$h(\alpha; x) = \text{Exp}_G(i(\alpha); x)$ on $D \times \mathfrak{h}$ has a pole at $\{\alpha=0\} \times \mathfrak{h}$, let $l \in \mathbf{N}$ be the smallest integer such that $\alpha^l h(\alpha; x)$ extends holomorphically to $D \times \mathfrak{h}$. Clearly $\phi(x) = (\alpha^l h(\alpha; x))|_{(\alpha=0)}$ is a nonzero solution of (put $\alpha(0) = (\lambda_0, k_0)$) the system of equations $(T_\xi(k_0) - (\xi, \lambda_0))\psi = 0 \ \forall \xi \in \mathfrak{h}$, and $\psi(0) = 0$. By Corollary 6.3 this implies $k_0 \notin K_+$ and the conclusion is that $Z \subset \mathfrak{h}^* \times (K_+)^c$. \square

PROPOSITION 6.8. *The meromorphic function $J = J_G$ defined by the formula*

$$J_G(\lambda, k; x) = \frac{1}{|G|} \sum_{g \in G} \text{Exp}(\lambda, k; gx) = \frac{1}{|G|} \sum_{g \in G} \text{Exp}_G(g\lambda, k; x)$$

has the following properties:

- (1) $J(\lambda, k; 0) = 1$.
- (2) $J(g\lambda, k; x) = J(\lambda, k; gx) = J(\lambda, k; x) \ \forall g \in G$.
- (3) If $(\lambda, k) \times \mathfrak{h}$ is not in the singular set of J , then $J(\lambda, k) \in \mathcal{O}(\mathfrak{h})^G$ satisfies (3.1).
- (4) $\text{Exp}_G(\lambda, k; x) = T(\lambda, k)J(\lambda, k; x)$.

Proof. Trivial consequences of the previous results in this section. \square

DEFINITION 6.9. J_G is called the Bessel function associated with G .

COROLLARY 6.10. *The singular set of both Exp and J is contained in $\mathfrak{h}^* \times (\Sigma \cap (K_+)^c) \times \mathfrak{h}$ (where $\Sigma = K - K^s$, see Remark 5.4).*

Proof. From the definition of J , Proposition 6.8(3), and Lemma 6.4(1) it follows that the singular sets of Exp and J coincide. Moreover, this set is of the form $Z \times \mathfrak{h}$ for some analytic set $Z \subset \mathfrak{h}^* \times K$. When we apply the argument as in the proof of Proposition 6.7(2), but with J instead of Exp we obtain the result that if $(\lambda_0, k_0) \in Z$ is a generic point, then there exists a nonzero solution ϕ of (3.1) with $\phi(0) = 0$. By Proposition 5.6 this implies that $k_0 \in \Sigma$, and hence that $Z \subset \mathfrak{h}^* \times \Sigma \times \mathfrak{h}$. On the other hand, by Proposition 6.8 we see that $Z \subset \mathfrak{h}^* \times (K_+)^c \times \mathfrak{h}$. \square

EXAMPLE 6.11 ($G = \mathbf{Z}/2\mathbf{Z}$). Let $G = \{\pm 1\}$ act on \mathbf{C} by $(\pm 1)x = \pm x$. Identify the complex number $\lambda \in \mathbf{C}$ with the linear functional $\alpha \rightarrow \lambda\alpha$ on \mathbf{C} , and denote by J_G the Bessel function associated with G . The relation between J_G and the classical Bessel function J_ν is given by:

$$\begin{aligned} J_G(\lambda, k; x) &= (\tfrac{1}{2}\lambda x)^{1/2-k} \Gamma(\tfrac{1}{2}+k) J_{k-1/2}(\sqrt{-1}\lambda x) \\ &= \Gamma(k+\tfrac{1}{2}) \sum_{l=0}^{\infty} \frac{(\tfrac{1}{2}\lambda x)^{2l}}{l! \Gamma(\tfrac{1}{2}+k+l)} \end{aligned}$$

It is not hard to see that in this case:

$$\begin{aligned}
 T(\lambda, k) &= 1 + \frac{1}{\lambda} T_1(k) \\
 &= 1 + \frac{1}{\lambda} \frac{\partial}{\partial x} + \sqrt{2} \frac{k}{\lambda} \Delta
 \end{aligned}$$

where $(\Delta f)(x) = (f(x) - f(-x))/x$, and thus $\Delta(J_G) \equiv 0$. Therefore:

$$\text{Exp}_G(\lambda, k; x) = J_G(\lambda, k; x) + \frac{1}{\lambda} \frac{d}{dx} J_G(\lambda, k; x).$$

REMARK 6.12. Just as in the case of hypergeometric functions associated with root systems, the Bessel function J_G sometimes occurs “in nature” as spherical function. More precisely, let \mathcal{G} be a noncompact real semisimple group with finite center, and $\mathcal{K} \subset \mathcal{G}$ a maximal compact subgroup. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the corresponding Cartan decomposition, and $\mathfrak{a} \subset \mathfrak{p}$ a maximal abelian subspace. Denote by $\Sigma \subset \mathfrak{a}^*$ the restricted root system, and put $m_\alpha = \dim_{\mathbf{R}}(\mathfrak{g}_\alpha)$ ($\forall \alpha \in \mathfrak{a}^*$). Identify \mathfrak{a} and \mathfrak{a}^* via the Killing form, and let $R \subset \mathfrak{a}$ be the normalized root system associated with the Weyl group W acting on \mathfrak{a} . Define $k_\beta = \frac{1}{4} \sum_{\alpha \in R} m_\alpha$ for all normalized roots $\beta \in R$. Then for all $\lambda \in \mathfrak{a}_\mathbf{C}^* = \mathfrak{h}^*$ we have $J_W(\lambda, k) = \phi_\lambda|_{\mathfrak{a}}$, the spherical function for the Euclidean symmetric space $\mathfrak{p} = \mathcal{K} \times \mathfrak{p}/\mathcal{K}$, restricted to \mathfrak{a} . This identity follows by taking radial parts of K -invariant constant coefficient differential operators on \mathfrak{p} (see [20], Ch. II, §3).

Note that, however, such an interpretation of J_G can only be given if G is a Weyl group. Even in that case, the set of multiplicities for which J_G is a spherical function for some Euclidean symmetric space is discrete (and in most cases even finite). In the case of the exponential function $\text{Exp}(\lambda, k; \alpha)$ I do not know of any direct interpretation of this sort (except for the trivial case $k=0$).

It is strongly suggested by these interpretations of J_G that if $\forall \alpha: k_\alpha \in \mathbf{R}_{\geq 0}$, one has the following analogy of the spherical transform on \mathfrak{p} : let $f \in C_c^\infty(\mathfrak{a})$ and define

$$\forall \lambda \in \mathfrak{h}^*: \hat{f}(\lambda) = \int_{\mathfrak{a}} f(x) J_G(-\lambda, k; x) \prod_{\alpha \in R_+} |\alpha^*(x)|^{2k_\alpha} dx.$$

Then the following inversion formula should hold:

$$\forall x \in \mathfrak{a}: f(x) = \int_{i\mathfrak{a}^*} \hat{f}(\lambda) J_G(\lambda, k; x) \prod_{\alpha \in R_+} |\alpha(\lambda)|^{2k_\alpha} d\lambda \tag{6.1}$$

(here dx and $d\lambda$ are suitably normalized Lebesgue measures) (for a proof of this formula in the case $\mathcal{X} \times \mathfrak{p}/\mathcal{X}$: see [20], Ch. IV, §9). For the exponential function $\text{Exp}_G(\lambda, k; x)$ one expects similarly: let $f \in C_c^\infty(\mathfrak{a})$ and define

$$\forall \lambda \in \mathfrak{h}^*: \tilde{f}(\lambda) = \int_{\mathfrak{a}} f(x) \text{Exp}_G(-\lambda, k; x) \prod_{\alpha \in R_+} |\alpha^*(x)|^{2k_\alpha} dx.$$

Then one should have:

$$\forall x \in \mathfrak{a}: f(x) = \int_{i\mathfrak{a}^*} \tilde{f}(\lambda) \text{Exp}_G(\lambda, k; x) \prod_{\alpha \in R_+} |\alpha(\lambda)|^{2k_\alpha} d\lambda \tag{6.2}$$

(where we assume $k_\alpha \geq 0 \forall \alpha$). Clearly, (6.1) follows from (6.2) by taking the average over G . Recently these important formulas were proven by Dunkl ([12]) (on some dense subspace of $L^2(\mathfrak{a}, \prod |\alpha^*(x)|^{2k_\alpha} dx)$ other than $C_c^\infty(\mathfrak{a})$).

REMARK 6.13. In the paper [11] Dunkl also obtains the function $\text{Exp}_G(\lambda, k; x)$ (denoted by $K(x, \lambda)$ in his notation), by completely different methods, and he derives many interesting properties for this kernel. It seems however that our methods using local analysis of differential operators and monodromy are best suited for the purpose of describing the singular behaviour of Exp_G as a function of the multiplicity parameter $k \in K$. We will also gain some insight into the meaning of these singularities (see Section 9).

7. Exponents at the origin

In this section we will study the singular behaviour of solutions of the Bessel differential equations near the origin (which is the most singular point for these equations). Our method is this. By the results of Section 5 we know that the monodromy representation is (generically) equivalent with the regular representation of the Hecke algebra. Therefore we can easily calculate the action of the center of the fundamental group $\pi_1(G \backslash \mathfrak{h}^{\text{reg}}) (\simeq A_G)$ on the space of local solutions of the Bessel equations (I am grateful to Prof. G. Lusztig for explaining this to me). On the other hand we can interpret a generator for the center of A_G topologically as a loop going once (or twice sometimes) around the origin in $G \backslash \mathfrak{h}^{\text{reg}}$ (see [7]). As a consequence we obtain a detailed description of the local exponents of solutions of the Bessel equations towards the origin. We give more details in the description of the exponents in this section than we will use in this paper. Yet we feel that these details are interesting enough to be presented here.

Finally we mention the fact that the results of this section also hold in the context of hypergeometric equations associated with root systems. This can be

used towards the evaluation of solutions of the hypergeometric equations at special points (these results will be given in a forthcoming paper).

Let $p: X \rightarrow \mathfrak{h}$ denote the blowing-up of \mathfrak{h} at $0 \in \mathfrak{h}$, i.e.

$$X = \{(x, l) \in \mathfrak{h} \times \mathbf{P}(\mathfrak{h}) \mid x \in l\}$$

and p projects onto the first factor. If $H \subset \mathfrak{h}$ is a subspace of codimension 1, and $x \in \mathfrak{h} - H$ then we can define an affine coordinate patch by

$$\begin{aligned} \phi_{x,H}: \mathbf{C} \times H &\rightarrow X \\ (t, h) &\rightarrow (t(x+h), \mathbf{C}(x+h)). \end{aligned} \tag{7.1}$$

Clearly we may cover X with patches like these, and $p(\phi_{x,H}(t, h)) = t(x+h)$. We use the following notations:

$$\begin{aligned} X^{\text{reg}} &= p^{-1}(\mathfrak{h}^{\text{reg}}), & Y &= p^{-1}(0), \\ Y^{\text{reg}} &= \{(0, l) \mid \alpha|_l \neq 0, \forall \alpha \in R\}, \\ D_{x,H} &= \phi_{x,H}(\mathbf{C} \times H) & \text{and} & \quad D_{x,H}^{\text{reg}} = D_{x,H} \cap (X^{\text{reg}} \cup Y^{\text{reg}}). \end{aligned}$$

Choose a base point $x_0 \in \alpha_+^{\text{reg}}$, and let $H = x_0^\perp$. Then

$$\phi_{x_0,H}^{-1}(X^{\text{reg}}) = \mathbf{C}^\times \times (H \cap \mathfrak{h}^{\text{reg}}).$$

Let $s \in \pi_1(G \setminus \mathfrak{h}^{\text{reg}}, \bar{x}_0)$ be the loop corresponding to the positive generator of $\pi_1(\mathbf{C}^\times, 1) \simeq \mathbf{Z}$, so that s can be represented by $\{t \rightarrow e^{2\pi\sqrt{-1}t} \cdot x_0\}$ ($t \in [0, 1]$). Recall Theorem 5.1. It is well known that the map $G \rightarrow A_G$ sending the reduced expression $g = r_{i_1} \cdots r_{i_l}$ to $\delta_g = \delta_{i_1} \cdots \delta_{i_l}$ is well defined (meaning that δ_g does not depend on the reduced expression for g) (cf. [22], Remark 3.19).

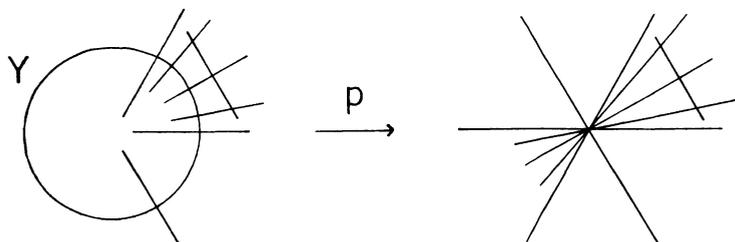


Fig. 1. The map p for $G = S_3$.

PROPOSITION 7.1 (Deligne, see [7]). $s = \delta_{g_0}^2 \in A_G \simeq \pi_1(G \setminus \mathfrak{h}^{\text{reg}}, \bar{x}_0)$, where $g_0 \in G$ is the longest element.

Proof. Assume (for sake of simplicity) that $g_0(x_0) = -x_0$. Let $g_0 = r_{i_1} \cdots r_{i_n}$ be a reduced expression for g_0 . Consider the following path in $\mathfrak{h}^{\text{reg}}$

$$\{r_{i_n} \cdots r_{i_2}(s_{i_1})\} \circ \cdots \circ \{r_{i_n} r_{i_{n-1}}(s_{i_{n-2}})\} \circ \{r_{i_n}(s_{i_{n-1}})\} \circ \{s_{i_n}\} = s^*$$

(from x_0 to $-x_0$). We claim that s^* is homotopic with the path

$$\{t \rightarrow e^{\pi\sqrt{-1}t} \cdot x_0\}_{t \in [0,1]} \text{ in } \mathfrak{h}^{\text{reg}}.$$

To see this, note that $r_{i_n} \cdots r_{i_i}(s_{i_{i-1}})$ is homotopic with (in $\mathfrak{h}^{\text{reg}}$) the path $s_{i_{i-1}}^*$ defined by ($t \in [0, 1]$):

$$t \rightarrow (1 - t)r_{i_n} \cdots r_{i_i}x_0 + tr_{i_n} \cdots r_{i_i}r_{i_{i-1}}x_0 + \sqrt{-1}(\sin \pi t)x_0$$

since $r_{i_n} \cdots r_{i_i}(\alpha_{i_{i-1}}) > 0$ ($\forall i$). So

$$\begin{aligned} s^* &\sim s_1^* \cdots s_n^* \subset (\mathfrak{a} + \sqrt{-1}\mathbf{R}x_0) \cap \mathfrak{h}^{\text{reg}} = \Sigma \\ &= (\mathfrak{a} + \sqrt{-1}\mathbf{R}x_0) \setminus \{x \in \mathfrak{a} \mid \alpha^*(x) = 0 \text{ for some } \alpha \in R\}. \end{aligned}$$

The path $\{t \rightarrow e^{\pi\sqrt{-1}t} \cdot x_0\}_{t \in [0,1]}$ also lies in this $(n + 1)$ dimensional real subspace of \mathfrak{h} , and is obviously homotopic with s^* in Σ since they both lie in the halfspace $\mathfrak{a} + \sqrt{-1}\mathbf{R}_+x_0$. This implies the result. \square

PROPOSITION 7.2 (Deligne, see [7]). $\delta_{g_0}^2 \in Z(A_G)$, the center of A_G .

Proof. If $r_{i_1} \cdots r_{i_i}$ is minimal and $\alpha_{i_{i+1}}$ is simple and such that $r_{i_1} \cdots r_{i_i}(\alpha_{i_{i+1}}) > 0$, then $r_{i_1} \cdots r_{i_{i+1}}$ is also minimal. Thus if a minimal expression cannot be extended (from the right) then it represents the longest element. Therefore there exists a minimal expression $g_0 = r_{i_1} \cdots r_{i_n}$ for any choice of r_{i_1} , a simple reflection. Let r_i be a simple reflection, and let $g_0 = r_{i_1} r_{i_2} \cdots r_{i_n}$ be minimal. Put $-g_0(\alpha_i) = \alpha_{\tau(i)}$, then also

$$g_0 = r_{i_2} \cdots r_{i_n} r_{\tau(i)} = r_{\tau(i)} r_{i_n} \cdots r_{i_2} = r_{i_n} \cdots r_{i_2} r_i,$$

and the corresponding expressions hold for $\delta_{g_0} \in A_G$. Hence

$$\begin{aligned} \delta_i \delta_{g_0}^2 &= \delta_i (\delta_{i_2} \cdots \delta_{i_n} \delta_{\tau(i)}) (\delta_{i_n} \cdots \delta_{i_2} \delta_i) \\ &= (\delta_i \delta_{i_2} \cdots \delta_{i_n}) (\delta_{\tau(i)} \delta_{i_n} \cdots \delta_{i_2}) \delta_i = \delta_{g_0}^2 \delta_i \quad \text{for all } i \in \{1, \dots, n\}. \end{aligned} \quad \square$$

REMARK 7.3. The two previous results are due to Deligne ([7]). In this paper Deligne proves a stronger result, namely that $Z(A_G)$ is generated by δ_{g_0} if $g_0 = -\text{id}$, and by $\delta_{g_0}^2$ otherwise (we do not need this stronger result for our purposes).

COROLLARY 7.4. Let $\{\rho_i\}_{i=1}^L$ be a complete set of representatives for the set of irreducible representations of G , and put $d(i) = \dim_{\mathbb{C}}(\rho_i)$. Put $M_{i,\alpha} = \dim_{\mathbb{C}}(\text{Ker}(\rho_i(r_\alpha) + 1))$. With these notations, we have:

- (1) If ρ_i also denotes the representation of $\mathfrak{H}_{\bar{Q}}$ corresponding with the representation ρ_i of G (recall $\mathfrak{H}_{\bar{Q}} \simeq \bar{Q}G$ (cf. the proof of Corollary 5.5)), then $\rho_i(T_{g_0}^2)$ acts by multiplication with

$$\left\{ k \rightarrow \exp \left(-\frac{2\pi\sqrt{-1}}{d(i)} \sum_{\alpha \in R} k_\alpha M_{i,\alpha} \right) = \varepsilon_i \in \mathcal{R}. \right.$$

- (2) Suppose $\{\varepsilon_i\}_{i=1}^L = \{\varepsilon_1, \varepsilon_2, \dots, \varepsilon_L\}$ (so L' is the number of distinct ε_i). Then $\mu(\delta_{g_0}^2) \in GL(|G|, \mathcal{R})$ has minimum polynomial $\prod_{i=1}^{L'} (T - \varepsilon_i) \in \mathcal{R}[T]$.

Proof. By Corollary 5.5 we know that μ induces the left regular representation of $\mathfrak{H}_{\bar{Q}}$ on $\bar{Q}^{|G|}$, so that (2) follows from (1). As for (1), let $T_{g_0} = T_{i_1} \dots T_{i_N}$ be a minimal expression. Then, if $\rho_i(T_{g_0}^2)$ acts by the scalar $f \in \bar{Q}$ (since $T_{g_0}^2 \in Z(\mathfrak{H}_{\bar{Q}})$), we obtain

$$\det(\rho_i(T_{g_0}^2)) = \prod_{j=1}^N \det(T_{i_j})^2 = \prod_{\alpha \in R_+} (q_\alpha)^{2M_{i,\alpha}} = f^{d(i)}.$$

From this equation we solve f (with $f(0) = 1$). □

Let us return to the situation at the beginning of this section. The Euler vector field

$$\theta = \sum_{i=1}^n x_i \frac{\partial}{\partial x_i}$$

$((x_1, \dots, x_n)$ an orthonormal basis for \mathfrak{a}) on \mathfrak{h} will play an important role. Note that, in the coordinates (t, h) (via $p \circ \phi_{x,H}$) one has:

$$\theta = t \frac{\partial}{\partial t}.$$

Let

$$\mathcal{W}^{\text{reg}} = \mathbb{C}[X^{\text{reg}}] \otimes \mathcal{H} = (\pi \circ p)^*(\mathbb{C}[G \setminus \mathfrak{h}^{\text{reg}}] \otimes V),$$

and let $\mathbb{C}[\mathfrak{h}^* \times K] \otimes \mathcal{W}^{\text{reg}}$ be endowed with the connection $\nabla_{\mathcal{W}} = (\pi \circ p)^*(\nabla)$. Let $1 = q_1^*, q_2^*, \dots, q_{|G|}^*$ be a basis for \mathcal{H}^* consisting of homogeneous elements such that $\deg(q_i) \leq \deg(q_{i+1}) (\forall i)$. On the coordinate patch $D_{x,H} \cap X^{\text{reg}}$ the elements $(B_i)_{x,H} = t^{\deg(q_i)} \partial_{q_i}$ form a basis for $\mathcal{W}^{\text{reg}}|_{D_{x,H} \cap X^{\text{reg}}}$. Let \mathcal{W} be the vector bundle on the space $X^{\text{reg}} \cup Y^{\text{reg}}$ such that $(B_i)_{x,H} (i = 1, \dots, |G|)$ form a basis of sections for $\mathcal{W}|_{D_{x,H}^{\text{reg}}}$. Let $\mathcal{W}_Y = \mathcal{W}|_{Y^{\text{reg}}}$.

LEMMA 7.5. *The connection $\nabla_{\mathcal{W}}$ has logarithmic poles along Y^{reg} with respect to the extension \mathcal{W} of \mathcal{W}^{reg} . The residue $\Gamma = \text{Res}_{Y^{\text{reg}}}(\nabla_{\mathcal{W}}) \in \text{End}(\mathcal{W}_Y)$ is independent of $\lambda \in \mathfrak{h}^*$.*

Proof. This is a local calculation on $D_{x_0,H}$. Let (y_2, \dots, y_n) be coordinates on H , and let (x_1, \dots, x_n) be coordinates on \mathfrak{h} such that

$$p \circ (\phi_{x_0,H})(t, y_2, \dots, y_n) = (t, ty_2, \dots, ty_n) = (x_1, \dots, x_n).$$

Then

$$\begin{cases} t \frac{\partial}{\partial t} = \theta = \sum_{i=1}^n x_i \frac{\partial}{\partial x_i} \\ \frac{\partial}{\partial y_i} = t \frac{\partial}{\partial x_i} = x_1 \frac{\partial}{\partial x_i} \quad (i \geq 2). \end{cases}$$

We have to show that

$$\nabla_{\mathcal{W}}(\theta), \nabla_{\mathcal{W}}\left(\frac{\partial}{\partial y_i}\right) \quad (i \geq 2)$$

do not present poles at $t = 0$ when expressed with respect to the basis $\{(B_i)_{x_0,H}\}$. Now if D is any differential operator on $\mathfrak{h}^{\text{reg}}$, we can apply D to sections of \mathcal{W}^{reg} by means of the structure defined by $\nabla_{\mathcal{W}}$. On $D_{x_0,H}^{\text{reg}} (= D_{x_0,H} \cap p^{-1}(\mathfrak{h}^{\text{reg}}))$ we define a matrix

$$(f_{ij}(D)) \in \mathfrak{gl}(|G|, \mathbb{C}[\mathfrak{h}^* \times K] \otimes \mathbb{C}[D_{x_0,H}^{\text{reg}}])$$

by the requirement

$$DB_j = \sum_i f_{ij}(D)B_i \quad (\forall j)$$

or equivalently (by definition of $\nabla_{\mathcal{W}}$):

$$D \cdot t^{\deg(q_j)} \partial_{q_j} \equiv \sum_{i=1}^{|G|} f_{ij}(D) t^{\deg(q_i)} \partial_{q_i} \pmod{\mathbf{J}} \quad (\forall j). \tag{7.2}$$

Note that, in the case where D is a vector field, the matrix of the covariant differentiation $\nabla_{\mathcal{W}}(D)$ with respect to $\{(B_i)_{x_0, H}\}_{i=1}^{|\mathcal{G}|}$ is just $(f_{ij}(D))$. So, since the left-hand side of (7.2) commutes with θ if D does so, it is enough to show the following. Let $E \in \mathbf{A}[\mathfrak{h}^{\text{reg}}]$, and $\text{ad } \theta(E) = 0$. If we write (according to (3.2))

$$E = \sum_{i=1}^{|\mathcal{G}|} g_i B_i + \sum_{j=1}^{|\mathcal{G}|} \sum_{l \geq 1} h_{l,j} B_j (t^{\text{deg}(p_l)} (D_{p_l} - p_l)) \tag{7.3}$$

(here $p_0 = 1, p_1, p_2, \dots$ is a homogeneous basis for $\mathbf{C}[\mathfrak{h}^*]^{\mathcal{G}}$), then g_i has no poles at $(t=0)$, and $g_i|_{(t=0)}$ is independent of $\lambda \in \mathfrak{h}^*$. To this end, let d be the largest integer such that we can write

$$g_i = \sum_{k \geq d} g_i^k, \quad h_{l,j} = \sum_{k \geq d} h_{l,j}^k$$

(expansion in $\text{ad } \theta$ eigenfunctions). Then if $d < 0$, (7.3) yields

$$0 = \sum_{i=1}^{|\mathcal{G}|} g_i^d B_i + \sum_{j=1}^{|\mathcal{G}|} \sum_{l \geq 1} h_{l,j}^d B_j (t^{\text{deg}(p_l)} D_{p_l}).$$

Again by (3.2), (specialized in $\lambda=0$) this implies

$$g_i^d = 0, \quad h_{l,j}^d = 0 \quad \forall (i, l, j)$$

which contradicts the assertion of maximality for d . Now it is obvious that $d=0$, and (7.3) yields

$$E = \sum_{i=1}^{|\mathcal{G}|} g_i^0 B_i + \sum_{j=1}^{|\mathcal{G}|} \sum_{l \geq 1} h_{l,j}^0 B_j (t^{\text{deg } p_l} D_{p_l})$$

by (3.2), specialized for $\lambda = 0$, this determines

$$g_i^0 \in \mathbf{C}[K] \otimes \mathbf{C}[(x_0 + H) \cap \mathfrak{h}^{\text{reg}}]. \tag{□}$$

DEFINITION 7.6. Let $D_{x_0, H}$ be a coordinate patch as above, with $x_0 \in \mathfrak{a}^{\text{reg}}$. Then $\nabla_{x_0, H}$ denotes the connection on $\mathbf{C}[K] \otimes \mathcal{W}|_{(D_{x_0, H} \cap Y^{\text{reg}})}$ defined by

$$\nabla_{x_0, H} \left(\frac{\partial}{\partial y_i} \Big|_{t=0} \right) = \nabla_{\mathcal{W}} \left(\frac{\partial}{\partial y_i} \Big|_{(t=0)} \right).$$

REMARK 7.7. It is easy to see that $\nabla_{x_0, H}$ is an integrable connection (cf. [6]). Note however that its definition depends on the local coordinates (t, y_2, \dots, y_n) , and that the $\nabla_{x_0, H}$ do not glue together to give a globally defined connection on

\mathcal{W}_Y . Also note that $\nabla_{x_0, H}$ is independent of $\lambda \in \mathfrak{h}^*$, similar to the fact that Γ is independent of $\lambda \in \mathfrak{h}^*$ (Lemma 7.5). Finally we mention the fact that $\Gamma \in \text{End}(\mathcal{W}_Y) \simeq \mathcal{W}_Y \otimes \mathcal{W}_Y^*$ is flat with respect to $\nabla_{x_0, H} \otimes \nabla_{x_0, H}^*$ (cf. [6]). In particular, the endomorphisms $\Gamma_*(y \in Y)$ are conjugate to each other.

In addition to the notation of Corollary 7.4, let

$$\{p_{ij} \mid i = 1, \dots, L, j = 1, \dots, d(i)\} \subset \{0, \dots, |R_+|\}$$

be the set of so-called ρ_i -exponents of G (i.e. the representation ρ_i occurs with positive multiplicity in $\mathcal{H}^{*, p_{ij}}$ ($j = 1, \dots, d(i)$), the homogeneous harmonic polynomials of degree p_{ij} (cf. [30], Section 2)). Denote by $v_{ij} \in \mathbb{C}[K]$ the following affine linear function on K :

$$v_{ij}(k) = p_{ij} - d(i)^{-1} \sum_{\alpha \in R} M_{i, \alpha} k_\alpha.$$

Let $\{v_{ij} \mid i = 1, \dots, L; j = 1, \dots, d(i)\} = \{v_1, v_2, \dots, v_{L''}\}$.

DEFINITION 7.8. The numbers $v_1(k), \dots, v_{L''}(k)$ are called the exponents at the origin of the system of Bessel differential equations (3.1). Also define the multiplicity of the exponent v_i by

$$m(i) = \sum_{i'} (\rho_{i'} : \mathcal{H}^{*, v_i(0)}) d(i')$$

where the sum is taken over all $i' \in \{1, \dots, L\}$ such that $\varepsilon_{i'} = \exp(2\pi\sqrt{-1} v_i)$ (see Corollary 7.4).

REMARK 7.9. The reason for these definitions is explained in Theorem 7.10 and Corollary 7.12. The following observations are obvious from the definition. The exponent that corresponds to the trivial representation will be called v_1 from now on, and clearly $v_1(k) = 0 (\forall k)$. Note that $m(1) = 1$. If $\text{Re}(k_\alpha) < 0 (\forall \alpha)$, then $\text{Re}(v_i(k)) > 0 \forall i \in \{2, \dots, L''\}$.

THEOREM 7.10. *If f is a section of \mathcal{W}^{reg} , then f can be considered as a differential operator on $\mathfrak{h}^{\text{reg}}$. Hence it makes sense to consider $(\text{ad } \theta)f$, which is again a section of \mathcal{W}^{reg} . Note that $(\text{ad } \theta)B_i = 0$ ($i = 1, \dots, |G|$) on every coordinate neighbourhood $D_{x, H}$, so that on $D_{x, H}$:*

$$\text{ad}(\theta)(f) = (\text{ad } \theta) \left(\sum_i f_i B_i \right) = \sum_i \theta(f_i) B_i.$$

- (1) *If $\lambda = 0$ then the space of $\nabla_{\mathcal{W}}$ -flat sections is invariant for $\text{ad } \theta$. Extend $\Gamma \in \text{End}(\mathcal{W}_Y)$ on $D_{x, H}^{\text{reg}}$ to $\tilde{\Gamma} \in \text{End}(\mathcal{W}|_{D_{x, H}^{\text{reg}}})$ by the requirement that with*

respect to the bases $(B_i)_{x,H}$ the matrix of $\tilde{\Gamma}$ is independent of t , and restricts to Γ on $Y \cap D_{x,H}^{\text{reg}}$. Then we have: $\text{ad } \theta = -\tilde{\Gamma}$ on $\mathcal{W}^\nabla|_{D_{x,H}^{\text{reg}}}$.

- (2) In the situation of (1), one has $\exp(2\pi\sqrt{-1}\tilde{\Gamma}^t) = \mu(s)$ on $(\mathcal{W}^*)^{\nabla^*}|_{D_{x,H}^{\text{reg}}}$ (see Proposition 7.1 and Lemma 5.2).
- (3) The minimum polynomial for Γ equals $\prod_{i=1}^{L'}(T - v_i)$ ($\lambda \in \mathfrak{h}^*$ arbitrary now). If $k \in K$ is generic then

$$\forall i \in \{1, \dots, L'\}: m(i) = \dim \text{Ker}(\Gamma(k) - v_i(k)).$$

Proof. (1) If $\lambda = 0$ then it is clear from (7.3) that

$$\nabla_{\mathcal{W}}(\theta) \quad \text{and} \quad \nabla_{\mathcal{W}}\left(\frac{\partial}{\partial y_i}\right) \quad (i \geq 2)$$

are independent of t when expressed, on $D_{x,H}$, with respect to the basis $\{(B_{i,x,H})_{i=1}^{|G|}\}$. Hence $\text{ad } \theta$ leaves invariant the space of flat sections. Moreover, by definition of $\tilde{\Gamma}$ one has $\nabla_{\mathcal{W}}(\theta) = \text{ad } \theta + \tilde{\Gamma}$ (with respect to the basis $(B_i)_{x,H}$), proving (1).

Next we prove (2). On $\mathcal{W}^*|_{D_{x,H}^{\text{reg}}}$ we obtain from (1) that $\text{ad}(\theta)$ acts on $(\mathcal{W}^*)^{\nabla^*}$ by means of the matrix $\tilde{\Gamma}^t$ (with respect to the basis (B^*)). Since the vector field $\sqrt{-1}\theta$ integrates to the loop s (as in Proposition 7.1) we obtain the desired result.

Finally consider (3). It is enough to prove this at a certain $y_0 \in Y^{\text{reg}}$ (see Remark 7.7) and by Lemma 7.5 we may assume $\lambda = 0$. Hence we may consider the endomorphism $\tilde{\Gamma}_{(t,y)}$ as in (1), instead of Γ_{y_0} . By (1) this endomorphism commutes with the monodromy action $\mu(0, k)$ on $\mathcal{W}_{(t,y)}$. Recall that $\mu(0, k)$ is the regular representation of $H_G(q)$ if $k \in K^s$. Thus from (2) we obtain that $\tilde{\Gamma}_{(t,y)}(k)$ is semisimple if $k \in K^s$ and that there exists a decomposition into irreducibles of $\mathcal{W}_{(t,y)}$ for the action of $\mu(0, k)$ such that $\tilde{\Gamma}_{(t,y)}$ is a constant times the identity on all the irreducible constituents of $\mathcal{W}_{(t,y)}$. Clearly the eigenvalues of $\tilde{\Gamma}_{(t,y)}$ are algebraic functions of k . So on a suitable open set $\Omega \subset K$ we may assume that the eigenvalues depend analytically on $k \in \Omega$. But then we see from (2) and Corollary 7.4(1) that the eigenvalues are in fact polynomials in k of degree 1 obtained from the ε_i by taking a proper logarithm and dividing by $2\pi\sqrt{-1}$. From the situation at $k = 0 \in K^s$ we see which logarithm we have to take on the irreducible components and this shows that v_i is an eigenvalue with geometric multiplicity $m(i)$. □

We are now in the position to study the behaviour of $\nabla^*_{\mathcal{W}}$ -flat sections of \mathcal{W}^* in a neighbourhood of Y . As a direct result we can describe the behaviour of solutions of (3.1) near the origin.

Let $x \in \mathfrak{h}^{\text{reg}}$ and H a hyperplane in \mathfrak{h} such that $x \notin H$. Let $(t, y_2, \dots, y_n) = (t, y)$ be coordinates on $D_{x,H}$ as before. So $x \in \mathfrak{h}$ has coordinates $(1, 0, \dots, 0)$, and the

origin $(0, \dots, 0)$ corresponds to the element $(\mathbf{C}x) \in Y$. Let $\{B_{ij}\}_{i=1}^{|G|}$ be the basis of $\mathcal{W}^*|_{D_{x,H}}$ as usual, and let $\varepsilon > 0$ be small enough to ensure that $\tilde{D}_{x,H} = \{(t, y) \mid |t| < \varepsilon \text{ and } |y_i| < \varepsilon \forall i\}$ has the property $\tilde{D}_{x,H} - X^{\text{reg}} = \{(0, y) \mid |y_i| < \varepsilon \forall i\}$.

THEOREM 7.11. *Let $(\lambda, k) \in \mathfrak{h}^* \times K$ be arbitrary. There exists a basis $\{f_{ij} \mid i = 1, \dots, L''; j = 1, \dots, m(i)\}$ (for the definition of $m(i)$ see Definition 7.8) of $\nabla_{\mathcal{W}^*}$ -flat sections in $\mathcal{W}^*|_{\tilde{D}_{x,H}^{\text{reg}}}$ of the form*

$$f_{ij}(t, y) = t^{v_i(k)} \sum_{l=0}^{n_{ij}(\lambda, k)} (\log t)^l v_{ijl}(t, y).$$

Here $v_{ijl}: \tilde{D}_{x,H} \rightarrow \Sigma_i^{|G|} \mathbf{CB}_i^*$ is holomorphic, and $\{\phi_{ij} \mid i = 1, \dots, L''; j = 1, \dots, m(i)\}$ is a basis of solutions of $(\text{ad } \theta - \Gamma'_{\mathbf{C}x}(k))\phi = 0$, if we put

$$\phi_{ij}(t) = t^{v_i(k)} \sum_{l=0}^{n_{ij}(\lambda, k)} (\log t)^l v_{ijl}(0, 0).$$

On the other hand, if f is a $\nabla_{\mathcal{W}^*}(\lambda, k)$ -flat section on $\tilde{D}_{x,H}$ of the form

$$f(t, y) = t^v \sum_{l=0}^n (\log t)^l v_l(t, y)$$

such that $(v_0(0, 0), \dots, v_n(0, 0)) \neq 0$, then $v \in \{v_1(k), \dots, v_{L''}(k)\}$.

Proof. First consider the equation $\nabla_{\mathcal{W}^*}(\theta)f = \text{ad}(\theta)(f) + Mf = 0$ on the line $\{(t, 0) \mid 0 < |t| < \varepsilon\}$. Recall that $M_{(0,0)} = -\Gamma'_{\mathbf{C}x}(k)$. Using Theorem 7.10(3), and standard theory of systems of first order linear differential equations in one variable (see for example [34]) we obtain a basis of solutions for this equation of the form:

$$f_{ij}^0(t) = t^{v_i(k)} \sum_{l=0}^{n_{ij}(\lambda, k)} (\log t)^l v_{ijl}^0(t)$$

where

$$\phi_{ij}(t) = t^{v_i(k)} \sum_{l=0}^{n_{ij}(\lambda, k)} (\log t)^l v_{ijl}^0(0)$$

runs through a basis of solutions of $(\text{ad } \theta)(\phi) = \Gamma'_{\mathbf{C}x}(k)\phi$. Now define $v_{ijl}(t, y)$ by

$$\begin{cases} \nabla_{\mathcal{W}^*} \left(\frac{\partial}{\partial y_p} \right) v_{ijl}(t, y) = 0 & (\forall p, i, j, l) \\ v_{ijl}(t, 0) = v_{ijl}^0(t) & (\forall i, j, l). \end{cases}$$

Hence $v_{ijl}(t, y)$ (t fixed) is a flat section of an integrable connection with holomorphic parameter t , and thus $v_{ijl}(t, y)$ is holomorphic in (t, y) . With this definition of $v_{ijl}(t, y)$ it is clear that f_{ij} is $\nabla_{\mathscr{W}}^*$ -flat, and that $\{f_{ij}\}$ forms a basis of $(\mathscr{W}^*)^{\nabla^*}$. Finally, if

$$f(t, y) = t^{\nu} \sum_{l=0}^n (\log t)^l v_l(t, y)$$

is $\nabla_{\mathscr{W}}^*$ -flat, then it is clear from series expansion at $t=0$ on the line $\{(t, 0) \mid |t| < \varepsilon\}$ that

$$\phi = t^{\nu} \sum_{l=0}^n (\log t)^l v_l(0, 0)$$

satisfies the “constant coefficient” equation $(\text{ad } \theta)\phi = \Gamma'_{\mathbb{C}^x}(k)\phi$. Clearly this equation implies (when $\phi \neq 0$ of course) that ν is an eigenvalue of $\Gamma'_{\mathbb{C}^x}(k)$. \square

COROLLARY 7.12. *Let $(\lambda, k) \in \mathfrak{h}^* \times K$ be arbitrary. There exists a basis $\{\phi_{ij} \mid i = 1, \dots, L''; j = 1, \dots, m(i)\}$ of solutions of (3.1) on $\tilde{D}_{x,H}$ of the form*

$$\psi_{ij}(t, y) = t^{\nu_i(k)} \sum_{l=0}^{n_{ij}(\lambda, k)} (\log t)^l u_{ijl}(t, y).$$

Here u_{ijl} is holomorphic on $\tilde{D}_{x,H}$, and $(u_{ij0}(0, 0), \dots, u_{ijn_{ij}(\lambda, k)}(0, 0)) \neq (0, \dots, 0)$. Moreover, if $\psi(t, y)$ is any solution of (3.1) on $\tilde{D}_{x,H}$ of the form

$$\psi(t, y) = t^{\nu} \sum_{l=0}^n (\log t)^l u_l(t, y)$$

with u_l holomorphic and $(u_l(0, 0))_{l=0}^n \neq 0$ then $\nu \in \{\nu_i(k)\}_{i=1}^{L''}$.

Proof. This is immediate from Theorem 7.11 if we use the isomorphism $(\mathscr{W}^*)^{\nabla^*}_p \rightarrow \mathscr{L}_p$ given by $v \rightarrow v(1)$ (see Corollary 3.6). Recall that the inverse of this map is given by $f \rightarrow \sum_{i=1}^{|G|} (B_i f) B_i^*$. \square

REMARK 7.13. Note that $\Gamma(k)B_1 = 0 \forall k$, and also $\Gamma'(k)B_1^* = 0 \forall k$. The first of these identities is immediate from the definition. The second one holds since (by definition) Γ is the matrix determined by the condition that for all i , there exist functions h_{ij} , regular at $t = 0$, such that

$$\Gamma B_i = \theta B_i - \sum_{j=1}^{|G|} \sum_{l \geq 1} h_{lj} B_j (t^{\text{deg}(p_l)} D_{p_l}) \tag{7.4}$$

(see Lemma 7.5). On the other hand,

$$\Gamma^l B_1^* = 0 \Leftrightarrow B_1^*(\Gamma B_i) = 0 \quad \forall i \Leftrightarrow (\Gamma B_i)(1) = 0 \quad \forall i.$$

Applying (7.4) to the function 1 we see that this is indeed true.

EXAMPLE 7.14. By Remark 7.13 we know that $\forall y \in Y^{\text{reg}}: \Gamma_y^l(k) \cdot B_1^* = 0 (\forall k)$. Hence $\phi_{11}(t) = B_1^*$ is a solution of

$$(\text{ad } \theta - \Gamma_{\mathcal{C}_x}^l) \phi = 0$$

and according to Theorem 7.11 and 7.12 there exists a solution $\psi_{11}(t, y)$ on $\tilde{D}_{x,H}$ of the form:

$$\psi_{11}(t, y) = \sum_{l=0}^{n_{11}(\lambda, k)} (\log t)^l u_{11l}(t, y)$$

of (3.1), with $u_{110}(0, 0) = 1$, and $u_{11l}(0, 0) = 0 \forall l > 0$. From the proof of Lemma 7.5 it is easy to see that

$$\nabla_{x,H}^* \left(\frac{\partial}{\partial y_i} \Big|_{t=0} \right) (B_i^*) = 0$$

(see Definition 7.6) and so, by construction of ϕ_{11} as in Theorem 7.11 and Corollary 7.12 we obtain that

$$\begin{cases} u_{110}(0, y) = 1 & \forall y \text{ with } |y_i| < \varepsilon \\ u_{11l}(0, y) = 0 & \forall y \text{ with } |y_i| < \varepsilon, \quad \forall l > 0. \end{cases}$$

We close this section by mentioning some simple consequences of Corollary 7.12. Let $\mathcal{L}(G \setminus \mathfrak{h})$ denote the \mathbf{C} -vector space $\{f \in \mathcal{O}(G \setminus \mathfrak{h}) \mid f \text{ satisfies (3.1)}\}$.

COROLLARY 7.15. *Let $\lambda \in \mathfrak{h}^*$ be arbitrary, and $k \in K$ such that $v_i(k) \notin \{0, 1, 2, \dots\} \forall i > 1$ (this is true for example if $\text{Re}(k_\alpha) > 1 \forall \alpha \in R$). If $f \in \mathcal{L}(G \setminus \mathfrak{h})(\lambda, k)$ then $f \equiv 0 \Leftrightarrow f(0) = 0$.*

Proof. Use Corollary 7.12. □

COROLLARY 7.16. *Let $(\lambda, k) \in \mathfrak{h}^* \times K$ satisfy one of the following conditions*

- (1) $v_i(k) \notin \{0, 1, 2, \dots\} \forall i > 0$.
- (2) $\lambda \in (\mathfrak{h}^*)^{\text{reg}}$.

Then $\dim_{\mathbf{C}}[\mathcal{L}(G \setminus \mathfrak{h})(\lambda, k)] = 1$.

Proof. By Proposition 5.6(2) we have $\dim[\mathcal{L}(G \setminus \mathfrak{h})(\lambda, k)] \geq 1$. Clearly the

assumption $\dim[\mathcal{L}(G \setminus \mathfrak{h})(\lambda, k)] > 1$ would imply the existence of $0 \neq f \in \mathcal{L}(G \setminus \mathfrak{h})(\lambda, k)$ such that $f(0) = 0$. So (1) is clear using Corollary 7.15. If $\lambda \in (\mathfrak{h}^*)^{\text{reg}}$ we can dispose of the intertwining isomorphisms $i_{\bar{s}}$ of Proposition 4.1. By repeated application of these we may assume that $\text{Re}(k_{\alpha}) \gg 0 \forall \alpha \in R$, in which case condition (1) holds again. This proves (2). \square

REMARK 7.17. If we look at Definition 7.8 and compare this to the calculation of Dunkl in ([8], p.176) we observe that the eigenvalues of the operator $\Sigma_{i=1}^u \xi_i^* T_{\xi}(k)$ on the space \mathcal{H}^* are precisely the exponents $v_i(-k)$ ($i = 1, \dots, L'$) (counted with multiplicity). From this observation one easily sees that (see Definition 6.1):

$$-K_+ = \{k \in K \mid \text{Re}(v_i(k)) > 0 \ (i = 2, \dots, L'')\}.$$

We will use the notation $K_- = -K_+$ in the sequel. Finally note that $v_i(k) \in \mathbf{Z} (\forall i)$ if $k_{\alpha} \in \mathbf{Z} \forall \alpha \in R$ (cf. [10], Lemma 2.1 and Corollary 2.2).

8. Evaluation at the origin

Let $x_0 \in \mathfrak{h}^{\text{reg}}$, and consider its image as base point for $G \setminus \mathfrak{h}^{\text{reg}}$. Let $K_- = -K_+$ (see Remark 7.17) and recall that if $k \in K_-$, then the exponents $v_i(k)$ satisfy: $v_i(k) \geq 0 \forall k$, and $v_i(k) = 0 \Rightarrow i = 1$. Therefore the following definition makes sense, provided that $k \in K_-$.

DEFINITION 8.1. Let $(\lambda, k) \in \mathfrak{h}^* \times K_-$. Let $E(=E(\lambda, k; x_0)) \in \mathcal{L}_{x_0}^*(\lambda, k)$ be defined by $E(f) = \lim_{t \rightarrow 0} f(tx_0)$ (analytic continuation via the path $(t \rightarrow tx_0)$). E is called “evaluation at the origin”.

PROPOSITION 8.2. $E(\lambda, k)$ is a nonzero global section of $\mathcal{L}^*(\lambda, k)$.

Proof. This follows immediately from Corollary 7.12 and Example 7.14. \square

Recall the pairing $\{\cdot, \cdot\}$ introduced in Corollary 3.12.

THEOREM 8.3. $\forall \phi \in \mathcal{L}(\lambda, k): \{\phi, J(-\lambda, 1-k)\} = a(k)E(\phi)$, for some $a \in \mathbf{C}(K)$ such that $a(k)$ has neither poles nor zeros inside K_- .

Proof. Let us recall the bases $\{B_i\}_{i=1}^{|G|}$ for \mathcal{W} on a coordinate patch of the form $D_{x_0, H}$ (as in Lemma 7.5), consisting of differential operators of homogeneous degree 0. With respect to this basis we may write (use Corollary 3.12):

$$\{\phi, J(-\lambda, 1-k)\} = \sum_i \beta_i(\lambda, k; t, y)(B_i(\phi)(t, y)) \tag{8.1}$$

with, as a result of Corollary 3.12 and Corollary 6.10, β_i a holomorphic function in the variables λ and k (use that $1-k \in K_+$ if $k \in K_- = -K_+$) and a merom-

orphic function of t and y . We claim that $(t=0)$ is not a hyperplane of poles for $\beta_i(\forall i)$. Since it is sufficient to prove this for generic (λ, k) we may assume that (λ, k) are fixed, and such that the monodromy representation is semisimple. In this situation it is clear that $\{\phi, J(-\lambda, 1-k)\} = 0$ for all $\phi \in \mathcal{L}(\lambda, k)$ that have no component in the trivial part of the monodromy representation. Now let $m \in \mathbf{Z}_+$ be minimal subject to the condition that $t^m \beta_i$ is regular at $t=0 \forall i$, and put $\mu_i = \mu_i(y) = \lim_{t \rightarrow 0} t^m \beta_i(t, y)$. The above remark implies that (use formula (8.1), Remark 7.9 and Theorem 7.11):

$$v \left(\sum_i \mu_i B_i \right) = 0$$

for all eigenvectors v of Γ' with nonzero eigenvalue. By Remark 7.13 this means that $\mu_i = 0$ when $i = 2, \dots, |G|$ (since Γ is semisimple, and B_1 is the only eigenvector with eigenvalue 0). The assumption $m > 0$ would imply that also $\mu_1 = B_1^*(\sum_i \mu_i B_i) = 0$ by a similar argument, but this contradicts the minimality of m . So we have not only proved our claim, but also the result that $\mu_i = \lim_{t \rightarrow 0} (\beta_i(t, y)) = 0 \forall i \geq 2$. Using formula (8.1) again, we obtain

$$\{\phi, J(-\lambda, 1-k)\} = \mu_1(\lambda, k)E(\phi)$$

where $\mu_1(\lambda, k)$ is polynomial in λ and rational in k . (Here we used the fact that all the coefficients of the power series expansion in t at $t=0$ of $J(\lambda, k, (t, y))$ are rational in k and polynomial in λ . This follows at once if from the recurrence relations defining them (using $\nabla_{\mathcal{W}}^*$ and Lemma 7.5.) Since for all $(\lambda, k) \in \mathfrak{h}^* \times K_-$ both $\{\cdot, J(-\lambda, 1-k)\}$ and E are nonzero sections of $\mathcal{L}^*(\lambda, k)$ (recall that $1-k \in K_+$ if $k \in K_-$), we see that $\mu_1(\lambda, k) \neq 0, \infty$ for $(\lambda, k) \in \mathfrak{h}^* \times K_-$. Hence $\mu_1(\lambda, k) = a(k) \in \mathbf{C}(k)$, having no poles or zeros in K_- . □

COROLLARY 8.4. $E(\lambda, k)$ depends holomorphically on $(\lambda, k) \in \mathfrak{h}^* \times K_-$.

9. The singular set of J_G and Exp_G

In this section we will prove a refinement of Corollary 6.10, namely we will give an exact description of the singular set of J_G (and thus of Exp_G – cf. the proof of Corollary 6.10), counted with multiplicity. At the same time we will discuss the relation of this result with the problem of calculating the Bernstein-Sato polynomial b of the discriminant of G explicitly. As a consequence we obtain a proof of the formula for b that was conjectured by Yano and Sekiguchi (see [35]). Finally we will show how this “almost” proves the Macdonald conjecture ([24], Conjecture 5.1) on the Mehta type integral. (In order to get a

complete proof we still need to calculate a certain constant by computer. This was done by Prof. F. Garvan.) We note that these results have previously been proven in [28] for all cases where G is a Weyl group, as an application of the calculus of hypergeometric shift operators. Thanks to the results of Dunkl and Heckman mentioned in Section 2 we know that shift operators do exist in the general case. But still the calculation given in [28] does not carry over to the general situation, since it is based on the evaluation of Jacobi-polynomials at the identity element (and Jacobi-polynomials are not present in the general case). This part of the proof will be replaced by the use of the evaluation map E (as defined in Section 8) in the present section.

Recall the notion of l -length $l(g)$ of $g \in G$ (see [23]). If $k \in K$ is a multiplicity function on $R = R(G)$, we define $k^{l(g)} = k_1^{l_1(g)} \dots k_m^{l_m(g)}$. Here $k = \sum_{i=1}^m k_i 1_{C_i}$, if $R = \prod_{i=1}^m C_i$ is the decomposition of R in minimal G orbits, and $l_i(g)$ is the number of simple reflections r_α with $\alpha \in C_i$ that occur in a minimal expression for g . We denote by $P_G(k) = \sum_{g \in G} k^{l(g)}$ the Poincaré polynomial of G (see [23]).

The regular representation of $H_G(q)$ contains the trivial representation precisely once, and it is not hard to give an explicit basis vector in $H_G(q)$ for this one dimensional subspace.

LEMMA 9.1 (See [13]). *The vector $v(q) = \sum_{g \in G} (-q)^{-l(g)} T_g \in H_G(q)$ satisfies $T_i v = v(\forall i)$ (where $T_i = T_{s_i}$).*

Proof. Easy calculation, similar to ([13], Lemma 1). □

DEFINITION 9.2. Recall the notations of Section 5. In particular, if $k \in K$ we define $q \in K$ by: $q(k) = -\exp(-2\pi\sqrt{-1}k)$. Clearly $q_\alpha \in \mathcal{R} \forall \alpha \in R$ (recall that \mathcal{R} is the ring of entire functions on $\mathfrak{h}^* \times K$). Let $A \in \mathcal{R}A_G$ be the element $A(k) = \sum_{g \in G} (-q)^{-l(g)} \delta_g$ (δ_g was defined just before Proposition 7.1).

PROPOSITION 9.3 *Let $(\lambda, k) \in \mathfrak{h}^* \times K_-$ (see Remark 7.17), and let $f \in \mathcal{L}_x(\lambda, k)$ (for some $x \in \mathfrak{h}^{*eg}$). Then (see Lemma 5.2 for the definition of μ):*

$$(\mu(\lambda, k)A(k))f = P_G(-q^{-1}) E(f) J_G(\lambda, k). \tag{9.1}$$

Proof. Clearly $(\mu(\lambda, k)A(k))f \in \mathcal{L}(\lambda, k)$ has trivial monodromy as a consequence of Lemma 9.1. If $k \in K^s$ this implies that $(\mu(\lambda, k)A(k))f$ equals some constant times $J_G(\lambda, k)$, as a direct result of Proposition 5.6(2) and Definition 6.9. Then formula (9.1) follows by applying E and using Proposition 8.2. When $k \in K_-$ arbitrary, we can still extend f by a holomorphic family of solutions $f(k') \in \mathcal{L}_x(\lambda, k')$, k' varying in a neighbourhood of k and such that $f(k) = f$. Hence formula (9.1) follows from the case $k' \in K^s$ by meromorphic continuation when we use Corollary 8.4. □

DEFINITION 9.4. Let $B \in \mathbf{C}[K]$ be the polynomial defined by

$$B(k) = G_R^-(k + 1_R)(1) = \left(\prod_{\alpha \in R_+} T_\alpha(k) \right) \left(\prod_{\alpha \in R_+} \alpha^* \right).$$

(Here $1_R: T \rightarrow \mathbf{C}$ denotes the function $1_R(\alpha) = 1 \forall \alpha \in R$ as usual). We denote the zero locus of B (counted with multiplicity) by $Z_B \subset K$. Also let $\tilde{b} \in \mathbf{C}[s]$ (s an indeterminate) denote the restriction of B given by $\tilde{b}(s) = B(s1_R)$, and let $Z_{\tilde{b}}$ be its zero locus in the s -plane.

PROPOSITION 9.5. $\tilde{b} \in \mathbf{C}[s]$ has degree $|R_+|$.

Proof. From its definition it is clear that we have to show that

$$\left(\prod_{\alpha \in R_+} \tilde{T}_\alpha \right) \left(\prod_{\alpha \in R_+} \alpha^* \right) \neq 0$$

where

$$\tilde{T}_\alpha = \lim_{s \rightarrow \infty} \frac{1}{s} T_\alpha(s1_R) = \sum_{\beta \in R_+} (\alpha, \beta) \Delta_\beta.$$

In order to see this we use Dunkl's results on harmonic polynomials and peak sets ([9], Th. 2.10, §3). Let E denote the peak set, and let $x_0 \in E$. Then

$$\begin{aligned} \left(\prod_{\alpha \in R_+} \tilde{T}_\alpha \right) \left(\prod_{\alpha \in R_+} \alpha^* \right) &= \frac{1}{|G|} \sum_{g \in G} \left(\prod_{\alpha \in R_+} \tilde{T}_\alpha \right) \left(\prod_{\alpha \in R_+} \alpha^* \right) (gx_0) \\ &= \frac{1}{|G|} \sum_{g \in G} \left(\prod_{\alpha \in R_+} \alpha^*(gx_0) \right) \left(\left(\prod_{\alpha \in R_+} \tilde{T}_\alpha^* \right) 1 \right) (gx_0) \\ &= \left(\prod_{\alpha \in R_+} \alpha^*(x_0) \right) \left(\left(\prod_{\alpha \in R_+} \tilde{T}_\alpha^* \right) 1 \right) (x_0) \neq 0. \end{aligned}$$

(Here \tilde{T}_α^* denotes the adjoint of \tilde{T}_α with respect to the inner product (on $\mathbf{C}[E]$) $(f, g) = \sum_{x \in E} f(x)g(x)$. In the last step we used Dunkl's results (Section 3, loc. cit.) that the map

$\mathbf{C}[\mathfrak{h}] \rightarrow \mathbf{C}[E]$, $p(x_1, \dots, x_n) \rightarrow p(\tilde{T}_1^*, \dots, \tilde{T}_n^*)1|_E$ is an isomorphism when restricted to the harmonic polynomials $\mathcal{H}^* \subset \mathbf{C}[\mathfrak{h}]$, together with his result $x_0 \in \mathfrak{h}^{\text{reg}}$. □

PROPOSITION 9.6. The set of poles of J_G equals (with multiplicity) $\mathfrak{h}^* \times Z \times \mathfrak{h}$, where $Z = \bigcup_{n \in \mathbf{Z}_+} (Z_B - n1_R)$.

Proof. First of all recall that J is holomorphic on $\mathfrak{h}^* \times K_+ \times \mathfrak{h}$ (Corollary 6.10). We use the functional equation

$$J(\lambda, k) = \frac{1}{B(k)} G_R^-(k + 1_R) J(\lambda, k + 1_R)$$

in order to extend J meromorphically to $\mathfrak{h}^* \times K \times \mathfrak{h}$. Hence we see that the set of poles of J is contained in $\mathfrak{h}^* \times Z \times \mathfrak{h}$. We obtain the equality by observing that if $\lambda \in (\mathfrak{h}^*)^{\text{reg}}$ then $G_R^-(k + 1_R) J(\lambda, k + 1_R) \neq 0$ (by Proposition 4.1). This completes the proof. \square

We will now determine \tilde{b} explicitly (up to a multiplicative constant) using Proposition 9.3 and Proposition 9.6. A key role is played by the well known factorization formula for P_G due to Bott [1], Solomon [29] (in the case of identical root labels) and Macdonald [23] (general case).

THEOREM 9.7. $Z = Z_P \cap (K_+)^c$, where $(K_+)^c$ denotes the set-theoretic complement of K_+ and Z_P denotes the zero locus of $k \rightarrow P_G(-q^{-1}(k))$ (counted with multiplicity).

Proof. We leave it to the reader to verify that all components of Z_P intersect the diagonal s -plane $C1_R \subset K$ (for instance by inspection of Macdonald’s list ([23], Section 2.2) which gives factorizations for the polynomials $P_G(q)$). Since $C1_R \subset K_+ \cup K_-$ this implies that the components of $Z_P \cap (K_+)^c$ all contain points of K_- . Therefore we may apply formula (9.1), and we obtain the result $Z \subset Z_P \cap (K_+)^c$ (use Proposition 9.6 and Corollary 8.4. Note that we may assume that $E(f) = 1$ in (9.1) since $E \in \mathcal{L}^*(\lambda, k)$ is nonzero (Proposition 8.2)). It is sufficient now to show that the intersections of Z and $Z_P \cap (K_+)^c$ with $C1_R$ coincide (with multiplicity, of course). Recall the theorem of Bott and Solomon (see [1], [29]): (t an indeterminate)

$$P_G(t1_R) = \prod_{i=1}^n \frac{t^{d_i} - 1}{t - 1}$$

where d_i denotes the i th primitive degree of G . Hence (the dot \cdot denotes intersection counted with multiplicity):

$$(Z_P) \cdot (C1_R) = \bigcup_{l \in Z} \bigcup_{i=1}^n \bigcup_{j=1}^{d_i-1} \{(l + j/d_i)1_R\}.$$

By Proposition 9.6, $(Z) \cdot (C1_R) = \bigcup_{l \in Z_+} (Z\tilde{b} - l1_R)$, thus by counting the number of points per unit interval (Proposition 9.5) we obtain $(Z_P) \cdot (C1_R) = (Z) \cdot (C1_R)$ in a half plane $\text{Re}(s) < M$ (for some $M \in Z_-$). Using (9.1) again this is clearly

equivalent with the condition:

$$\exists f \in \mathcal{L}_x(\lambda, k) \text{ such that } (\mu(\lambda, k)A(k))f \neq 0. \tag{*}$$

Hence (*) is satisfied when $k = s1_R$ with $\text{Re}(s) < M$. Moreover, if we take $\lambda \in (\mathfrak{h}^*)^{\text{reg}}$ then condition (*) is periodic in $k \in K$ with period 1_R by Proposition 4.1, and thus condition (*) holds for all $k = s1_R$, with $s \in \mathbb{C}$. Hence by formula (9.1) we have:

$$\begin{aligned} Z \cdot (\mathbf{C}1_R) &= (Z_P \cap K_-) \cdot (\mathbf{C}1_R) = (Z_P \cap K_+^c) \cdot (\mathbf{C}1_R) \\ &= \bigcup_{l \in \mathbb{N}} \bigcup_{i=1}^n \bigcup_{j=1}^{d_i-1} \{(-l + j/d_i)1_R\}. \end{aligned} \quad \square$$

This proves the conjecture of Yano and Sekiguchi on the b -function for the discriminant of a finite Coxeter group (see [35]):

THEOREM 9.8. *The Bernstein-Sato polynomial $b \in \mathbb{C}[s]$ of the discriminant $I_G \subset (G \setminus \mathfrak{h})$ is, for any finite Coxeter group G , given by*

$$b(s) = \prod_{i=1}^n \prod_{j=1}^{m_i} \left(s + \frac{1}{2} + \frac{j}{d_i} \right) \tag{9.2}$$

($m_i = d_i - 1$, the G -exponents and $n = \text{rank}(G)$).

Proof. The reasoning in [28], Section 7 shows that $\tilde{b}(s + \frac{1}{2}) = c \cdot b(s)$ (for some constant c). By the proof of Theorem 9.7, we also have:

$$(Z) \cdot (\mathbf{C}1_R) = \bigcup_{l \in \mathbb{Z}_+} (-l1_R + Z\tilde{b}) = \bigcup_{l \in \mathbb{N}} \bigcup_{i=1}^n \bigcup_{j=1}^{d_i-1} \{-l1_R + (j/d_i)1_R\}.$$

Hence,

$$Z\tilde{b} = \bigcup_{i=1}^n \bigcup_{j=1}^{d_i-1} \{-(j/d_i)1_R\}$$

and the factorization (9.2) follows. □

The above results also yield information on Macdonald's conjecture on the Mehta type integral for finite reflection groups ([24], Conjecture 5.1):

THEOREM 9.9. *Consider the Mehta type integral*

$$M(s) = \int_{\mathfrak{a}} I^s(x) d\gamma(x)$$

where $d_\gamma(x) = (2\pi)^{-n/2} e^{-1/2\|x\|^2} dx$ denotes the Gaussian measure. There exists a constant $c \in \mathbf{R}_+$ such that

$$M(s) = c^s \prod_{i=1}^n \frac{(d_i s)!}{s!}.$$

Proof. Using Corollary 3.10 and Definition 9.4 we obtain the functional equation

$$M(s+1) = \tilde{b}(s)M(s)$$

where, according to Theorem 9.8, for some nonzero constant:

$$\tilde{b}(s) = c|G| \prod_{i=1}^n \prod_{j=1}^{d_i-1} (d_i s + j).$$

This leads to the stated result (copy the proof which was given in ([28], Section 7) for the case where G is a Weyl group). □

This would prove Macdonald’s conjecture ([24], Conjecture 5.1) for these integrals if we could show that in fact $c = 1$. Equivalently, we have to show that:

$$\left(\prod_{\alpha \in \mathbf{R}_+} \partial_\alpha \right) \left(\prod_{\alpha \in \mathbf{R}_+} \alpha^* \right) = \prod_{i=1}^n (d_i!). \tag{9.3}$$

Unfortunately, I do not have an intrinsic explanation for (9.3). But when we use the classification of irreducible finite Coxeter groups it suffices to verify (9.3) for the groups H_3 and H_4 only (since the conjecture was proved for dihedral groups by direct computation in [24], and for Weyl groups in [28]). As was already mentioned in the Introduction, Prof. F. Garvan has informed me that he has indeed been able to verify (9.3) for these cases by computer (after he simplified (9.3) somewhat using certain symmetries). (In the case of H_3 this computer calculation was carried out independently by Dr. H. Finkelnberg using “Maple”.) Thus, provided that we accept these computer calculations as a proof, we have:

COROLLARY 9.10. *Macdonald’s conjecture on the Mehta type integral associated with a finite Coxeter group is true.*

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