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On secant spaces to projective curves

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In this note we give an affirmative answer (in a stronger form) to a question raised in a recent paper ([1], problem 1.4). The main tool for the proof will be [1], th. 1.2. To state the question, we need to introduce a notation.

Let \( r, d, e, n \) be integers with \( 2 \leq n + 1 \leq r \) and \( e \geq n + 1 \), \( C \) a smooth complete connected curve and \( T \) a \( \Gamma g_2 \) on \( C \); let \( C^{(e)} \) be the symmetric product; set (as in [1]) \( V_{ne}(T) := \{ D \in C^{(e)} : D \text{ imposes at most } n + 1 \text{ conditions to } T \} \). Set

\[
t(r, n, e) := (n + 1 - e)(r - n) + e
\]

By a standard determinantal description, every irreducible component of \( V_{ne}(T) \) has dimension at least \( t(r, n, e) \) (see e.g. [1], §1).

The question will be answered proving (over any algebraically closed base field) the following theorem 0.1.

**THEOREM 0.1.** Fix integers \( r, d, e, n \), with \( e \geq n + 1, r > n \geq 2, d \geq 2e - 1 \). Let \( C \) be a smooth complete connected curve and \( \Gamma g_2 \) on \( C \) which is not a complete linear system. Assume

\[
(n + 1 - e)(r - n) + e \geq 0
\]

Then \( V_{ne}(\Gamma) \) is not empty.

Note that the bound "\( t(r, n, e) \geq 0 \)" required in (2) for non-complete linear systems is weaker than the bound required in [1], th. 1.2, in the case of complete linear systems, and that it is "sharp". The condition "\( d \geq 2e - 1 \)" will be used only to apply the statement of [1], th. 1.2; hence any improvement of [1], th. 1.2, related to this condition should give a corresponding improvement of 0.1 (see the related discussion in [1], 1.3). The fact that the bound (2) in 0.1 is better than the one in [1], th. 1.2, occurs essentially for numerical reasons. Indeed theorem 0.1 will be proven by a reduction to the case proven in [1], th. 1.2.
A few words on the case of positive characteristic. Theorem 1.2 of [1] was claimed only in characteristic 0. However that proof (and in particular [1], lemma 1.2.1) works without changes in positive characteristic if all the references used there are justified (mainly [2]). The key tool for [1], lemma 1.2.1, and for [2] is a section of [3] which works in any characteristic (and this was explicitly remarked in [3], Remark 2.8).

Proof of 0.1. We may easily reduce to the case (which will be assumed from now on) that $\Gamma$ has no base point. Since $\Gamma$ is not complete, it corresponds to a hyperplane of a $g^r_{d+1}$, $\Phi$ (or, in geometric language, “the $\mathbb{P}^r$ corresponding to $\Gamma$ is seen as the projection from a point $u$ of the $\mathbb{P}^{r+1}$ corresponding to $\Phi$”). If $V^e_{a+1}(\Phi) \neq \emptyset$, then it is obvious that $V^a_\infty(\Gamma) \neq \emptyset$. Hence we may assume that $V^e_{a}(\Phi) = \emptyset$. This implies that if we take any degree $e$ effective divisor $Z$ of $C$, there is at most one $(n + 1)$-dimensional subspace of $\Phi$ containing it. Set $k := r - n$. Note that by (2) we have:

$$t(n + 1, r + 1, e) = t(n, r, e) + k \geq k \quad (3)$$

If $\Phi$ is complete, by (2) we may apply [1], th. 1.2, and find $V^e_{a+1}(\Phi) \neq \emptyset$. If $\Phi$ is not complete, we may work by induction on the codimension of $\Gamma$ in the complete linear system $|\Gamma|$; in both cases we may assume $V^e_{a+1}(\Phi) \neq \emptyset$. As remarked in [1], by the determinental description of $V^e_{n+1}(\Phi)$ every irreducible component, $T$, of $V^e_{n+1}(\Phi)_{\text{red}}$ has dimension at least $t(n + 1, r + 1, e) \geq k$ by (2). Fix any such $T$ and let $S$ be the “complete integral subvariety of $\mathbb{P}^{r+1}$ which is the union of all $(n + 1)$-dimensional linear spaces parametrized by $T$”; $S$ is complete because $T$ is complete. It is sufficient to check that $u \in S$. Hence we may assume by contradiction $S \neq \mathbb{P}^{r+1}$, i.e. $\dim(S) \leq n + k$. Since $\dim(S) < n + k$, using a suitable incidence variety and counting dimensions we see that for a general $x \in S$ there is at least a 1-dimensional family, $T(x)$, of elements of $T$. Take as $x$ a smooth point of $S$. Every $L \in T(x)$ is contained in the Zariski tangent space $T_xS$, i.e. in a fixed hyperplane. Since by definition of linear system the image of $C$ by the map corresponding to $\Phi$ spans $\mathbb{P}^{r+1}$, we see that the union of the effective divisors contained in these linear spaces is supported by at most $d$ points of $C$. Since a set with $d$ elements has finitely many subsets, the contradiction comes from the assumption “$V^e_{a}(\Phi) = \emptyset$”, i.e. by the fact that every $L \in T$ is uniquely determined by the degree $e$ effective divisor of $C$ contained in $L$.

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References

